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Fourth-order difference equation satisfied by the co-recursive of q-classical orthogonal polynomials

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Abstract

We derive the fourth-order q-difference equation satisfied by the co-recursive of q-classical orthogonal polynomials. The coefficients of this equation are given in terms of the polynomials ϕ and ψ appearing in the q-Pearson difference equation $\mathscr{D}_q(\phi\rho) = \psi\rho$ defining the weight ρ of the q-classical orthogonal polynomials inside the q-Hahn tableau. Use of suitable change of variable and limit processes allow us to recover the results known for the co-recursive of the classical continuous and classical discrete orthogonal polynomials. Moreover, we describe particular situations for which the co-recursive of classical orthogonal polynomials are still classical and express these new families in terms of the starting ones. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let $(P_n)_n$ be the monic orthogonal family defined by the three-term recurrence relation

$$P_{n+1} = (x - \beta_n)P_n - \gamma_n P_{n-1}, \ n \geqslant 1, \ P_0 = 1, \ P_1 = x - \beta_0, \tag{1}$$

where β_n and γ_n are complex numbers with $\gamma_n \neq 0$ and $\gamma_0 \equiv 1$. The rth associated $(P_n^{(r)})_n$ and the co-recursive [5] $(P_n^{[\mu]})_n$ of P_n are the monic polynomial families obtained by modifying relation (1),

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and defined, respectively, by the relations

$$P_{n+1}^{(r)} = (x - \beta_{n+r})P_n^{(r)} - \gamma_{n+r}P_{n-1}^{(r)}, \quad n \geqslant 1, \quad P_0^{(r)} = 1,\ P_1^{(r)} = x - \beta_r, \tag{2}$$

$$P_{n+1}^{[\mu]} = (x - \beta_n)P_n^{[\mu]} - \gamma_n P_{n-1}^{[\mu]}, \quad n \geqslant 1, \quad P_0^{[\mu]} = 1,\ P_1^{[\mu]} = x - \beta_0 - \mu, \tag{3}$$

where μ, in general, is a complex number.

The family $(P_n^{[\mu]})_n$ which is orthogonal by Favard's theorem [6], belongs in general to the Laguerre–Hahn class [3,8–10,15,22] and therefore any polynomial $P_n^{[\mu]}$ satisfies a fourth-order linear differential or difference equation.

The fourth-order difference equation satisfied by $P_n^{[\mu]}$ was given in [25,28] for classical continuous orthogonal polynomials and for classical discrete orthogonal polynomials in [19,26].

In this work, we use relations between the P_n, $P_n^{[\mu]}$, $P_n^{(1)}$ and the fourth-order q-difference equation satisfied by the associated orthogonal polynomials of the Laguerre–Hahn class [10,12,13] to derive the factorized form of the fourth-order q-difference equation satisfied by the co-recursive of *all* q-classical orthogonal polynomials.

Moreover, we use suitable change of variable and formal limit processes to deduce the fourth-order difference equation (resp. differential) equation for the co-recursive of *all* classical orthogonal polynomials of a discrete and continuous variables, respectively. We also use these difference, q-difference or differential equations to prove that under certain conditions, the co-recursive of Jacobi, Hahn, little q-Jacobi and big q-Jacobi polynomials are still classical and express the new polynomials families in terms of the starting ones. q-classical orthogonal polynomials involved in this work belong to the q-Hahn class introduced by Hahn [16]. They are represented by the basic hypergeometric series appearing at the level ${}_3\phi_2$ and not at the level ${}_4\phi_3$ of the Askey–Wilson orthogonal polynomials [14,17].

The difference or differential equation obtained in the framework of this paper can be used, for instance, to solve connection coefficients and linearization problems [2,20,21,26], to represent finite modifications inside the Jacobi matrices of the classical starting family [27] and also to prove the existence of the classical orthogonal families for which the co-recursive are still classical and express these families in terms of the starting ones.

The orthogonality weight ρ for q-classical orthogonal polynomials is defined by a Pearson-type q-difference equation

$$\mathscr{D}_q(\phi\rho) = \psi\rho, \tag{4}$$

where the q-difference operator $\mathscr{D}_q$ is defined [16] by

$$\mathscr{D}_q f(x) = \frac{f(qx) - f(x)}{(q-1)x}, \quad x \neq 0, \quad 0 < q < 1. \tag{5}$$

ϕ is a polynomial of degree at most two and ψ is polynomial of degree 1.

