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Darboux theory of integrability in $\mathbb{C}^n$ taking into account the multiplicity

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ABSTRACT

Darboux in 1878 provided a theory on the existence of first integrals of polynomial systems based on the existence of sufficient invariant algebraic hypersurfaces, called now the Darboux theory of integrability. In 1979 Jouanolou successfully improved the Darboux theory of integrability characterizing the existence of rational first integrals, for this he used sophisticated tools of algebraic geometry. The aim of this paper is to improve the classical result of Darboux and the new one of Jouanolou taking into account the multiplicity of the invariant algebraic hypersurfaces. Additionally our proof of the improved result of Jouanolou is much simpler and elementary than the original one. Some examples show that the improved Darboux theory of integrability with multiplicity is much useful than the classical one.

1. Introduction

In many branches of applied mathematics, physics and, in general, in applied sciences appear nonlinear ordinary differential equations. If a differential equation or vector field defined in $\mathbb{R}^n$ or $\mathbb{C}^n$ has a first integral, then its study can be reduced in one dimension; of course working with real or complex time, respectively. Therefore a natural question is: *Given a vector field in* $\mathbb{R}^n$ *or* $\mathbb{C}^n$*, how to recognize if this vector field has a first integral*? This question has no a good answer up to now.

In this paper we shall study the existence of Darboux first integrals of polynomial vector fields in $\mathbb{R}^n$ or $\mathbb{C}^n$, and in particular of rational first integrals. The best answer to this question using Darboux first integral was given by Darboux [5,6] in 1878, and for the rational first integral the best

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answer was given by Jouanolou [9] in 1979. The Darboux theory of integrability provides a link between the integrability of polynomial vector fields and the number of invariant algebraic hypersurfaces that they have.

The Darboux theory of integrability has been successfully applied to study some physical models (see, for instance, [12,13,18,19]), centers, limit cycles and bifurcation problems of planar systems (see, for instance, [8,11,16]). This theory needs a big number of invariant algebraic hypersurfaces for higher dimensional systems, and this causes some difficulties for applying it. The goal of this paper is to provide an improved version of the Darboux theory of integrability, which takes into account not only the invariant algebraic hypersurfaces but also their multiplicity. Some examples show that our improved version of Darboux theory of integrability is much useful than the classical ones.

The paper is organized as follows. In Section 2 we provide the notation and definitions, and we state our main results. In Section 3 we work with the notion of functionally independence and first integrals. Section 4 provides a proof of Theorem 1, which presents a characterization of the algebraic multiplicity of an invariant algebraic hypersurface. The results of Section 3 and Theorem 1 will be used in Section 5 for proving Theorem 3, which improves the Darboux theory of integrability in $\mathbb{C}^n$ taking into account the multiplicities of the invariant algebraic hypersurfaces. Finally in Section 6 we apply our theory to some examples.

2. Definitions and statement of the main result

Since any polynomial differential system in $\mathbb{R}^n$ can be thought as a polynomial differential system inside $\mathbb{C}^n$ we shall work only in $\mathbb{C}^n$. If our initial differential system is in $\mathbb{R}^n$, once we get a complex first integral of this system inside $\mathbb{C}^n$ taking the square of the modulus of this complex integral we have a real first integral. Moreover if that complex first integral is rational, the square of its modulus also is rational. In short in the rest of the paper we shall work in $\mathbb{C}^n$.

As usual $\mathbb{C}[x] = \mathbb{C}[x_1, \ldots, x_n]$ denotes the ring of all complex polynomials in the variables $x_1, \ldots, x_n$. We consider the *polynomial vector field* in $\mathbb{C}^n$

$$\mathcal{X} = \sum_{i=1}^{n} P_i(x)\frac{\partial}{\partial x_i}, \quad x = (x_1, \ldots, x_n) \in C^n, \tag{1}$$

where $P_i = P_i(x) \in \mathbb{C}[x]$ have no common factor for $i = 1, \ldots, n$. The integer $d = \max\{\deg P_1, \ldots, \deg P_n\}$ is the *degree* of the vector field $\mathcal{X}$. Usually for simplicity the vector field $\mathcal{X}$ will be represented by $(P_1, \ldots, P_n)$.

Let $f = f(x) \in \mathbb{C}[x]$. We say that $\{f = 0\} \subset \mathbb{C}^n$ is an *invariant algebraic hypersurface* of the vector field $\mathcal{X}$ if there exists a polynomial $L_f \in \mathbb{C}[x]$ such that

$$\mathcal{X}(f) = \sum_{i=1}^{n} P_i\frac{\partial f}{\partial x_i} = fL_f.$$

The polynomial L_f is called the *cofactor* of $f = 0$. Note that from this definition the degree of L_f is at most $d - 1$, and also that if an orbit $x(t)$ of the vector field $\mathcal{X}$ has a point on $f = 0$, then the whole orbit is contained in $f = 0$. This justifies the name of invariant algebraic hypersurface, because it is invariant by the flow of the vector field $\mathcal{X}$.

If $f, g \in \mathbb{C}[x]$ are coprime, we write $(f, g) = 1$. Suppose that $(f, g) = 1$, we say that $\exp(g/f)$ is an *exponential factor* of the vector field $\mathcal{X}$ if there exists a polynomial $L_e \in \mathbb{C}[x]$ of degree at most $d - 1$ such that

$$\mathcal{X}\big(\exp(g/f)\big) = \exp(g/f)L_e.$$

The polynomial L_e is called the *cofactor* of the exponential factor. It is easy to prove that if $\exp(g/f)$ is an exponential factor, then $f = 0$ is an invariant algebraic hypersurface.

