The structure of spherical graphs

J.H. Koolen\textsuperscript{a}, V. Moulton\textsuperscript{b}, D. Stevanovi\textsuperscript{c,d}

\textsuperscript{a}Division of Applied Mathematics, KAIST, Daejeon 305-701, Republic of Korea
\textsuperscript{b}The Linnaeus Centre for Bioinformatics, Uppsala University, BMC, Box 598, 751 24 Uppsala, Sweden
\textsuperscript{c}Department of Mathematics, Faculty of Science, University of Nis, Visegradska 33, 18000 Nis, Yugoslavia
\textsuperscript{d}Forschungsschwerpunkt Mathematisierung, University of Bielefeld, Pf. 100131, 33501 Bielefeld, Germany

Dedicated to the memory of Prof. J.J. Seidel

Abstract

A spherical graph is a graph in which every interval is antipodal. Spherical graphs are an interesting generalization of hypercubes (a graph \( G \) is a hypercube if and only if \( G \) is spherical and bipartite). Besides hypercubes, there are many interesting examples of spherical graphs that appear in design theory, coding theory and geometry e.g., the Johnson graphs, the Gewirtz graph, the coset graph of the binary Golay code, the Gosset graph, and the Schl"afli graph, to name a few. In this paper we study the structure of spherical graphs. In particular, we classify a subclass of these graphs consisting of what we call the strongly spherical graphs. This allows us to prove that if \( G \) is a triangle-free spherical graph then any interval in \( G \) must induce a hypercube, thus providing a proof for a conjecture due to Berrachedi, Havel and Mulder.

\( © \) 2003 Published by Elsevier Ltd.

1. Introduction

In this paper, all graphs are simple and finite. Given a connected graph \( G = (V, E) \) and vertices \( u, v \in V \) let \( d_G(u, v) = d(u, v) \) denote the length of any geodesic (i.e., shortest path) between \( u \) and \( v \) in \( G \). In addition, define the interval between \( u \) and \( v \) to be the subset of \( V \) given by

\[ [u, v]_G = [u, v] = \{ w \in V : d(u, v) = d(u, w) + d(w, v) \}, \]

and denote by \( G[u, v] \) the subgraph of \( G \) induced on \( [u, v] \). Note that \( [u, v] \) consists of those vertices in \( V \) that lie on some geodesic between \( u \) and \( v \). We call a graph \( G = (V, E) \) spherical if given any interval \( [u, v] \) in \( V \) with \( u, v \in V \) and any \( w \in [u, v] \), there exists

E-mail addresses: jhk@amath.kaist.ac.kr (J.H. Koolen), vincent.moulton@lcb.uu.se (V. Moulton), dragance@pmf.pmf.ni.ac.yu (D. Stevanovi\textsuperscript{c}).
precisely one vertex \( x \in [u, v] \) with \([w, x] = [u, v]\). We call \( x \) the antipodal vertex or antipode of \( w \) in \([u, v]\).

Spherical graphs were introduced in [2] and represent an interesting generalization of hypercubes. In particular, a graph \( G \) is a hypercube if and only if \( G \) is spherical and bipartite [2]. Spherical graphs enjoy several nice properties; for example, they are regular and the direct product of two connected graphs is spherical if and only if both of the graphs are spherical. Moreover, there are several important examples of spherical graphs which—extending the list given in [2]—include the following:

(i) \( n \)-cubes, \( n \geq 1 \).
(ii) Cocktail party graphs \( CP(n), n \geq 2 \) (i.e., the complement of \( n \) copies of \( K_2 \)).
(iii) Johnson graphs \( J(n, t), n \geq 2t \geq 2 \) [3, p. 114].
(iv) Halved \( n \)-cubes, \( n \geq 2 \) [3, p. 264].
(v) Folded \((2n + 1)\)-cubes, \( n \geq 2 \) [3, p. 264].
(vi) Folded halved \((4n + 2)\)-cubes, \( n \geq 1 \) [3, p. 264].
(vii) Strongly regular graphs [3, p. 3] with parameters \( \lambda \leq 1 \) and \( \mu = 2 \), e.g., the Gewirtz graph [4] (with \( \lambda = 0, \mu = 2 \) and degree 10) and the Berlekamp–van Lint–Seidel graph [1] (with \( \lambda = 1, \mu = 2 \) and degree 22).
(viii) Complete graphs, \( K_n, n \geq 2 \).
(ix) The coset graph of the binary Golay code [1], [3, p. 361].
(x) The Gosset graph [3, p. 103].
(xi) The Schläfli graph [3, p. 103].

Although several results are given in [2] concerning the structure of spherical graphs, there is still much to be understood on this topic. As a first step in this direction, we will provide a complete classification of a subclass of the spherical graphs, the strongly spherical graphs. As we shall see, the subgraph induced on any interval of a spherical graph is strongly spherical (Lemma 2.1)—indeed this motivated the definition of strongly spherical graphs—and hence our classification of the strongly spherical graphs provides us with the structure of the intervals of a spherical graph. We are confident that in future it will be possible to use this information to obtain a complete classification of the spherical graphs.