The monic polynomials $P_n(x;q)$, orthogonal with respect to ρ satisfy the second-order q-difference equation

$$[\phi(x)\mathscr{D}_q\mathscr{D}_{1/q} + \psi(x)\mathscr{D}_q + \lambda_{q,n}\mathscr{I}_d]y(x) = 0, \tag{6}$$

an equation which can be written in the q-shifted form

$$[(\phi_{(1)} + \psi_{(1)}t_{(1)})\mathscr{G}_q^2 - ((1+q)\phi_{(1)} + \psi_{(1)}t_{(1)} - \lambda_{q,n}t_{(1)}^2)\mathscr{G}_q + q\phi_{(1)}\mathscr{I}_d]y(x) = 0 \tag{7}$$

with

$$\lambda_{q,n} = -[n]_q \left\{ \psi' + [n-1]_{1/q} \frac{\phi''}{2q} \right\}, \quad [n]_q = \frac{1-q^n}{1-q},$$

$$\phi_{(i)} \equiv \phi(q^i x), \;\; \psi_{(i)} \equiv \psi(q^i x), \;\; t_{(i)} \equiv t(q^i x), \;\; t(x) = (q-1)x \tag{8}$$

and the geometric shift $\mathscr{G}_q$ defined by

$$\mathscr{G}_q^i f(x) = f(q^i x), \quad \mathscr{G}_q^0 \equiv \mathscr{I}_d (\equiv \text{identity operator}). \tag{9}$$

2. Fourth-order difference equation for the co-recursive q-classical orthogonal polynomials

2.1. Known materials

In the first step, we recall the main result we shall need for this work.

Theorem 1 (Foupouagnigni [10,12,13]). *Let $(P_n)_n$ be a polynomial family orthogonal with respect to the classical weight ρ satisfying $\mathscr{D}_q(\phi\rho) = \psi\rho$, where ϕ is a polynomial of degree at most two and ψ is a first degree polynomial. The first associated $P_{n-1}^{(1)}$ of P_{n-1} satisfies*

$$(\phi_{(1)} + \psi_{(1)} t_{(1)}) \mathscr{Q}^*_{2,n-1} [P_{n-1}^{(1)}(x;q)] = [e\mathscr{G}_q + f\mathscr{I}_d] P_n(x;q) \tag{10}$$

with

$$\mathscr{Q}^*_{2,n-1} = \phi_{(2)} \mathscr{G}_q^2 - ((1+q)\phi_{(1)} + \psi_{(1)} t_{(1)} - \lambda_{q,n} t_{(1)}^2) \mathscr{G}_q + q(\phi + \psi\, t) \mathscr{I}_d,$$

$$e = \left(\frac{\phi''}{2} - \psi' \right) ((1+q)\phi_{(1)} + \psi_{(1)} t_{(1)} - \lambda_{q,n} t_{(1)}^2) t_{(1)},$$

$$f = -\left(\frac{\phi''}{2} - \psi' \right) ((q+1)\phi_{(1)} + \psi_{(1)} t_{(1)}) t_{(1)}. \tag{11}$$

2.2. The q-difference equation

In the second step, we use Eqs. (10) and (11) taking into account the relation [7,8]

$$P_n^{[\mu]} = P_n - \mu P_{n-1}^{(1)} \tag{12}$$

and the fact that $P_n(x;q)$ satisfies Eq. (7). This give after some computations:

Theorem 2. *The co-recursive $P_n^{[\mu]}$ of the q-classical orthogonal polynomials P_n, orthogonal with respect to the q-classical weight ρ satisfying $\mathscr{D}_q(\phi\rho) = \psi\rho$, where ϕ is a polynomial of degree at most two and ψ a first degree polynomial satisfies*:

$$(\phi_{(1)} + \psi_{(1)} t_{(1)}) \mathscr{Q}^*_{2,n-1} [P_n^{[\mu]}(x;q)] = [\bar{e}_q \mathscr{G}_q + \bar{f}_q \mathscr{I}_d] P_n(x;q) \tag{13}$$

with

$$\bar{e}_q = \left(-\left(\frac{\phi''}{2} - \psi'\right)\mu t_{(1)} - \phi_{(1)} - \psi_{(1)}t_{(1)} + \phi_{(2)}\right)((1+q)\phi_{(1)} + \psi_{(1)}t_{(1)} - \lambda_{n,q}t_{(1)}^2),$$

$$\bar{f}_q = \left(\frac{\phi''}{2} - \psi'\right)\mu((q+1)\phi_{(1)} + \psi_{(1)}t_{(1)})t_{(1)} - q\phi_{(1)}\phi_{(2)} + q(\phi + \psi\, t)(\phi_{(1)} + \psi_{(1)}t_{(1)}). \tag{14}$$

Remark 3. Since $P_n^{[0]} = P_n$, the Eq. (13) taken for $\mu = 0$, coincide with (7).

