Relative multiplicities of graded algebras

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ABSTRACT

Without any finiteness assumption we define a sequence of relative multiplicities for a pair \( A \subset B \) of standard graded Noetherian algebras that extends the notion of relative multiplicities of Simis, Ulrich and Vasconcelos and unifies them with the \( j \)-multiplicity of ideals introduced by Achilles and Manaresi as well as the \( j \)-multiplicity of modules defined by Ulrich and Validashti. Using our relative multiplicities, we give numerical criteria for integrality and birationality of the extension \( A \subset B \).

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1. Introduction

Let \( R \) be a Noetherian ring and \( A \subset B \) be a homogeneous inclusion of standard graded Noetherian rings with \( A_0 = B_0 = R \). We call the extension \( A \subset B \) birational if \( A_\mathfrak{p} = B_\mathfrak{p} \) for all minimal primes \( \mathfrak{p} \) of \( A \). If \( R \) is local with maximal ideal \( \mathfrak{m} \), the extension \( A \subset B \) is called weakly birational if \( A_\mathfrak{p} = B_\mathfrak{p} \) for all minimal primes \( \mathfrak{p} \) of \( A \) containing \( \mathfrak{m} \) with \( \dim A/\mathfrak{p} = \dim B \). We would like to study numerical criteria for integrality and birationality of the extension \( A \subset B \). An example of such an extension arises when two submodules \( U \subset E \) of a free module \( F = R^e \) are given. In this case, \( U \) and \( E \) generate subalgebras \( R[U] \subset R[E] \) in the symmetric algebra \( \text{Sym}(F) = R[x_1, \ldots, x_e] \) of \( F \), called the Rees algebras of \( U \) and \( E \). Let \( G \) denote the associated graded ring of \( B \) with respect to the ideal \( A_1 \). In [11] Simis, Ulrich and Vasconcelos have shown that \( A \subset B \) is integral or birational if certain modules over \( G \) have small dimension. To be precise, they express the integrality and birationality or the integrality of the extension \( A \subset B \) in terms of the positivity of the codimension of the annihilator \( 0 :_G B_1 G \) or of its stable value \( 0 :_G (B_1 G)^\infty \). Similar ideas have been used by Katz, Kirby, Kleiman, Rees and Thorup [5,7–10]. If \( R \) is local and \( B_1/A_1 \) has finite length, an infinite series of relative
multiplicities are defined for the extension $A \subset B$ [11]. These numbers eventually become stable and can be used to detect birationality and integrality. We generalize these results further to the case where the length $\lambda_R(B_1/A_1)$ is not finite, by introducing a series of relative multiplicities $j^t(A|B)$ associated to the numerical functions

$$
\Sigma^t(n) := \lambda_R\left( \Gamma_m\left( \frac{B_n}{A_1B_{n-1}} \oplus \frac{A_1B_{n-1}}{A_2B_{n-2}} \oplus \cdots \frac{A_{n-t}B_t}{A_{n-t+1}B_{t-1}} \right) \right)
$$

for $n \geq t$, where $\Gamma_m$ denotes the zeroth local cohomology with respect to the maximal ideal $m$ of $R$. In fact, the function $\Sigma^t(n)$ can be expressed as a polynomial of the form

$$
\frac{j^t(A|B)}{(\dim B - 1)!} n^{\dim B - 1} + \text{lower terms}
$$

for $n$ large. We suppress $t$ when $t$ is 1. One can show that $j^1(A|B)$ has a stable value denoted by $j_\infty(A|B)$. If $\lambda_R(B_1/A_1)$ is finite, then the relative multiplicity $j^t(A|B)$ is equal to the relative multiplicity $e_t(A, B)$ in the sense of Simis, Ulrich and Vasconcelos [11], which was also considered by Kirby and Rees [7]. Likewise, $j_\infty(A|B)$ is equal to the relative multiplicity $e_\infty(A, B)$, that was introduced by Kleiman and Thorup [8, 5.6 and 5.7]. It turns out that similar to the Artinian case, the vanishing of $j_\infty(A|B)$ is related to the integrality of the extension, whereas $j(A|B)$ detects both integrality and birationality. We also prove that being weakly birational is equivalent to the equality of the above two relative multiplicities. One may be tempted to define a relative multiplicity based on the simpler function $\Lambda(n) := \lambda_R(\Gamma_m(B_n/A_n))$ instead [13]. However, this function need not be polynomial eventually [2]. Our main results are as follows.

**Theorem 1.1.** If $A \subset B$ is a finite and birational extension, then $j(A_q|B_q)$ vanishes for all primes $q$ of $R$. The converse holds if $B$ is universally catenary and locally equidimensional at every prime of $R$.

**Theorem 1.2.** If $B$ is integral over $A$, then $j_\infty(A_q|B_q)$ vanishes for all primes $q$ of $R$. The converse holds if $B$ is universally catenary and locally equidimensional at every prime of $R$.

**Theorem 1.3.** If $R$ is local, then the extension $A \subset B$ is weakly birational if and only if $j_\infty(A|B)$ is equal to $j(A|B)$.