Before defining strongly spherical graphs, we first recall the definition of a related type of graph. We call a connected graph \( G = (V, E) \) with diameter at least 1 antipodal if for every vertex \( u \in V \) there exists some vertex \( v \in V \) with \([u, v] = V\). If such a vertex exists then it is easy to see that it is unique, and we call \( v \) the antipode of \( u \) in \( G \). Note that there is unique antipodal graph of diameter 1, namely the edge—which also corresponds to the halved 2-cube and the Johnson graph \( J(2, 1) \)—and that an antipodal graph with diameter 2 must be a cocktail party graph. In addition, if \( G \) is an antipodal graph, then it is straightforward to see that the map which sends each vertex to its antipode is an automorphism of \( G \).

We say that a connected graph is strongly spherical if it is both antipodal and spherical. Note that there are antipodal graphs that are not spherical (e.g., the \((2n)\)-gon, \( n \geq 3 \)) and spherical graphs that are not antipodal (e.g., the halved \((2n + 1)\)-cube, \( n \geq 1 \)). As with spherical graphs, the properties of being antipodal or strongly spherical are both preserved under taking direct products. In fact, for strongly spherical graphs we can say even more.
The main result of this paper, which we prove in Section 4, is as follows (note that an antipodal Johnson graph must be of the form $J(2n, n)$, $n \geq 1$, and that an antipodal halved cube is a halved $2n$-cube, $n \geq 1$).

**Theorem 1.1.** Suppose that $G$ is a strongly spherical graph. Then $G$ is isomorphic to the direct product of graphs $G_1 \times \cdots \times G_m$, where $m$ is some positive integer and for $i \in \{1, \ldots, m\}$ the graph $G_i$ is either an antipodal Johnson graph, an antipodal halved cube, a Gosset graph or a cocktail party graph.

In [2] it was conjectured that if $G$ is a triangle-free spherical graph, then any interval of $G$ must induce a hypercube. As a consequence of Theorem 1.1, we see that this conjecture is true. In particular, since the Johnson graph $J(2n, n)$ ($n \geq 2$), the halved $2n$-cube ($n \geq 2$), the cocktail party graph $CP(n)$ ($n \geq 3$), and the Gosset graph all contain a triangle, it follows by Theorem 1.1 that the following result holds.

**Corollary 1.2.** Suppose that $G$ is a triangle-free spherical graph.

(i) If $G$ is antipodal (i.e., $G$ is strongly spherical), then $G$ is a hypercube.
(ii) For any two vertices $x, y$ of $G$ the subgraph induced by $G$ on $[x, y]$ is a $d(x, y)$-cube. In particular, $G$ is interval-regular (see Section 2 for the definition of this last term).

**Remark 1.3.** The last corollary also holds in the case when, instead of assuming that $G$ is triangle-free, we assume that every edge of $G$ is contained in at most one triangle.

As a direct consequence of this last corollary together with [3, Proposition 4.3.6] we obtain the following result concerning the structure of spherical graphs which contain no triangles or pentagons.

**Corollary 1.4.** Suppose that $G$ is a triangle- and pentagon-free spherical graph with degree $k \geq 1$. Then there exists a map from the vertex set of the $k$-cube $Q_k$ to the vertex set of $G$ that preserves distances up to $3$ (i.e., any pair of vertices that are at most $3$ apart in $Q_k$ is mapped to a pair of vertices in $G$ that are the same distance apart). In particular, the number of vertices of $G$ must equal some power of $2$.

In view of this corollary, we suspect that it should not be too hard to classify the triangle-free spherical graphs (i.e., find a complete list of all such graphs). This would be of interest since, for example, there is no classification yet known for the class of triangle-free, interval-regular, distance-regular graphs, which is precisely the class of triangle-free spherical graphs that are in addition distance-regular (note that hypercubes only cover triangle-free, interval-regular, distance-regular graphs that are in addition pentagon-free). In particular, if $G$ is a triangle-free spherical graph with degree $k \geq 3$ and diameter at least $3$ and $G$ is not the direct product of two other graphs, then we conjecture that there exists a map from the vertex set of $Q_k$ to the vertex set of $G$ that preserves distances up to $2$ (note that this conjecture does not hold in the case the diameter of $G$ is $2$, e.g., the Gewirtz graph provides a counter-example). If this conjecture held then, for example, the preimages of this map would be certain special codes in the hypercube, $|V|$ would be a power of two, and the eigenvalues of $G$ would be integral.

We conclude this introduction with a brief summary of the contents of this paper. In the next section we prove that spherical graphs are degree-regular. In Section 3 we classify...
the locally-connected strongly spherical graphs (see Proposition 3.2). This classification follows mainly from the fact that, as we shall show, locally-connected strongly spherical graphs are distance-regular. In Section 4, we prove the main theorem by, in essence, showing that any strongly spherical graph must be the direct product of two locally connected strongly spherical graphs and applying the classification of these graphs derived in the previous section.

2. Regularity of spherical graphs

We begin this section with a key observation. Recall that if $G$ is a connected graph, then a subgraph $H$ of $G$ is an isometric subgraph if $d_H(x, y) = d_G(x, y)$ holds for all $x, y \in V(H)$.