In the third step we use Eq. (7) for $P_n(x;q)$, (10) and (11) to obtain after some computations with Maple V.4 [4] the operator $\mathcal{Q}_{2,n}^{[\mu]}$ annihilating the right-hand side of (9):

Theorem 4. *The co-recursive $P_n^{[\mu]}$ of the q-classical orthogonal polynomials P_n, orthogonal with respect to the weight ρ satisfying $\mathcal{D}_q(\phi\rho) = \psi\rho$ (degree of $\phi \leqslant 2$ and degree of $\psi = 1$) satisfies a fourth-order q-difference equation given in the factorized form as*

$$\mathcal{Q}_{2,n}^{[\mu]} \frac{\mathcal{Q}_{2,n-1}^{*}}{[t(qx)]^2} (P_n^{[\mu]}) = 0 \tag{15}$$

with

$$\mathcal{Q}_{2,n}^{[\mu]} = A_2(x)\mathcal{G}_q^2 + A_1(x)\mathcal{G}_q + A_0(x)\mathcal{I}_d, \tag{16}$$

where A_j, $j = 0,\ldots,2$ are polynomials of fixed degrees.

Proof. The proof is obtained by applying twice the operator $\mathcal{G}_q$ to Eq. (13) and using the second-order difference equation satisfied by P_n (see (7)). Notice that we have decided not to give the polynomials coefficients A_j, $j = 0,\ldots,2$ since they are space consuming. However, the operators $\mathcal{Q}_{2,n-1}^{*}$ and $\mathcal{Q}_{2,n}^{[\mu]}$ are given below for the discrete q-Hermite II case. □

2.3. Example

For the discrete q-Hermite II polynomials ($\phi(x) = 1, \psi(x) = x/(1-q)$), the operators $\mathcal{Q}_{2,n-1}^{*}$ and $\mathcal{Q}_{2,n}^{[\mu]}$ are given by

$$\mathcal{Q}_{2,n-1}^{*} = \mathcal{G}_q^2 + (\eta q^2 x^2 - q - 1)\mathcal{G}_q - q(x-1)(x+1)\mathcal{I}_d, \quad \mathcal{Q}_{2,n}^{[\mu]} = A_2(x)\mathcal{G}_q^2 + A_1(x)\mathcal{G}_q + A_0(x)\mathcal{I}_d$$

with the notation $\eta = q^n$ and

$$\begin{aligned}
A_2 = {} & -(q^3x-1)(q^3x+1)(-1 + q - x^2q^5\eta - x^2q^6\eta + xq^6\eta\mu \\
& + xq^5\eta\mu + 2xq^3\eta\mu + xq^2\eta\mu + 2xq^4\eta\mu - 2xq^2\mu - 2xq^5\mu - xq^3\mu \\
& - 2xq^4\mu + q^4\mu^2 + q^2\mu^2 + q^3\mu^2 + q\mu^2 + x^2q^6 - q\eta\mu^2 - q^3\eta\mu^2 - q^4\eta\mu^2 \\
& - q^2\eta\mu^2 + 2x^2q^4 + q^3 - q^4 - x^2q^4\eta - x^2q^3\eta + x^4q^7\eta^2 + xq\eta\mu \\
& - x^3q^5\eta^2\mu - x^3q^7\eta^2\mu + x^2q^5\eta^2\mu^2 - qx\mu - x^2q^5\mu^2 + x^3q^5\mu + x^3q^6\mu \\
& + x^2q^2 - q^6x^4),
\end{aligned}$$

$$\begin{aligned}A_1 = {} & q^{12}\eta(q\eta^2-1)x^6 - q^{10}\eta\mu(-q^2+\eta^2q^3+\eta^2-1)x^5\\ & + q^6(q^4\eta^3\mu^2 - \eta^2q^3 - q\eta^2 + q^4 + q^4\eta - q^5\eta^2 + 2q^2\eta - q^6\eta^2 + q^5 - q^2\eta^2\\ & - q^4\eta^2 - q^4\eta\mu^2 + \eta)x^4 + q^4\mu(q^6\eta^2 + 2\eta^2q^3 + 2q^4\eta^2 + q^2\eta^2 + q^8\eta^2\\ & - q^3\eta - 2q^2\eta - q^4 + q\eta^2 - q^6\eta - q^6 - q\eta + q^7\eta^2 + 2q^5\eta^2 + \eta^2 - q^7 - \eta\\ & - 2q^4\eta - q^5)x^3 - q^2(\eta + 2q^5 + 2q^2 + q^8\eta + q^2\eta^2\mu^2 + q^4\eta^2\mu^2 - q^2\eta\\ & - q^6\eta - 2q^4\eta - 2q^3\eta - 2q^5\eta + q^7\eta^2\mu^2 - q^7\eta + q^6\eta^2\mu^2 + q^5\eta^2\mu^2\\ & - q^5\eta\mu^2 - q^3\eta\mu^2 - q^4\eta\mu^2 - q^2\eta\mu^2 + q^3\eta^2\mu^2 + 2q^3 + 2q^4 - q^7\mu^2\\ & - q\eta - q^6\mu^2)x^2 - q\mu(q+1)(q^2+1)(q^4\eta - 2q^3 + 2q^2\eta - q - 1 + \eta)x\\ & + (q+1)(q^4 + q^4\eta\mu^2 - q^4\mu^2 - q^3\mu^2 + q^3\eta\mu^2 - q^3 - q^2\mu^2 + q^2\eta\mu^2\\ & - q\mu^2 + q\eta\mu^2 - q + 1),\end{aligned}$$

