Characterizations of Biorthogonal Wavelets Which Are Associated with Biorthogonal Multiresolution Analyses

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We first characterize the Riesz wavelets which are associated with multiresolution analyses (MRAs) and the Riesz wavelets whose duals are also Riesz wavelets. The characterizations show that if a Riesz wavelet is associated with an MRA, then it has a dual Riesz wavelet. We then improve Wang’s characterization for a pair of biorthogonal wavelets to be associated with biorthogonal MRAs by showing that one of the two conditions in his characterization is redundant.

1. INTRODUCTION

In this paper we deal with the following two problems:

(A) When is a Riesz wavelet associated with a multiresolution analysis (MRA)?

(B) When is a pair of biorthogonal wavelets associated with biorthogonal multiresolution analyses?

Before we review the history of the problems, we introduce several definitions in order to clarify them. If $X$ and $Y$ are two closed subspaces of a Hilbert space $H$, then we say that $X$ and $Y$ are complementary subspaces if $H = X + Y$ and $X \cap Y = \{0\}$. In this case we write $H = X \perp Y$. If each $X_i$, $i \in \mathbb{Z}$, is a closed subspace of $H$ such that $X_i \cap X_j = \{0\}$, $i \neq j$.

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i \neq j \text{ and if each } j \in H \text{ can be written as a norm-convergent series } f = \sum_{i \in \mathbb{Z}} x_i, x_i \in X_i, \text{ then we write } H = \bigoplus_{i \in \mathbb{Z}} X_i. \text{ We reserve the notation } \oplus \text{ for the orthogonal sum.}

For } \psi \in L^2(\mathbb{R}) \text{ let } \psi_{jk}(x) := 2^{j/2}\psi(2^j x - k), j, k \in \mathbb{Z}. \text{ We say that } \psi \text{ is an orthogonal wavelet if } \{\psi_{jk} : j, k \in \mathbb{Z}\} \text{ is an orthonormal basis for } L^2(\mathbb{R}), \text{ and that } \psi \text{ is a Riesz wavelet if } \{\psi_{jk} : j, k \in \mathbb{Z}\} \text{ is a Riesz basis for } L^2(\mathbb{R}), \text{ respectively. } (\psi, \hat{\psi}) \text{ is said to be a pair of biorthogonal wavelets if } \{\psi_{jk} : j, k \in \mathbb{Z}\} \text{ is a Riesz basis for } L^2(\mathbb{R}) \text{ and } \{\hat{\psi}_{jk} : j, k \in \mathbb{Z}\} \text{ is the dual Riesz basis in the sense that } \langle \psi_{jk}, \hat{\psi}_{j'k'} \rangle = \delta_{jj'}\delta_{kk'}. \text{ We refer to } [7, 10, 12, 13] \text{ for the relevant definitions of the Riesz basis and its basic properties. A family of closed subspaces } \{V_j : j \in \mathbb{Z}\} \text{ of } L^2(\mathbb{R}) \text{ is said to be a multiresolution analysis if it satisfies the following conditions } [7, 10, 13]:

(i) } V_j \subset V_{j+1} \text{ for each } j \in \mathbb{Z};
(ii) } D(V_j) = V_{j+1} \text{ and } T_1(V_0) = V_0;
(iii) } \bigcup_{j \in \mathbb{Z}} V_j = L^2(\mathbb{R}) \text{ and } \bigcap_{j \in \mathbb{Z}} V_j = \{0\};
(iv) \text{ There exists a scaling function } \varphi \in V_0 \text{ such that } \{T_k \varphi : k \in \mathbb{Z}\} \text{ is an orthonormal basis for } V_0,

where } D : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}) \text{ is the unitary dyadic dilation operator defined by } Df(x) := 2^{j/2}f(2x), \text{ and } T_r \text{ is the translation operator defined by } T_r f(x) := f(x - r), r \in \mathbb{R}.

In Condition (iv), we note that “an orthonormal basis” can be replaced by “a Riesz basis.” In fact, if } \{T_k \varphi : k \in \mathbb{Z}\} \text{ is a Riesz basis and } \varphi^\perp \text{ is defined by }

\hat{\varphi}^\perp(x) = \frac{\hat{\varphi}(x)}{(\sum_{k \in \mathbb{Z}} |\hat{\varphi}(x + 2\pi k)|^2)^{1/2}},

then } \varphi^\perp \in V_0 \text{ and } \{T_k \varphi^\perp : k \in \mathbb{Z}\} \text{ is an orthonormal basis of } V_0 \text{ [10], where } \wedge \text{ denotes the Fourier transform defined via } \hat{f}(x) := \int_\mathbb{R} f(t)e^{-ixt} \, dt \text{ for } f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \text{ and extended to be } \sqrt{2\pi} \text{ times a unitary operator from } L^2(\mathbb{R}) \text{ onto } L^2(\mathbb{R}). \text{ Finally, two MRAs } \{V_j\}_{j \in \mathbb{Z}} \text{ and } \{\hat{V}_j\}_{j \in \mathbb{Z}} \text{ are said to be biorthogonal MRAs if their respective Riesz scaling functions } \varphi \text{ and } \hat{\varphi} \text{ satisfy } \langle T_k \varphi, T_l \hat{\varphi} \rangle = \delta_{kl} \text{ for each } k, l \in \mathbb{Z}.