Let $\mathbb{C}_m[x]$ be the $\mathbb{C}$-vector space of polynomials in $\mathbb{C}[x]$ of degree at most m. Then it has dimension $R = \binom{n+m}{n}$. Let $v_1, \ldots, v_R$ be a base of $\mathbb{C}_m[x]$. Denote by M_R the $R \times R$ matrix

$$\begin{pmatrix} v_1 & v_2 & \ldots & v_R \\ \mathcal{X}(v_1) & \mathcal{X}(v_2) & \ldots & \mathcal{X}(v_R) \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{X}^{R-1}(v_1) & \mathcal{X}^{R-1}(v_2) & \ldots & \mathcal{X}^{R-1}(v_R) \end{pmatrix}, \tag{2}$$

where $\mathcal{X}^{k+1}(v_i) = \mathcal{X}(\mathcal{X}^k(v_i))$. The *mth extactic hypersurface* of $\mathcal{X}$ is given by the equation $\det M_R = 0$. We also call $\det M_R$ the *mth extactic polynomial*. From the properties of the determinant we note that the extactic hypersurface is independent of the choice of the base of $\mathbb{C}_m[x]$. Observe that if $f = 0$ is an invariant algebraic hypersurface of degree m of $\mathcal{X}$, then f divides $\det M_R$. This is due to the fact that if f is a member of a base of $\mathbb{C}_m[x]$, then f divides the whole column in which f is located.

An algebraic hypersurface $f = 0$ is *irreducible* if f is an irreducible polynomial in $\mathbb{C}[x]$. We say that an irreducible invariant algebraic hypersurface $f = 0$ of degree m has *defined algebraic multiplicity k* or simply *algebraic multiplicity k* if $\det M_R \not\equiv 0$ and k is the maximum positive integer such that f^k divides $\det M_R$; and it has *no defined algebraic multiplicity* if $\det M_R \equiv 0$. If $f = 0$ is an invariant algebraic hypersurface and $f = f_1^{n_1} \cdot \cdots \cdot f_s^{n_s}$ with f_i irreducible and $n_i \in \mathbb{N}$, then $f_i = 0$ is an irreducible invariant algebraic hypersurface (see for instance [10]).

We remark that the matrix (2) already appears in the work of Lagutinskii (see also Dobrovol'skii et al. [7]). For a modern definition of the mth extactic hypersurface and a clear geometric explanation of its meaning, the readers can look at Pereira [15]. Christopher et al. [4] used the extactic curve to study the algebraic multiplicity of invariant algebraic curves of planar polynomial vector fields, and prove the equivalence of the algebraic multiplicity with other three ones: the infinitesimal multiplicity, the integrable multiplicity and the geometric multiplicity.

Let $\mathcal{D}$ be an open subset of $\mathbb{C}^n$ having full Lebesgue measure in $\mathbb{C}^n$. A non-constant holomorphic function $H : \mathcal{D} \to \mathbb{C}$ is a *first integral* of the polynomial vector field $\mathcal{X}$ on $\mathcal{D}$ if it is constant on all orbits $x(t)$ of $\mathcal{X}$ contained in $\mathcal{D}$; i.e. $H(x(t)) =$ constant for all values of t for which the solution $x(t)$ is defined and contained in $\mathcal{D}$. Clearly H is a first integral of $\mathcal{X}$ on $\mathcal{D}$ if and only if $\mathcal{X}(H) = 0$ on $\mathcal{D}$. Of course a *rational first integral* is a first integral given by a rational function, defined in the open subset of $\mathbb{C}^n$ where its denominator does not vanish. A *Darboux first integral* is a first integral of the form

$$\left(\prod_{i=1}^{r} f_i^{l_i}\right) \exp(g/h),$$

where f_i, g and h are polynomials, and the l_i's are complex numbers.

Our first result presents a characterization under suitable assumptions of the algebraic multiplicity of an invariant algebraic hypersurface using the number of exponential factors of $\mathcal{X}$ associated with the invariant algebraic hypersurface. This characterization extends the algebraic multiplicity deduced in [4] for invariant algebraic curves of $\mathbb{C}^2$ to invariant algebraic hypersurfaces of $\mathbb{C}^n$. The result will play the key point for proving Theorem 3.

Theorem 1. *For a given irreducible invariant algebraic hypersurface $f = 0$ of degree m of $\mathcal{X}$, assume that $\mathcal{X}$ restricted to $f = 0$ has no rational first integral. Then f has a defined algebraic multiplicity k if and only if the vector field $\mathcal{X}$ has $k-1$ exponential factors* $\exp(g_i/f^i)$, *where g_i is a polynomial of degree at most im and $(g_i, f) = 1$, for $i = 1, \ldots, k-1$.*

We remark that if $\mathcal{X}$ is a planar vector field, then clearly Theorem 1 always holds without the assumption on the non-existence of rational first integrals on $f = 0$. For higher dimensional systems the assumption is necessary as the following example shows.

The real polynomial differential system

$$\dot{x} = 1, \qquad \dot{y} = y(y - 2z), \qquad \dot{z} = -z(y - z)$$

has $z = 0$ as an invariant plane of multiplicity 2 and its restriction to $z = 0$ has the rational first integral $x + 1/y$. But this system has no exponential factor associated with $z = 0$ as it is proved in Appendix A.