One can show that for the converse statements in above, we only need to consider finitely many primes of $R$. The paper is organized in the following way. In Section 2 we provide some background on multiplicities of standard graded algebras over a Noetherian local ring. In Section 3 we introduce the sequence of relative multiplicities and we explore their basic properties. We describe our results on criteria for integrality and birationality in Section 4. In Section 5 we prove an inequality between the multiplicities of graded algebras over an Artinian local ring. Section 6 is devoted to the case of an extension of Rees algebras of two modules.

### 2. Preliminaries

We recall the notion of $j$-multiplicity for graded modules as introduced and developed in [4, 6.1]. Let $R$ be a Noetherian ring with a fixed maximal ideal $m$, and $T$ a standard graded Noetherian $R$-algebra, i.e., a graded $R$-algebra with $T_0 = R$ that is generated by finitely many homogeneous elements of degree one. Let $S$ be a finitely generated graded module over $T$. Notice that $\Gamma_m(S) \subset S$ is a graded $T$-submodule, where $\Gamma_m$ denotes the zeroth local cohomology with respect to the maximal ideal $m$ of $R$. In particular, $\Gamma_m(S)$ is finitely generated over $T$. Thus there exists a fixed power $m^2$ of $m$ that annihilates it, and then $\Gamma_m(S)$ can be regarded as a finitely generated graded module over $T/m^2T$, a standard graded Noetherian algebra over the Artinian local ring $R/m^2$. Hence $\Gamma_m(S)$ has a Hilbert function that is eventually polynomial of degree at most $\dim S - 1$ and gives the multiplicity $e(\Gamma_m(S))$. 

Now let $D$ be any integer with $D \geq \dim S$. One defines the $j$-multiplicity $j_D(S)$ of the graded module $S$ to be $e(\Gamma_m(S))$ when $D = \dim \Gamma_m(S)$ and zero otherwise. Also notice that

$$j_D(S) = (D - 1)! \lim_{n \to \infty} \frac{\lambda_R(\Gamma_m(S_n))}{n^{D-1}}.$$ 

If $D = \dim S$ we simply write $j(S)$ instead of $j_D(S)$. If the graded components of $S$ have finite length, then we write $e_D(S)$ for $j_D(S)$. Note that for $D \geq \dim S$ we have $\dim S/mS < D$ if and only if $\dim \Gamma_m(S) < D$. Therefore, $j_D(S) = 0$ if and only if $\dim S/mS < D$. Consider the set of prime ideals $q \subset R$ such that $j(S_q)$ does not vanish. Then this set is finite, for if $q$ is a prime with $j(S_q) \neq 0$, then $\dim S_q/qS_q = \dim S_q$. Therefore, $q$ is contracted from a minimal prime in $\text{Supp}_R(S)$. We refer to [4] for basic properties of $j$-multiplicities.

3. Relative multiplicities

Let $R$ be a Noetherian ring. Let $A \subset B$ be an inclusion of homogeneous standard graded $R$-algebras with $A_0 = B_0 = R$ and $M$ be a finitely generated graded $B$-module. Let $G$ denote the associated graded algebra of $B$ with respect to the ideal $I := A_1B$. Similarly, consider the associated graded module $G(M)$ of $M$ with respect to the ideal $I$, as a module over $G$. We endow $G$ and $G(M)$ with the internal grading as introduced in [12, Definition 2.3], where one considers the grading induced by $B$ and $M$ instead of the usual grading on $G$ and $G(M)$ respectively. Therefore, for a non-negative integer $t$, the graded components of the module $B_tG(M)$ under the internal grading are

$$[B_tG(M)]_n = \bigoplus_{i=0}^{\infty} [i^iB_tM + i^{i+1}M/i^{i+1}M]_n,$$

for $n \geq t$. Moreover, if $M$ is generated in degree zero, for $n \geq t$ we can write

$$[B_tG(M)]_n = \bigoplus_{i=0}^{n-t} [i^iM/i^{i+1}M]_n.$$

Thus $G$ becomes a standard graded Noetherian $R$-algebra and $B_tG(M)$ is a finitely generated graded module over $G$ for any non-negative integer $t$. Let, in addition, $R$ be local with maximal ideal $m$. For an integer $D \geq \dim M$ we define the $t$-th relative $j$-multiplicity of $A$ and $B$ with coefficients in $M$ as

$$j^*_D(A\mid B, M) := j_D(B_tG(M)) = e_D(\Gamma_m(B_tG(M))).$$

If $M = B$ we simply write $j^*_D(A\mid B)$, and we will suppress $D$ when it is equal to $\dim M$. We will also omit $t$ when $t = 1$. Note that if $M$ is generated in degree zero, the $n$-th graded component of the module $B_tG(M)$ for $n \geq t$ is

$$[B_tG(M)]_n = \frac{B_nM_0}{A_1B_{n-1}M_0} \oplus \frac{A_1B_{n-1}M_0}{A_2B_{n-2}M_0} \oplus \cdots \oplus \frac{A_{n-t}B_tM_0}{A_{n-t+1}B_{t-1}M_0}.$$

Thus the Hilbert function of $\Gamma_m(B_tG(M))$ for $n \geq t$ is

$$\Sigma^t(n) := \lambda_R(\Gamma_m([B_tG(M)]_n)) = \sum_{i=0}^{n-t} \lambda_R(\Gamma_m\left(\frac{A_iB_{n-i}M_0}{A_{i+1}B_{n-i-1}M_0}\right)).$$
The corresponding Hilbert polynomial has degree at most $D - 1$ and is of the form

$$j_D^j (A|B, M) n^{D-1} + \text{lower terms.}$$

It follows that

$$j_D^j (A|B, M) = \lim_{n \to \infty} \frac{(D - 1)!}{n^{D-1}} \cdot \sum_{i=0}^{n-t} \lambda_R \left( \Gamma_m \left( \frac{A_i B_{n-i} M_0}{A_{i+1} B_{n-i-1} M_0} \right) \right).$$