**Lemma 2.1.** Suppose that $G$ is a spherical graph. If $x, y \in V(G)$, then the subgraph induced on $[x, y]$ is an isometric subgraph of $G$. In particular, this subgraph is spherical and, hence, strongly spherical.

**Proof.** Suppose $u, z \in [x, y]$. By definition, there exists a vertex $v \in [x, y]$ with $[u, v] = [x, y]$. But $z \in [u, v]$ and hence $[u, z] \subseteq [u, v] = [x, y]$. It immediately follows that the distance between $u$ and $z$ in $G[x, y]$ equals $d_G(u, z)$ and, since $u, z$ were arbitrary vertices in $[x, y]$, this completes the proof of the lemma. □

Before proving the main result of this section, we first recall some definitions concerning regularity of graphs. Given a connected graph $G = (V, E)$ with diameter $D$ and some $x \in V$, we define

$$N_i^G(x) = N_i(x) = \{y \in V : d(x, y) = i\},$$

where $0 \leq i \leq D$ is an integer, and denote by $G_i(x)$ the subgraph induced by $G$ on $N_i(x)$. We also put $N^G(x) = N(x) = N^G_0(x)$ and $G(x) = G_1(x)$. In addition,

- $G$ is called degree-regular if for any $u, v \in V$ we have $|N_i(u)| = |N_i(v)|$ for all $0 \leq i \leq D$ (so, in particular, any degree-regular graph is also regular);
- $G$ is called interval-regular if, for each pair of vertices $u, v \in V$ with $d(u, v) = i$, the number of vertices in $N_{i-1}(v)$ that are adjacent to $u$ equals $i$ ($1 \leq i \leq D$) (cf. [5]);
- $G$ is called distance-regular if it is regular (denote its degree by $k$) and the following holds: there are natural numbers $b_0 = k, b_1, \ldots, b_{D-1}$ and $c_1 = 1, c_2, \ldots, c_D$ (called intersection numbers) such that for each pair of vertices in $G$ satisfying $d(u, v) = i$ we have

$$\begin{align*}
(1) & \text{ the number of vertices in } N_{i-1}(v) \text{ adjacent to } u \text{ is } c_i \quad (1 \leq i \leq D), \\
(2) & \text{ the number of vertices in } N_{i+1}(v) \text{ adjacent to } u \text{ is } b_i \quad (0 \leq i \leq D-1).
\end{align*}$$

We put $a_j := k - b_j - c_j$ for $1 \leq j \leq D - 1$, and $a_D := k - c_D$.

The following result generalizes [2, Lemma 2(ii)] in which it is shown that any spherical graph is regular.

**Proposition 2.2.** A spherical graph is degree-regular.

**Proof.** Suppose that $G$ is a spherical graph with diameter $D$, and that $u$ and $v$ are adjacent vertices of $V(G)$. Suppose in addition that $x$ is an arbitrary vertex in $N_i(u) \cap N_{i-1}(v)$ for
some $1 \leq i \leq D$. By Lemma 2.1 $G[x, u]$ is antipodal. Since $d(x, v) = i - 1$, $d(x, u) = i$ and $d(u, v) = 1$, the vertex $v$ is contained in $[x, u]$. Let $x'$ be the antipode of $v$ in $G[x, u]$ so that, in particular, $[x, u] = [x', v]$ holds. Then, since $d(x', v) = i$, $x \in [x', v]$, and $d(x, u) = i$, it follows that $x'$ is adjacent to $x$, and since $x' \in [x, u]$ it follows that $x' \in N_i(v) \cap N_{i-1}(u)$ must hold. Moreover, $x'$ is the only vertex in $N_i(v) \cap N_{i-1}(u)$ that is adjacent to $x$, since if $x'' \in N_i(v) \cap N_{i-1}(u)$ is adjacent to $x$, then $x'' \in [x, u]$ and, as $d(x'', v) = i$, it follows that $x''$ is the antipode of $v$ in $G[x, u]$ and so $x' = x''$.

In view of these considerations, it follows that the mapping from $N_i(u) \cap N_{i-1}(v)$ to $N_i(v) \cap N_{i-1}(u)$ that takes any $x$ in $N_i(u) \cap N_{i-1}(v)$ to its unique neighbour $x'$ in $N_i(v) \cap N_{i-1}(u)$ is an injection and hence, using symmetry, we conclude that $|N_i(u) \cap N_{i-1}(v)| = |N_i(v) \cap N_{i-1}(u)|$ holds. As $N_i(u)$ can be partitioned in the following way

$$N_i(u) = (N_i(u) \cap N_{i-1}(v)) \cup (N_i(u) \cap N_i(v)) \cup (N_i(u) \cap N_{i+1}(v)),$$

it immediately follows that $|N_i(v)| = |N_i(u)|$ must also hold. Since $G$ is connected it thus follows that $G$ is degree-regular.

3. Locally connected strongly spherical graphs

In this section, we determine the structure of locally connected strongly spherical graphs. We begin with a result concerning the local structure of a strongly spherical graph.

Proposition 3.1. Suppose that $G$ is a strongly spherical graph with diameter at least 2. If $x \in V(G)$ and $C$ is a connected component of $G(x)$, then $C$ is an isometric subgraph of $G$. In particular $C$ is spherical.

