Commutativity conditions for rings: 1950–2005

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Abstract

During the last 55 years there have been many results concerning conditions that force a ring to be commutative. These results were stimulated by Jacobson’s famous result and were extensively developed by Herstein. This paper will survey the area by organizing the results according to whether they come from variations on Herstein’s conditions, depend on general polynomial conditions, depend on the presence of a derivation, or whether a ring has special properties that make commutativity more easily accessible. Finally, the most recent conditions concern product sets and lead to results in a new area of inquiry.

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1. Introduction

When I was searching the Math Reviews back in the late 1960s I came across a few references to fascinating conditions under which a ring would be commutative. The original work was by Jacobson who proved that if for every \( x \) in a ring \( R \) there exists a positive integer \( n(x) \) such that \( x^{n(x)} = x \) then \( R \) is commutative. Jacobson used the newly discovered method of deconstructing a ring via its radical. This stimulated research for similar results, but by 1970 I knew of only a few papers in this area. When I came back to this topic after

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an absence of over three decades I discovered that algebraists had not been idle. There are now a 100 or more papers in which conditions are given that determine commutativity for a ring or a special type of ring.

Much of the initial thrust of the work in this area was either authored by Herstein or inspired by his work. A significant contributor has been Bell who individually, or with coauthors, has written over 25 articles, the most recent of which have broken new ground. Another strong contributor has been Yaqub with a variety of coauthors. In this paper, I will attempt to summarize many of the interesting new discoveries that have occurred and to put them in historical perspective.

First, we give a few definitions. A ring $R$ is commutative if $xy = yx$ for all $x, y \in R$. The center of $R$, is defined as $Z = Z(R) = \{ x \in R | xy = yx \text{ for all } y \in r \}$. The commutator of $x$ and $y$ is $xy - yx$, denoted as $[x, y]$. The commutator ideal is the ideal generated by all commutators. An element, $x$, of a ring $R$ is nilpotent if there exists a positive integer $n$ such that $x^n = 0$. An ideal, $I$, is nil if every element of $I$ is nilpotent. A ring is said to be semisimple if it has no non-zero nil ideals. A ring $R$ is a prime ring if the radical of $R$ is prime. A ring $R$ is $n$-torsion free if $nx = 0$ implies $x = 0$ for $x \in R$.

2. Herstein-type conditions

One of the first mathematicians to follow-up Jacobson’s result was Herstein. Starting in 1951 [34] he proved first that if there exists a positive integer $n$ in a ring $R$ such that $x^n - x$ is in the center, then $R$ is commutative. In 1953 [35] he strengthened the result by weakening the condition so that the exponent $n$ depends on $x$ and is not global. He then [36] considerably improved the theorem by showing that if for each $x \in R$ there exists a polynomial $p_x(x)$ such that $x^2 p_x(x) - x \in Z$ then $R$ is commutative. This last result seems to have been ahead of its time since no similar results appeared for almost 20 years. Herstein used the newly discovered structure of semisimple rings which are a subdirect sum of primitive rings. Since the conditions are carried down to projections, he could use the conditions to show that such primitive rings are division rings and then concentrate on the division ring case.

In 1955 and 1957, Herstein [37,38] proved that for a ring, $R$, to be commutative the following conditions are necessary and sufficient:

- H1: For all $x$ and $y$ in $R$ there exists $n(x, y) \geq 2$ such that $(x^n - x)y = y(x^n - x)$;
- H2: For all $x$ and $y$ in $R$ there exists $n(x, y) \geq 2$ such that $xy - yx = (xy - yx)^n$;

If a ring is semisimple then the following are necessary and sufficient for commutativity:

- H3: For all $x$ and $y$ in $R$ there exists $n(x, y) \geq 2$ such that $x^n y = y x^n$;
- H4: For all $x$ and $y$ in $R$ there exists $n(x, y) \geq 2$ such that $(xy)^n = x^n y^n$.