$$\begin{aligned}A_0 = {} & q(-1 + q - x^2q^5\eta - x^2q^6\eta + xq^6\eta\mu + 2xq^5\eta\mu + xq^3\eta\mu\\ & + xq^2\eta\mu + 2xq^4\eta\mu - x^4q^{10} - xq^2\mu - 2xq^5\mu - 2xq^3\mu - xq^4\mu\\ & + q^4\mu^2 + q^2\mu^2 - x^3q^8\eta^2\mu + x^2q^7\eta^2\mu^2 + x^4q^{11}\eta^2 - x^2q^7\eta + q^3\mu^2\\ & - x^3q^{10}\eta^2\mu - x^2q^8\eta + xq^7\eta\mu + q\mu^2 + 2x^2q^6 - q\eta\mu^2 - q^3\eta\mu^2\\ & - q^4\eta\mu^2 - q^2\eta\mu^2 + x^2q^4 + q^3 - q^4 + x^2q^8 - x^2q^7\mu^2 + x^3q^9\mu - 2xq^6\mu\\ & + x^3q^8\mu).\end{aligned}$$

2.4. Some applications on co-recursive classical orthogonal polynomials

2.4.1. q-classical orthogonal polynomials

For the little q-Jacobi polynomials and for the big q-Jacobi polynomials, the coefficients $\bar{e}_q$ and $\bar{f}_q$ (see (14)) vanishes under certain conditions. This implies that $P_n^{[\mu]}$ satisfies a second-order (instead of fourth-order) difference equation of hypergeometric type, and is therefore q-classical.

- For the little q-Jacobi polynomials $p_n(x;a,b|q)$ [2,17]

$$\phi(x) = x(x-1),\ \psi(x) = \frac{1-aq+(abq^2-1)x}{q-1}, \tag{17}$$

  $\bar{e}_q = \bar{f}_q = 0$ when $ab = 1$ and $\mu = (1-a)/(q-1)$.
- For the big q-Jacobi polynomials $P_n(x;a,b,c;q)$ [2,17]

$$\phi(x) = (x-qa)(x-qc),\ \psi(x) = \frac{cq + aq(1-(b+c)q) + (abq^2-1)x}{q-1}, \tag{18}$$

  $\bar{e}_q = \bar{f}_q = 0$ when $ab = 1$ and $\mu = (q(1-a)(c-1))/(q-1)$.
- Computations involving the coefficients β_n and γ_n (see (1) and (22)) generate the following relations:

Proposition 5. *The monic little q-Jacobi and the monic big q-Jacobi polynomials are related with their respective co-recursive ones by*

$$p_n^{[(1-a)/(q-1)]}\left(x;a,\frac{1}{a}|q\right)=a^n\,p_n\left(\frac{x}{a},\frac{1}{a},a|q\right), \tag{19}$$

$$P_n^{[(q(1-a)(c-1))/(q-1)]}\left(x;a,\frac{1}{a},c;q\right)=a^nP_n\left(\frac{x}{a},\frac{1}{a},a,c;q\right). \tag{20}$$

Proof. The proposition is proven using the three-term recurrence relation (3) satisfied by the co-recursive orthogonal polynomials taking into account the following relations:

$$\beta_0\left(a,\frac{1}{a}|q\right)+\frac{1-a}{q-1}=a\beta_0\left(\frac{1}{a},a|q\right),$$

$$\beta_n\left(a,\frac{1}{a}|q\right)=a\beta_n\left(\frac{1}{a},a|q\right),\ n\geqslant 1,$$

$$\gamma_n\left(a,\frac{1}{a}|q\right)=a^2\gamma_n\left(\frac{1}{a},a|q\right),\ n\geqslant 1,$$

$$\beta_0\left(a,\frac{1}{a},c;q\right)+\frac{q(1-a)(c-1)}{q-1}=a\beta_0\left(\frac{1}{a},a,c;q\right),$$

$$\beta_n\left(a,\frac{1}{a},c;q\right)=a\beta_n\left(\frac{1}{a},a,c;q\right),\ n\geqslant 1,$$

$$\gamma_n\left(a,\frac{1}{a},c;q\right)=a^2\gamma_n\left(\frac{1}{a},a,c;q\right),\ n\geqslant 1. \tag{21}$$

