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On the rationality of Cantor and Ahmes series

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ABSTRACT

We give criteria for the rationality of Cantor series $\sum_{n=1}^{\infty}\frac{b_n}{a_1\cdots a_n}$ and series $\sum_{n=1}^{\infty}\frac{b_n}{a_n}$ where $a_1, a_2, \cdots$ and $b_1, b_2, \cdots$ are integers such that $a_n > 0$ and the series converge. We precisely say when $\sum_{n=1}^{\infty}\frac{b_n}{a_1\cdots a_n}$ is rational (i) if $\{a_n\}_{n=1}^{\infty}$ is a monotonic sequence of integers and $b_{n+1} - b_n = o(a_{n+1})$ or $\limsup_{n\to\infty}(\frac{b_{n+1}}{a_{n+1}} - \frac{b_n}{a_n}) \le 0$, and (ii) if $\frac{b_{n+1}}{a_{n+1}-1} \le \frac{b_n}{a_n-1}$ for all large n. We give similar criteria for the rationality of Ahmes series $\sum_{n=1}^{\infty}\frac{1}{a_n}$ and more generally series $\sum_{n=1}^{\infty}\frac{b_n}{a_n}$. For example, if $b_n > 0$ and $\limsup_{n\to\infty} A_{n-1}(\frac{b_{n+1}a_n}{a_{n+1}} - \frac{b_n}{a_n}) \le 0$, where $A_n = \mathrm{lcm}(a_1, a_2, \cdots, a_n)$, then $\sum_{n=1}^{\infty}\frac{b_n}{a_n}$ is rational if and only if $a_{n+1} = \frac{b_{n+1}}{b_n}a_n(a_n-1)+1$ for large n.

On the other hand, we show that such results are impossible without growth restrictions. For example, we show that for any integers $d > c > 1$ there is a sequence $\{b_n\}_{n=1}^{\infty}$ such that every number x from some interval can be represented as $x = \sum_{n=1}^{\infty}\frac{b_n}{a_1\cdots a_n}$ with $a_n \in \{c, d\}$ for all n.

1. INTRODUCTION

Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two integer sequences with $a_n > 0$ for all n. Put $S = \sum_{n=1}^{\infty}\frac{b_n}{a_1\cdots a_n}$ and $R_N = \sum_{n=N}^{\infty}\frac{b_n}{a_N\cdots a_n}$ for $N = 1, 2, \cdots$. Most proofs are based on the following fact. If S is a rational number, $S = r/q$ with $r \in \mathbb{Z}, q \in \mathbb{N}$ say, then $qR_N \in \mathbb{Z}$ for every N.

In Section 2 we present some basic results. In Theorem 2.1 we generalize a result of Oppenheim as follows: if $a_n > 1$ for all $n, b_n = O(a_n)$ and $\{\frac{b_n}{a_n}\}_{n=1}^{\infty}$ has an irrational limit point, then $S \notin \mathbb{Q}$. Oppenheim required additionally that $0 \le b_n < a_n$ for all n.

Let $\{a_n\}_{n=1}^{\infty}$ be a nondecreasing sequence with $a_n > 1$ for all n. In [5] Hančl and Tijdeman showed that S is rational if and only if $\frac{b_n}{a_n-1}$ is eventually constant provided that (i) $b_n = n$ and $a_n \to \infty$, or (ii) $a_n = n, b_{n+1} - b_n = o(n)$ or (iii) $b_n = o(a_n^2), b_n \geq 0, b_{n+1} - b_n < \epsilon a_n$ for $n \geq n_1(\epsilon)$. In Section 3 we present a common generalization of (i) and (ii) and we show that the condition $b_n = o(a_n^2)$ in (iii) is superfluous.

Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be arbitrary sequences of positive integers. In Section 4 a rationality criterion for S is given (i) if $\{a_n\}_{n=1}^{\infty}$ is a nondecreasing sequence and $\limsup_{n\to\infty}(\frac{b_{n+1}}{a_{n+1}} - \frac{b_n}{a_n}) \leq 0$, and (ii) if $a_n b_{n+1} - a_{n+1} b_n \leq b_{n+1} - b_n$ for all n. Furthermore we generalize and refine rationality criteria of Sylvester [7], Badea [1], [2] and Erdös and Straus [4] for Ahmes series $\sum_{n=1}^{\infty} \frac{1}{a_n}$ and more generally series $\sum_{n=1}^{\infty} \frac{b_n}{a_n}$. For example, if $\limsup_{n\to\infty} A_{n-1}(\frac{b_{n+1}a_n}{a_{n+1}} - \frac{b_n}{a_n}) \leq 0$, where $A_n = \operatorname{lcm}(a_1, a_2, \cdots, a_n)$, then $\sum_{n=1}^{\infty} \frac{b_n}{a_n}$ is rational if and only if $a_{n+1} = \frac{b_{n+1}}{b_n} a_n(a_n - 1) + 1$ for large n.

In Section 5 some variants of a construction of Hančl and Tijdeman [5] are presented. We show that if $k > 1$ is an integer and $\{b_n\}_{n=1}^{\infty}$ a monotonically nondecreasing sequence, then every number x from some interval can be represented as $x = \sum_{n=1}^{\infty} \frac{b_n}{a_1 \cdots a_n}$ with $a_n \in \{k, k+1, \cdots, k^2\}$. Furthermore, we show that there exists a sequence $\{b_n\}_{n=2}^{\infty}$ such that every number x from some interval can be represented as $x = \sum_{n=2}^{\infty} \frac{b_n}{a_2 \cdots a_n}$ with $a_n \in \{n, n+1\}$ for every n. Finally, for any integers $d > c > 1$, we construct a sequence $\{b_n\}_{n=1}^{\infty}$ such that every number x from some interval can be represented as $x = \sum_{n=1}^{\infty} \frac{b_n}{a_1 \cdots a_n}$ with $a_n \in \{c, d\}$. These construtions show that the results in Sections 3 and 4 do not hold without growth restrictions.