Proof. Suppose that $G$ is a strongly spherical graph with diameter $D \geq 2$. We prove the proposition using induction on $D$.

If $D = 2$, then $G$ is a cocktail party graph and so the proposition clearly holds for $G$.

Suppose $D \geq 3$ and that the proposition holds for any strongly spherical graph with diameter that is at least 2 and strictly less than $D$.

Let $G$ be a strongly spherical graph with diameter $D$. Let $x \in V(G)$. Suppose that $C$ is a connected component of $G(x)$ and that $C$ is not an isometric subgraph of $G$. In order to complete the proof we show that this leads to a contradiction.

Since $C$ is not an isometric subgraph of $G$, there must be a geodesic $x_1, x_2, x_3, x_4$ in $C$.

By Lemma 2.1 $G[x_1, x_4]$ is isometric with diameter 2. Hence it is a cocktail party graph $CP(n)$ with $n \geq 2$. Note that if $n > 2$, then there must exist some vertex in $V(G)$ that is adjacent to $x$, $x_1$ and $x_4$, which is clearly impossible. Hence $n = 2$.

Suppose $y \in N^G(x_1) \cap N^G(x_4)$ with $y$ distinct from $x$. Let $x'$ be the antipode of $x$ in $G$. Let $y, z_1, z_2, \ldots, z_{D-3}, x'$ be a geodesic in $G$, and put $z = z_{D-3}$. Clearly $d(x_1, z) = d(x_4, z) = D - 2$ and $d(z, x) = D - 1$.

Now, suppose $d(x_2, z) = D - 2$. Suppose $x''$ is the antipode of $x$ in $G[x_2, x_4]$. Then $d(x'', z) = D - 3$ and, since $x_3$ is adjacent to $x''$, it follows that $d(x_3, z) = D - 2$ must hold. Hence, by applying the inductive hypothesis to $G[x, z]$, it follows that any connected component of $G(x) \cap G[x, z]$ must be an isometric subgraph of $G(x, z)$. But then $x_1, x_2, x_3, x_4$ would be a geodesic between $x_1$ and $x_4$ in $G[x, z]$, which is a contradiction.
Hence, by symmetry, \( d(x_2, z) = d(x_3, z) = D - 1 \). Now, suppose \( y' \in V(G) \) with \( y' \) distinct from \( x_3 \), adjacent to \( x_2 \) and \( x_4 \), but not adjacent to \( x_3 \), so that \( d(y', z) = D - 1 \). Note that a vertex \( y' \) with these properties must exist since the subgraph induced by \( G \) on \( \{x_2, x_4\} \) is a cocktail party graph.

Suppose \( z' \) is the antipode of \( z \) in \( G \), so that \( d(z, z') = D \). Then \( z' \) must be adjacent to \( x_2, x_3 \) and \( y' \), but not to \( x_4 \). Hence \( x_2, x_4 \) and \( z' \) are common neighbours of \( y' \) and \( x_3 \), but \( x_4 \) is not adjacent to either \( z' \) or \( x_2 \). But this is a contradiction since the subgraph induced by \( G \) on \( \{y', x_3\} \) is a cocktail party graph. This final contradiction completes the proof of the proposition. □

It would be interesting to know whether or not this last proposition also holds for spherical graphs.

Recall that a graph \( G \) is locally connected if for every vertex \( x \in V(G) \) the graph \( G(x) \) is connected. We now give a classification of the locally connected strongly spherical graphs.

**Proposition 3.2.** Suppose that \( G \) is a locally connected strongly spherical graph with diameter \( D \geq 2 \). Then the following statements hold:

- If \( D = 2 \), then \( G \) is a cocktail party graph.
- If \( D = 3 \), then \( G \) is either the Gosset graph, \( J(6, 3) \) or the halved 6-cube.
- If \( D \geq 4 \), then \( G \) is either \( J(2D, D) \) or the halved 2D-cube.

**Proof.** We prove the proposition using induction on \( D \).

The proposition clearly holds if \( D = 2 \). So suppose \( D \geq 3 \) and that the proposition holds for any locally connected strongly spherical graph with diameter that is at least \( 2 \) and strictly less than \( D \).

Let \( G \) be a locally connected strongly spherical graph with diameter \( D \). Put \( V := V(G) \).

**Claim 1.** If \( x, y \in V \) with \( d(x, y) \geq 2 \), then \( G[x, y] \) is locally connected.

**Proof of Claim 1.** Put \( L := G[x, y] \). We must show that \( L(z) \) is connected for all \( z \in V(L) \). Since \( G \) is spherical, for every \( z \in V(L) \) there exists some vertex \( z' \in [x, y] \) with \( [z, z'] = [x, y] \). Thus, since we can interchange the roles of \( z \) and \( x \), it suffices to show that \( L(x) \) is connected.