$\beta_n(a,b|q)$ and $\gamma_n(a,b|q)$ (resp. $\beta_n(a,b,c;q)$ and $\gamma_n(a,b,c;q)$) denote the coefficients of the three-term recurrence relation (see (1) satisfied by the little q-Jacobi (resp. big q-Jacobi) polynomials. The coefficients β_n and γ_n of the three-term recurrence relation satisfied by the monic q-classical polynomials family, orthogonal with respect to the weight ρ are given in terms of the polynomials ϕ and ψ appearing in Eq. (4) by [2,10,13]

$$\begin{aligned}\beta_n(q,\phi,\psi)=&-\eta((-(q+1)(-1+\eta)(-q+\eta)\phi_1-(q-1)(-\eta q^2+q-q\eta+\eta^2)\psi_0)\phi_2\\&-\eta(q-1)(q+1)(-1+\eta)\psi_1\phi_1-\eta^2(q-1)^2\psi_0\psi_1)/\\&((-1+\eta)(\eta+1)\phi_2+\eta^2(q-1)\psi_1)(-(-q+\eta)(q+\eta)\phi_2-\eta^2(q-1)\psi_1),\end{aligned} \tag{22}$$

$$\begin{aligned}\gamma_n(q,\phi,\psi)=&-(-1+\eta)((-\eta+q^2)\phi_2-(q-1)\eta\psi_1)((-q+\eta)^2(q+\eta)^2\phi_0\phi_2^2\\&+(-q\eta(-q+\eta)^2\phi_1^2-q\eta(q-1)(-q+\eta)^2\psi_0\phi_1\\&+2\eta^2(q-1)(-q+\eta)(q+\eta)\psi_1\phi_0+q^2\eta^2(q-1)^2\psi_0^2)\phi_2\\&-\eta^2q(q-1)(-q+\eta)\psi_1\phi_1^2-q\eta^3(q-1)^2\psi_0\psi_1\phi_1\\&+\eta^4(q-1)^2\psi_1^2\phi_0)\eta q/(((-q+\eta^2)\phi_2+\eta^2(q-1)\psi_1)\\&((q-\eta)(q+\eta)\phi_2-\eta^2(q-1)\psi_1)^2((q^3-\eta^2)\phi_2-\eta^2(q-1)\psi_1))\end{aligned} \tag{23}$$

with $\phi(x)=\phi_2x^2+\phi_1x+\phi_0,\ \psi(x)=\psi_1x+\psi_0.$ □

2.4.2. Classical continuous orthogonal polynomials

Since $\lim_{q\to 1}\mathscr{D}_q = \mathrm{d}/\mathrm{d}x$, from Eqs. (13)–(15) and by formal limit processes, we recover the following known results [25]:

1. The co-recursive $\tilde{P}_n^{[\mu]}$ of the polynomial $\tilde{P}_n$, orthogonal with respect to classical weight $\tilde{\rho}$ satisfying $(\tilde{\phi}\tilde{\rho})' = \tilde{\psi}\tilde{\rho}$ (degree of $\tilde{\phi} \leqslant 2$ and degree of $\tilde{\psi} = 1$) satisfies [25]

$$\mathscr{Q}_{2,n-1}^{*,c}(\tilde{P}_n^{[\mu]}) = \left((2\tilde{\phi}' - 2\tilde{\psi} - \mu(\tilde{\phi}'' - 2\tilde{\psi}'))\frac{\mathrm{d}}{\mathrm{d}x} + (\tilde{\phi}'' - \tilde{\psi}')\mathscr{I}_d\right)\tilde{P}_n, \tag{24}$$

where

$$\mathscr{Q}_{2,n-1}^{*,c} = \tilde{\phi}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + (2\tilde{\phi}' - \tilde{\psi})\frac{\mathrm{d}}{\mathrm{d}x} + (\tilde{\lambda}_n - \tilde{\psi}' + \tilde{\phi}'') \tag{25}$$

with $\tilde{\lambda}_n = -n((n-1)(\tilde{\phi}''/2) + \tilde{\psi}')$.

The operator $\mathscr{Q}_{2,n}^{[\mu,c]}$ annihilating the right-hand side of (24) is obtained from the second derivative of (24) and using the second-order differential equation satisfied by the classical continuous orthogonal polynomials $\tilde{P}_n$

$$\left(\tilde{\phi}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \tilde{\psi}\frac{\mathrm{d}}{\mathrm{d}x} + \tilde{\lambda}_n\mathscr{I}_d\right)\tilde{P}_n = 0 \tag{26}$$

in order to eliminate $(\mathrm{d}^2/\mathrm{d}x^2)\tilde{P}_n$ in the equation obtained from derivation of (24). The fourth-order differential equation satisfied by $\tilde{P}_n^{[\mu]}$ reduces in the factorized form as