2. A CRITERION AND SOME BASIC PROPERTIES

In this section we study necessary and sufficient conditions under which the Cantor series

$$(1) \qquad S = \sum_{n=1}^{\infty} \frac{b_n}{a_1 \cdots a_n}$$

is rational, where $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are two sequences of integers with a_n positive for all n. We do so by studying the N-th partial sum S_N and the N-th remainder R_N defined by

$$(2) \qquad S_N = \sum_{n=1}^{N-1} \frac{b_n}{a_1 \cdots a_n}, \quad R_N = \sum_{n=N}^{\infty} \frac{b_n}{a_N \cdots a_n}.$$

Throughout the paper we assume without further mention that $\sum_{n=1}^{\infty} \frac{b_n}{a_1 \cdots a_n}$ converges when we discuss its rationality. Hence it suffices to consider the value of $\lim_{k\to\infty} S_{n_k} (= S)$ for some subsequence $\{n_k\}_{k=1}^{\infty}$ of the positive integers. The following results are crucial.

Lemma 2.1. ([5]). (i) *If there is a constant c such that* $b_n = c(a_n - 1)$ *for* $n \geq n_0$, *then* $S \in \mathbb{Q}$.

(ii) *If* $S = r/q$ *for some* $r \in \mathbb{Z}, q \in \mathbb{N}$, *then* $qR_n \in \mathbb{Z}$ *for all n.*

For a subsequence $\{n_k\}_{k=1}^{\infty}$ of the positive integers, put $n_0 = 1$,

(3) $$a_k^\star = a_{n_k - 1} a_{n_k - 2} \cdots a_{n_{k-1}},$$

(4) $$b_k^\star = b_{n_k - 1} + b_{n_k - 2} a_{n_k - 1} + \cdots + b_{n_{k-1}} a_{n_k - 1} a_{n_k - 2} \cdots a_{n_{k-1}+1}.$$

Then, for $k = 1, 2, \cdots$,

(5) $$S = \sum_{k=1}^{\infty} \frac{b_k^\star}{a_1^\star \cdots a_k^\star}, \quad S_{n_k} = \sum_{j=1}^{k} \frac{b_j^\star}{a_1^\star \cdots a_j^\star}, \quad R_{n_k} = \sum_{j=k+1}^{\infty} \frac{b_j^\star}{a_{k+1}^\star \cdots a_j^\star}.$$

The next lemma presents a sufficient condition for the rationality of S.

Lemma 2.2. *If there exists a subsequence $\{n_k\}_{k=1}^{\infty}$ of the positive integers such that $R_{n_k} = R_{n_{k+1}}$ for $k = 1, 2, \cdots$, then S is rational.*

Proof. Put $R = R_{n_1}$. Using the notation (3)-(4) we have

$$R = R_{n_{k-1}} = \frac{b_k^\star}{a_k^\star} + \frac{1}{a_k^\star} R_{n_k} = \frac{b_k^\star}{a_k^\star} + \frac{1}{a_k^\star} R.$$

Hence $R = \frac{b_k^\star}{a_k^\star - 1} \in \mathbb{Q}$, or $b_k^\star = 0, a_k^\star = 1$ for $k = 1, 2, \cdots$. Since the series for S converges, we have $a_1^\star \cdots a_k^\star \to \infty$ as $k \to \infty$ unless $b_k^\star = 0$ for $k \geq k_0$. In the latter case $S \in \mathbb{Q}$. In the former case we obtain

$$S = \lim_{k \to \infty} S_{n_k} = \frac{b_1^\star}{a_1^\star} + \lim_{k \to \infty} \sum_{j=2}^{\infty} \frac{(a_j^\star - 1)R}{a_1^\star \cdots a_j^\star}$$
$$= \frac{b_1^\star}{a_1^\star} + R - \lim_{k \to \infty} \frac{R}{a_1^\star \cdots a_k^\star} = \frac{b_1^\star}{a_1^\star} + R \in \mathbb{Q}. \quad \square$$

The case $n_k = k$ for all k of the following result was repeatedly used by Hančl and Tijdeman in [5].

Proposition 2.1. *If $\{R_n\}_{n=1}^{\infty}$ is bounded from below and there exists a subsequence $\{n_k\}_{k=1}^{\infty}$ of the positive integers with $R_{n_{k+1}} - R_{n_k} < \epsilon$ for $k \geq k_0(\epsilon)$, then S is rational if and only if $R_{n_k} = R_{n_{k+1}}$ for all large k.*

Proof. Assume $S = r/q$ for some $r \in \mathbb{Z}, q \in \mathbb{N}$. Then $qR_n \in \mathbb{Z}$ for all n by Lemma 2.1. Therefore for $K \geq k_0(1/q)$ we have $R_{n_K} \geq R_{n_{K+1}} \geq R_{n_{K+2}} \geq \cdots$. Since $\{qR_{n_k}\}_{k=K}^{\infty}$ is an integer sequence bounded from below, we have $R_{n_k} = R_{n_{k+1}}$ for k sufficiently large.

The sufficiency of the condition follows from Lemma 2.2. $\square$

Remark 2.1. In a similar way we can prove that the conclusion of Proposition 2.1 holds if there exists an integer sequence $\{n_k\}_{k=1}^{\infty}$ with $R_{n_{k+1}} - R_{n_k} \to 0$ as $k \to \infty$. This idea is used in the proof of the following theorem.

Oppenheim [6] proved: let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences of integers such that $a_n > 1$ and $0 \le b_n < a_n$ for all n and that $\{\frac{b_n}{a_n}\}_{n=1}^{\infty}$ has an irrational limit point. Then $S = \sum_{n=1}^{\infty} \frac{b_n}{a_1 \cdots a_n}$ is irrational. We show here that the condition $0 \le b_n < a_n$ can be relaxed to $b_n = O(a_n)$.

Theorem 2.1. *Suppose that $a_n > 1$ for all n, that $b_n = O(a_n)$ and that $\{\frac{b_n}{a_n}\}_{n=1}^{\infty}$ has an irrational limit point α. Then S is irrational.*

Proof. Suppose $S = r/q$ for some $r \in \mathbb{Z}, q \in \mathbb{N}$. Then, by Lemma 2.1, $qR_n \in \mathbb{Z}$ for every n. Suppose $|\frac{b_n}{a_n}| \le M$ for every n. Consider a subsequence $\{n_k\}_{k=1}^{\infty}$ of the positive integers such that $\frac{b_{n_k}}{a_{n_k}} \to \alpha$ as $k \to \infty$. Since $\alpha \notin \mathbb{Q}$, we have $\lim_{k\to\infty} a_{n_k} = \infty$. Observe that

$$R_{n_k} = \frac{b_{n_k}}{a_{n_k}} + \frac{1}{a_{n_k}} R_{n_k+1}.$$

Since

$$|R_{n_k+1}| \le \left|\frac{b_{n_k+1}}{a_{n_k+1}}\right| + \left|\frac{b_{n_k+1}}{a_{n_k+1}a_{n_k+2}}\right| + \cdots \le M\left(1 + \frac{1}{2} + \frac{1}{4} + \cdots\right) = 2M,$$

we obtain

$$\lim_{k\to\infty} qR_{n_k} = q \lim_{k\to\infty} \frac{b_{n_k}}{a_{n_k}} + q \lim_{k\to\infty} \frac{R_{n_k+1}}{a_{n_k}} = q\alpha.$$