The relative multiplicities unify several notions of multiplicity. In the above setting, $j_D^0 (A|B, M) = j_D^0 (I, M)$ and $j_D^j (A|B, M) = j_D^j (I, M)$ defined by Ulrich and Validashti in [12]. In particular, if we set $A := R[E]$ and $B := \text{Sym}(F)$, where $E$ is a submodule of a free module $F$, then $j(A|B)$ coincides with the $j$-multiplicity $j(E)$ of $E$ in [12]. See also Section 6 in this regard. Moreover, if $\lambda_R(F/E)$ is finite, then $j(A|B)$ is equal to the Buchsbaum–Rim multiplicity $br(E)$ of $E$.

If $M = B$ and $\lambda_R(B_1/A_1)$ is finite, then $\Sigma^j(n)$ reduces to $\lambda_R(B_n/A_n-t+1B_{t-1})$. In this case, the normalized leading coefficient of the arising polynomial is called $e_t(A, B)$ by Simis, Ulrich and Vasconcelos in [11]. Thus $j^t(A|B) = e_t(A, B)$.

Let $a$ be an ideal in $R$ and $N$ a finitely generated $R$-module. Consider the inclusion of algebras $A := R[\alpha] \subset B := R[\tau]$ and set $M := N \otimes_R B$. Then $\Sigma^j(n)$ becomes

$$\sum_{i=0}^{n-t} \lambda_R \left( \Gamma_m \left( a^i N/a^{i+1} N \right) \right),$$

which is a sum transform of the Hilbert function of the associated graded module $\text{gr}_a(N)$ after applying $\Gamma_m$. Thus it gives rise to a polynomial of degree at most $\dim N$, whose normalized leading coefficient is called the $j$-multiplicity $j(a, N)$ of an ideal $a$ with coefficient module $N$ introduced by Achilles and Manaresi in [1]. Hence $j^j(A|B, M)$ is equal to $j(a, N)$ for all $t \geq 0$ in this case.

**Remark 3.1.** By the definition of relative multiplicities, $j_D^j (A|B, M)$ vanishes if and only if $\dim \Gamma_m(B_1 G(M)) < D$. Thus, the set of primes $q$ of $R$ for which $j_D^j (A_q|B_q, M_q) \neq 0$ is finite, and that such primes are contracted from minimal primes in the support of $B_1 G(M)$.

We observe that $B_t G(M)$ is a descending chain of submodules of $G(M)$ with respect to $t$. Thus from the definition of relative multiplicities we obtain that $j_D^j (A|B, M)$ is a non-increasing sequence of non-negative integers,

$$j_D^0 (A|B, M) \geq j_D^1 (A|B, M) \geq \cdots \geq 0.$$ 

Hence it will be constant for $t$ large, and we denote its stable value by $j_D^\infty (A|B, M)$. With a similar proof to that of [12, 3.2], we can show the following statement.

**Theorem 3.2.** If $\dim M/mM < D$, then the sequence of relative multiplicities $j_D^j (A|B, M)$ is constant.

Note that the chain of ideals

$$0 :_G \Gamma_m(B_1 G(M)) \subset 0 :_G \Gamma_m(B_2 G(M)) \subset \cdots$$

eventually stabilizes, and we denote the stable ideal by $0 :_G \Gamma_m(B_1 G(M))^\infty$. The following result gives another way of seeing the constancy of the higher relative multiplicities.
Proposition 3.3. For \( t \) large,
\[
j_D^t(A|B, M) = e_D(G(M))/(0 : G \Gamma_m(B_tG(M)))G(M).
\]
In particular,
\[
j_D^\infty(A|B, M) = e_D(G(M)/(0 : G \Gamma_m(B_1G(M))^\infty))G(M).
\]

Proof. Choose \( t \) large so that \( \text{Supp}_G(B_tG(M)) \) is stabilized. By the associativity formula for multiplicities of graded modules, it is enough to show
\[
\left( \Gamma_m(B_tG(M)) \right)_\Omega \simeq \left( G(M)/(0 : G \Gamma_m(B_tG(M)))G(M) \right)_\Omega
\]
for every \( \Omega \in \text{Supp}_G(B_tG(M)) \) of dimension \( D \). We may further restrict to those \( \Omega \) with \( \Omega \cap R = m \).
Also note that \( B_1G_\Omega = G_\Omega \), since otherwise \( B_1G_\Omega \subset QG_\Omega \) and then \( B_1G_\Omega \) is nilpotent on \( G(M)_\Omega \), which contradicts our assumption that \( \Omega \in \text{Supp}_G(B_tG(M)) \) for \( t \) large. Now the left-hand side becomes
\[
\left( \Gamma_m(B_tG(M)) \right)_\Omega \simeq \Gamma_{mG_\Omega}(B_tG(M)_\Omega) = B_tG(M)_\Omega = G(M)_\Omega.
\]

For the right-hand side we obtain
\[
\left( G(M)/(0 : G \Gamma_m(B_tG(M)))G(M) \right)_\Omega \simeq (0 : G_\Omega \Gamma_{mG_\Omega}(B_tG(M)_\Omega))G(M)_\Omega = 0
\]
and the result follows. \( \Box \)

We can restate Theorem 2.5 of [12] in terms of relative multiplicities.