$$\mathscr{Q}_{2,n}^{[\mu,c]}\mathscr{Q}_{2,n-1}^{*,c}[\tilde{P}_n^{[\mu]}] = 0, \tag{27}$$

where $\mathscr{Q}_{2,n}^{[\mu,c]}$ is a second-order linear differential operator with polynomial coefficients. The operators $\mathscr{Q}_{2,n-1}^{*,c}$ and $\mathscr{Q}_{2,n}^{[\mu,c]}$ for the co-recursive Laguerre polynomials ($\tilde{\phi}(x) = x$, $\tilde{\psi}(x) = 1 + \alpha - x$) are given by

$$\mathscr{Q}_{2,n-1}^{*,c} = x\frac{\mathrm{d}^2}{\mathrm{d}x^2} + (x + 1 - \alpha)\frac{\mathrm{d}}{\mathrm{d}x} + (n+1)\mathscr{I}_d, \quad \mathscr{Q}_{2,n}^{\mu,c} = B_2\frac{\mathrm{d}^2}{\mathrm{d}x^2} + B_1\frac{\mathrm{d}}{\mathrm{d}x} + B_0\mathscr{I}_d \tag{28}$$

with

$$B_2 = 4x(x-\alpha+\mu)^2 n + x(x - 4x\alpha + 2x\mu + 2x^2 + 2\alpha + 2\alpha^2 - 2\mu - 2\mu\alpha),$$

$$\begin{aligned}B_1 = &-4(x^2 + x\mu - 2x\alpha - \mu\alpha - 2\mu + \alpha^2 + 2\alpha)(x - \alpha + \mu)n\\ &- 6x\alpha^2 + 6\alpha x^2 + x + 4\alpha - 2\mu\alpha^2 + 4\mu x\alpha - 2x^3 - x^2 - 2x^2\mu + 4x\mu\\ &+ 6\alpha^2 - 4\mu + 2\alpha^3 - 5x\alpha - 6\mu\alpha,\end{aligned}$$

$$B_0 = 4(x-\alpha+\mu)^2 n^2 + (12\alpha + 2\mu\alpha + 2x^2 - 4\mu^2 - 12\mu + 3x + 2\alpha^2 - 4x\alpha - 2x\mu)n.$$

The fourth-order difference equation for the co-recursive Laguerre polynomials given above coincide obviously with those obtained in [18] (with μ replaced by $-\mu$) and in [25], checking carefully because of few misprints already corrected in [28, p. 304].

2. For the Jacobi polynomials $\tilde{P}_n(\alpha, \beta; x)$,

$$\tilde{\phi}(x) = 1 - x^2,\ \tilde{\psi}(x) = -(\alpha + \beta + 2)x + \beta - \alpha, \quad \alpha > -1, \beta > -1, \tag{29}$$

the right-hand side of Eq. (24) vanishes when $\alpha + \beta = 0$ and $\mu = \alpha - \beta = 2\alpha$. We therefore deduce the following relation between the Jacobi polynomials and it is co-recursive.

$$\tilde{P}_n^{[2\alpha]}(\alpha, -\alpha; x) = \tilde{P}_n(-\alpha, \alpha; x), \ -1 < \alpha < 1. \tag{30}$$

2.4.3. Classical discrete orthogonal polynomials

In this subsection we extend the result obtained for the co-recursive of q-classical orthogonal polynomials to the co-recursive of classical orthogonal polynomials of a discrete variable by using suitable change of variables and formal limit processes (for more details, see [10]).

- Difference equation linking classical discrete and its co-recursive
  In the first step, we substitute in (13) and (14) ϕ, ψ, P_n and $P_n^{[\mu]}$ by $\mathscr{T}_{\omega/(1-q)}\bar{\phi}, \mathscr{T}_{\omega/(1-q)}\bar{\psi}$, $\mathscr{T}_{\omega/(1-q)}\bar{P}_n$ and $\mathscr{T}_{\omega/(1-q)}\bar{P}_n^{[\mu]}$ respectively; and use the relation [15,23]

  $$\mathscr{T}_{\omega/(1-q)}A_{q,\omega} = \mathscr{G}_q\mathscr{T}_{\omega/(1-q)} \tag{31}$$

  to get an equation involving $\bar{P}_n$ and it's co-recursive. The operators $\mathscr{T}_a$ and $A_{q,\omega}$ are defined by [10,23] $\mathscr{T}_aP(x) = P(x+a)$, $A_{q,\omega}P(x) = P(qx+\omega)$. In the second step we multiply the above mentioned equation by $\mathscr{T}_{-\omega/(1-q)}$ and take the limit of the last equation as q and ω go to one and we recover the following result (see [10] for more details on this formal limit processes and change of variables) obtained for the co-recursive of classical orthogonal polynomials of a discrete variable [26].