Recall that $qR_{n_k} \in \mathbb{Z}$. Thus α is rational. □

Corollary 2.1. *Suppose $\lim_{n\to\infty} \frac{b_n}{a_n}$ exists and is irrational. Then S is irrational.*

3. THE CASE $b_{n+1} - b_n = o(a_{n+1})$

Let $\{a_n\}_{n=1}^{\infty}$ be a nondecreasing sequence of integers with $a_n > 1$ for all n. Hančl and Tijdeman [5] showed that $S = \sum_{n=1}^{\infty} \frac{b_n}{a_1 \cdots a_n}$ is rational if and only if $\frac{b_n}{a_n - 1}$ is constant for n greater than some n_0 provided that (i) $b_n = n$ and $a_n \to \infty$ (Theorem 6.2), or (ii) $a_n = n, b_{n+1} - b_n = o(n)$ (Corollary 4.2) or (iii) $b_n = o(a_n^2), b_n \ge 0, b_{n+1} - b_n < \epsilon a_n$ for $n \ge n_1(\epsilon)$. In this section we present a common generalization of (i) and (ii) in Theorem 3.1 and we show that the condition $b_n = o(a_n^2)$ in (iii) can be dropped in Theorem 3.2.

Theorem 3.1. *Let $\{a_n\}_{n=1}^{\infty}$ be a monotonic integer sequence with $a_n > 1$ for all n and $\{b_n\}_{n=1}^{\infty}$ an integer sequence such that $b_{n+1} - b_n = o(a_{n+1})$. Then the sum $S = \sum_{n=1}^{\infty} \frac{b_n}{a_1 \cdots a_n}$ is rational if and only if $\frac{b_n}{a_n - 1}$ is constant from some n_0 on.*

Proof. In view of Lemma 2.1(i) one direction is obvious. Therefore it suffices to prove the other direction. Suppose $S = r/q$ for some $r \in \mathbb{Z}, q \in \mathbb{N}$. Then, by Lemma 2.1(ii), $qR_n \in \mathbb{Z}$ for every n. By the definition of R_n we have

$$R_{n+1} = a_n R_n - b_n \qquad (n = 1, 2, \cdots) \tag{6}$$

and, by the convergence assumption $S - S_n = \frac{R_n}{a_1 \cdots a_{n-1}} \to 0$ as $n \to \infty$,

(7) $\qquad R_n = o(a_1 a_2 \cdots a_{n-1})$.

It follows from (6) that

(8) $\qquad R_{n+2} - R_{n+1} = (R_{n+1} - R_n)a_{n+1} + R_n(a_{n+1} - a_n) - (b_{n+1} - b_n)$.

Since $a_{n+1} \geq a_n, q(R_{n+1} - R_n) \in \mathbb{Z}$ and $b_{n+1} - b_n < \frac{a_{n+1}}{4q}$ for $n \geq n_1$, we see that $R_{m+1} > R_m \geq 0$ for some $m \geq n_1$ implies $R_{m+2} > R_{m+1}$. Moreover,

$$R_{m+2} - R_{m+1} > (R_{m+1} - R_m)a_{m+1} - \frac{a_{m+1}}{4q}.$$

Hence by (6) with $n = m + 1$,

$$\begin{aligned} R_{m+3} - R_{m+2} &> a_{m+2}(R_{m+2} - R_{m+1}) - \frac{a_{m+2}}{4q} \\ &> a_{m+2}a_{m+1}(R_{m+1} - R_m) - \frac{a_{m+2}a_{m+1}}{4q} - \frac{a_{m+2}}{4q}. \end{aligned}$$

By induction we get, using that $a_n > 1$ for all n,

$$R_{m+r+1} - R_{m+r} \geq$$
$$(R_{m+1} - R_m)a_{m+1} \cdots a_{m+r} - \frac{1}{4q}(a_{m+1} \cdots a_{m+r} + a_{m+2} \cdots a_{m+r} + \cdots + a_{m+r}) >$$
$$\frac{1}{q}a_{m+1} \cdots a_{m+r} - \frac{1}{4q}a_{m+1} \cdots a_{m+r}(1 + \frac{1}{2} + \frac{1}{4} + \cdots) = \frac{1}{2q}a_{m+1} \cdots a_{m+r}.$$

Therefore

$$\lim_{n \to \infty} \frac{R_{n+1}}{a_1 \cdots a_n} = \frac{1}{a_1 \cdots a_m} \lim_{r \to \infty} \frac{R_{m+r+1}}{a_{m+1} \cdots a_{m+r}} \neq 0,$$

which contradicts (7). Thus $R_{m+1} \leq R_m$ if $R_m \geq 0, m \geq n_1$. By replacing b_n with $-b_n$ for all n, we see that also $R_{m+1} \geq R_m$ if $R_m \leq 0, m \geq n_1$. If R_n is constant from some n_0 on, then S is rational by Lemma 2.2. Thus we may assume that $\{R_n\}_{n=1}^{\infty}$ has infinitely many sign changes. Let $m \geq n_1$ be such that $R_m \leq 0, R_{m+1} > 0$. By (6) we have $b_m < 0$. Hence, $b_{m+1} < b_m + \frac{a_{m+1}}{4q} \leq \frac{a_{m+1}}{4q}$. From (6) and $a_n > 1$ with $n = m + 1$ and Lemma 2.1 we get

$$R_{m+2} - R_{m+1} = (a_{m+1} - 1)R_{m+1} - b_{m+1} > \frac{a_{m+1}}{2q} - \frac{a_{m+1}}{4q} > 0.$$

On applying (8) for $n = m + 1, m + 2, \cdots$ we obtain by induction that

$$R_{m+i+1} - R_{m+i} > (R_{m+i} - R_{m+i-1})a_{m+i} - \frac{a_{m+i}}{4q}$$

and reasoning as before we again arrive at a contradiction with (7). □

Theorem 3.2. *Let $\{a_n\}_{n=1}^{\infty}$ be a monotonic integer sequence with $a_n > 1$ for all n. Let $\{b_n\}_{n=1}^{\infty}$ be a sequence of positive integers such that* $\limsup_{n \to \infty} \frac{b_{n+1} - b_n}{a_n} \leq 0$. *Then $S = \sum_{n=1}^{\infty} \frac{b_n}{a_1 \cdots a_n}$ is rational if and only if $\frac{b_n}{a_n - 1}$ is constant from some n_0 on.*

Proof. Since $R_n \geq 0$ for all n, it suffices to follow the first part of the proof of Theorem 3.1. $\square$

Example 3.1. $\sum_{n=1}^{\infty} \frac{(n+1)!}{(2n)!} \notin \mathbb{Q}$. Apply Theorem 3.1 with $b_n = n+1$ and $a_n = 4n+2$.