Herstein continued to work in this area, introducing in 1975 [39] the concept of the hypercenter of a ring $R$, $T(R) = \{ x \in R | \text{ for all } y \in R \text{ there exists } n(y) \text{ such that } [x, y^n] = 0 \}$. He proved that if a ring has no nil ideals then the hypercenter equals the center. In 1976 [40], he showed that if for each pair $x, y$ of elements of a ring with no non-zero nil ideals there exist integers $n(x, y)$ and $m(x, y)$ such that $x^m y^n = y^n x^m$, then the ring is commutative. In 1980 [42], he proved that if $[[x^m, y^n], z^q] = 0$ where $m, n, q$, depend on $x, y, z$, then the commutator ideal is nil.
In 1980, Klein et al. [51] defined the \( k \)th commutator recursively by \([a, b]_k =[[a, b]_{k-1}, b]\) and \([a, b]_1 = [a, b]\). They conjectured that the commutator ideal of a ring, \( R \), is nil if there exists a positive integer \( k > 1 \) and for all \( x, y \) in \( R \) there exist integers \( m = m(x, y) \) and \( n = n(x, y) \) such that \([x^m, y^n]_k = 0\). They were able to prove this if they restricted the size of \( m \) with a ceiling \( M \). In 1989, Chuang and Lin [32] generalized the concept of the hypercenter to the \( k \)th hypercenter defined to be \( \{x \in R \mid \text{for all } y \in R \text{ there exists } n(y) \text{ such that } [x, y^n]_k = 0\} \). Using this definition they further defined a ring \( R \) to be a \( C_k \)-ring if for every pair \( x, y \) there are integers \( m = m(x, y) \) and \( n = n(x, y) \) such that \([x^m, y^n]_k = 0\). If, in addition, the value of \( k \) depends on the pair \( x \), \( y \) and is not global for the ring, the ring is called a \( C \)-ring. Significantly, they proved the stronger version of Herstein’s 1976 result and another related result.

**Theorem 1** (Chuang and Lin [32]). If a \( C_k \)-ring has no non-zero nilpotents then the ring is commutative.

**Theorem 2** (Chuang and Lin [32]). If \( R \) is a \( C \)-ring and if \( N_r(R) = 0 \), where \( N_r(R) \) is the sum of the nil left ideals of \( R \), then \( R \) is commutative.

While the two theorems above culminate one line of development, the overall efforts have not been linear as is seen in other results below. Starting in 1968 Bell and others showed that many conditions similar to the conditions of Herstein are equivalent to a ring being commutative, or, at least, that its commutator ideal is nil. Bell [8] used the properties of Duo Rings (every one-sided ideal is two-sided) to prove that if for every pair in a ring there exists an \( n > 1 \) such that \([x, y]^n = [x, y]\), then the ring is commutative. He also showed that if there is a fixed \( n > 1 \) for a ring \( R \) such that \( x^n - x \) is central for all \( x \in R \), then \( R \) is commutative. In 1973, Bell [9] proved that if there exists a fixed positive integer \( n > 1 \) and \( R \) is a ring generated by the \( n \)th powers of its elements and if \( R \) satisfies the identity \( x^ny - yx^n = xy^n - y^n x \), then \( R \) is commutative. In 1976, Bell [11] proved the strong result that a ring, \( R \), is commutative if and only if for all \( x \) and \( y \in R \) there exist integers \( m, n \geq 1 \) for which \( xy = y^nx \). Note that this was an improvement over Herstein’s result since it was proven without requiring that no non-zero nil ideals exist. In 1985, Abu-Khuzam and Yaqub [7] extended Herstein’s 1980 work, proving that if \( R \) is semisimple ring and if for each \( x, y, z \in R \) there exists an integer \( n = n(x, y, z) \) such that \((xyz)^n - x^n y^n z^n\) belongs to the center of \( R \), then \( R \) is commutative.