This example shows that the additional assumption on the non-existence of the rational first integral on the invariant algebraic hypersurface for polynomial vector fields of dimension larger than 2 is necessary. So if we are not in the assumptions of Theorem 1, it is possible that the number of exponential factors does not depend on the algebraic multiplicity of the invariant algebraic hypersurface.

The classical Darboux theory of integrability in $\mathbb{C}^n$ with $n \geqslant 2$ is summarized in the next theorem.

Theorem 2. *Assume that the polynomial vector field $\mathcal{X}$ in $\mathbb{C}^n$ of degree $d > 0$ has irreducible invariant algebraic hypersurfaces $f_i = 0$ for $i = 1, \dots, p$. Then the following statements hold.*

(a) *If $p \geqslant N + 1$, then the vector field $\mathcal{X}$ has a Darboux first integral, where $N = \binom{n+d-1}{n}$.*
(b) *If $p \geqslant N + n$, then the vector field $\mathcal{X}$ has a rational first integral.*

Statement (a) of Theorem 2 is due to Darboux [5,6], and statement (b) of Theorem 2 was proved by Jouanolou [9]. For a short proof of statement (b) see [2,3] for $n = 2$ and [14] for $n \geqslant 2$.

The following theorem improves the Darboux theory of integrability taking into account not only the invariant algebraic hypersurfaces but also their algebraic multiplicities.

Theorem 3. *Assume that the polynomial vector field $\mathcal{X}$ in $\mathbb{C}^n$ of degree $d > 0$ has irreducible invariant algebraic hypersurfaces.*

(i) *If some of these irreducible invariant algebraic hypersurfaces has no defined algebraic multiplicity, then the vector field $\mathcal{X}$ has a rational first integral.*
(ii) *Suppose that all the irreducible invariant algebraic hypersurfaces $f_i = 0$ have defined algebraic multiplicity q_i for $i = 1, \dots, p$. If $\mathcal{X}$ restricted to each hypersurface $f_i = 0$ having multiplicity larger than 1 has no rational first integral, then the following statements hold.*
   (a) *If $\sum_{i=1}^{p} q_i \geqslant N + 1$, then the vector field $\mathcal{X}$ has a Darboux first integral, where N is the number defined in Theorem 2.*
   (b) *If $\sum_{i=1}^{p} q_i \geqslant N + n$, then the vector field $\mathcal{X}$ has a rational first integral.*

Statement (i) follows from the second part of Theorem 3 of Pereira [15] (see also Theorem 5.3 of [4] for dimension 2), we will not prove it. We state it here for completeness. Under the assumption (b) of Theorem 3 any orbit of the vector field $\mathcal{X}$ is contained in an invariant algebraic hypersurface. We remark that if the vector field $\mathcal{X}$ is 2-dimensional, then the assumption on the non-existence of rational first integral of $\mathcal{X}$ restricted to the invariant algebraic curves is not necessary.

3. Preliminary results on first integrals

Assume that $H_j(x)$ for $j = 1, \dots, m$ are holomorphic first integrals of system (1) defined in a full Lebesgue measurable subset $\mathcal{D}_1$ of $\mathbb{C}^n$. For each $x \in \mathcal{D}_1$ let $r(x)$ be the rank of the m vectors $\nabla H_1(x), \dots, \nabla H_m(x)$ in $\mathbb{C}^n$, where $\nabla H_k(x)$ denotes the gradient of the function $H_k(x)$ with respect to x. We say that $H_1, \dots, H_m$ are *functionally independent* in $\mathcal{D}_1$ if $r(x) = m$ for all $x \in \mathcal{D}_1$ except perhaps for a subset of Lebesgue measure zero.

We say that $H_1, \dots, H_m$ are *k-functionally independent* in $\mathcal{D}_1$ if there exist k of these $H_1, \dots, H_m$ which are functionally independent in $\mathcal{D}_1$, and any $k+1$ elements of $\{H_1, \dots, H_m\}$ are not functionally independent in any positive Lebesgue measurable subset of $\mathcal{D}_1$.

It is easy to check that if m first integrals $H_1, \dots, H_m$ of a polynomial vector field in $\mathbb{C}^n$ are k-functionally independent then $k \leqslant n-1$.

The following theorem proved in [14] is the first key point for proving Theorem 3(b).

Theorem 4. *For $k < m$ we assume that $H_1, \dots, H_m$ are k-functionally independent first integrals of the polynomial vector field $\mathcal{X}$ given by* (1) *and defined in a full Lebesgue measurable subset of $\mathbb{C}^n$. Without loss of generality we can assume that $H_1, \dots, H_k$ are functionally independent.*

(a) *For each $s \in \{k+1, \dots, m\}$ there exist holomorphic functions $C_{s1}(x), \dots, C_{sk}(x)$ defined in a full Lebesgue measurable subset of $\mathbb{C}^n$ such that*

$$\nabla H_s(x) = C_{s1}(x)\nabla H_1(x) + \dots + C_{sk}(x)\nabla H_k(x). \tag{3}$$

(b) *For every $s \in \{k+1, \dots, m\}$ and $j \in \{1, \dots, k\}$ the function $C_{sj}(x)$* (*if not a constant*) *is a first integral of system* (1)*.*

4. Proof of Theorem 1

For proving Theorem 1 we need the following lemma. Denote by $\mathbb{C}[x][\varepsilon]$ the ring of polynomials in ε with coefficients polynomials in x, and by $\mathbb{C}[x][\varepsilon]/\langle\varepsilon^k\rangle$ the set formed by elements of $\mathbb{C}[x][\varepsilon]$ modulus the ideal generated by ε^k.