Suppose that \( u, v \) are non-adjacent vertices of \( L(x) \). We show that \( u \) and \( v \) have a common neighbour in \( L(x) \), from which the claim follows. By **Proposition 3.1**, \( G(x) \) is an isometric subgraph of \( G \). Hence, as \( d(u, v) = 2 \) and \( u, v \) are vertices of \( G(x) \), there is a vertex \( w \) of \( G(x) \) that is a common neighbour of \( u, v \). But \( [u, v] \subseteq [x, y] \) and thus \( w \in [x, y] \). Hence \( w \) is a vertex of \( L(x) \) and \( u, v, w \) is a path in \( L(x) \). This completes the proof of Claim 1. □

**Claim 2.** There exists a natural number \( q_{D-1} \) so that \( |N(x) \cap N_{D-2}(y)| = q_{D-1} \) holds for all \( x, y \in V \) with \( d(x, y) = D - 1 \).

**Proof of Claim 2.** Suppose that \( x, y \in V \) with \( d(x, y) = D - 1 \). Since \( G \) and \( G(y) \) are spherical they are also regular. Let \( k \) be the degree of \( G \) and \( \lambda \) be the degree of \( G(y) \).

Suppose that \( y' \) is the antipode of \( y \) in \( G \). Since the map sending each vertex in \( G \) to its antipode is an automorphism of \( G \), it follows that \( G(y') \) is regular with degree \( \lambda \). Clearly
x ∈ N(y'). But λ = |N(y') ∩ N(x)|. Thus, since N(y') = N_{D−1}(y) and y is adjacent to x, it follows that |N(x) ∩ N_{D−2}(y)| = k−λ−1 must hold. Hence Claim 2 follows if we can show that the degree of G(z) is λ for any vertex z ∈ V.

To this end, suppose that w is adjacent to y. Since y has degree λ in G(w), it follows that (the necessarily regular graph) G(w) has degree λ. But G is connected and so it immediately follows that the degree of G(z) is λ for any vertex z ∈ V. This completes the proof of Claim 2.

Now suppose x, y ∈ V with d(x, y) = D−1. Put H := G[x, y]. Note that H is strongly spherical and by Claim 1 it is locally connected. Thus, by induction, H is isomorphic to either a cocktail party graph in the case D = 3, the Gosset graph, the Johnson graph J(6, 3) or the halved 6-cube in the case D = 4, or the Johnson graph J(2(D − 1), D − 1) or the halved 2(D − 1)-cube in the case D ≥ 5.

Since G is a regular antipodal graph it follows by Claim 2 that H has degree q_{D−1}. Now suppose that u, v ∈ V are also vertices in G at distance D − 1. Then by Claim 2 G[u, v] has degree q_{D−1} and so by induction it follows that G[u, v] is isomorphic to H. In particular, since G is a regular antipodal graph, it follows that G must be distance-regular (see Section 2 for the definition of this term).

Let a_{i}, b_{i}, c_{i}, 1 ≤ i ≤ D, denote the intersection numbers of G. Note that c_{D−1} = q_{D−1} and that, by definition, a_{1} is the degree of any local graph of G and that c_{2} is the number of common neighbours of any two vertices in V that are at distance 2.

Claim 3. Let a'_{i} be the degree of any local graph of H. Then

\[ a'_{i} = a_{1} − c_{2} + 2. \] (1)

Proof of Claim 3. Suppose that x, z are adjacent vertices in V, that x′ is the antipodal of x in G, and that Δ is the graph induced by G on N_{D−2}^{G}(x′) ∩ N^{G}(z) = N_{2}^{G}(x) ∩ N^{G}(z). Since Δ is the local graph of G[z, x′] at z, it follows that Δ is regular with degree a'_{i}. But, on the other hand, if w ∈ N_{2}^{G}(x) ∩ N^{G}(z) then |N^{G}(x) ∩ N^{G}(z) ∩ N^{G}(w)| = c_{2} − 2 (since w has a unique neighbour that is adjacent to x and distance 2 from z). Thus Δ has degree a_{1} − c_{2} + 2. This completes the proof of Claim 3.

We now complete the proof of the theorem.

In the case D = 3, so that H is isomorphic to a cocktail party graph, G must be a Taylor graph with a_{1} = 2c_{2} − 4 by (1). Hence by [3, Corollary 1.15.3] it follows that G is either the Johnson graph J(6, 3), the halved 6-cube or the Gosset graph.

In the case D = 4, so that H is isomorphic to the Gosset graph, it follows by (1) that G is an antipodal distance-regular graph with degree 52, intersection numbers c_{3} = 27, c_{2} = 10 and diameter 4. However such a graph does not exist since if this were the case then it would follow that |N^{G}_{2}(x)| = 52.27/10, which is not an integer.

Finally, in the case D ≥ 4, so that H is isomorphic to either J(2(D − 1), D − 1) or the halved (D − 1)-cube, then by using (1) again it follows that G has the same intersection numbers as J(2D, D) or the halved D-cube, respectively. This completes the proof of the theorem since both the Johnson graphs and the halved D-cubes are determined by their intersection numbers [6].
4. Proof of Theorem 1.1

We begin by presenting a sketch of the proof since it is quite long. Suppose that $G$ is a strongly spherical graph, and that $x$ is any vertex in $V = V(G)$. Let $C$ be any connected component of $G(x)$. We must show that there is some vertex $y$ in $V$ with $N(y) \subseteq C$.