In 1988, Quadri and Khan [66] proved that if there exist positive integers \( m \) and \( n \) such that \( xy - y^m x^n \) commutes with \( x \), then \( R \) is commutative. Kezlan [48] proved that in a ring with unity \( R \) if there exists an \( n \geq 1 \) and for all \( x, y \in R \) there exists \( m = m(x, y) \) such that \( xy - y^m x^n \) commutes with \( x \), then \( R \) is commutative. In 1998, Kezlan [49] further proved the same result when the roles of \( m \) and \( n \) were reversed.

In a unique direction, as researchers looked for keys to unlock commutativity, they considered rings that satisfy a Herstein-type condition for a range of integers. In particular, Bell, expanding on a result of Herstein and the work of others, defined \((n, k)\)-rings to be rings in which \( n \) and \( k \) are global for the whole ring and for every \( x, y \), \( (xy)^m = x^m y^n \) for all integers \( m, n \leq m \leq n + k - 1 \). In a special case predating the definition, Herstein [40] had shown that \((n, 1)\)-rings have nil commutator ideal. Later, in another special case,
Luh [54] proved that primary \((n, 3)\)-rings must be commutative. The most general result was achieved by Bell [13] who proved that an \((n, 2)\)-ring for which \(R^+\) is \(n\)-torsion-free is commutative.

3. Polynomial constraints

Conditions that are equivalent to commutativity for a ring can be generalized in many directions, as we will explore below. Since the conditions above relating \(x\) and \(y\) are polynomial constraints, it is natural to consider rings which satisfy general polynomial identities where the polynomial is not defined. An early result of this type was by Bell [10] who proved that if \(R\) is a ring in which for every ordered pair \((x, y)\) of elements of \(R\) there exists a polynomial \(p(X) \in \mathbb{Z}[X]\) such that \(xy = yxp(x)\), then \(R\) is commutative. Outcalt and Yaqub [62] showed that if \(R\) is a ring with left identity such that (A) for each \(x \in R\) there exists an \(n = n(x) \geq 1\) and a polynomial, \(p(X)\), dependent on \(x\), such that \(x^n = xnp(x)\) and (B) \(x - y \in N\), the set of nilpotent elements, implies that either \(x^2 = y^2\) or \(x\) and \(y\) commute with all elements of \(N\), then \(R\) is commutative. Bell [16] later proved that a ring is commutative if for every pair \(x, y \in R\) there exists a polynomial \(p(X) \in t^2\mathbb{Z}[t]\) depending on \(x\) and \(y\), such that \([x, y] = p(xy) - p(yx)\), then the ring is commutative.

Another natural extension of the previous ideas is to use polynomials to construct general expressions whose evaluation is in the center for all elements, as Herstein had done with his 1953 paper. This thread was not picked up again until 1979 when Outcalt and Yaqub [63] generalized Herstein’s result by replacing the center with a set \(A\) of elements closed under addition and multiplication and with the elements satisfy \(x^n = x^{n+1}p(x) \in A\). To prove the result they also needed the ring to be periodic and to satisfy the condition that \(x - y \in A\) implies \(x^2 = y^2\) or \(x\) and \(y\) commute with all elements of \(A\). In 1981 Bell [15] proved the same theorem for periodic rings, but without the restriction that \(A\) be closed under addition. In 1983, Cherubini and Varisco [30] removed the need for \(R\) to be periodic. In 1985 [31] they further improved the result by dropping the commutativity of \(A\) and proving:

**Theorem 3.** Let \(R\) be a ring with a non-empty subset \(A\) such that for every \(x \in R\) there exists a polynomial \(p(x)\) with integer coefficients such that \(x - x^2 p(x) \in A\). Furthermore for every \(x, y \in R\), the condition that \(x - y \in A\) implies either \(x^2 = y^2\) or both \(x\) and \(y\) belong to the centralizer of \(A\). Then \(R\) is either commutative or isomorphic to a subdirect sum of nil rings of bounded index \(2\) and/or commutative local rings whose nilpotent elements satisfy the identities \(x^2 = 0 = 2x\).