Lemma 5. *Let $f = 0$ be an irreducible invariant algebraic hypersurface having degree m of the vector field $\mathcal{X}$ of degree d. The following hold.*

(a) *If for any $\varepsilon > 0$ there exist*

$$F = f_0 + f_1\varepsilon + \dots + f_{k-1}\varepsilon^{k-1} \in \mathbb{C}[x][\varepsilon]/\langle\varepsilon^k\rangle,$$
$$L_F = L_0 + L_1\varepsilon + \dots + L_{k-1}\varepsilon^{k-1} \in \mathbb{C}[x][\varepsilon]/\langle\varepsilon^k\rangle,$$

with $f_0 = f, f_1, \dots, f_{k-1} \in \mathbb{C}_m[x]$, f_1 not a multiple of f, and $L_0, L_1, \dots, L_{k-1} \in \mathbb{C}_{d-1}[x]$ such that

$$\mathcal{X}(F) = FL_F \quad \text{in } \mathbb{C}[x][\varepsilon]/\langle\varepsilon^k\rangle, \tag{4}$$

then $\mathcal{X}$ has $k-1$ exponential factors $\exp(g_i/f^i)$ *with g_i coprime with f and of degree at most im for $i = 1, \dots, k-1$.*

(b) *Assume that $\mathcal{X}$ restricted to $f = 0$ has no rational first integral. Then the inverse of statement* (a) *holds.*

The results of Lemma 5 can be found in [4] for polynomial vector fields in the plane without the assumption of statement (b). Here we show that these results extend to polynomial vector fields of arbitrary dimension but with this additional assumption.

We must say that for higher dimensional systems the assumption in (b) is necessary. For instance, the following system

$$\dot{w} = (y+z)w^2, \qquad \dot{y} = \left(1 + y^2 + yz\right)w, \qquad \dot{z} = z + bw^2 - cwz + \left(yz + z^2\right)w \tag{5}$$

has the exponential factors $E_1 = \exp(y/w)$ and $E_2 = \exp((1/2 + y^2/2 + wz)/w^2)$ with respectively the cofactors $L_{E_1} = 1$ and $L_{E_2} = bw - cz$. Choose $F = f_0 + f_1\varepsilon$ and $L_F = L_0 + L_1\varepsilon$ with $f_0 = w$, $L_0 = w(y+z)$, $f_1 = y$ and $L_1 = L_{E_1}$, we have $\mathcal{X}_1(F) = FL_F$ in $\mathbb{C}[w,y,z][\varepsilon]/\langle\varepsilon^2\rangle$, where $\mathcal{X}_1$ denotes the vector field defined by (5). But easy calculations show that there are not $f_2 \in \mathbb{C}[w,y,z]$ and $L_2 \in \mathbb{C}_2[w,y,z]$ such that $\mathcal{X}_1(F) = FL_F$ in $\mathbb{C}[w,y,z][\varepsilon]/\langle\varepsilon^3\rangle$ with $F = f_0 + f_1\varepsilon + f_2\varepsilon^2$ and $L_F = L_0 + L_1\varepsilon + L_2\varepsilon^2$.

Proof of Lemma 5. (a) As usual we denote by $\mathbb{C}(x)$ the field of rational functions in x. We use $\mathbb{C}(x)\{\varepsilon\}$ to denote the set of all series in ε with coefficients in $C(x)$. Denote by $\mathcal{F}_f$ the field formed by all quotients of two elements from $\mathbb{C}[x]/\langle f\rangle$, where $\langle f\rangle$ is the ideal generated by f in $\mathbb{C}[x]$.

Taking the logarithm of F we have

$$\log F = \log f_0 + \varepsilon\frac{g_1}{f_0} + \cdots + \varepsilon^{k-1}\frac{g_{k-1}}{f_0^{k-1}} \quad \text{in } \mathbb{C}(x)\{\varepsilon\}/\langle\varepsilon^k\rangle,$$

with

$$g_i = \frac{(-1)^{i-1}}{i} f_1^i + f_0 G_i(f_0,\ldots,f_i), \tag{6}$$

where G_i is a homogeneous polynomial of degree $i-1$ in $f_0,\ldots,f_i$. Observe that $\deg g_i \leqslant im$ and $(g_i, f) = 1$ because $(f_1, f) = 1$. Then it follows from Eq. (4) that $\exp(g_i/f^i)$, $i = 1,\ldots,k-1$, are $k-1$ exponential factors of $\mathcal{X}$.

(b) Let $F_i := \exp(g_i/f^i)$, $i = 1,\ldots,k-1$, be the $k-1$ exponential factors of $\mathcal{X}$ with cofactors L_{F_i}. We will prove this part by induction.

For $k = 2$, taking $f_0 = f$, $f_1 = g_1$, $L_0 = L_f$ and $L_1 = L_{F_1}$, then $F = f_0 + f_1\varepsilon$ and $L_F = L_0 + L_1\varepsilon$ are suitable for the lemma.

Assume that for the number of exponential factors $2 \leqslant l-1 < k-1$ there exist

$$F^{(l)} = f_0 + f_1\varepsilon + \cdots + f_{l-1}\varepsilon^{l-1} \quad \text{in } \mathbb{C}[x][\varepsilon]/\langle\varepsilon^l\rangle,$$
$$L_{F^{(l)}} = L_0 + L_1\varepsilon + \cdots + L_{l-1}\varepsilon^{l-1} \quad \text{in } \mathbb{C}[x][\varepsilon]/\langle\varepsilon^l\rangle,$$

such that Eq. (4) holds in $\mathbb{C}[x][\varepsilon]/\langle\varepsilon^l\rangle$, where $f_i \in \mathbb{C}_m[x]$ and $L_i \in \mathbb{C}_{d-1}[x]$ for $i = 0,1,\ldots,l-1$, and $(f_1, f) = 1$.