4. THE CASE OF POSITIVE b_n

In this section we assume $b_n > 0$ for all n, but in most results we drop the requirement that $\{a_n\}_{n=1}^{\infty}$ is monotonic. This will enable us to derive rationality results on series $\sum_{n=1}^{\infty} \frac{b_n}{a_n}$ too.

Theorem 4.1 also deals with such series. Its proof is based on the proofs of Erdös and Straus [4], but it is much simpler and more general.

Theorem 4.1. *Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences of positive integers such that the series $S := \sum_{n=1}^{\infty} \frac{b_n}{a_n}$ converges. Let A_n denote the lowest common multiple of the numbers $a_1, \cdots, a_n$. Suppose $\limsup_{n\to\infty} A_{n-1}(\frac{b_{n+1}a_n}{a_{n+1}} - \frac{b_n}{a_n}) \leq 0$. Then S is rational if and only if $a_{n+1} = \frac{b_{n+1}}{b_n} a_n(a_n - 1) + 1$ for large n.*

Proof. Suppose $S = r/q$ with $r, q \in \mathbb{N}$. Put $R_n^* = \sum_{k=n+1}^{\infty} \frac{b_k}{a_k}$. Then $qA_nR_n^* = A_nr - q\sum_{k=1}^{n} \frac{A_nb_n}{a_n} \in \mathbb{N}$ for all n. By the assumptions of the theorem, for every $\epsilon > 0$, there is an $n_1(\epsilon)$ such that

$$\frac{b_{n+1}a_n}{a_{n+1}} - \frac{b_n}{a_n} \leq \frac{\epsilon}{A_{n-1}} \quad \text{and} \quad \frac{b_n}{a_n} \leq \epsilon,$$

which implies $a_n < \epsilon a_{n+1}$, for $n > n_1(\epsilon)$. We have, assuming that $\epsilon < \frac{1}{2}$,

$$\begin{aligned} a_nR_n^* - R_{n-1}^* &= \left(\frac{b_{n+1}a_n}{a_{n+1}} - \frac{b_n}{a_n}\right) + \left(\frac{b_{n+2}a_n}{a_{n+2}} - \frac{b_{n+1}}{a_{n+1}}\right) + \left(\frac{b_{n+3}a_n}{a_{n+3}} - \frac{b_{n+2}}{a_{n+2}}\right) \\ &+ \cdots \leq \left(\frac{b_{n+1}a_n}{a_{n+1}} - \frac{b_n}{a_n}\right) + \frac{a_n}{a_{n+1}}\left(\frac{b_{n+2}a_{n+1}}{a_{n+2}} - \frac{b_{n+1}}{a_{n+1}}\right) \\ &+ \frac{a_n}{a_{n+1}}\frac{a_{n+1}}{a_{n+2}}\left(\frac{b_{n+3}a_{n+2}}{a_{n+3}} - \frac{b_{n+2}}{a_{n+2}}\right) + \cdots \\ &< \frac{\epsilon}{A_{n-1}} + \frac{\epsilon^2}{A_n} + \frac{\epsilon^3}{A_{n+1}} + \cdots < \frac{2\epsilon}{A_{n-1}}. \end{aligned}$$

Choose $\epsilon = \frac{1}{4q}$. It follows that the integer $qA_{n-1}a_nR_n^* - qA_{n-1}R_{n-1}^*$ is less than 1, hence ≤ 0, for $N > n_1$. Therefore $a_1 \cdots a_nR_n^* \leq a_1 \cdots a_{n-1}R_{n-1}^*$ for $n > n_1$. Since $qa_1 \cdots a_nR_n^* \in \mathbb{N}$ and the sequence $\{a_1 \cdots a_nR_n^*\}_{n=1}^{\infty}$ is non-increasing for $n > n_1$, we obtain that the sequence is ultimately constant, whence

$$a_nR_n^* = R_{n-1}^*$$

for $n > n_2$. Observe that $a_nR_n^* = R_{n-1}^* = \frac{b_n}{a_n} + R_n^*$. So $R_n^* = \frac{b_n}{a_n(a_n-1)}$ and $\frac{b_{n+1}}{a_{n+1}-1} = a_{n+1}R_{n+1}^* = R_n^* = \frac{b_n}{a_n(a_n-1)}$ for $n > n_2$. This implies that $a_{n+1} = \frac{b_{n+1}}{b_n}a_n(a_n - 1) + 1$ for $n > n_2$.

On the other hand, suppose $a_{n+1} = \frac{b_{n+1}}{b_n}a_n(a_n - 1) + 1$ and $a_n > 1$ for

$n \geq n_0$. Then, by induction, $\sum_{k=n_0}^{n} \frac{b_k}{a_k} = \frac{b_{n_0}}{a_{n_0}(a_{n_0}-1)} - \frac{b_n}{a_n(a_n-1)}$ for $n \geq n_0$. Hence $\sum_{k=n_0}^{\infty} \frac{b_k}{a_k} \in \mathbb{Q}$. $\square$

Theorem 4.1 implies several old results on Ahmes series. Case (i) of Corollary 4.1 is due to Badea [1], [2]. The special case with $b_n = 1$ for all n already occurs in a paper of Sylvester [7]. Case (iv) with $b_n = 1$ for all n is Theorem 1 of Erdös and Straus [4] and case (v) with the same restriction is an improvement of Theorem 3 of that paper. We show that the condition (i) of their Theorem 3 can be dropped.