Theorem 3.4. Let \( 0 \to M' \to M \to M'' \to 0 \) be a short exact sequence of finitely generated graded \( B \)-modules, and \( D \) an integer with \( D \geq \dim M \). Then
\[
j_D^t(A|B, M) = j_D^t(A|B, M') + j_D^t(A|B, M'').
\]

Also note that in the proof of part (a) in Theorem 3.3 of [12], we are indeed showing the following triangle inequality.

Theorem 3.5. Let \( R \) be a Noetherian local ring and \( A \subset B \subset C \) inclusions of homogeneous standard graded algebras with \( A_0 = B_0 = C_0 = R \). Let \( M \) be a finitely generated graded \( C \)-module generated in degree zero, and \( D \) an integer with \( D \geq \dim M \). If \( B_1M_0/A_1M_0 \) has finite length over \( R \), then
\[
j_D(A|C, M) \geq j_D(B|C, M) + j_D(A|B, M_0B).
\]

In the above inequality, \( M_0B \) denotes the \( B \)-module generated by the degree zero part of \( M \). Unfortunately, the equality may not hold in general. See for instance the discussion at the end of Section 6. Finally, the example discussed in Remark 2.4 in [12] yields the following instance of the relative \( j \)-multiplicities of algebras.
Example 3.6. Let $R := k[[y_1, \ldots, y_d]]$. Consider the homogeneous inclusion of graded $R$-algebras $A := R[x_1y_1, \ldots, x_dy_d] \subset B := R[x_1, \ldots, x_d]$, where $R$ is sitting in degree zero and $x_i$ have degree one. Then $j(A|B) = 1$. Therefore $j^t(A|B) = 1$ for all non-negative integers $t$, see for instance Theorem 3.2.

4. Integral and birational extensions

In this section, we will show that the relative multiplicities can be used to detect finite and birational extensions. We retain the notations of Section 3. In addition, we assume that the $B$-module $M$ is generated in degree zero. We say $B$ is integral over $A$ on $M$ if the ideal $A_1B$ is a reduction of the ideal $B_1B$ on $M$, equivalently, $B_{n+1}M = A_1B_nM$ for a non-negative integer $n$. Let $G$ denote the associated graded ring of $B$ with respect to the ideal $A_1B$. In [11, 2.3, 2.5] Simis, Ulrich and Vasconcelos translate the integrality and birationality of the extension $A \subset B$ in terms of the positivity of the codimension of the annihilator $0 :_G B_1G$ or of its stable value $0 := (B_1G)^\infty$. We will need the following version of their results that allows a coefficient module.

Theorem 4.1. Let $t$ be a positive integer and $M$ a faithful $B$-module. Then the height of ideal $B_1G + (0 :_G B_tG(M))/B_1G$ is positive if and only if $M_\mathfrak{P} = \sum_{i=0}^{t-1} M_iA_\mathfrak{P}$ for all minimal primes $\mathfrak{P}$ of $A$. Also the ideal $0 :_G B_tG(M)$ has positive height if and only if $B$ is integral over $A$ on $M$ and $M_\mathfrak{P} = \sum_{i=0}^{t-1} M_iA_\mathfrak{P}$ for all minimal primes $\mathfrak{P}$ of $A$.

Proof. First observe that $G/B_1G$ can be identified with $A$ as $[G/B_1G]_n \simeq A_n$ for all $n$. Thus by Nakayama’s Lemma,

$$V(B_1G + (0 :_G B_tG(M))/B_1G) = \text{Supp}_{G/B_1G}(B_tG(M)/B_{t+1}G(M))$$

$$= \text{Supp}_A(B_tG(M)/B_{t+1}G(M)).$$

Note that $B_tG(M)/B_{t+1}G(M) \simeq \sum_{i=0}^{t} M_iA/\sum_{i=0}^{t-1} M_iA$ as $A$-modules. Since $M$ is generated in degree zero and $B$ is standard graded, we also have

$$\text{Supp}_A\left(\sum_{i=0}^{t} M_iA/\sum_{i=0}^{t-1} M_iA\right) = \text{Supp}_A\left(M/\sum_{i=0}^{t-1} M_iA\right).$$

This proves the first part. If $0 :_G B_tG(M)$ has positive height, then $B_tG(M) = 0$ for every minimal prime $\mathfrak{Q}$ of $G$. Therefore $B_1G$ is contained in every minimal prime of $G$ in the support of $G(M)$. Since $M$ is faithful $B$-module, the annihilator of $G(M)$ is nilpotent. Hence $B_1G$ is nilpotent and a power of $B_1G$ annihilates $G(M)$. Therefore, a power of $B_1B/A_1B$ annihilates $M/A_1M$, showing that $B$ is integral over $A$ on $M$. Now the asserted equivalence follows from previous part. □

Note that the chain of ideals

$$0 :_G B_1G(M) \subset 0 :_G B_2G(M) \subset \cdots$$

eventually stabilizes, and we denote the stable ideal by $0 :_G B_1G(M)^\infty$. We have the following corollary of Theorem 4.1.