Proposition 6 (Ronveaux et al. [26]). *The co-recursive $\bar{P}_n^{[\mu]}$ of the classical orthogonal polynomials of a discrete variable $\bar{P}_n$, orthogonal with respect to the discrete classical weight $\bar{\rho}$ satisfying $\Delta(\bar{\phi}\bar{\rho}) = \bar{\psi}\bar{\rho}$, where $\bar{\phi}$ is a polynomial of degree at most two and $\bar{\psi}$ a first degree polynomial satisfies*:

$$(\bar{\phi}_{[1]} + \bar{\psi}_{[1]})\mathscr{Q}^{*,d}_{2,n-1}[\bar{P}_n^{[\mu]}(x;q)] = [\bar{e}_d\mathscr{T} + \bar{f}_d\mathscr{I}_d]\bar{P}_n(x;q) \tag{32}$$

with

$$\begin{aligned}
\mathscr{Q}^{*,d}_{2,n-1} &= \bar{\phi}_{[2]}\mathscr{T}^2 - (2\bar{\phi}_{[1]} + \bar{\psi}_{[1]} - \bar{\lambda}_n)\mathscr{T} + (\bar{\phi} + \bar{\psi})\mathscr{I}_d,\\
\bar{e}_d &= \left(-\left(\frac{\bar{\phi}''}{2} - \bar{\psi}'\right)\mu - \bar{\phi}_{[1]} - \bar{\psi}_{[1]} + \bar{\phi}_{[2]}\right)(2\bar{\phi}_{[1]} + \bar{\psi}_{[1]} - \bar{\lambda}_n),\\
\bar{f}_d &= \left(\bar{\phi}''2 - \bar{\psi}'\right)\mu(2\bar{\phi}_{[1]} + \bar{\psi}_{[1]}) - \bar{\phi}_{[1]}\bar{\phi}_{[2]} + (\bar{\phi} + \bar{\psi})(\bar{\phi}_{[1]} + \bar{\psi}_{[1]}),
\end{aligned} \tag{33}$$

where $\bar{\lambda}_n = -n((n-1)\frac{\bar{\phi}''}{2} - \bar{\psi}')$, $\bar{\phi}_{[j]} \equiv \bar{\phi}(x+j)$, $\bar{\psi}_{[j]} \equiv \bar{\psi}(x+j)$ and the operators Δ and $\mathscr{T}$ are defined respectively by $\Delta P(x) = P(x+1) - P(x), \mathscr{T}P(x) = P(x+1)$.

The factorized form of the fourth-order difference equation satisfied by the co-recursive of the classical discrete orthogonal polynomials $P_n^{[\mu]}$

$$\mathscr{Q}^{[\mu,d]}_{2,n}\mathscr{Q}^{*,d}_{2,n-1}P_n^{[\mu]} = 0 \tag{34}$$

is obtained by applying twice the operator $\mathscr{T}$ to (32) and using the second-order difference equation satisfied by the classical discrete orthogonal polynomials [24].

$$(\bar{\phi}\Delta\nabla + \bar{\psi}\Delta + \bar{\lambda}_n \mathscr{I}_d)P_n = 0 \tag{35}$$

in order to eliminate the factor $\mathscr{T}^2 P_n$ appearing after applying the operator $\mathscr{T}$ to (32). Here, the operator ∇ is defined by $\nabla P(x) = P(x-1) - P(x)$. Since the operator $\mathscr{Q}_{2,n}^{[\mu,d]}$ is space consuming, we give the difference equations for the Charlier case.

Example.

- For Charlier orthogonal polynomials $c_n^{(a)}$ [24] ($\phi(x) = x$, $\psi(x) = a - x$, $a > 0$), the operators $\mathscr{Q}_{2,n-1}^{*,d}$ and $\mathscr{Q}_{2,n}^{[\mu,d]}$ are given, respectively, by

$$\mathscr{Q}_{2,n-1}^{*,d} = (x+2)\mathscr{T}^2 - (x+1+a-n)\mathscr{T} + a\mathscr{I}_d, \quad \mathscr{Q}_{2,n}^{[\mu,d]} = D_2\mathscr{T}^2 + D_1\mathscr{T} + D_0\mathscr{I}_d$$