Corollary 4.1. *Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences of positive integers such that the series $S := \sum_{n=1}^{\infty} \frac{b_n}{a_n}$ converges. Then $a_{n+1} = \frac{b_{n+1}}{b_n} a_n(a_n - 1) + 1$ for large n if and only if S is rational provided that at least one of the following conditions is satisfied:*

(i) $a_{n+1} \geq \frac{b_{n+1}}{b_n} a_n^2 - \frac{b_{n+1}}{b_n} a_n + 1$,
(ii) $a_{n+1} \geq \frac{b_{n+1}}{b_n} a_n^2 + O(b_{n+1}\, a_n)$,
(iii) $a_{n+1} \geq \frac{b_{n+1}}{b_n} a_n^2(1 - \epsilon_n)$ *where* $\sum_{n=1}^{\infty} |\epsilon_n| < \infty$,
(iv) $a_{n+1} \geq \frac{b_{n+1}}{b_n} a_n^2(1 + o(1))$ *and* $\{\frac{A_n b_{n+1}}{a_{n+1}}\}$ *is bounded,*
(v) $a_{n+1} \geq \frac{b_{n+1}}{b_n} a_n^2(1 + o(\frac{a_n}{A_{n-1} b_n}))^{-1}$.

Proof. Condition (v) is just a rewriting of the limsup condition. If $\{\frac{A_n b_{n+1}}{a_{n+1}}\}$ is bounded, then condition (iv) implies condition (v).

Suppose condition (iii) holds. Then

$$a_{n+1} \geq \frac{b_{n+1}}{b_n} a_n^2(1 - \epsilon_n) \geq \frac{b_{n+1}}{b_n} a_n a_{n-1}(1 - \epsilon_n)(1 - \epsilon_{n-1}) \geq \cdots \geq \frac{b_{n+1}}{b_n} a_n a_{n-1} \cdots a_2 a_1^2 \prod_{k=1}^{n} (1 - \epsilon_k).$$

Since $C := \prod_{k=1}^{\infty}(1 - \epsilon_k)$ converges and is positive, we obtain

$$\frac{A_n b_{n+1}}{a_{n+1}} \leq \frac{a_1 \cdots a_n b_{n+1}}{a_{n+1}} \leq \frac{b_1}{a_1 C}.$$

Of course $\epsilon_n \to 0$ as $n \to \infty$. Thus case (iii) follows from (iv).

Cases (i) and (ii) follow immediately from (iii) since $\sum_{n=1}^{\infty} \frac{1}{a_n} \leq \sum_{n=1}^{\infty} \frac{b_n}{a_n}$ converges. $\square$

By varying the proof of Theorem 4.1 we derive a result on Cantor series. Theorem 4.2 shows that in Corollary 4.1 of [5] the conditions $b_n = O(a_n^2)$ and $b_{n+1} - b_n = o(a_n)$ can be replaced with the single condition $\frac{b_{n+1}}{a_{n+1}} \leq \frac{b_n}{a_n} + o(1)$.

Theorem 4.2. *Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of positive integers such that $\{a_n\}_{n=1}^{\infty}$ is ultimately monotonic and* $\limsup_{n\to\infty}(\frac{b_{n+1}}{a_{n+1}} - \frac{b_n}{a_n}) \leq 0$. *Then* $S = \sum_{n=1}^{\infty} \frac{b_n}{a_1 \cdots a_n} \in \mathbb{Q}$ *if and only if $\frac{b_n}{a_n - 1}$ is constant for n larger than some n_0.*

Proof. For every $\epsilon > 0$ there is an $n_1(\epsilon)$ such that $\frac{b_{n+1}}{a_{n+1}} - \frac{b_n}{a_n} < \epsilon$ for $n \geq n_1(\epsilon)$. Suppose $S = r/q$ with $r, q \in \mathbb{N}$. Choose $\epsilon = \frac{1}{4q}$. We have

$$R_{n+1}-R_n=\left(\frac{b_{n+1}}{a_{n+1}}-\frac{b_n}{a_n}\right)+\left(\frac{b_{n+2}}{a_{n+1}a_{n+2}}-\frac{b_{n+1}}{a_na_{n+1}}\right)$$
$$+\left(\frac{b_{n+3}}{a_{n+1}a_{n+2}a_{n+3}}-\frac{b_{n+2}}{a_na_{n+1}a_{n+2}}\right)+\cdots$$
$$<\left(\frac{b_{n+1}}{a_{n+1}}-\frac{b_n}{a_n}\right)+\frac{1}{a_{n+1}}\left(\frac{b_{n+2}}{a_{n+2}}-\frac{b_{n+1}}{a_{n+1}}\right)+\frac{1}{a_{n+1}a_{n+2}}\left(\frac{b_{n+3}}{a_{n+3}}-\frac{b_{n+2}}{a_{n+2}}\right)+\cdots$$
$$<\frac{1}{4q}\left(1+\frac{1}{2}+\frac{1}{4}+\cdots\right)\leq\frac{1}{2q}.$$

The fact that $q(R_{n+1}-R_n)$ is an integer implies $R_{n+1}\leq R_n$ for all $n\geq n_1(\frac{1}{4q})$. Since $R_n>0$ for all n, we obtain $R_{n+1}=R_n$ for n larger than some n_0. Hence, by (6) and $b_n>0$, we find that $\frac{a_n-1}{b_n}$ is constant for $n>n_0$.

The assertion in the other direction follows from Lemma 2.1. □

In the following variant of Theorem 4.2 the monotonicity of $\{a_n\}_{n=1}^{\infty}$ is no longer required. Note that the proof has a different structure.

Theorem 4.3. *Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences of positive integers such that $a_nb_{n+1}-a_{n+1}b_n\leq b_{n+1}-b_n$ for all large n. Then $\sum_{n=1}^{\infty}\frac{b_n}{a_1\cdots a_n}$ is rational if and only if $\frac{a_n-1}{b_n}$ is constant for $n\geq n_0$.*

Proof. One direction follows from Lemma 2.1. Suppose $S=r/q$ with $r\in\mathbb{Z},q\in\mathbb{N}$. If $R_{n+1}\leq R_n$ for all but finitely many n, then the assertion follows as in the last few lines of the proof of Theorem 4.2. So let m be an integer with $R_{m+1}>R_m$. From (6) we obtain $R_{m+2}-a_{m+1}R_{m+1}=\frac{b_{m+1}}{b_m}(R_{m+1}-a_mR_m)$ which we rewrite as

$$\begin{aligned}&R_{m+2}-R_{m+1}\\ &\quad=R_{m+1}\left(a_{m+1}+\frac{b_{m+1}}{b_m}-a_m\frac{b_{m+1}}{b_m}-1\right)+(R_{m+1}-R_m)a_m\frac{b_{m+1}}{b_m}.\end{aligned}\tag{9}$$

