An observation on certain point-line configurations in classical planes

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Abstract

We define a cotangency set (in the projective plane over any field) to be a set of points that satisfy two conditions (A) and (B). The main result says that a cotangency set can never contain a quadrangle. A number of profound-sounding consequences involving Hermitian curves are really observations that follow quickly from the theorem by way of elementary arguments.

Let \( \Pi = \text{PG}(2, \mathbb{F}) \), the projective plane over the (commutative) field \( \mathbb{F} \). Let \( \mathcal{S} \) be a set of points in \( \Pi \) and assume that there is a 1-1 (injective) mapping \( f \) from \( \mathcal{S} \) into the set of lines of \( \Pi \) satisfying the two properties,

(A) if \( P \) is in \( \mathcal{S} \) then \( f(P) \) does not contain \( P \);

(B) if \( P_1, P_2 \) are distinct points of \( \mathcal{S} \) then the points \( P_1, P_2 \), and the intersection \( f(P_1) \cap f(P_2) \) of \( f(P_1) \) with \( f(P_2) \) lie on a line.

We refer to \( \mathcal{S} \) as a cotangency set, for reasons that will become apparent from the applications. Our main observation is the following theorem.

**Theorem 1.** If \( \mathcal{S} \) is any cotangency set then it cannot contain a quadrangle.

**Proof.** If all points of \( \mathcal{S} \) lie on a line there is nothing to prove, so let \( P, Q, R \) be any triangle in \( \mathcal{S} \). From axioms (A) and (B) for a cotangency set it follows that
94 A.A. Bruen, J.C. Fisher

\( f(P) \) cannot contain a point of \( \mathcal{S} \); consequently we can set \( f(P) = \ell_\infty \), the line at infinity. We turn now to a method we learned from Descartes. Set \( P = (0,0) \), \( Q = (0,1) \), and \( R = (1,0) \). From axiom (B), \( f(Q) \) is the line \( x = \alpha \) and \( f(R) \) is the line \( y = \beta \) for some nonzero \( \alpha, \beta \in \mathbb{F} \). Since \( Q, R, \) and \( f(Q) \cap f(R) \) are collinear we deduce

\[ \alpha + \beta = 1. \]  

(1)

Let \( W \) be any point of \( \mathcal{S} \) not on \( PQ \) or \( PR \). We shall show that \( W \in QR \). We have already observed that \( W \) is not on \( f(P) \) so \( W = (u, v) \) with \( u, v \neq 0 \). Since \( P, W, \) and \( f(P) \cap f(W) \) are collinear we get

\[ f(W) \text{ is the line } y = mx + b, \quad m = v/u. \]  

(2)

Similarly, \( Q, W, \) and \( f(Q) \cap f(W) \) are collinear; i.e., \( (0,1), (u, v) \) and \( (\alpha, am + b) \) are collinear, from which we obtain

\[ b = 1 - \alpha/u. \]  

(3)

Finally, \( (1,0), (u, v) \), and the intersection of \( y = \beta \) with \( y = mx + b \) (i.e., \( R, W, \) and \( f(R) \cap f(W) \)) are collinear. With (1), (2), and (3) this gives \( u + v = 1 \), which implies that \( W \) is on the line joining \( Q \) and \( R \) as desired. \( \square \)

We now look at some applications. The correlations of the plane provide familiar examples for our mappings \( f \): a correlation is an incidence-preserving mapping defined on the points and lines of \( \Pi \) that takes each point of the plane to a line and vice versa. (Note, in contrast, that Theorem 1 is concerned only with how \( f \) acts on a subset of the points of \( \Pi \)—no conditions are placed on the lines or on the remaining points.) A point or line is called absolute if it is incident with its image under the correlation. Recall that the absolute points of a correlation could form any type of conic (degenerate or not) or (should the underlying field admit nontrivial automorphisms) a Hermitian curve or even some more general object; the duals of these point sets are the possible sets of absolute lines. A polarity is a correlation of period two (which is equivalent to \( P \in f(Q) \) implies \( Q \in f(P) \)), so that its absolute points and lines form a self-dual configuration. See [2, § IV.3] for details. The axioms for cotangency sets become, in the language of correlations:

When \( f \) is a correlation then \( \mathcal{S} \) is a cotangency set if (A) no point of \( \mathcal{S} \) lies on its image under \( f \) and (B) any two points of \( \mathcal{S} \) lie on an absolute line.