Assume that we have l exponential factors. Taking

$$L^*_{F^{(l)}} = L_0 + L_1\varepsilon + \cdots + L_{l-1}\varepsilon^{l-1} + L_l^*\varepsilon^l,$$

with $L_l^* = -f_0^{-1}\sum_{i=1}^{l-1} f_i L_{l-i}$, easy calculations show that

$$\mathcal{X}\big(F^{(l)}\big) = F^{(l)} L^*_{F^{(l)}} \quad \text{in } \mathbb{C}(x)\{\varepsilon\}/\langle\varepsilon^{l+1}\rangle.$$

Taking the logarithm of $F^{(l)}$ in $\mathbb{C}(x)\{\varepsilon\}/\langle\varepsilon^{l+1}\rangle$ we get that

$$\log F^{(l)} = \log f_0 + \frac{g_1}{f_0}\varepsilon + \cdots + \frac{g_{l-1}}{f_0^{l-1}}\varepsilon^{l-1} + \frac{g_l^*}{f_0^l}\varepsilon^l,$$

where g_i for $i = 1,\ldots,l-1$ and g_l^* have the form as those given in (6). It follows that $\mathcal{X}(g_l^*/f_0^l) = L_l^*$, and consequently $\mathcal{X}(g_l^*) = lL_0 g_l^* - f_0^{l-1}\sum_{i=1}^{l-1} f_i L_{l-i}$.

Using $\mathcal{X}(f_1) = f_1L_0 + f_0L_1$ and the fact that $\exp(g_l/f^l)$ is an exponential factor of $\mathcal{X}$, we can prove that $\mathcal{X}(g_l/f_1^l) = 0$ on $f = 0$. This implies that g_l/f_1^l is a first integral of $\mathcal{X}$ in $\mathcal{F}_f$. By the assumption we get that g_l/f_1^l is a constant on $f = 0$, denoted by c_l. Clearly $c_l \neq 0$. From (6) we have $g_l^* = (-1)^{l-1}l^{-1}f_1^l$ on $f = 0$. It follows that $g_l - (-1)^{l-1}lc_l g_l^* = 0$ on $f = 0$. By Hilbert's Nullstellensatz [1] and the fact that $f_0 = f$ is irreducible, there exists a polynomial g_{l-1}^* such that

$g_l = (-1)^{l-1} l c_l (g_l^* + f_0 g_{l-1}^*)$. It is necessary that g_{l-1}^* has degree at most $(l-1)m$. Moreover we get from $\mathcal{X}(g_l) = l L_0 g_l + L_{F_l} f^l$ and the expression of $\mathcal{X}(g_l^*)$ that

$$\mathcal{X}\big(g_{l-1}^*\big) = (l-1) L_0 g_{l-1}^* + f_0^{l-2}\left(\sum_{i=1}^{l-1} f_i L_{l-i} + (-1)^{l-1} f_0 L_{F_l}/(l c_l)\right).$$

Recall that L_{F_l} is the cofactor of the exponential factor $F_l = \exp(g_l/f^l)$. Using this last equality we can verify that $\mathcal{X}(g_{l-1}^*/f_1^{l-1}) = 0$ on $f = 0$. This implies that there exists a constant c_{l-1} and a polynomial g_{l-2}^* of degree at most $(l-2)m$ such that $g_{l-1}^* = (-1)^{l-2}(l-1)c_{l-1} g_{l-1} + f_0 g_{l-2}^*$. Moreover we have

$$\mathcal{X}\big(g_{l-2}^*\big) = (l-2) L_0 g_{l-2}^* + f_0^{l-3}\left(\sum_{i=1}^{l-1} f_i L_{l-i} + (-1)^{l-1} f_0 L_{F_l}/(l c_l) - (-1)^{l-2}(l-1) c_{l-1} f_0 L_{l-1}\right).$$

Repeating this last process for $i = 3, \ldots, l-1$, we get $g_{l+1-i}^* = (-1)^{l-i}(l+1-i) c_{l+1-i} g_{l+1-i} + f_0 g_{l-i}^*$ with c_{l+1-i} a constant and g_{l-i}^* a polynomial of degree at most $(l-i)m$. The polynomial g_{l-i}^* satisfies

$$\mathcal{X}\big(g_{l-i}^*\big) = (l-i) L_0 g_{l-i}^* + f_0^{l-1-i}\Bigg(\sum_{i=1}^{l-1} f_i L_{l-i} + (-1)^{l-1} f_0 L_{F_l}/(l c_l) - (-1)^{l-2}(l-1) c_{l-1} f_0 L_{l-1} - \cdots - (-1)^{l-i}(l+1-i) c_{l+1-i} f_0 L_{l+1-i}\Bigg).$$

Take $f_l = g_1^*$ and

$$F^{l+1} = f_0 + \varepsilon f_1 + \cdots + \varepsilon^l f_l,$$
$$L_{F^{l+1}} = L_0 + \varepsilon L_1 + \cdots + \varepsilon^l L_l,$$

with $L_l = (-1)^{l-1} L_{F_l}/(l c_l) - (-1)^{l-2}(l-1) c_{l-1} L_{l-1} - \cdots - (-1) 2 c_2 L_2$. Then we have

$$\mathcal{X}\big(F^{l+1}\big) = F^{l+1} L_{F^{l+1}} \quad \text{in } \mathbb{C}[x][\varepsilon]/\big\langle \varepsilon^{l+1}\big\rangle.$$