The inequality $a_mb_{m+1}-a_{m+1}b_m\leq b_{m+1}-b_m$ is equivalent to $a_{m+1}+\frac{b_{m+1}}{b_m}-a_m\frac{b_{m+1}}{b_m}-1\geq 0$. Hence

$$R_{m+2}-R_{m+1}\geq(R_{m+1}-R_m)a_m\frac{b_{m+1}}{b_m}>0.$$

On applying induction we obtain

$$R_{m+r+1}-R_{m+r}\geq(R_{m+1}-R_m)a_m\cdots a_{m+r-1}\cdot\frac{b_{m+r}}{b_m}\quad(r=1,2,\cdots).$$

By the convergence condition we obtain,

$$\lim_{n\to\infty}\frac{b_n}{a_n}=\lim_{r\to\infty}\frac{b_{m+r}}{a_{m+r}}\leq\frac{a_1\cdots a_{m-1}b_m}{R_{m+1}-R_m}\lim_{r\to\infty}\frac{R_{m+r+1}}{a_1a_2\cdots a_{m+r}}=0.$$

Let $0<\epsilon\leq\frac{1}{2}$. Then $\frac{b_n}{a_n}\leq\epsilon$ and $a_n\geq 2$ for $n\geq n_1(\epsilon)$. Hence, for $n\geq n_1(\epsilon)$,

$R_n = \frac{b_n}{a_n} + \frac{b_{n+1}}{a_n a_{n+1}} + \cdots \le \epsilon(1 + \frac{1}{a_n} + \frac{1}{a_n a_{n+1}} + \cdots) \le 2\epsilon$. Since $qR_n \in \mathbb{Z}$ by Lemma 2.1, we obtain $R_n = 0$ for $n \ge n_1(\frac{1}{2q})$, which is impossible. □

Remark. The following argument shows that Theorem 4.3 implies Badea's result (i) of Corollary 4.1. On applying Theorem 4.3 with $B_n := a_1 a_2 \cdots a_n$ in place of b_n, we find that $\sum_{n=1}^{\infty} \frac{b_n}{a_n}$ is rational if and only if $\frac{a_n - 1}{B_n}$ is constant for $n \ge n_0$. Hence $(a_n - 1)B_{n+1} = (a_{n+1} - 1)B_n$ for $n \ge n_0$ and the equality can be rewritten as $a_{n+1} = \frac{b_{n+1}}{b_n} a_n^2 - \frac{b_{n+1}}{b_n} a_n + 1$.

In a similar way we obtain the following refinement of Badea's result.

Corollary 4.2. *Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences of positive integers such that the series $\sum_{n=1}^{\infty} \frac{b_n}{a_n}$ converges and has a rational sum. Let A_n denote the lowest common multiple of the numbers $a_1, \cdots, a_n$. If*

$$a_{n+1} \ge \frac{b_{n+1}}{b_n} a_n \left(\frac{A_n}{A_{n-1}} - 1 \right) + \gcd(A_n, a_{n+1})$$

for all large n. Then

$$a_{n+1} = \frac{b_{n+1}}{b_n} a_n \left(\frac{A_n}{A_{n-1}} - 1 \right) + \gcd(A_n, a_{n+1})$$

for $n \ge n_0$.

Proof. Put $A_0 = 1$, $B_n^\star = \frac{b_n A_n}{a_n}$ for $n = 1, 2, \cdots$. Then

$$\sum_{n=1}^{\infty} \frac{b_n}{a_n} = \sum_{n=1}^{\infty} \frac{B_n^\star}{A_1 \cdot \frac{A_2}{A_1} \cdots \frac{A_n}{A_{n-1}}}.$$

Note that

$$\frac{A_n}{A_{n+1}} \left(\frac{A_{n+1}}{A_n} - 1 \right) = 1 - \frac{\gcd(a_{n+1}, A_n)}{a_{n+1}}. \tag{10}$$

On applying Theorem 4.3 with A_n/A_{n-1} in place of a_n and $B_n^\star$ in place of b_n and using (10), we find that $\sum_{n=1}^{\infty} \frac{b_n}{a_n}$ is rational if and only if $\frac{A_n/A_{n-1} - 1}{B_n^\star} = \frac{A_{n+1}/A_n - 1}{B_{n+1}^\star}$ for $n \ge n_0$. By (10) the equality is equivalent with

$$a_{n+1} = \frac{b_{n+1}}{b_n} a_n \left(\frac{A_n}{A_{n-1}} - 1 \right) + \gcd(A_n, a_{n+1}). \quad □$$

The following proposition shows that under the conditions of Corollary 4.2 in case $b_n = 1$ for all n and $\limsup \frac{a_n^2}{a_{n+1}} \le 1$ it follows that the gcd equals 1 from some n_1 on so that $a_{n+1} = a_n^2 - a_n + 1$ for all larger n.

Proposition 4.1. *Let the notation be as in Corollary 4.2. If $b_n = 1$ and $a_{n+1} = a_n(\frac{A_n}{A_{n-1}} - 1) + \gcd(A_n, a_{n+1})$ for all n and there are infinitely many n such that $\gcd(A_n, a_{n+1}) > 1$, then $\limsup_{n \to \infty} \frac{a_n^2}{a_{n+1}} > 1$.*

Proof. Note that $a_n \to \infty$ as $n \to \infty$. If $a_n | A_{n-1}$, then $A_n = A_{n-1}$, hence $a_{n+1} = \gcd(A_n, a_{n+1})$ and so $a_{n+1} | A_n = A_{n-1}$. This would imply that $\{a_n\}_{n=1}^{\infty}$ is bounded which is excluded. Therefore $A_n > A_{n-1}$ and $\gcd(A_n, a_{n+1}) \leq \frac{a_{n+1}}{2}$ for all $n > 1$. If $\gcd(A_n, a_{n+1}) = \frac{a_{n+1}}{2}$, then $a_{n+1} = a_n + \frac{a_{n+1}}{2}$ whence $\frac{a_n^2}{a_{n+1}} = \frac{a_n}{2}$. If $1 < \gcd(A_n, a_{n+1}) < \frac{a_{n+1}}{2}$, then $a_{n+1} < \frac{a_n^2}{2} + \frac{a_{n+1}}{3}$ whence $\frac{a_n^2}{a_{n+1}} \geq \frac{4}{3}$. So if there are infinitely many n such that $\gcd(A_n, a_{n+1}) > 1$, then $\limsup_{n\to\infty} \frac{a_n^2}{a_{n+1}} > 1$. $\square$