**Proof of Claim 1.** Note first that there is some vertex $z$ in $V$ with $d(x, z) = d(x, y) - 1$ holds. Moreover, $d(u, c) = 2$ holds for all $c \in V(C)$ since $C$ is a connected component of $G(x)$. Hence $d(u', c) = d(x, y) - 2 = d(x, u') - 1$ for all $c \in V(C)$ and so $V(C) \subseteq N_{d(x, y) - 1}(u') \cap N(x)$. This contradicts the choice of $y$, which completes the proof of Claim 1. \hfill $\square$

Choose $y \in V$ with $N_{d(x, y) - 1}(y) \cap N(x) = V(C)$, whose existence is guaranteed by Claim 1. Let $H := G[x, y]$. Note $H(x) = C$ so that, in particular, $H(x)$ is connected.

**Claim 2.** The graph $H$ is locally connected.

**Proof of Claim 2.** We must show that $H(u)$ is connected for all $u \in V(H)$. We will show that this holds in the case $u$ is adjacent to $x$, from which Claim 2 follows (since both $G$ and $H(x)$ are connected).

Suppose $u$ is a vertex of $H(x)$. We show that for any $v \in H(u)$ there is a path in $H(u)$ connecting $v$ to $x$, so that, in particular, $H(u)$ is connected. Clearly it suffices to show that this holds in the case $v$ is a vertex of $H(x)$.

Suppose that $v$ is such a vertex. Since $G[x, v]$ is a cocktail party graph, $v$ and $x$ have a common neighbour $w$ of $H(x)$ with $d(u, w) = 2$. Moreover, since $H(x)$ is connected and $u$ and $w$ are vertices of $H(x)$, by Proposition 3.1 there must exist some vertex $z$ of $H(x)$ which is a common neighbour of $u$ and $w$. But then $z$ must also be a common neighbour of $x$ and $v$, i.e., $x, z, v$ is a path in $H(u)$. This completes the proof of Claim 2. \hfill $\square$

Denote the antipode of any vertex $z \in V$ in $G$ by $z'$. Note that if $y = x'$ held, then we would have $G = H$ which, in view of Claim 2 and Proposition 3.2, would complete the proof of the theorem. So, from now on we assume $y \neq x'$.

Let $L := [x', y]$. In Claim 8 below, we will prove that $G$ is isomorphic to $H \times L$. However, we first require some further definitions and results.
Claim 3. \( G(y) = H(y) \cup L(y) \).

**Proof of Claim 3.** We first show that if \( z \) is a vertex of \( G(x) - H(x) \), then \( d(y, z) = d(y, x) + 1 \). We prove that this holds using induction on \( d(x, x') \).

If \( d(x, x') \leq 2 \) then this clearly holds, since \( G \) must be either an edge (in the case \( d(x, x') = 1 \)) or a cocktail party graph (in the case \( d(x, x') = 2 \)).

Suppose \( d(x, x') \geq 3 \). If \( x \) is adjacent to \( y \), then since \( N^H(x) = \{ y \} \), we immediately see that if \( z \) is a vertex of \( G(x) - H(x) \), then \( d(y, z) = d(y, x) + 1 \). So assume \( d(x, y) \geq 2 \).

Suppose \( z \) is a vertex of \( G(x) - H(x) \) and \( u \) is a vertex of \( H(x) \). Then \( d(u, z) = 2 \) since \( H(x) \) is a connected component of \( G(x) \). Thus there is some vertex \( v \in V \) that is a common neighbour of \( u \) and \( z \) and that is at distance 2 from \( x \). Moreover, \( v \notin [u, y] \), since otherwise \( z \in [x, y] \).

We now show \( d(v, y) = d(u, y) + 1 \). Let \( K := G[u, y] \). Note that the subgraph \( S \) that is induced by \( G \) on \( V(K(u) \cap H(u)) \) is a connected component of \( K(u) \), as \( u \neq y \), and equals \( K(u) = \overline{K_{d(x, u)} - 1} \). Hence, by induction \( d(v, y) = d(u, y) + 1 \), since \( K \) is an isometric subgraph of \( G \), \( d(u, x') < d(x, x') \), and \( S \) is a connected component of \( K(u) \).

Thus \( v, x \in [u, y] \) and, since \( v, x \) are neighbours of both \( u \) and \( y \), it follows that \( z \) is contained in \( [u, y] \) and so \( d(u, y') = d(x, y') + 1 = d(z, y') + 2 \) holds. Thus \( d(x, y') = d(z, y') + 1 \), so that \( d(y, z) = d(x, x') + 1 \) also holds.

Now, let \( R := G[x, y] \). Then, in view of the fact that \( d(y, z) = d(y, x) + 1 \) holds for all vertices \( z \) of \( G(x) - H(x) \), it follows that \( G(x) \) is the disjoint union of \( H(x) \) and \( R(x) \). Moreover, since \( R \) is regular and \( z \in R(x') \) implies \( z' \in L(y) \), we have \( |R(x)| = |R(y)| = |L(y)| \) and, since \( H \) is regular, we have \( |H(x)| = |H(y)| \). But \( G \) is regular, and so the degree of \( x \) in \( G \) equals the degree of \( y \) in \( G \). Hence

\[
|G(y)| = |G(x)| = |H(x)| + |R(x)| = |H(y)| + |L(y)|
\]

which completes the proof of Claim 3. \( \Box \)

For any \( z \in V(L) \) we denote by \( \bar{z} \) its antipodal in \( L \), define \( z^* := (\bar{z})' \) and \( H^z := G[z, z^*] \).