By induction we complete the proof of statement (b), and consequently this completes the proof of the lemma. □

Proof of Theorem 1. Using Lemma 5 and taking into account the additional necessary assumption on the non-existence of rational first integrals of the vector field $\mathcal{X}$ restricted to the irreducible invariant algebraic hypersurface $f = 0$, we can prove the sufficient part of Theorem 1 in the same way as the proof of Proposition 5.7 of [4], and the necessary part of Theorem 1 in the same way as the proof of Proposition 5.8 and Theorem 5.10 of [4]. □

5. Proof of Theorem 3

As we have said statement (i) is essential the second part of Theorem 3 of Pereira [15], now we only prove statement (ii).

Let $f_i(x) = 0$ for $i = 1, \ldots, p$ be an irreducible invariant algebraic hypersurface having the algebraic multiplicity q_i. Theorem 1 shows that for each f_i we have $q_i - 1$ exponential factors $\exp(g_{ij}/f^j)$, $j = 1, \ldots, q_i - 1$. Denote by k_r, for $r = 1, \ldots, K := \sum_{i=1}^p q_i$, the K cofactors of the p invariant algebraic hypersurfaces and of the $K - p$ exponential factors. Recall that each cofactor is a polynomial of degree at most $d-1$, where d is the degree of the vector field $\mathcal{X}$. To simplify the notations we denote by F_r, with $r \in \{1, \ldots, K\}$, the irreducible invariant algebraic hypersurface or the exponential factor of the vector field $\mathcal{X}$ having cofactor k_r. Then we have $\mathcal{X}(\log F_r) = k_r$ for $r = 1, \ldots, K$.

Since every polynomial in $\mathbb{C}[x]$ is uniquely determined by its coefficients, and so it can be seen as a vector. Under this notation $\mathbb{C}_{d-1}[x]$ forms a $\mathbb{C}$-vector space of dimension N. Recall that $N := \binom{n+d-1}{n}$.

(a) If $K \geqslant N+1$, then the K cofactors are linearly dependent. So there exist constants $\sigma_1, \ldots, \sigma_K \in \mathbb{C}$ not all zero such that $\sum_{r=1}^{K} \sigma_r k_r = 0$. This last equation can be written as $\mathcal{X}(\log(F_1^{\sigma_1} \ldots F_K^{\sigma_K})) = 0$. So $\log(F_1^{\sigma_1} \ldots F_K^{\sigma_K})$ is a first integral of $\mathcal{X}$. This proves statement (a).

(b) Let ρ be the dimension of the vector subspace of $\mathcal{V}$ generated by $\{k_1(x), \ldots, k_{N+n}(x)\}$. Then we have $\rho \leqslant N$. Now in order to simplify the proof and the notation we shall assume that $\rho = N$ and that $k_1(x), \ldots, k_N(x)$ are linearly independent in $\mathcal{V}$. If $\rho < N$ the proof would follow exactly equal using the same arguments.

For each $s \in \{1, \ldots, n\}$ there exists a vector $(\sigma_{s1}, \ldots, \sigma_{sN}, 1) \in \mathbb{C}^{N+1}$ such that

$$\sigma_{s1} k_1(x) + \cdots + \sigma_{sN} k_N(x) + k_{N+s}(x) = 0. \tag{7}$$

Eq. (7) can be written as

$$\mathcal{X}\big(\log\big(F_1^{\sigma_{s1}} \ldots F_N^{\sigma_{sN}} F_{N+s}\big)\big) = 0.$$

This implies that the functions $H_s = \log(F_1^{\sigma_{s1}} \ldots F_N^{\sigma_{sN}} F_{N+s})$ for $s = 1, \ldots, n$ are holomorphic first integrals of the vector field $\mathcal{X}$, defined on a convenient full Lebesgue measurable subset Ω_1 of $\mathbb{C}^n$.

We claim that the n first integrals H_i's are functionally dependent on any positive Lebesgue measurable subset of Ω_1. Otherwise there exists a positive Lebesgue measurable subset Ω_2 of Ω_1 where they are functionally independent, then from the definition of first integral we have

$$\frac{\partial H_i(x)}{\partial x_1} P_1(x) + \cdots + \frac{\partial H_i(x)}{\partial x_n} P_n(x) = 0, \quad \text{for } i = 1, \ldots, n \text{ and for all } x \in \Omega_2,$$

and from the functionally independence this last homogeneous linear system of dimension n only has the trivial solution $P_i(x) = 0$ for $i = 1, \ldots, n$ on Ω_2, and consequently the vector field $\mathcal{X} \equiv 0$ in $\mathbb{C}^n$, in contradiction with the fact that $\mathcal{X}$ has degree $d > 0$. So the claim is proved.

We define $r(x) = \operatorname{rank}\{\nabla H_1(x), \ldots, \nabla H_n(x)\}$ and $m = \max\{r(x)\colon x \in \Omega_1\}$. Then there exists an open subset Θ of Ω_1 such that $m = r(x)$ for each $x \in \Theta$ and $m < n$. Without loss of generality we can assume that $\{\nabla H_1(x), \ldots, \nabla H_m(x)\}$ has the rank m for all $x \in \Theta$. Therefore, by Theorem 4(a) for each $x \in \Theta$ there exist $C_{k1}(x), \ldots, C_{km}(x)$ such that

$$\nabla H_k(x) = C_{k1}(x) \nabla H_1(x) + \cdots + C_{km}(x) \nabla H_m(x), \quad k = m+1, \ldots, n. \tag{8}$$

By Theorem 4(b) it follows that the function $C_{kj}(x)$ (if not a constant) for $j \in \{m+1, \ldots, n\}$ is a first integral of the vector field $\mathcal{X}$ defined on Θ.