5. CONSTRUCTIONS

For any monotonically non-decreasing sequence $\{b_n\}_{n=1}^{\infty}$ of positive integers such that $T := \sum_{n=1}^{\infty} b_n 2^{-n}$ converges and for any number $S \in (\frac{T}{2}, T)$, Hančl and Tijdeman [5] constructed a sequence $\{a_n\}_{n=1}^{\infty}$ with $a_n \in \{2, 3, 4\}$ for every n such that $S = \sum_{n=1}^{\infty} \frac{b_n}{a_1 \ldots a_n}$. Here we extend this result to any integer $k > 1$ where $a_n \in \{k, k+1, \cdots, k^2\}$. Moreover we show that there exist rapidly growing sequences $\{b_n\}_{n=1}^{\infty}$ for which a restriction $a_n \in \{k, k+1\}$ suffices. We give some further examples in the same vein.

For given sequence $\{b_n\}_{n=1}^{\infty}$ and positive integer k put $T_N = \sum_{n=N}^{\infty} b_n k^{N-n}$ for $N \geq 1$.

Theorem 5.1. *Let $k > 1$ be an integer. Let $\{b_n\}_{n=1}^{\infty}$ be any sequence of positive integers such that $T = \sum_{n=1}^{\infty} b_n k^{-n}$ converges and $b_n \leq (1 - 1/k) T_{n+1}$ for all n. Let $S \in (\frac{T}{k+1}, T]$. Then there exist $a_n \in \{k, k+1, \cdots, k^2\}$ such that $S = \sum_{n=1}^{\infty} \frac{b_n}{a_1 \ldots a_n}$.*

Remark 5.1. If $\{b_n\}_{n=1}^{\infty}$ is monotonically non-decreasing, then $T_{N+1} \geq T_N \geq \sum_{n=N}^{\infty} b_N k^{N-n} = \frac{k}{k-1} b_N$ so that the condition is satisfied. So Theorem 5.1 applies to all monotonic sequences $\{b_n\}_{n=1}^{\infty}$ for which the series T converges.

Proof. Put $S_1 = S$ and for $n = 1, 2,$

$$a_n = \begin{cases} k & \text{if } \frac{T_n}{k+1} < S_n \leq \frac{T_n}{k} \\ k+1 & \text{if } \frac{T_n}{k+2} < S_n \leq \frac{T_n}{k+1} \\ \cdots & \\ k^2 - 1 & \text{if } \frac{T_n}{k^2} < S_n \leq \frac{T_n}{k^2-1} \\ k^2 & \text{if } \frac{T_n}{k(k+1)} < S_n \leq \frac{T_n}{k^2} \end{cases}$$

and $S_{n+1} = a_n S_n - b_n$. Note that $T_n = b_n + \frac{T_{n+1}}{k}$, that $\frac{k}{k+1} T_n < a_n S_n \leq T_n$ and that $\frac{k}{k+1} T_n - b_n = \frac{1}{k+1}(T_{n+1} - b_n) \geq \frac{1}{k(k+1)} T_{n+1}$. By induction it follows that $\frac{T_n}{k(k+1)} < S_n \leq \frac{1}{k} T_n$ and $\sum_{n=1}^{N} \frac{b_n}{a_1 \cdots a_n} = S - \frac{S_{N+1}}{a_1 \cdots a_N}$ for all N. Since $\frac{S_{N+1}}{a_1 \cdots a_N} \leq \frac{T_{N+1}}{k^{N+1}} = \sum_{n=N+1}^{\infty} b_n k^{-n} \to 0$ as $N \to \infty$, we have $S = \sum_{n=1}^{\infty} \frac{b_n}{a_1 \ldots a_n}$. $\square$

The next theorem and the subsequent example show that for some sequences $\{b_n\}_{n=1}^{\infty}$ the range for the a_n can be restricted to two consecutive numbers.

Theorem 5.2. *Let $k > 1$ be an integer, $\{b_n\}_{n=1}^{\infty}$ any sequence of positive integers* *such that $T = \sum_{n=1}^{\infty} b_n k^{-n}$ converges and $T_{N+1} \geq (k+1)b_N$ for $N > 1$. Let $S \in (\frac{k^2 T}{(k+1)^2}, T]$. Then there exist $a_n \in \{k, k+1\}$ such that $S = \sum_{n=1}^{\infty} \frac{b_n}{a_1 \ldots a_n}$.*

Proof. Put $S_1 = S$ and for $n = 1, 2,$

$$a_n = \begin{cases} k & \text{if } \frac{T_n}{k+1} < S_n \leq \frac{T_n}{k} \\ k+1 & \text{if } \frac{k}{(k+1)^2} T_n < S_n \leq \frac{1}{k+1} T_n \end{cases}$$

and $S_{n+1} = a_n S_n - b_n$. By induction it follows that $\frac{k}{(k+1)^2} T_n < S_n \leq \frac{T_n}{k}$ and that $\sum_{n=1}^{N} \frac{b_n}{a_1 \cdots a_n} = S - \frac{S_{N+1}}{a_1 \cdots a_N}$ for all N. Since $\frac{S_{N+1}}{a_1 \cdots a_N} \leq \frac{T_{N+1}}{k^{N+1}} \to 0$ as $N \to \infty$, we have $S = \sum_{n=1}^{\infty} \frac{b_n}{a_1 \ldots a_n}$. □

Example 5.1. For $k > 1$, put $b_n = [(k - \frac{1}{3})^n]$ for $n = 1, 2, \cdots$. Then $T = \sum_{n=1}^{\infty} b_n k^{-n}$ converges and

$$T_{n+1} \geq \left(k - \frac{1}{3}\right)^{n+1} \cdot \sum_{m=0}^{\infty} \frac{(k - \frac{1}{3})^m}{k^m} - 2 = 3k(k - \frac{1}{3})^{n+1} - 2.$$

Since $(k - \frac{1}{3})^n (k+1) < 3k(k - \frac{1}{3})^{n+1} - 2$ for $n = 1, 2, \cdots$, we have $b_n \leq \frac{T_{n+1}}{k+1}$ for all n. Thus $\{b_n\}_{n=1}^{\infty}$ satisfies the condictions of Theorem 5.2.