Claim 4. Suppose that \( u \) and \( v \) are vertices of \( L \). Then for all \( w \in V(H^u) \) and \( z \in V(H^v) \) we have \( d(w, z) \geq d(u, v) \).

**Proof of Claim 4.** Note that

\[
d(v^u, u) = d(v, u^u) = d(v^u, v) + d(v, u) \quad (2)
\]

holds, which follows since

\[
d(x, x') = d(x, y) + d(y, x')
\]

\[
= d(u, u')
\]

\[
\geq d(u', v^u) + d(v, v^u) + d(v, u)
\]

\[
= d(u, v) + d(v, v^u)
\]

\[
= d(\bar{v}, v) + d(v, v^u)
\]
Proof of Claim 5.\\nSuppose \( w \in V(H^u) \) and \( z \in V(H^t) \). By (2) we have
\[
d(v^*, v) + d(v, u) = d(v^*, u) \leq d(v^*, z) + d(z, w) + d(w, u),
\]
and hence
\[
d(v^*, v) - d(v^*, z) + d(u, v) = d(z, v) + d(v, u) \leq d(z, w) + d(w, u).
\]
Similarly we have
\[
d(w, u) + d(u, v) \leq d(w, z) + d(z, v).
\]
Thus \( d(w, z) \geq d(u, v) \). This completes the proof of Claim 4. \( \square \)

By Claim 4 we see that if \( u \) and \( v \) are distinct vertices of \( L \), then \( V(H^u) = [u, u^*] \) and \( V(H^v) = [v, v^*] \) are disjoint.

Claim 5. Suppose that \( st \) is an edge of \( L \). Let \( P := G[s^*, t] \) and denote the subgraph induced by \( G \) on \( V(H^s) \cup V(H^t) \) by \( Q \). Then \( P \) equals \( Q \). Moreover, \( E(P) - E(H^s) - E(H^t) \) consists of \( |H^s| = |H^t| \) disjoint edges so that, in particular, the map which takes each vertex in \( H^s \) to its (necessarily unique) neighbour in \( H^t \) is an isomorphism between \( H^s \) and \( H^t \).

Proof of Claim 5. We first show that \( Q \) is a subgraph of \( P \). Since \( s^* \) is the antipodal in \( G \) of \( \tilde{s} \) it follows that the edge \( st \) must be contained in some geodesic between \( s^* \) and \( \tilde{s} \). Moreover, as \( s \) and \( t \) are both in \( V(L) \) and \( \tilde{s} \) is the antipodal of \( s \) in \( L \) it follows that there is a geodesic in \( L \) of the form
\[
\gamma : s^*, t^*, w_1, \ldots, w_p, s, t, z_1, \ldots, z_q, \tilde{s}
\]
with \( z_i, w_j \in V(G), 1 \leq i \leq p, 1 \leq j \leq q \). Hence it follows that \( [s^*, s] \) is contained in \( [s^*, t] \). In addition, as \( t^* \) is adjacent to \( s^* \) it also follows that \( [t^*, t] \) is contained in \( [s^*, t] \). Hence \( Q \) is a subgraph of \( P \).

Suppose \( x \) is a vertex of \( H^t \). Denote the antipode of any \( v \in V(P) \) in \( P \) by \( \tilde{v} \). We show that \( \tilde{x} \) is a vertex of \( H^s \). Since \( x \in H^t \) and \( \gamma \) is a geodesic, it follows that there must be a geodesic of the form
\[
t, s, z_1, \ldots, z_l, x, w_1, \ldots, w_m, s^*
\]
with \( z_i, w_j \in V(G), 1 \leq i \leq l, 1 \leq j \leq m \). Hence
\[
s^*, t^*, z_1, \ldots, z_l, \tilde{x}, w_1, \ldots, w_m, t
\]
is a geodesic in \( G \), since the antipodes of \( s \) and \( t \) in \( P \) are \( t^* \) and \( s^* \), respectively. Thus, since the map
\[
\phi : V(P) \to V(P) : x \mapsto \tilde{x}
\]
is an automorphism of \( P \), it induces an isomorphism between \( H^s \) and \( H^t \). In particular, \( H^s \) has the same degree as \( H^t \).
Now, suppose \( x^* \) is the antipode of \( x \) in \( H' \). Then since \( \phi \) is an automorphism of \( P \) and \( d(x, x^*) + 1 = d(x, \tilde{x}) \), it follows that \( x^* \) is adjacent to \( \tilde{x} \). Hence each vertex in \( V(H^u) \) in \( Q \) has degree that is at least its degree in \( H' \) plus one. The same clearly holds for each vertex in \( H' \). Therefore, since \( t \) has degree equal to one plus its degree in \( H' \), it follows that every vertex in \( Q \) has degree at least its degree in \( P \). Therefore, as \( Q \) is a subgraph of \( P \), we see that \( P \) equals \( Q \) and, moreover, that \( E(P) - E(H^u) - E(H^t) \) consists of \(|H^u| = |H'|\) disjoint edges. This completes the proof of Claim 5. \( \square \)

Claim 6. Suppose that \( u \) is a vertex of \( L \). For all \( w \in V(H^u) \) there exists a unique vertex \( z \in V(H) \) with \( d(w, z) = d(u, y) \).