From the construction of the H_i's we know that each ∇H_i is a vector of rational functions. Since the vectors $\{\nabla H_1(x), \ldots, \nabla H_m(x)\}$ are linearly independent for each $x \in \Theta$, solving system (8) we get a unique solution $(C_{k1}(x), \ldots, C_{km}(x))$ on Θ for every $k = m+1, \ldots, n$. Clearly each function $C_{kj}(x)$ for $j \in \{1, \ldots, m\}$ is rational and by Theorem 4(b) it satisfies

$$\frac{\partial C_{kj}}{\partial x_1} P_1 + \cdots + \frac{\partial C_{kj}}{\partial x_n} P_n = 0 \quad \text{on } \Theta.$$

Since Θ is an open subset of $\mathbb{C}^n$ and $C_{kj}(x)$ is rational, it should satisfy the last equation in $\mathbb{C}^n$ except possibly a subset of Lebesgue measure zero where C_{kj} is not defined. Hence if some of the functions $C_{kj}(x)$'s is not a constant, it is a rational first integral of the vector field $\mathcal{X}$.

Now we shall prove that some function C_{kj} is not a constant. Eq. (8) implies that if all functions $C_{k1},\dots,C_{km}$ are constants, then $H_k(x)=C_{k1}H_1(x)+\dots+C_{km}H_m(x)+\log C_k$, where C_k is a constant. So we have $F_1^{\sigma_{k1}}\dots F_N^{\sigma_{kN}}F_{N+k}=C_k(F_1^{\sigma_{11}}\dots F_N^{\sigma_{1N}}F_{N+1})^{C_{k1}}\dots(F_1^{\sigma_{m1}}\dots F_N^{\sigma_{mN}}F_{N+m})^{C_{km}}$ for $k\in\{m+1,\dots,n\}$. This is in contradiction with the fact that $F_1,\dots,F_{N+n}$ are irreducible and pairwise different Darboux polynomials and pairwise different exponential factors. Hence we must have a non-constant function $C_{k_0j_0}(x)$ for some $j_0\in\{1,\dots,m\}$ and some $k_0\in\{m+1,\dots,n\}$. This completes the proof of Theorem 3.

6. Examples

In this section we provide some examples showing the application of our improved Darboux theory of integrability.

Example 1. The real planar quadratic polynomial differential system

$$\dot{x}=x^2-1,\qquad \dot{y}=-3+y-x^2+xy \tag{9}$$

was studied in [17]. Choosing 1, x, y as a base of $\mathbb{R}_1[x,y]$ we get the 1th extactic polynomial $-(x-1)^3(x+1)(x+3)$. This means that the invariant lines $x=1$ and $x=-1$ have multiplicities 3 and 1, respectively. Since $3+1=4=\binom{2+2-1}{2}+1$, our improved Darboux theory of integrability (see Theorem 3) shows that system (9) has a Darboux first integral.

Associated with $x=1$ we have exponential factors $e^{(y-2)/(x-1)}$ with cofactor $1-x$ and $e^{(y-2)^2/(x-1)^2}$ with cofactor $4-2y$. Obviously, the invariant lines $x=1$ and $x=-1$ have cofactors $x+1$ and $x-1$. So using the proof of statement (a) of Theorem 3 we get a Darboux first integral $H=(x+1)e^{(y-2)/(x-1)}$.

Example 2. Consider the real 3-dimensional polynomial differential system

$$\dot{x}=-x^2,\qquad \dot{y}=1-2xy+3xz,\qquad \dot{z}=z(1-2x). \tag{10}$$

In the definition of extactic polynomial, choosing $1,x,y,z$ as a base of $\mathbb{R}_1[x,y,z]$ we can prove easily that the invariant plane $z=0$ has multiplicity 1 and that the invariant plane $x=0$ has multiplicity 4. In addition, we can check that system (10) restricted to $x=0$ has no rational first integral. Since $1+4=5=\binom{3+2-1}{3}+1$, it follows from Theorem 3 that system (10) has a Darboux first integral.

For obtaining the explicit expression of the Darboux first integral, we compute the exponential factors. In fact associated with the invariant plane $x=0$ we have the three exponential factors $E_1=\exp(1/x)$ with cofactor $L_{E_1}=1$, $E_2=\exp((1-2xy)/x^2)$ with cofactor $L_{E_2}=2y-6z$ and $E_3=\exp((1-3y+9x^2z)/x^3)$ with cofactor $L_{E_3}=-9z$. On the other hand corresponding to the invariant planes $x=0$ and $z=0$ the cofactors are $L_x=-x$ and $L_z=1-2x$, respectively. So again using the proof of statement (a) of Theorem 3 we obtain the Darboux first integral $H=x^2e^{1/x}z^{-1}$.

Example 3. The real 3-dimensional polynomial differential system

$$\dot{x}=-x^2,\qquad \dot{y}=y(1-2x-y+2z),\qquad \dot{z}=z(1-2x+3y+2z) \tag{11}$$

has three invariant planes $x=0$ with multiplicity 2, $y=0$ with multiplicity 2, and $z=0$ with multiplicity 1. The number of these invariant planes taking into account their multiplicity is 5. Hence, we get from Theorem 3 that system (11) has a Darboux first integral.