with

$$\begin{aligned}
D_0 = {} & (R+a+1)(\mu+2a+R)(\mu-2+2a+R)n^2 \\
& + (R+a+1)(2R-1)(\mu+2a+R)(\mu-2+2a+R)n + (R+a+1) \\
& \quad (2\mu Ra + 2a + 4Ra^2 + \mu R^2 + R^3 + 2R - 3a^2 - \mu a - 3R^2 - 6Ra - \mu R + 4R^2 a), \\
D_1 = {} & -(\mu+1+2a+R)(\mu-2+2a+R)n^3 \\
& - 3R(\mu+1+2a+R)(\mu-2+2a+R)n^2 + (-2 - \mu + 6\mu Ra + 2a\mu^2 \\
& - 4a + 8Ra^2 + 3\mu R^2 + 2R^3 + \mu^2 R - R + a^2 + 8\mu a^2 + \mu a + 8a^3 \\
& - 8a^2R^2 - 2R^4 - 8R^3 a + 5R^2 + Ra - 2\mu^2 R^2 - 8\mu R^2 a + \mu^2 + 6R^2 a \\
& - 4\mu R^3)n + \mu Ra - 2a + 7Ra^2 + 2R^3 - 2R + a^2 + 2\mu a^2 + \mu a + 4a^3 \\
& - 4a^2R^2 - R^4 - 4R^3 a + R^2 + Ra + \mu R - 2\mu R^2 a + 5R^2 a - \mu R^3, \\
D_2 = {} & -a(\mu+1+2a+R)(\mu-1+R+2a)n^2 \\
& - a(2R+1)(\mu+1+2a+R)(\mu-1+R+2a)n \\
& - a(2\mu Ra + 4Ra^2 + \mu R^2 + R^3 - R + a^2 + \mu a + 2Ra + \mu R + 4R^2 a),
\end{aligned}$$

  where $R = -x - a - 2$. It should be mentioned that the above results on co-recursive Charlier orthogonal polynomials coincide with those given in [19,26] (replace μ by $-\mu$ when comparing our results with those obtained in [19]).
- For Hahn polynomials $H_n(\alpha,\beta,N;x)$ [24],

$$\phi(x) = x(N+\alpha-x), \quad \psi(x) = -(\alpha+\beta+2)x + (\beta+1)(N-1), \quad \alpha > -1, \quad \beta > -1, \tag{36}$$

  $\bar{e}_d = \bar{f}_d = 0$ when $\alpha+\beta = 0$ and $\mu = N\alpha$. Since N is an integer and α a real number, the complex number μ, in this case, is real.

Proposition 7. *The co-recursive Hahn family $H_n(\alpha,-\alpha,N,x)^{[N\alpha]}$ belongs to the Hahn family and obeys the relation*

$$H_n(\alpha,-\alpha,N,x)^{[N\alpha]} = H_n(-\alpha,\alpha,N,x-\alpha), \quad -1 < \alpha < 1. \tag{37}$$

Proof. The above relation is derived using the three-term recurrence relation satisfied by the co-recursive Hahn polynomials and the following relations between the coefficients $\beta_n^H(\alpha,\beta,N)$ and $\gamma_n^H(\alpha,\beta,N)$ of the three-term recurrence relation (see (1)).

$$\begin{aligned}
&\beta_0^H(\alpha,-\alpha,N)+N\alpha=\beta_0^H(-\alpha,\alpha,N)+\alpha,\\
&\beta_n^H(\alpha,-\alpha,N)=\beta_n^H(-\alpha,\alpha,N)+\alpha,\quad n\geqslant 1,\\
&\gamma_n^H(\alpha,-\alpha,N)=\gamma_n^H(-\alpha,\alpha,N),\quad n\geqslant 1,
\end{aligned}\tag{38}$$

where $\beta_n^H(\alpha,\beta,N)$ and $\gamma_n^H(\alpha,\beta,N)$ are the coefficients of the three-term recurrence relation satisfied by the monic Hahn polynomials given in [1,24]

$$\beta_n^H(\alpha,\beta,N)=\frac{\alpha-\beta+2N-2}{4}+\frac{(\beta^2-\alpha^2)(\alpha+\beta+2N)}{4(\alpha+\beta+2n)(\alpha+\beta+2n+2)},$$

$$\gamma_n^H(\alpha,\beta,N)=\frac{n(N-n)(\alpha+n)(\beta+n)(\alpha+\beta+n)(\alpha+\beta+N+n)}{(\alpha+\beta+2n-1)(\alpha+\beta+2n)^2(\alpha+\beta+2n+1)}.$$

It should be mentioned that the coefficient $\beta_0^H(\alpha,-\alpha,N)$ is given by

$$\beta_0^H(\alpha,-\alpha,N)=\lim_{\beta\to-\alpha}\beta_0^H(\alpha,\beta,N)=\frac{(1-\alpha)(N-1)}{2}.\qquad\Box$$

2.5. Concluding remark

The difference equations involved in this work can be used to solve linearization problems and compute connection coefficients, etc. (see the Introduction), they can be used also in order to derive the difference equations satisfied by the co-recursive associated classical orthogonal polynomials. Co-recursive associated orthogonal polynomials, which are defined as the co-recursive of the rth associated orthogonal polynomials, are very useful (see, for example, [18,19]). Works on difference equation for the co-recursive associated of *all* classical orthogonal polynomials are under investigation [11].

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