Corollary 4.2. The ideal $0 :_G B_1G(M)$ has positive height if and only if $B$ is integral over $A$ on $M$ and $M_\mathfrak{P} = M_0A_\mathfrak{P}$ for all minimal primes $\mathfrak{P}$ of $A$. The ideal $0 :_G B_1G(M)^\infty$ has positive height if and only if $A \subset B$ is integral on $M$. 

Lemma 4.3. Let \( t \) be a non-negative integer. Then height of \( 0 : G B_t G(M) \) is positive if and only if the ideal \( 0 : G B_t G(M)_q \) has positive height for all primes \( q \) of \( R \), equivalently, for all primes \( q \) of \( R \) that are contracted from a minimal prime of the support of \( B_t G(M) \).

Proof. To prove the forward direction assume the height of \( 0 : G B_t G(M) \) is positive. Then the height of \( 0 : G B_t G(M)_q \) is positive for all primes \( q \) of \( R \). Hence \( 0 : G B_t G(M)_q \) has positive height since it contains \( 0 : G B_t G(M)_q \). To prove the other direction, let \( \Omega \) be a minimal prime in the support of \( B_t G(M) \), and let \( q = \Omega \cap R \). Then \( q G \Omega \) is nilpotent on \( B_t G(M)_\Omega \). Thus

\[
\left( \Gamma_{qG} (B_t G(M)) \right)_\Omega = \Gamma_{qG} (B_t G(M)_{\Omega}) = B_t G(M)_{\Omega}.
\]

Therefore

\[
\left( 0 : G \Gamma_{qG} (B_t G(M)) \right)_\Omega = 0 : G \Gamma_{qG} (B_t G(M)_{\Omega}) = 0 : G B_t G(M)_{\Omega} \neq G_{\Omega}.
\]

Hence \( 0 : G \Gamma_{qG} (B_t G(M)_q) \) is contained in \( \Omega_q \). Thus \( \Omega \) has positive height. \( \square \)

Note that there are finitely many primes \( q \) of \( R \) that are contracted from a minimal prime of \( 0 : G B_t G(M)^\infty \). Thus, there exists a non-negative integer \( t \) such that the chains of ideals \( 0 : G B_t G(M) \) and \( 0 : G \Gamma_q G(B_t G(M)) \) become stable at the same time for all such primes \( q \). Therefore, we have the following corollary of Lemma 4.3.

Corollary 4.4. The height of \( 0 : G B_t G(M)^\infty \) is positive if and only if the ideal \( 0 : G \Gamma_q G(B_t G(M)_q)^\infty \) has positive height for all primes \( q \) of \( R \), equivalently, for all primes \( q \) of \( R \) that are contracted from a minimal prime of \( 0 : G B_t G(M)^\infty \).

Using Lemma 4.3, we can generalize [11, 3.3] to the case where \( R \) may not be Artinian and give a criterion for the integrality and birationality of the extensions.

Theorem 4.5. If \( A \subset B \) is integral on \( M \) and \( M_P = \sum_{i=0}^{t-1} M_iA_P \) for all minimal primes \( P \) of \( A \), then \( j^i(A_q|B_q, M_q) = 0 \) for all primes \( q \) of \( R \). Conversely, if \( B \) is universally catenary, \( M \) is locally equidimensional at every prime of \( R \), and \( j^i(A_q|B_q, M_q) = 0 \) for all primes \( q \) of \( R \) that are contracted from a minimal prime of the support of \( B_t G(M) \), then \( A \subset B \) is integral on \( M \) and \( M_P = \sum_{i=0}^{t-1} M_iA_P \) for all minimal primes \( P \) of \( A \).

Proof. We may assume \( M \) is a faithful \( B \)-module. By Theorem 4.1, \( B \) is integral over \( A \) on \( M \) and \( M_P = \sum_{i=0}^{t-1} M_iA_P \) for all minimal primes \( P \) of \( A \) if and only if the ideal \( 0 : B_t G(M) \) has positive height. By Lemma 4.3, this is equivalent to the height of \( 0 : G_q \Gamma_q G(M)_q \) being positive for all primes \( q \) of \( R \), equivalently, for all primes \( q \) of \( R \) that are contracted from a minimal prime of the support of \( B_t G(M) \). Therefore, the result follows from the fact that \( j^i(A_q|B_q, M_q) = 0 \) if and only if

\[
\dim G_q/0 : G_q \Gamma_q G(M)_q < \dim G(M)_q. \quad \square
\]

Corollary 4.6. If \( A \subset B \) is a finite and birational extension, then \( j(A_q|B_q) = 0 \) for all primes \( q \) of \( R \). The converse holds if \( B \) is universally catenary and locally equidimensional at every prime of \( R \).

We can also use the stabilized relative \( j \)-multiplicities to generalize [11, 3.5(b)] and give a criterion for the integrality of the extensions.