Suppose } \psi \text{ is a wavelet of any kind (orthonormal or Riesz wavelet). We let, for } j \in \mathbb{Z}, \ W_j := \overline{\text{span}}\{\psi_{jk} : k \in \mathbb{Z}\}, V_j := \langle k<\gamma \rangle W_k. \text{ Then it is easy to see that Conditions (i)–(iii) are satisfied. We say that a wavelet } \psi \text{ is associated with an MRA if there exists a scaling function } \varphi \in V_0 \text{ such that } \{T_k \varphi : k \in \mathbb{Z}\} \text{ is a Riesz basis for } V_0.

Hernández and Weiss [13], Gripenberg [11], and Wang [19] showed that an orthogonal wavelet } \psi \text{ is associated with an MRA if and only if }

\sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}} |\hat{\psi}(2^j(x + 2\pi k))|^2 = 1 \quad \text{ for a.e. } x \in T, \tag{1.1}

where } T \text{ is the circle group which can conveniently be identified with } [0, 2\pi). \text{ Further details of the history on this subject can be found in } [13, \text{ pp. 393–395}]. \text{ An example of an orthogonal wavelet not associated with an orthogonal MRA can be found in Example 4.1 and also in } [19, \text{ p. 77}]. \text{ On the other hand, Papadakis [16] showed that an orthonormal wavelet is always associated with a “generalized frame MRA,” where a countable collection of scaling functions generates a frame for } V_0, \text{ and Kim et al. [15] dealt with the problem of the association of a “semi-orthogonal” frame wavelet to a “generalized frame MRA.” Wang [19] showed that for a pair of biorthogonal wavelets } (\psi, \hat{\psi}) \text{ to be}
associated with biorthogonal MRAs it is necessary and sufficient that the following two conditions are satisfied:

(a) \( \sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}} \hat{\psi}(2^j(x + 2\pi k)) \overline{\hat{\psi}(2^j(x + 2\pi k))} \neq 0 \) for a.e. \( x \in \mathbb{T} \);

(b) There exists a constant \( C_{\psi, \tilde{\psi}} > 0 \) such that

\[
\sum_{j=1}^{\infty} \left( \sum_{k \in \mathbb{Z}} \left| \hat{\psi}(2^j(x + 2\pi k)) \right|^2 \right)^{1/2} \left( \sum_{k \in \mathbb{Z}} \left| \overline{\hat{\psi}(2^j(x + 2\pi k))} \right|^2 \right)^{1/2} \leq C_{\psi, \tilde{\psi}} \sum_{j=1}^{\infty} \left| \sum_{k \in \mathbb{Z}} \hat{\psi}(2^j(x + 2\pi k)) \overline{\hat{\psi}(2^j(x + 2\pi k))} \right|
\]

for a.e. \( x \in \mathbb{T} \).

The paper is organized in the following manner. In Section 2 we first characterize those Riesz wavelets which are associated with MRAs (Theorem 2.6) and then characterize Riesz wavelets whose duals are also Riesz wavelets (Theorem 2.8). The characterizations show that if a Riesz wavelet is associated with an MRA, then it has a dual Riesz wavelet (Corollary 2.9). In Section 3 we give a series of equivalent conditions so that a pair of biorthogonal Riesz wavelets be associated with biorthogonal MRAs (Theorem 3.5) by showing that Condition (b) in Wang’s characterization is redundant. Finally, we show that the two conditions in Theorem 2.6 (d) are independent by two examples in Section 4.

2. RIESZ WAVELETS ASSOCIATED WITH MRAs

In this section, we give a series of necessary and sufficient conditions so that a Riesz wavelet \( \psi \) be associated with an MRA in Theorem 2.6. These conditions are related to the dimension of a sequence space generated by \( \hat{\psi} \), and the shift-invariance of \( \bigoplus_{j<0} W_j \). The following proposition is well known. See [10, Sect. 5.3.1], for example.