4. Derivations and commutativity

Another technique for investigating rings is the use of derivations. (A derivation is an additive map satisfying \(d(xy) = d(x)y + xd(y)\).) To indicate how strongly related a
derivation is to commutativity we say a derivation (or other function) \( d : R \to R \) is commuting if \( d(x)x = xd(x) \) for all \( x \in R \) and centralizing if \( xd(x) - d(x)x \) is in the center, \( Z \), of \( R \) for all \( x \in R \). A derivation is called centralizing on a subset \( S \) if \( [x, d(x)] \) is in the center, \( Z \), of \( R \) for all \( x \in S \). Even before there were results for derivations, there were results using automorphisms, precursors to the results concerning derivations. In an early result Luh [53] proved that if a prime ring \( R \) has a non-trivial commuting automorphism, then \( R \) is a commutative integral domain. In 1976, Mayne [55] strengthened this by proving that a prime ring with a non-trivial centralizing automorphism is an integral domain. In 1982, Mayne [56,57] showed that if a ring \( R \) has a non-zero ideal \( U \) and a non-trivial automorphism or derivation \( d \) such that \( d(U) \subseteq U \) for all \( u \in U \), then \( R \) is commutative. In 1984, he [58] proved the even stronger result in which the function need only be centralizing on a non-zero ideal \( U \) and need not map \( U \) into \( U \). A little after Mayne’s results, 1987, Bell and Martindale [26] proved that if \( R \) is a prime ring, if \( U \) is a non-trivial left ideal of \( R \), and if there exists a non-trivial endomorphism \( T \) of \( R \) which is one to one and centralizing on \( U \), then \( R \) is commutative. Herstein [41] connected commutativity and derivations in 1978, proving that if a prime ring \( R \) has a derivation \( d \neq 0 \) such that \( d(x)d(y) = d(y)d(x) \) for all \( x, y \in R \) then, if \( \text{char} R \neq 2 \), then \( R \) is a commutative integral domain, and if \( \text{char} R = 2 \), then \( R \) is commutative or an order in a simple algebra which is 4-dimensional over its center. In 1990, Vukman [69] proved that if \( R \) is a prime ring with characteristic different from 2 and 3, and with a non-zero centralizing derivation on \( R \), then \( R \) is commutative. Bresar [29] proved that if \( R \) is a prime ring and \( U \) a non-zero left ideal of \( R \), and if derivations \( d \) and \( g \) exist on \( R \) such that \( d(u)u - ug(u) \in Z \) for all \( u \in U \) and \( d \neq 0 \), then \( R \) is commutative. More recently (1996) Lee and Lee [52] proved that if \( R \) is a prime ring with non-zero ideal \( I \), \( n \) is a positive integer, \( d \) is a derivation on \( R \) such that \( d^n(I) \subseteq Z \), the center of \( R \), then either \( d^n = 0 \) or \( R \) is commutative. Along the same lines Bell et al. [25] proved:

**Theorem 4.** If \( R \) is a prime ring, if \( L \) is a left ideal of \( R \), and if \( d \) is a derivation on \( R \) such that \( d^n(L) \subseteq Z \) for some positive integer \( n \), then \( R \) is commutative or \( d^n = 0 \). Rehman [67] found similar results using generalized derivations on prime rings in 2002.

5. Special rings

In an effort to discover new conditions that are equivalent to commutativity, many authors have looked at rings with additional constraints beyond the standard simple, semisimple, or prime constraints. These range from being periodic to \( n \)-torsion free to (left, right) \( s \)-unital.

