Oscillation Criteria for Second Order Nonlinear Perturbed Differential Equations

CHEH-CHIH YEH

Department of Mathematics, Central University,
Chung-Li, Taiwan, Republic of China

Submitted by George Adomian
Received August 10, 1987

Some criteria are obtained which guarantee that all solutions of the second order nonlinear perturbed differential equations

\[(r(t) y'(t))'' + p(t) y'(t) + Q(t, y(t)) = H(t, y(t), y'(t))\]

are oscillatory. © 1989 Academic Press, Inc.

A regular (infinitely continuable) function which is defined for all large \(t\) is called oscillatory if it has no last zero, otherwise it is called non-oscillatory.

Consider the second order linear differential equations

\[y''(t) + p(t) y(t) = 0 \quad (E_1)\]

with a sign variable coefficient \(p(t) \in C_{[t_0, \infty)}\). Kamenev [8] proved that every regular solution of (\(E_1\)) is oscillatory if

\[
\limsup_{t \to \infty} t^{1-n} \int_{t_0}^{t} (t-s)^{n-1} p(s) \, ds = \infty, \text{ for some integer } n \geq 3
\]

which generalized the well-known oscillation theorem of Wintner [13]. Recently numerous studies addressing several areas of this subject have appeared. For general interest we refer to Grace and Lalli [2–5], Grace, Lalli, and Yeh [6], Philos [9, 10], Singh [11, 12], Wong [14, 15], Yan [16, 17], and Yeh [18, 19].

In 1980, Grace and Lalli [1] discussed the oscillatory property of solutions of the perturbed nonlinear differential equation

\[(r(t) y'(t))' + M(t, y(t)) = F(t, y(t), y'(t)) \quad (E_2)\]
and proved the following interesting result:

**Theorem A.** Let

(a) \( r(t) \in C(J = [t_0, \infty), (0, \infty)) \);
(b) \( M \in C(J \times R, R = (-\infty, \infty)) \) and \( F \in C(J \times R^2, R) \) satisfy

\[
M(t, y) \geq m(t) f_1(y), \quad F(t, y, y') \leq b(t) g(y') f_2(y)
\]

for \( y \neq 0 \), where \( m \in C(J, R) \), \( b \in C(J, R^+ = [0, \infty)) \), \( g, f_i \in R(R, R) \), \( i = 1, 2 \), with \( xf_i(x) > 0 \) for \( i = 1, 2 \), \( f_2(x) \leq f_1(x) \), \( f_1'(x) \geq 0 \) for all \( x \neq 0 \), and \( 0 < g(x) \leq c \) for some constant \( c \);
(c) \( \int^{\infty} (1/r(s)) \, ds = \infty \);
(d) \( \int^{\infty} (m(s) - cb(s)) \, ds = \infty \).

Then all regular solutions of (E₂) are oscillatory.

Theorem A contains Theorem 1 of [7] and Theorem 6(i) of [15].

The purpose of this paper is to establish some sufficient conditions under which all regular solutions of the second order nonlinear perturbed differential equations

\[
(r(t) y'(t))' + p(t) y'(t) + Q(t, y(t)) = H(t, y(t), y'(t)) \quad (E)
\]

are oscillatory by using Kamenev's technique.

In the sequel we assume that

(A) \( r, p \in C(\mathcal{I} = [T, \infty), R) \), \( r(t) > 0 \) for all \( t \geq T > 0 \);
(B) \( Q \in C(I \times R, R) \) and \( H \in C(I \times R^2, R) \) satisfy

\[
yQ(t, y) \geq yq(t) f_1(y) \quad \text{and} \quad yH(t, y, y') \leq yh(t) g(y') f_2(y)
\]

for all \( y \neq 0 \), where \( q, h \in C(I, R) \) and \( g, f_1, f_2 \in C(R, R) \) such that

(i) \( yf_1(y) > 0 \) and \( yf_2(y) \geq 0 \) for \( y \neq 0 \);
(ii) \( h(t) > 0 \) for \( t \geq T \);
(iii) \( 0 < g(y') \leq c \) for some constant \( c \).