Theorem 4.7. If \( B \) is integral over \( A \) on \( M \), then \( j^\infty(A_q|B_q, M_q) = 0 \) for all primes \( q \) of \( R \). Conversely, if \( B \) is universally catenary, \( M \) is locally equidimensional at every prime of \( R \), and \( j^\infty(A_q|B_q, M_q) = 0 \) for all primes \( q \) of \( R \) that are contracted from a minimal prime of \( 0 : G B_t G(M)^\infty \), then \( B \) is integral over \( A \) on \( M \).
Proof. We may assume $M$ is a faithful $B$-module. By Corollary 4.2, $B$ is integral over $A$ on $M$ if and only if $0 : C_1 B_1 G(M)_q^{\infty}$ has positive height. By Corollary 4.4, this holds if and only if the height of $0 : G_q (B_1 G(M)_q)^{\infty}$ is positive for all primes $q$ of $R$, equivalently, for all primes $q$ of $R$ that are contracted from a minimal prime of $0 : C_1 B_1 G(M)_q^{\infty}$. Therefore the result follows from the fact that $j^\infty(A_q | B_q, M_q) = 0$ if and only if
\[
\dim G_q / 0 : G_q \Gamma_q G_q (B_1 G(M)_q)^{\infty} < \dim G(M)_q. \tag*{\square}
\]

Notice that for the converse statements in Theorems 4.5 and 4.7 we only need to check the vanishing of the relative multiplicities for finitely many primes of $R$. The following result is an attempt to recover [11, 3.7] and characterize weakly birational extensions.

**Theorem 4.8.** Let $R$ be local with maximal ideal $m$. Then $M_\mathfrak{p} = \sum_{t=0}^{l-1} M_t A_\mathfrak{p}$ for all minimal primes $\mathfrak{p}$ of $A$ containing $m$ if and only if
\[
j^s(A|B, M) = j^t(A|B, M)
\]
for some $s > t$, equivalently, for all $s > t$.

**Proof.** We may assume $M$ is a faithful $B$-module. Let $\mathcal{U}$ denote the set of all primes $\Omega$ in the support of $B_t G(M)$ with $\Omega \cap R = m$ and $\dim G/\Omega = \dim B$. First observe that by the associativity formula for multiplicity of graded modules, applied to the graded modules $B_t G(M)$ and $B_t G(M)$, $j^s(A|B, M) = j^t(A|B, M)$ if and only if $B_t G(M)_\Omega = B_t G(M)_\Omega$ for all primes $\Omega$ in $\mathcal{U}$. This holds if and only if $B_t G(M)_\Omega = G_\Omega$ for all primes $\Omega$ in $\mathcal{U}$. For if $B_t G(M)_\Omega = B_t G(M)_\Omega$ and $B_1 G_\Omega$ is a proper ideal of $G_\Omega$, then
\[
B_t G(M)_\Omega = B_{s-t} B_t G(M)_\Omega = \cdots = B_{k(s-t)} B_t G(M)_\Omega
\]
for all positive integers $k$. But $B_{k(s-t)} G(M)_\Omega = 0$ for $k$ large, which contradicts $\Omega$ being in the support of $B_t G(M)$. On the other hand, in the proof of Theorem 4.1 we have shown
\[
V\left( B_1 G + (0 : C_1 B_1 G(M)) / B_1 G \right) = \text{Supp}_A \left( M / \sum_{t=0}^{l-1} M_t A \right).
\]
Therefore, $M_\mathfrak{p} = \sum_{t=0}^{l-1} M_t A_\mathfrak{p}$ for all minimal primes $\mathfrak{p}$ of $A$ with $\mathfrak{p} \cap R = m$ and $\dim A/\mathfrak{p} = \dim B$ if and only if $B_1 G + (0 : C_1 B_1 G(M))$ is not contained in any prime $\Omega$ with $\Omega \cap R = m$ and $\dim G/\Omega = \dim B$. Hence the result follows from the above-mentioned observation. \(\square\)

**Corollary 4.9.** If $R$ is local, then the extension $A \subset B$ is weakly birational if and only if $j^\infty(A|B) = j(A|B)$.

It is interesting to compare our results with the following statement that is in fact proved in [12, 3.4], generalizing Flenner and Manaresi’s criterion for integral dependence of ideals in [3].

**Theorem 4.10.** Let $R$ be a universally catenary Noetherian ring and $A \subset B \subset C$ inclusions of homogeneous standard graded algebras such that $A_q = B_q = C_q = R$. Let $M$ be a graded $C$-module generated by finitely many homogeneous elements of degree zero. If $A \subset B$ is integral on $M$, then $j(A_q | C_q, M_q) = j(B_q | C_q, M_q)$ for all primes $q$ of $R$. The converse holds if $M$ is equidimensional locally at every maximal ideal of $R$, and $A_p = C_p$ for every prime $p$ of $R$ that contracts from a minimal prime of $\text{Supp}_C(M)$.

Note that if $B$ and $C$ are equal, then $j(B_q | C_q, M_q) = 0$ for all primes $q$ of $R$. Thus the above result implies Theorem 4.5 for $t = 1$, under the additional assumption that $A_p = B_p$ for every prime $p$ of $R$ that contracts from a minimal prime of $\text{Supp}_B(M)$.
5. Comparing multiplicities

Throughout this section, in addition to the notations of Section 3, we assume that \( R \) is an Artinian local ring, \( B \) is equidimensional and universally catenary with the homogeneous maximal ideal \( n \), and the extension \( A \subset B \) is birational. We would like to compare the multiplicities of \( A \) and \( B \) and give a lower bound for the difference in terms of the \( j \)-multiplicity of certain ideals.