Many of the theorems above require that the ring contain a unity, but rings need not have a unit and mathematicians still want to be able to work with them. In an effort to have a class of rings that are close to having a unit, the concept of \( s \)-unital was developed: A ring, \( R_s \), is called \( s \)-unital if for every \( x \in R \), \( x \in xR \cap Rx \). An easy consequence that was exploited is that for every finite set \( F \) in an \( s \)-unital ring there exists a pseudo-identity \( e \) such that \( ef = f = fe \) for all \( f \in F \). These rings were introduced in 1980 by Abu-Khuzam et al. [6] in a paper in which they proved that if \( n \) is a fixed positive integer and \( R \) is an \( s \)-unital ring in which every commutator is \( n \)-torsion-free and if \( R \) satisfies the identities
Also in 1990 Grosen et al. [33] introduced the concept of a weakly periodic ring as one that in a ring, the set of nilpotent elements, is commutative, and if for each \( x, y \in R \) there exists a word of the type with length at least three with \( xy = w(x, y) \), then \( R \) is commutative. In 1987, Abu-Khuzam [3] proved that if \( R \) is periodic ring, \( N \), the set of nilpotent elements, is commutative, and if for each \( x \in R \) and \( a \in A \) there exists \( n = n(x, a) \) such that \( [x^n, [x^n, a]] = 0 \) and \( [x^{n+1}, [x^{n+1}, a]] = 0 \) then \( R \) is commutative. In 1988, Bell [17] strengthened this result by using the \( k \)th commutator to prove:

**Theorem 5.** If \( R \) is an \( s \)-unital ring, if \( m \) and \( n \) are fixed positive integers, and if \( [x^n - y^m x, x] = 0 \) for all \( x \) and \( y \in R \), then \( R \) is commutative.

In 1990 Abujabal [1] proved a similar result for left \( s \)-unital \( (x \in xR) \) rings. He showed that if \( m > 1, n \), and \( k \) are fixed non-negative integers, if \( R \) is left \( s \)-unital, and if \( [x^n y - y^m x^k, x] = 0 \) for all \( x, y \in R \), then \( R \) is commutative.

A ring \( R \) is called periodic if for all \( x \in R \) there exists different positive integers \( n, m \) such that \( x^n = x^m \). In 1977, Bell [12] introduced the idea of a periodic ring and using a set of criterions on words showed that if for a ring \( R \) there was one type of word (out of nine possible types) such that for each \( x, y \in R \) there exists a word of the type with length at least three with \( xy = w(x, y) \), then \( R \) is commutative. In 1987, Abu-Khuzam [3] proved that if \( R \) is periodic ring, \( N \), the set of nilpotent elements, is commutative, and if for each \( x \in R \) and \( a \in A \) there exists \( n = n(x, a) \) such that \( [x^n, [x^n, a]] = 0 \) and \( [x^{n+1}, [x^{n+1}, a]] = 0 \) then \( R \) is commutative.

In 1988, Bell [17] strengthened this result by using the \( k \)th commutator to prove:

**Theorem 6.** If \( R \) is a periodic ring in which \( N \) is commutative and if for each \( x \in R \) and \( a \in N \) there exist positive integers \( j, k, m, n \) such that \( (n, m) = 1 \) and \( [a, x^n]_j = [a, x^m]_k = 0 \) then \( R \) is commutative.

In 1990, Bell and Guerriero [20] took this line in a different direction by proving a periodic ring with only finitely many non-central subrings of zero divisors is finite or commutative. Also in 1990 Grosen et al. [33] introduced the concept of a weakly periodic ring as one in which every element of the ring can be written as the sum of a nilpotent element and a potent element, where an element \( x \) is potent if there exists \( n = n(x) > 1 \) such that \( x^n = x \). They used some results about weakly periodic rings to prove that if in a ring \( R \) with 1 there exists an integer \( n \) such that \( [x - x^n, y - y^n] = 0 \) for all \( x, y \in R \) and if \( [x^n, y^n] \in Z \), then \( R \) is commutative. In 2003, Rosin and Yaqub [68] extended this idea to subweakly periodic rings, rings in which every element of \( R \setminus J \) can be written as the sum of a potent element and a nilpotent element.