Whenever the field \( \mathbb{F} \) admits an involutory automorphism \( \sigma \) there exist Hermitian curves. A Hermitian curve, or hyperconic as it was formerly called [4, § V.3], can be defined as a set of points that is projectively equivalent to the zeros of \( axx'' + byy'' + czz'' = 0 \), where \( a, b, c \) are nonzero elements of the subfield of \( \mathbb{F} \) fixed by \( \sigma \). Its points and tangents are the absolute points and lines of a Hermitian polarity. Let \( \mathcal{H} \) be a Hermitian curve. The tangents to \( \mathcal{H} \) from an
exterior point $P$ meet $\mathcal{H}$ in the points of a Baer subline of the polar of $P$ (which is the line to which $P$ is mapped by the polarity). From Theorem 1 we obtain the following corollary.

**Corollary 2** (cf. [6, p. 507]). Let $\mathcal{H}$ be a Hermitian curve in $\Pi = \text{PG}(2, \mathbb{F})$ and $\mathcal{F}$ be any set of points in $\Pi \setminus \mathcal{H}$. Assume that the line joining any 2 points of $\mathcal{F}$ is a tangent to $\mathcal{H}$. Then $\mathcal{F}$ cannot contain a quadrangle.

**Proof.** For $P$ in $\mathcal{F}$ let $f(P)$ be its polar line (where $f$ is the polarity defined by $\mathcal{H}$). Since $P$ is not on $\mathcal{H}$, axiom (A) for a cotangency set is satisfied. For $P_1 \neq P_2$ in $\mathcal{F}$ set $P_3 = P_1 P_2 \cap \mathcal{H}$ (the point of tangency). From the properties of the polarity, $f(P_1)$ and $f(P_2)$ both pass through $P_3$. Axiom (B) for a cotangency set then follows as does the corollary. □

**Remark.** For the case of a finite field ($\mathbb{F} = \text{GF}(q^2)$), Corollary 2 follows by duality from a result of O'Nan [6, p. 507].

**Corollary 3.** Let $\mathcal{H}$ be a Hermitian curve in $\Pi = \text{PG}(2, q^2)$ and $\mathcal{F}$ be a set of points in $\Pi \setminus \mathcal{H}$ that is maximal with respect to the property that the line joining any two points of $\mathcal{F}$ is a tangent to $\mathcal{H}$. Then there are two possibilities.

(a) $\mathcal{F}$ is the set of $q^2$ points off $\mathcal{H}$ that lie on a line tangent to $\mathcal{H}$.

(b) $\mathcal{F}$ consists of $q + 2$ points off $\mathcal{H}$, of which $q + 1$ form a Baer subline of a line $\ell$ that is tangent to $\mathcal{H}$, and the other point is off $\ell$.

**Corollary 4** (cf. [5]). Let $\mathcal{H}$ be a hermitian curve in $\Pi = \text{PG}(2, q^2)$ with $q \geq 3$ and $\mathcal{F}$ be a set of $q^2$ points in $\Pi \setminus \mathcal{H}$ such that the line joining any two points of $\mathcal{F}$ is tangent to $\mathcal{H}$. Then the points of $\mathcal{F}$ all lie on the same tangent to $\mathcal{H}$.