It is possible to vary the construction in such a way that $a_n \to \infty$ as $n \to \infty$. The following example illustrates this observation. It provides a monotonic sequence $\{b_n\}_{n=1}^{\infty}$ such that every number S from some interval can be represented as $S = \sum_{n=1}^{\infty} \frac{b_n}{a_1 \ldots a_n}$ with $a_n \in \{n, n+1\}$ for every n. In some sense this is a counterpart to Theorems 3.1, 4.1 and 4.2 which show that the rationality of such sums is very restricted if $\{b_n\}_{n=1}^{\infty}$ satisfies some growth condiction.

Example 5.2. *Put $b_n = (n-2)!$ for $n \geq 2$. Every number $S \in (\frac{146}{75}, \frac{8}{3}]$ can be represented as $S = \sum_{n=2}^{\infty} \frac{b_n}{a_2 \cdots a_n}$ with $a_n \in \{n, n+1\}$ for $n = 2, 3, \cdots$.*

Proof. We have $T := \sum_{n=2}^{\infty} \frac{b_n}{n!} = 1$ and $T_N := \sum_{n=N}^{\infty} \frac{b_n}{n!} N! = \frac{N!}{N-1}$ for $N \geq 2$. Put $a_2 = 2, a_3 = 3, S_4 = S - \frac{2}{3}$ and for $n = 4, 5, \cdots$.

$$a_n = \begin{cases} n & \text{if } \frac{T_n}{n+1} < S_n \leq \frac{T_n}{n} \\ n+1 & \text{if } \frac{n}{(n+1)^2} T_n < S_n \leq \frac{T_n}{n+1} \end{cases}$$

and $S_{n+1} = a_n S_n - b_n$. Observe that $S = \frac{b_2}{a_2} + \frac{b_3}{a_2 \cdot a_3} + S_4$ and $S_4 \in (\frac{32}{25}, 2] = (\frac{4T_4}{25}, \frac{T_4}{4}]$ and that $T_n = b_n + \frac{T_{n+1}}{n+1}$ for $n = 2, 3, \cdots$. By induction it follows that $\frac{n}{(n+1)^2} T_n < S_n \leq \frac{T_n}{n}$ for $n = 5, 6, \cdots$ and that $\sum_{n=2}^{N} \frac{b_n}{a_2 \cdots a_n} = S - \frac{S_{N+1}}{a_2 \cdots a_N}$ for all N. Since $\frac{S_{N+1}}{a_2 \cdots a_N} \leq \frac{T_{N+1}}{(N+1)!} \to 0$ as $N \to \infty$, we have $\sum_{n=2}^{\infty} \frac{b_n}{a_2 \ldots a_n} = S$. □

A natural question is whether Theorem 5.2 only holds for the choice from two consecutive integers. The last example shows that for all positive integers $d > c > 1$ there exist sequences $\{b_n\}_{n=1}^{\infty}$ for which the choice $a_n \in \{c, d\}$ suffices.

Example 5.3. Let c and d be integers with $d > c > 1$. Let ϵ be a number with

$0 < \epsilon < \frac{cd-c}{d^2-c}$. Put $b_n = (d-1)^n$. Then every number $S \in (\frac{c(d-1)(d-1+\epsilon)}{d^2\epsilon}, \frac{d-1}{\epsilon}]$ can be represented as $S = \sum_{n=1}^{\infty} \frac{b_n}{a_1 \ldots a_n}$ with $a_n \in \{c, d\}$ for every n.

Proof. We have $T := \sum_{n=1}^{\infty} \frac{b_n}{(d-1+\epsilon)^n} = \frac{d-1}{\epsilon}$ and $T_N := \sum_{n=N}^{\infty} \frac{b_n}{(d-1+\epsilon)^{n-N}} = \frac{(d-1)^N(d-1+\epsilon)}{\epsilon}$. Put $S_1 = S$ and for $n = 1, 2, \cdots$,

$$a_n = \begin{cases} c & \text{if} \quad \frac{T_n}{d} < S_n \le \frac{T_n}{d-1+\epsilon} \\ d & \text{if} \quad \frac{c}{d^2} T_n < S_n \le \frac{T_n}{d} \end{cases}$$

and $S_{n+1} = a_n S_n - b_n$. Hence $\frac{c}{d^2} T_1 < S_1 \le \frac{1}{d-1+\epsilon} T_1$ and $T_n = b_n + \frac{T_{n+1}}{d-1+\epsilon}$ for every n. By induction it follows that $\frac{c}{d^2} T_n < S_n \le \frac{T_{n+1}}{d-1+\epsilon}$ and $\sum_{n=1}^{N} \frac{b_n}{a_1 \cdots a_n} = S - \frac{S_{N+1}}{a_1 \cdots a_N}$. Since $\frac{S_N}{a_1 \cdots a_{N-1}} = \frac{b_N}{a_1 \cdots a_N} + \frac{S_{N+1}}{a_1 \cdots a_N}$ and $b_N = \frac{\epsilon T_{N+1}}{(d-1)(d-1+\epsilon)} \ge \frac{\epsilon S_{N+1}}{d-1}$, we have $\frac{S_{N+1}}{a_1 \cdots a_N} \le \frac{(d-1)S_N}{(d-1+\epsilon)a_1 \cdots a_{N-1}}$ for every N. Hence $\lim_{N\to\infty} \frac{S_{N+1}}{a_1 \cdots a_N} = 0$ and $\sum_{n=1}^{\infty} \frac{b_n}{a_1 \cdots a_n} = S$. □

The following questions are open.

Question: Let $k \ge 2$ be any integer, $\{b_n\}_{n=1}^{\infty}$ be any sequence of positive integers such that $T = \sum_{n=1}^{\infty} b_n k^{-n}$ converges, $a \ge k$ and $b \ge k$ be two integers with $a \ne b$. Is there any fixed interval $(u, v), u < v$ such that for every prescribed value S in this interval there is a sequence $\{a_n\}_{n=1}^{\infty}$ with $a_n = a$ or b, and $S = \sum_{n=1}^{\infty} \frac{b_n}{a_1 \cdots a_n}$? Are there infinitely many different $\{a_n^{(i)}\}_{n=1}^{\infty}, i = 1, 2, \cdots$ with $a_n^{(i)} = a$ or b, and $S = \sum_{n=1}^{\infty} \frac{b_n}{a_1^{(i)} \cdots a_n^{(i)}}$?

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