**Proposition 5.1.** Under the above assumptions, \( e(B) = e(G) \) and \( e(A) = j(G_n) \).

**Proof.** First note that \( e(B) = e(G) \) since

\[
\lambda_R \left( (G) n \right) = \sum_{i=0}^{n} \lambda_R \left( \frac{A_1 B_{n-i}}{A_{i+1} B_{n-i-1}} \right) = \lambda_R \left( B_n \right).
\]

By Corollary 4.9 and Proposition 3.3, we can write

\[
e(B) - e(A) = j(A|B) = j^\infty(A|B) = e_D \left( G/0 :_G (B_1 G)^\infty \right)
\]

where \( D := \dim B \). Note that \( 0 :_G (B_1 G)^\infty = \Gamma_{B_1 G}(G) = \Gamma_n(G) \). Therefore, by the additivity of the multiplicity along the short exact sequence

\[
0 \to \Gamma_n(G) \to G \to G/0 :_G (B_1 G)^\infty \to 0
\]

we obtain

\[
e_D \left( G/0 :_G (B_1 G)^\infty \right) = e_D(G) - e_D \left( \Gamma_n(G) \right).
\]

Thus \( e(B) - e(A) \) is equal to \( e_D(G) - e_D(\Gamma_n(G)) \). Now the result follows as \( e_D(G) = e(B) \) and \( e_D(\Gamma_n(G)) = j(G_n) \). Here we are using the fact that the maximal homogeneous ideals of \( G \) with respect to the usual grading and the internal grading are the same, and since \( R \) is an Artinian local ring, the multiplicity of a graded module over \( G \) is the same as the local multiplicity with respect to the maximal homogeneous ideal of \( G \).  \( \square \)

Note that \( j(G_n) \) can also be written as \( j(A_1 B_n) \) in the sense of Achilles and Manaresi in [1]. We will need the following result from [11, 6.3].

**Proposition 5.2.** Let \( \mathfrak{Q} \) be a minimal prime of \( G \) and set \( \mathfrak{P} = \mathfrak{Q} \cap B \). Then

\[
e(G/\mathfrak{Q}) \geq e(G_{\mathfrak{P}}/\mathfrak{Q}G_{\mathfrak{P}}) \cdot e(B/\mathfrak{P}).
\]

The following result gives a lower bound for the multiplicity of \( B \) in terms of the \( j \)-multiplicities of the ideal \( A_1 B \), locally at certain primes of \( B \). In the following, \( \ell(\cdots) \) denotes the analytic spread of an ideal.

**Theorem 5.3.**

\[
e(B) \geq \sum_{\mathfrak{P}} j(A_1 B_{\mathfrak{P}}) \cdot e(B/\mathfrak{P}),
\]

where the sum is taken over all primes \( \mathfrak{P} \) containing \( A_1 B \) with \( \ell(A_1 B_{\mathfrak{P}}) = \dim B_{\mathfrak{P}} \).
Proof. By the associativity formula for multiplicities and by Proposition 5.2 we have

\[ e(G) = \sum_{\Omega \in \text{Min}(G)} \lambda(G_{\Omega}) \cdot e(G/\Omega) \]

\[ \geq \sum_{\Omega \in \text{Min}(G)} \lambda(G_{\Omega}) \cdot (e(G_{/\Omega}G_{/\Omega}) \cdot e(B/\mathfrak{P})) \]

where \( \mathfrak{P} = \Omega \cap B \). Now by reordering the terms and using the associativity formula for the \( j \)-multiplicity we have

\[ e(G) \geq \sum_{\Omega \in \text{Min}(G)} e(B/\mathfrak{P}) \cdot \left( \sum_{\Omega \in \text{Min}(G), \Omega \cap B = \mathfrak{P}} \lambda(G_{\Omega}) \cdot e(G_{/\Omega}G_{/\Omega}) \right) \]

\[ = \sum_{\mathfrak{P}} e(B/\mathfrak{P}) \cdot \left( \sum_{\Omega \in \text{Min}(G), \Omega \cap B = \mathfrak{P}} \lambda(G_{\Omega}) \cdot e(G_{/\Omega}G_{/\Omega}) \right) \]

\[ = \sum_{\mathfrak{P}} e(B/\mathfrak{P}) \cdot j(G_{\mathfrak{P}}) \]

where the sums in above are taken over all primes \( \mathfrak{P} \) containing \( A_1B \) that are contracted from a minimal prime of \( G \), equivalently, all primes \( \mathfrak{P} \) containing \( A_1B \) with \( \ell(A_1B_{/\mathfrak{P}}) = \dim B_{/\mathfrak{P}} \). Now the result follows as \( e(G) = e(B) \) and \( j(G_{/\mathfrak{P}}) \) is equal to \( j(A_1B_{/\mathfrak{P}}) \) in the sense of Achilles and Manaresi [1].

The following corollary is a generalization of [11, 6.4]. It gives a lower bound for the difference of the multiplicities of \( A \) and \( B \).

**Corollary 5.4.**

\[ e(B) \geq e(A) + \sum_{\mathfrak{P}} j(A_1B_{/\mathfrak{P}}) \cdot e(B/\mathfrak{P}), \]

where the sum is taken over all primes \( \mathfrak{P} \neq n \) that contain the ideal \( A_1B \) such that \( \ell(A_1B_{/\mathfrak{P}}) = \dim B_{/\mathfrak{P}} \).