In 1980, Abu-Khuzam [2] proved that if \( R \) is an \( n(n - 1) \)-torsion free ring and satisfies the identity \( (xy)^n = x^n y^n \), then \( R \) is commutative. In 1987, Bell and Yaqub [27] showed that it is only necessary for the commutators to be \( n \)-torsion free, proving that if \( R \) is a ring such that \( n[x, y] = 0 \) implies \( [x, y] = 0 \) then \( R \) is commutative if either \( (xy)^n = (yx)^n \) for all \( x, y \in R \setminus N \) or \( (xy)^n = (yx)^n \) for all \( x, y \in R \setminus J \). In 1991, Abu-Khuzam et al. [5] proved that in a ring, \( R \), whose commutators are \( n \)-torsion free, \( R \) is commutative if it satisfies \( [x^n, y^n] = 0 \) and \( (xy)^{n+1} - x^{n+1} y^{n+1} \in Z \) for all \( x, y \in R \). In 2003, Abu-Khuzam and Bell [4] introduced a concept closely related to periodicity: a ring is a \( c^* \)-ring if for each \( x \in R \), either \( x \) is periodic or there exists a positive integer \( K \) such that \( x^k \in Z \) for all \( k \geq N \). They showed that a reduced \( c^* \)-ring is commutative and combining another concept, showed that a torsion-free \( c^* \)-ring with 1 is commutative. So the area of these special rings remains active if somewhat specialized.
6. \textit{C}*-\textit{Algebras}

A strand of development in the theory of \textit{C}*-\textit{algebras} parallels the developments in rings. In 1955, Ogasawara \cite{60} showed that if $a \geq b$ always implies $a^2 \geq b^2$ for all $a$, $b$ in a \textit{C}*-\textit{algebra} then the \textit{C}*-\textit{algebra} is commutative. In 1979 both Kato \cite{45} and Nakamoto \cite{59} gave characterization of commutativity of a \textit{C}*-\textit{algebra} in terms of the spectrum of elements of the algebra: A unital \textit{C}*-\textit{algebra} $A$ is commutative if and only if $\sigma(x) = \pi_{n}(x)x \in A$, where $\sigma(x)$ is the spectrum of $x$ and $\pi_{n}(x)$ is the normal approximate spectrum of $x$. A \textit{C}*-\textit{algebra} $A$ is commutative if and only if $\forall a, b \in A$, $\sigma(x) \subset \{ \lambda \}; \text{ dist}(\lambda, \sigma(b) \leq \|a - b\|)$.

There was lull in developments in this area until 2000 when Wu \cite{70} prove that a \textit{C}*-\textit{algebra} is commutative if and only if $e^{x+y} = e^{y}e^{x}$ in $A + C$ for all positive elements $x$, $y$ in $A$. In a similar vein, Jeang and Ko \cite{43} proved that if $A$ is a \textit{C}*-\textit{algebra}, if $f$, $g$ are two non-constant continuous functions defined on intervals $I_{1}$ and $I_{2}$, respectively, and if $f(x)g(y) = g(y)f(x)$ for all self-adjoint elements $x$ and $y$ of $A$ with $\sigma(x) \subset I_{1}$ and $\sigma(y) \subset I_{2}$, then $A$ is commutative. Later Ji and Tomiyama \cite{44} proved a more extensive result:

\textbf{Theorem 7.} Let $A$ be a \textit{C}*-\textit{algebra}. The following are equivalent:

1. $A$ is commutative.
2. every continuous monotone function on the positive axis becomes operator monotone on $A$;
3. there exists a continuous monotone function on the positive axis which is not matrix monotone of order 2 but operator monotone on $A$.