**Theorem 1.** Let

(C₁) \( f_1'(y) \geq k > 0 \) and \( f_2(y)/f_1(y) \leq K \) for \( y \neq 0 \), where \( K \geq 0 \).
If there exists a function $\rho \in C^1(I, (0, \infty))$ such that

\[ (C_2) \quad \limsup_{t \to \infty} t^{-2} \int_T^t (t-s)^{-2} \rho(s) \left\{ (t-s)^2 (q(s) - cK\rho(s)) \right. \]
\[ - \frac{1}{4\kappa} \int_0^r r(s) \left( (t-s) \left( \frac{p(s)}{r(s)} - \frac{\rho'(s)}{\rho(s)} \right) + \alpha \right)^2 \] 
\[ \left. \right\} ds = \infty \]

for some $\alpha > 0$, then every regular solution of (E) is oscillatory.

Proof. Let $y(t)$ be a nonoscillatory solution of (E) for which, without loss of generality, we may assume that $y(t) \neq 0$ for $t \geq T$. Define

$$ w(t) := \frac{r(t) y'(t)}{f_i(y(t))} \rho(t). $$

Then $w(t)$ satisfies

$$ w'(t) - \frac{\rho(t)}{f_i(y)} H(t, y, y') + \frac{p(t)}{r(t)} w(t) + \frac{p(t)}{f_i(y)} Q(t, y) $$
\[ - \frac{p'(t)}{\rho(t)} w(t) + \frac{f_i'(t)}{\rho(t) r(t)} w^2(t) = 0. \]

This and (B), (C1) imply

$$ w'(t) + \left( \frac{p(t)}{r(t)} - \frac{\rho'(t)}{\rho(t)} \right) w(t) + \rho(t)(q(t) - cK\rho(t)) $$
\[ + \frac{k}{\rho(t) r(t)} w^2(t) \leq 0. \]

Thus

$$ \int_T^t (t-s)^2 w'(s) ds + \int_T^t (t-s)^2 \left[ \frac{k}{\rho(s) r(s)} w^2(s) + \left( \frac{p(s)}{r(s)} - \frac{\rho'(s)}{\rho(s)} \right) w(s) \right] ds $$
\[ + \int_T^t (t-s)^2 \rho(s)(q(s) - cK\rho(s)) ds \leq 0. \]

Since

$$ \int_T^t (t-s)^2 w'(s) ds = \alpha \int_T^t (t-s)^{2-1} w(s) ds - w(T)(t-T)^2, $$
we obtain

\[ t^{-2} \int_{T}^{t} (t-s)^2 \rho(s)(q(s) - cKh(s)) \, ds \]

\[ \leq w(T) \left( \frac{t-T}{t} \right)^{2} - t^{-2} \int_{T}^{t} \left( \frac{k(t-s)^{2}}{\rho(s) r(s)} \right) w^{2}(s) \]

\[ + \left[ (t-s)^{2} \left( \frac{p(s)}{r(s)} - \frac{\rho'(s)}{\rho(s)} \right) + \alpha(t-s)^{2-1} \right] w(s) \, ds \]

\[ = w(T) \left( \frac{t-T}{t} \right)^{2} + \frac{1}{4kt^{2}} \int_{T}^{t} \left[ (t-s)^{2} \left( \frac{p(s)}{r(s)} - \frac{\rho'(s)}{\rho(s)} \right) + \alpha \right]^{2} \]

\[ \times \rho(s)(t-s)^{2-2} \, ds - t^{-2} \int_{T}^{t} \left[ \left( \frac{k(t-s)^{2}}{\rho(s) r(s)} \right)^{1/2} w(s) \right] \]

\[ + \frac{(t-s)(\rho(s)/r(s) - \rho'(s)/\rho(s)) + \alpha}{2(k/\rho(s) r(s))^{1/2}} (t-s)^{(1/2)(2-2)} \, ds. \]

Hence

\[ t^{-2} \int_{T}^{t} (t-s)^{2-2} \rho(s) \left\{ (t-s)^{2} (q(s) + cKh(s)) \right. \]

\[ - (4k)^{-1} r(s) \left. \left[ (t-s) \left( \frac{p(s)}{r(s)} - \frac{\rho'(s)}{\rho(s)} \right) + \alpha \right]^{2} \right) \, ds \]

\[ \leq w(T) \left( \frac{t-T}{t} \right)^{2} \to w(T) \]

as \( t \to \infty \), which contradicts Condition (C_{3}). Thus our proof is complete.