**Remarks.** The condition that $q \geq 3$ in Corollary 4 is required because when $q = 2$, $q + 2 = q^2$ and Case (b) of the preceding corollary would serve as a counterexample. Corollary 4 is the main result of Faina in [5]; we are grateful to G. Korchmaros for pointing this out to us. Of course there exist unitals in $\text{PG}(2, q^2)$ which are not Hermitian curves. For any unital $\mathcal{U}$, through any exterior point there are $q + 1$ tangents to $\mathcal{U}$; we shall label their points of tangency $T_1, \ldots, T_{q+1}$. Often the $T_i$ are collinear. If for every point not on $\mathcal{U}$ the corresponding $T_i$ are collinear, then the unital is a Hermitian curve [7]; on the other hand, all known unitals have many such exterior points. Consequently, even though Corollary 2 would not hold were the Hermitian curve $\mathcal{H}$ replaced by an arbitrary unital $\mathcal{U}$ (there are many counterexamples), it is true that if $\mathcal{U}$ is any unital of $\Pi = \text{PG}(2, \mathbb{F})$ and $\mathcal{F}$ is a set of points in $\Pi \setminus \mathcal{U}$ for which (a) the tangents from any point of $\mathcal{F}$ all meet $\mathcal{U}$ in collinear points and (b) the join of any pair of points of $\mathcal{F}$ is a tangent to $\mathcal{U}$, then $\mathcal{F}$ cannot contain a quadrangle. (The proof simply requires that the mapping $f$ be defined on $\mathcal{F}$, and that is guaranteed by (a)).
We cannot resist one further corollary, even though a proof without recourse to Theorem 1 is quite simple. It follows from Theorem 1 just as Corollary 2 did; we need only replace $\mathcal{H}$ by the conic $\mathcal{C}$ and interpret $f$ as the polarity induced by it.

**Corollary 5.** Let $q$ be odd and $\mathcal{I}$ be a set of points in $\text{PG}(2, q)$ that is disjoint from a conic $\mathcal{C}$; assume that the line joining any two points of $\mathcal{I}$ is a tangent to $\mathcal{C}$. Then $\mathcal{I}$ consists of the vertices of a triangle circumscribed about $\mathcal{C}$, or its points on a line tangent to $\mathcal{C}$.

We conclude with a related open problem. A recent paper of Blokhuis et al. [3] shows that a unital in $\text{PG}(2, q^2)$ is Hermitian if and only if it is in the code generated by the lines of $\text{PG}(2, q^2)$, confirming a conjecture of Assmus and Key [1, p. 61. Pursuing this idea further, let $\mathcal{H}$ be a Hermitian curve in $\text{PG}(2, q^2)$. For each point of $\mathcal{H}$ there is a tangent line, and in the code these are linearly independent. Next let $\mathcal{I}$ be a set of points disjoint from $\mathcal{H}$ with the following properties.

1. The line joining any two points of $\mathcal{I}$ is a secant to $\mathcal{H}$.
2. No point of $\mathcal{I}$ lies on the polar of a point of $\mathcal{I}$.

For each point $P$ of $\mathcal{I}$ we shall construct a codeword $w$ whose support contains $P$ but no other point of $\mathcal{I} \cup \mathcal{H}$. This implies that the points of $\mathcal{I} \cup \mathcal{H}$ are linearly independent. The codeword $w$ is constructed as follows. Let the tangents from $P$ meet $\mathcal{H}$ in the $q + 1$ collinear points $T_i$. Then $w$ is the sum of the $q + 1$ lines $PT_1, \ldots, PT_{q+1}$ minus the line $f(P)$, where, as before, $f(P)$ is the polar line of $P$ (which contains the $T_i$).

If $\mathcal{I}$ is composed of half the points of $\ell \setminus \mathcal{H}$, where $\ell$ is a secant to $\mathcal{H}$ and the points are chosen to satisfy (2), then $\mathcal{I}$ satisfies conditions (1) and (2) and $|\mathcal{I}| = (q^2 - q)/2$.

**Question.** Can one do any better? (For $q = 2$ the answer is no, since the dimension of the code is 10.)

**References**