Recently, Osaka et al. \cite{61} showed that for general \textit{C}*-\textit{algebras} the classes of monotone functions coincide with the standard classes of matrix and operator monotone functions, giving exact characterizations of \textit{C}*-\textit{algebras} with a given class of monotone functions and providing a monotonicity characterization of subhomogeneous \textit{C}*-\textit{algebras}, that is the \textit{C}*-\textit{algebras} with possible dimensions of irreducible representations not exceeding a certain number. They used this result to generalize previously known monotonicity-based characterizations of commutativity of \textit{C}*-\textit{algebras} to characterizations of subhomogeneity of \textit{C}*-\textit{algebras}, and showed how these results can be applied for a number of \textit{C}*-\textit{algebras}.

7. \textit{B}$_{k}$-rings, \textit{Q}$_{n}$-rings, \textit{P}$_{n}$-rings

Bell and Klein pioneered an interesting new approach to commutativity in rings by using restrictions on the size of products of finite subsets. In 1988, they \cite{21} introduced the concept of a ring $R$ being $(m, n)$ redundant, the condition that every subset $S$ with $|S| = m$ is $n$-redundant, that is $|S^n| < |S|^n$. They proved that every $(2,2)$-redundant ring with 1 is commutative. Another similar way to generalize commutativity is to ask that $XY = YX$ for all finite sets of some size. Bell \cite{14} defined \textit{P}$_{n}$-rings to be rings in which this equality holds for every $n$-set ($|S| = n$) in $R$. Using results about \textit{P}$_{n}$-groups, he proved:

\textbf{Theorem 8.} For $n \geq 2$ a \textit{P}$_{n}$-ring is commutative if any of the following hold: $R$ is torsion-free; the set of nilpotents = $\{0\}$; $R$ has 1 and is $(n - 1)!$-torsion-free; $R$ has 1 and $R \neq T$,
the set of torsion elements; \( R \) has 1 and \( T \) is commutative; \( R \) is semiprime and \( T \) is commutative; the zero divisors lie in the center; or the periodic elements lie in the center.

In 2003, Bell and Klein [24] extended this study to \( P_\infty \)-rings, rings in which \( XY = YX \) for all infinite subsets of the ring, and they proved that every \( P_\infty \)-ring is either finite or commutative.

A similar “near-commutative” idea concerns the number of elements in the set \( K^2 \) of products from the set \( K \). If a set \( K \) has \( k \) elements and is commutative under multiplication then there are at most \( \binom{n}{k} \) elements in \( K^2 \). If the elements of \( K \) do not all commute then the number of elements will be larger. In 2001, Bell and Klein [22] developed the concept of a \( B_k \) ring in which every subset of size \( k \) satisfies the inequality \( |K^2| \leq \binom{k+1}{2} \). (\( B_2 \) rings are equivalent to (2, 2)-redundant rings.) A more restrictive condition is that \( |K^2| \leq \binom{k}{2} \) and a ring satisfying this condition is defined to be a \( C_k \)-ring. They proved, not surprisingly, that such rings are commutative. They also proved

**Theorem 9 (Bell and Klein [22]).** If \( R \) is a semi-prime \( B_k \)-ring then \( R \) is either commutative or finite.

Another condition on the size of sets is that \( xS = Sx \) for all \( x \in R \) and all subsets \( S \) with \( |S| < n \). Bell and Klein [23] define such ring to be a \( Q_n \) ring and prove that any \( Q_n \) ring \( R \) with 1 and \( |R| > n \) is commutative. They have continued this investigation in [19,28,50].

This work approaches the condition for commutativity from a completely new direction with no requirements on individual element. Even some of the areas that have been explored for decades leave room for further work. In particular can Theorems 1 and 2 be extended to less restrictive rings. Will Theorem 4 ever apply to more general rings and not be restricted to prime rings? Similarly will Theorem 5 be true for rings which are not \( s \)-unital? We can expect continued efforts in this area and new directions will be explored.

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