**Corollary 2.** If the condition (C_{2}) is replaced by

\[ \limsup_{t \to \infty} t^{-2} \int_{T}^{t} (t-s)^{2} \rho(s)(q(s) - cKh(s)) \, ds = \infty \]

and

\[ (C_{3}) \quad \lim_{t \to \infty} t^{-2} \int_{T}^{t} \rho(s)(t-s)^{2-2} \left[ (t-s) \left( \frac{p(s)}{r(s)} - \frac{\rho'(s)}{\rho(s)} \right) + \alpha \right]^{2} \, ds < \infty, \]

then every regular solution of (E) is oscillatory.
THEOREM 3. Let the condition \((C_1)\) hold. If there exist a positive function 
\(\rho \in C^1(I)\) and a positive constant \(\alpha\) such that

\[
(C_4) \quad \limsup_{t \to \infty} t^{-\alpha} \int_T^t \left\{(t-s)^2 \rho(s)(q(s) - cKh(s)) \right. \\
- (4k)^{-1} \left[ (t-s) \frac{p(s) p(s)}{r(s)} + \alpha p(s) - (t-s) \rho'(s) \right]^2 \\
\times (t-s)^{\gamma - 1} \frac{r(s)}{\rho(s)} \right\} ds = \infty,
\]

then every regular solution of (E) is oscillatory.

Proof. Let \(y(t)\) be a nonoscillatory solution of (E) for which, without loss of generality, we may assume that \(y(t) \neq 0\) for \(t \geq T\). Define

\[
w(t) := \frac{r(t) y'(t)}{f_i(y(t))}.
\]

Then \(w(t)\) satisfies

\[
w'(t) = \frac{H}{f_i(y(t))} - \frac{p(t)}{r(t)} w(t) \\
- \frac{Q(t, y(t))}{f_i(y(t))} \left[ \frac{f_i(y(t))}{r(t)} \right] w^2(t).
\]

This and (B), \((C_1)\) imply

\[
w'(t) + \frac{p(t)}{r(t)} w(t) + q(t) - cKh(t) + \frac{k}{r(t)} w^2(t) \leq 0.
\]

Hence

\[
\int_T^t (t-s)^2 \rho(s) w'(s) ds + \int_T^t (t-s)^2 \rho(s) \frac{p(s)}{r(s)} w(s) ds \\
+ k \int_T^t (t-s)^{\gamma - 1} \frac{p(s)}{r(s)} w^2(s) ds \\
+ \int_T^t (t-s)^2 \rho(s)(q(s) - cKh(s)) ds \leq 0.
\]
Since
\[
\int_{T}^{t} (t-s)^{2} \rho(s) w'(s) \, ds = \alpha \int_{T}^{t} (t-s)^{r-1} \rho(s) w(s) \, ds
\]
\[
- \int_{T}^{t} (t-s)^{2} \rho'(s) w(s) \, ds
\]
\[
- w(T)(t-T)^{a} \rho(T),
\]
we get
\[
\int_{T}^{t} (t-s)^{2} \rho(s) (q(s) - cKh(s)) \, ds
\]
\[
\leq (t-T)^{2} \rho(T) w(T) - \int_{T}^{t} k(t-s)^{2} \rho(s) w^{2}(s) \, ds
\]
\[
- \int_{T}^{t} (t-s)^{r-1} w(s) \left[ (t-s) \frac{\rho(s) p(s)}{r(s)} \right] \, ds.
\]
This implies
\[
t^{-a} \int_{T}^{t} \left\{ (t-s)^{2} \rho(s) (q(s) - cKh(s)) - (4k)^{-1}
\]
\[
\times \left[ (t-s) \frac{\rho(s) p(s)}{r(s)} + \alpha \rho(s) - (t-s) \rho'(s) \right]^{2} (t-s)^{r-2} \frac{r(s)}{\rho(s)} \right\} \, ds
\]
\[
\leq t^{-2} (t-T)^{2} \rho(T) w(T) - t^{-a}
\]
\[
\times \int_{T}^{t} \left\{ (t-s)^{(1/2)x} \left( \frac{kp(s)}{r(s)} \right)^{1/2} \right\} \, ds + \frac{1}{2} k^{-1/2}
\]
\[
\times \left[ (t-s) \frac{\rho(s) p(s)}{r(s)} + \alpha \rho(s) - (t-s) \rho'(s) \right]
\]
\[
\times (t-s)^{(1/2)(\alpha-2)} \left( \frac{r(s)}{\rho(s)} \right)^{1/2} \right\}^{2} \, ds
\]
\[
\leq t^{-a} (t-T)^{2} \rho(T) w(T).
\]
Taking the upper limit as \( t \to \infty \) and using (C₄), we obtain a contradiction. This contradiction proves our theorem.
COROLLARY 4. If the condition \((C_4)\) in Theorem 3 is replaced by