Proof of Claim 6. Suppose that \( z_1 = u, z_2, \ldots, z_m = y \) is a geodesic in \( L \) between \( u \) and \( y \). Applying Claim 4 to the graph induced by \( G \) on \([z_i^*, z_{i+1}]\) for each \( 1 \leq i \leq m - 1 \), in view of Claim 5 it immediately follows that if \( w \in V(H^u) \), then there is some vertex \( z \in V(H) \) with \( d(w, z) = d(u, y) \).

We now show that this vertex \( z \) is unique. Suppose \( t \in V(H) \) with \( d(z, w) = d(t, w) = d(u, y) \). Let \( \tilde{z} \) denote the antipode of \( z \) in \( H \) and \( \tilde{w} \) denote the antipode of \( w \) in \( H' \). By interchanging the roles of \( H \) and \( L \) we see that

\[
d(w, \tilde{z}) = d(w, \tilde{w}) + d(\tilde{w}, \tilde{z}) = d(w, \tilde{w}) + d(w, z)
\]

holds. But we also have

\[
d(w, \tilde{z}) \leq d(t, w) + d(\tilde{z}, t) = d(z, w) + d(\tilde{z}, t).
\]

Hence \( z = t \). This completes the proof of Claim 6. \( \square \)

Claim 7. Let \( K \) denote the subgraph induced by \( G \) on the set

\[
\bigcup_{w \in V(L)} V(H^w).
\]

Then \( K \) equals \( G \).

Proof of Claim 7. Suppose \( u \in V(L) \). In view of Claim 5, the fact that \( L \) is regular and the fact that, for \( w, w' \in V(L) \) neighbours of \( u \), the set \( V(H^u) \cap V(H^w) \) is non-empty if and only if \( w = w' \), it follows that \( K \) is regular and that the degree of \( u \) in \( K \) equals the degree of \( y \) in \( K \). But by Claim 3 the degree of \( y \) in \( K \) is equal to the degree of \( y \) in \( G \). Thus, since \( K \) is a subgraph of \( G \) and \( G \) is regular and connected, it follows that \( K \) equals \( G \). This completes the proof of Claim 7. \( \square \)

In view of Claims 4 and 6, the map from \( V(H^u) \) to \( V(H) \) which takes each vertex \( w \) of \( L \) to the vertex in \( H \) that is closest to \( w \) in \( G \) is an isomorphism between \( H^u \) and \( H \). Moreover, by interchanging the roles \( H \) and \( L \), and defining for each \( a \in V(H) \) a graph \( L^a \) in an analogous way to the way in which \( H^u \) was defined for \( u \in L \), we see that the map that takes each vertex \( z \) in \( L^a \) to the vertex in \( L \) that is nearest to \( z \) is an isomorphism between \( L^a \) and \( L \).

By Claim 7, for any vertex \( v \in V \) there must be some \( u \in L \) with \( v \in V(H^u) \). Hence, since \( V(H^u) \) and \( V(H^v) \) are disjoint for \( u, v \in V(L) \) distinct, it follows that
\{V(H^u) : u \in V(L)\} is a partition of \(V\). Interchanging the roles of \(H\) and \(L\), we see that 
\{V(L^a) : a \in V(H)\} is also a partition of \(V\).

In view of these observations, it is now straightforward to see that the map \(\Phi : V \to V(H) \times V(L)\), which takes each vertex \(v \in V\) to the pair \((u, l)\) where \(u\) is the unique vertex in \(H\) that is closest to \(v\) and \(l\) is the unique vertex of \(L\) that is closest to \(v\), is a bijection.

**Claim 8.** The map \(\Phi\) induces an isomorphism between \(G\) and \(H \times L\).

**Proof of Claim 8.** The claim follows in view of the fact that any edge of \(G\) is either an edge of \(H^u\) for some \(u \in V\) or an edge of \(L^a\) for some \(a \in V\). \(\square\)

We now finish the proof of the theorem. By Claim 8 we know that \(G\) is isomorphic to \(H \times L\) where, by Claim 2, \(H\) is locally connected. In view of Proposition 3.2 and the fact that \(L\) is strongly spherical, it is straightforward to complete the proof of the theorem using an inductive argument on the diameter of \(G\).

**Acknowledgements**

The authors thank the anonymous referee for the helpful comments. V. Moulton thanks the Swedish Research Council (VR) for its support. D. Stevanović thanks the DIMACS Center, Rutgers University, USA for its support. This work was done when J. Koolen was visiting Com2MaC, POSTECH, Pohang, South Korea and he thanks for the support of KOSEF-Com2MaC

**References**