**Proof.** The result follows from Theorem 5.3 and Proposition 5.1.

Xie [14] has recently generalized the above result by giving the extra terms required to make it an equality.

6. **Relative multiplicities of modules**

Let \( R \) be a Noetherian local ring with maximal ideal \( m \). Let \( U \subset E \) be submodules of a free module \( F := R^e \) and \( N \) a finitely generated \( R \)-module. Then the forms defined by \( U \) and \( E \) in the symmetric algebra \( \text{Sym}(F) = R[x_1, \ldots, x_e] \) of \( F \) generate the homogeneous standard graded subalgebras

\[ A := R[U] \subset B := R[E] \subset \text{Sym}(F) \]

called the Rees algebras of \( U \) and \( E \). Consider the \( B \)-module \( M := B \otimes_R N \). Notice that \( M \) is a finitely generated graded \( B \)-module generated in degree zero. For a non-negative integer \( t \) we define the \( t \)-th relative \( j \)-multiplicity of the pair of modules \( U \subset E \) on \( N \) as

\[ j^t(U|E, N) := j^t(A|B, M). \]
where $j^t(A|B, M)$ is the relative multiplicity defined in Section 3. If $N = R$ we simply write $j^t(U|E)$ and we will suppress $t$ when $t = 1$. Note that with $I = UB$ and for $0 \leq i \leq n - 1$ we have

$$\left[ \frac{I^i M}{I^{i+1} M} \right]_n = \frac{U^i E^{n-i} N}{U^{i+1} E^{n-i-1} N},$$

where the products are taken in the $B$-module $M$. Thus, under the internal grading, the Hilbert function of $\Gamma_m(B_t G(M))$ for $n \geq t$ is

$$\Sigma^t(n) := \sum_{i=0}^{n-t} \lambda_R \left( \Gamma_m \left( \frac{U^i E^{n-i} N}{U^{i+1} E^{n-i-1} N} \right) \right).$$

The corresponding Hilbert polynomial has degree at most $\dim M - 1$ and is of the form

$$\frac{j^t(U|E, N)}{(\dim M - 1)!} \sum_{n=0}^{\dim M - 1} + \text{lower terms}.$$

If $E = F$, then $j(U|E, N)$ is equal to $j(U, N)$ in the sense of Ulrich and Validashti [12]. Similar to the relative multiplicities of graded algebras in Remark 3.1, the set of primes $q$ of $R$ for which $j^t(U_q|E_q, N_q) \neq 0$ is finite. Also $j^t(U|E, N)$ is a decreasing sequence of non-negative integers with respect to $t$. Hence it will be constant for $t$ large, and we denote its stable value by $j^\infty(U|E, N)$. We say a pair of submodules $U \subset E$ of a free module $F$ is (weakly) birational if their Rees algebras are (weakly) birational. The following results are a translation of Theorems 4.5, 4.7 and 4.8 for modules.

**Theorem 6.1.** Let $R$ be a Noetherian ring and $U \subset E$ finitely generated submodules of a free module $F$. If $U \subset E$ is integral and birational, then $j(U_q|E_q) = 0$ for all primes $q$ of $R$. The converse holds if $R$ is locally equidimensional and universally catenary and $E = F_p$ for every minimal prime $p$ of $R$.

**Theorem 6.2.** Let $R$ be a Noetherian ring and $U \subset E$ finitely generated submodules of a free module $F$. If $U \subset E$ is integral, then $j^\infty(U_q|E_q) = 0$ for all primes $q$ of $R$. The converse holds if $R$ is locally equidimensional and universally catenary and $E = F_p$ for every minimal prime $p$ of $R$.

**Theorem 6.3.** Let $R$ be a Noetherian local ring and $U \subset E$ finitely generated submodules of a free module $F$. Then $U \subset E$ is weakly birational if and only if $j^\infty(U|E) = j(U|E)$.

Relative multiplicities do not satisfy the triangle equation in general, by which we mean equality may not hold in the triangle inequality of Theorem 3.5. For example, let $I \subset J$ be ideals in a Noetherian local ring $R$ with maximal ideal $m$. For a non-zero divisor $x$ in $R$, all $i \geq 0$ and all $n > i$ we have

$$\frac{(xl)^i (xJ)^{n-i}}{(xl)^{i+1} (xJ)^{n-i-1}} \simeq \frac{i^i J^{n-i}}{i+1 J^{n-i-1}}.$$

Hence $j(xl|xJ) = j(I|J)$. Moreover, if $I$ and $J$ are $m$-primary, we obtain

$$\Sigma^1(n) = \lambda_R \left( J^n / I^n \right) = \lambda_R (R/I^n) - \lambda_R (R/J^n).$$

Therefore $j(xl|xJ) = j(I|J) = e(I) - e(J)$. If $R$ has depth at least 2, one can show that $j(xl|R) = j(xl) = e(I) + e(I R)$ where $R := R/xR$. See for instance [6, 3.12]. Similarly $j(xJ|R) = e(J) + e(J R)$. Thus, for the triple $xl \subset xJ \subset R$ the triangle equation
\[ j(x|I|R) = j(x|J|R) + j(x|I|J) \]

holds if and only if \( e(I\bar{R}) = e(J\bar{R}) \), but this may not be true in general.

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