\[
\lim_{t \to \infty} \sup t^{-2} \int_{T}^{t} (t-s)^{2} \rho(s)(q(s) - cKh(s)) \, ds = \infty
\]

and

\[
\lim_{t \to \infty} t^{-2} \int_{T}^{t} \left[ \frac{\rho(s) p(s)}{r(s)} + \alpha \rho(s) - (t-s) \rho'(s) \right]^{2} \times (t-s)^{2-\frac{r(s)}{\rho(s)}} \, ds < \infty,
\]

then every regular solution of \((E)\) is oscillatory.

THEOREM 5. Let \(q(t) \geq 0, f_{1}(y)/y \geq k_{1} > 0, \) and \(f_{2}(y)/y \leq k_{2} \) for \(y \neq 0,\) where \(k_{1}\) and \(k_{2}\) are constants. If there exists a positive function \(\rho \in C^{1}(I)\) such that

\[
(C_{5}) \quad \lim_{t \to \infty} \sup t^{-2} \int_{T}^{t} \frac{\rho(s)(t-s)^{2} - 2}{4} \left\{ (t-s)^{2} (k_{1} q(s) - c k_{2} h(s)) \right. \\
- \frac{r(s)}{4} \left[ (t-s) \left( \frac{p(s)}{r(s)} - \frac{\rho'(s)}{\rho(s)} \right) + \alpha \right]^{2} \right. \left. \right\} \, ds = \infty,
\]

for some \(\alpha > 0,\) then every regular solution of \((E)\) is oscillatory.

Proof. Assume that \(y(t)\) is a nonoscillatory solution of \((E)\) such that \(y(t) \neq 0 \) for \(t \geq T.\) Define

\[
w(t) := \frac{\rho(t) r(t) y'(t)}{y(t)}
\]

Then

\[
w'(t) - \frac{\rho(t)}{y(t)} H(t, y(t), y'(t)) + p(t) \frac{\rho(t) y'(t)}{y(t)}
+ \frac{\rho(t)}{y(t)} Q(t, y(t)) - \frac{\rho'(t)}{\rho(t)} w(t) + \frac{w^{2}(t)}{\rho(t) r(t)} = 0
\]

which implies

\[
w'(t) + \frac{w^{2}(t)}{\rho(t) r(t)} + \left( \frac{\rho(t)}{r(t)} - \frac{\rho'(t)}{\rho(t)} \right) w(t)
+ \rho(t)(k_{1} q(t) - c k_{2} h(t)) \leq 0.
\]
As in the proof of Theorem 1, we can prove our Theorem 5; hence, we omit the details.

**Corollary 6.** If Condition \((C_5)\) in Theorem 5 is replaced by \((C_3)\) and

\[
(C_6) \quad \limsup_{t \to \infty} t^{-\alpha} \int_{t}^{\infty} \rho(s)(t-s)^{\alpha} (k_1 q(s) - ck_2 h(s)) \, ds = \infty,
\]

then every regular solution of \((E)\) is oscillatory.

**Remark 1.** Theorem A requires the condition

\[
\int_{0}^{\infty} \frac{1}{r(t)} \, dt = \infty,
\]

but it is not required in our results.

**Remark 2.** Our results also extend some of the results in [3, 5, 6, 8, 17-19].

**References**