Some computations show that associated with the multiple invariant planes $x=0$ and $y=0$ system (11) has respectively the exponential factors $e^{1/x}$ with cofactor $L_1=1$, and $e^{z/y}$ with cofactor $L_2=4z$. Combining the three invariant planes we obtain the Darboux first integral $H=y^3zx^{-8}e^{-4/x}e^{-2z/y}$.

The above examples show that our improved Darboux theory of integrability is much more useful than the classical one.

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Appendix A

Proposition A.1. *The real polynomial differential system*

$$\dot{x}=1, \qquad \dot{y}=y(y-2z), \qquad \dot{z}=-z(y-z) \tag{A.1}$$

has no exponential factors associated to the invariant plane $z=0$ of multiplicity 2.

Proof. Assume that system (A.1) has an exponential factor associated with $z=0$. Let $E=\exp(g/z^s)$ be the exponential factor with cofactor L, where g is a polynomial of degree at most $s \geqslant 1$, and L is a polynomial of degree at most 1.

By the definition of exponential factor we get that g should satisfy the equation

$$\frac{\partial g}{\partial x}+y(y-2z)\frac{\partial g}{\partial y}-z(y-z)\frac{\partial g}{\partial z}=-s(y-z)g+Lz^s. \tag{A.2}$$

Set $L=L_0+L_1$, where L_0 is a constant and L_1 is a homogeneous polynomial of degree 1. In what follows we also denote by g_i the homogeneous polynomial of degree i of g.

If $\deg g=s$, equating the homogeneous part of degree $s+1$ of (A.2) we get

$$y(y-2z)\frac{\partial g_s}{\partial y}-z(y-z)\frac{\partial g_s}{\partial z}=-s(y-z)g_s+L_1z^s.$$

Using the Euler's formula for homogeneous functions this last equation is equivalent to

$$x(y-z)\frac{\partial g_s}{\partial x}+y(2y-3z)\frac{\partial g_s}{\partial y}=L_1z^s. \tag{A.3}$$

This implies that x divides $y(2y-3z)\partial g_s/\partial y-L_1z^s$. Hence we must have $\partial g_s/\partial y=0$ because y^2 does not divide L_1z^s. It follows that Eq. (A.3) holds only if $L_1=0$ and $\partial g_s/\partial x=0$. This proves that $g_s=az^s$ with a a constant. But in this case the exponential factor E is essentially the same with g of degree less than s.

If $\deg g=s-1$, equating the homogeneous part of degree s of Eq. (A.2) we get that $L=L_0$ and g_{s-1} satisfies the following equation

$$y(y-2z)\frac{\partial g_{s-1}}{\partial y}-z(y-z)\frac{\partial g_{s-1}}{\partial z}=-s(y-z)g_{s-1}+L_0z^s. \tag{A.4}$$

If $s=1$, it is easy to show that $g_{s-1}=0$. So system (A.1) has no exponential factor. For $s>1$, set $g_{s-1}=zp_{s-2}(x,y,z)+q_{s-1}(x,y)$ with p_{s-2} and q_{s-1} homogeneous polynomials of degrees $s-2$ and $s-1$, respectively. Then we obtain from (A.4) with $z=0$ that $y\partial q_{s-1}/\partial y=-sq_{s-1}$. This last equation has only the solution $q_{s-1}(x,y)=0$. So we have $g_{s-1}=zp_{s-2}(x,y,z)$.

From Eq. (A.4) we get that

$$y(y-2z)\frac{\partial p_{s-2}}{\partial y}-(y-z)\left(p_{s-2}+z\frac{\partial p_{s-2}}{\partial z}\right)=-s(y-z)p_{s-2}+L_0 z^{s-1}. \tag{A.5}$$

If $s=2$, we get from (A.5) that $p_{s-2}=0$, and so system (A.1) has no exponential factor. For $s>2$, set $p_{s-2}=zp_{s-3}(x,y,z)+q_{s-2}(x,y)$ with p_{s-3} and q_{s-2} homogeneous polynomials of degrees $s-3$ and $s-2$, respectively. Then we obtain from (A.5) with $z=0$ that $y\partial q_{s-2}/\partial y=-(s-1)q_{s-2}$. This equation has only the solution $q_{s-2}(x,y)=0$. So we have $p_{s-2}=zp_{s-3}(x,y,z)$. Moreover p_{s-3} satisfies the following equation

$$y(y-2z)\frac{\partial p_{s-3}}{\partial y}-(y-z)\left(2p_{s-3}+z\frac{\partial p_{s-3}}{\partial z}\right)=-s(y-z)p_{s-3}+L_0 z^{s-2}.$$

This equation has a similar form than (A.5).

By induction we can prove that $g_{s-1}=bz^{s-1}$ with b a constant. Substituting g_{s-1} into Eq. (A.4) yields $b=0$. So g has degree at most $s-2$.

If $\deg g=k\leqslant s-2$, then Eq. (A.2) implies that $L=0$. Moreover, we have

$$y(y-2z)\frac{\partial g_k}{\partial y}-z(y-z)\frac{\partial g_k}{\partial z}=-s(y-z)g_k. \tag{A.6}$$

Working in a similar way to the proof of (A.4) we can prove that (A.6) has only the solution $g_k=0$.

The above proof shows that system (A.1) has no exponential factor associated with the invariant plane $z=0$. This completes the proof of the proposition. $\square$

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