**Proposition 2.1.** For \( \varphi \in L^2(\mathbb{R}) \), \( \{ T_k \varphi : k \in \mathbb{Z} \} \) is a Riesz basis of its closed linear span with Riesz bounds \( A \) and \( B \) if and only if

\[
A \leq \sum_{k \in \mathbb{Z}} |\hat{\varphi}(x + 2\pi k)|^2 \leq B, \quad \text{for a.e. } x \in \mathbb{T}.
\]

Moreover, it is an orthonormal basis if and only if \( A = B = 1 \).

In our characterizations of those Riesz wavelets that are associated with MRAs we use the theory of shift-invariant subspaces [5, 6, 14, 17], the rudiments of which we briefly review now. A closed subspace \( S \) of \( L^2(\mathbb{R}) \) is said to be shift-invariant if \( T_k f \in S \) for any \( f \in S \) and \( k \in \mathbb{Z} \). Let \( \Phi \subset L^2(\mathbb{R}) \). Then \( S := S(\Phi) := \operatorname{span}(T_k \varphi : \varphi \in \Phi, k \in \mathbb{Z}) \) is clearly shift-invariant. In this case we say that \( S \) is the shift-invariant subspace generated by \( \Phi \). It is established in [5, Sect. 3] that a shift-invariant subspace of \( L^2(\mathbb{R}) \) is always countably generated. For \( f \in L^2(\mathbb{R}) \), let

\[
\hat{f}_k \varphi := (\hat{f}(x + 2\pi k))_{k \in \mathbb{Z}},
\]

which is in \( \ell^2(\mathbb{Z}) \) for a.e. \( x \in \mathbb{T} \) and let \( \hat{\Lambda}_k \varphi := \{ \hat{f}_k \varphi : f \in A \} \) for \( A \subset L^2(\mathbb{R}) \). We only need the following three propositions from shift-invariant space theory.


PROPOSITION 2.2. If $S = S(\Phi)$ for a countable $\Phi$, then for a.e. $x \in T$

$$\hat{S}_{\|x\|} = \overline{\text{span}} \hat{\Phi}_{\|x\|}.$$

Proof. See [5, Proposition 3.1] or [14, Theorem 2.1].

PROPOSITION 2.3. $S$ is a shift-invariant subspace with $\dim \hat{S}_{\|x\|} = 1$ for a.e. $x \in T$ if and only if there exists $f \in S$ such that $\{T_k f : k \in \mathbb{Z}\}$ is an orthonormal basis for $S$.

Proof. See [5, Theorem 2.10].

PROPOSITION 2.4. Let $S$ be a shift-invariant subspace and let $S_1$ be a shift-invariant subspace of $S$. Then $S_2 := S \ominus S_1$ is also shift-invariant.

Proof. See [5, Corollary 3.4].

Suppose that $\psi$ is a Riesz wavelet. Let $W_j := \overline{\text{span}} \{\psi_{jk} : k \in \mathbb{Z}\}$ and $V_j := \oplus_{k<j} W_k$, $j \in \mathbb{Z}$, be the closed subspaces defined as in Section 1. We claim that $S(V_0) = S((D^j \psi : j < 0))$. Indeed, $S((D^j \psi : j < 0)) \subset S(V_0)$, since $\{D^j \psi : j < 0\} \subset V_0$. Since $T_k D^j = D^j T_k^j$ and $\{2^j k : k \in \mathbb{Z}\} \subset \mathbb{Z}$ for $j < 0$, we have

$$S((D^j \psi : j < 0)) = \overline{\text{span}}(T_k D^j \psi : j < 0, k \in \mathbb{Z})$$

$$= \overline{\text{span}}(D^j T_k^j \psi : j < 0, k \in \mathbb{Z})$$

$$\supset \overline{\text{span}}(D^j T_k^j \psi : j < 0, k \in \mathbb{Z}).$$

Hence, $S((D^j \psi : j < 0))$ contains $V_0$; so it contains $S(V_0)$ since it is shift-invariant. Therefore, $S(V_0) = S((D^j \psi : j < 0))$. By Proposition 2.2, we note that for a.e. $x \in T$

$$(S(V_0))_{\|x\|} = \overline{\text{span}}((D^j \psi)_{\|x\|} : j < 0)$$

$$= \overline{\text{span}}((\hat{\psi}(2^j(x + 2\pi k)))_{k \in \mathbb{Z}} : j > 0).$$

This proves the following lemma.

LEMMA 2.5. Suppose that $\psi$ is a Riesz wavelet. Let $W_j := \overline{\text{span}} \{\psi_{jk} : k \in \mathbb{Z}\}$ and $V_0 := \oplus_{k<j} W_k$. Let $F(x) := \overline{\text{span}} \{\hat{\psi}(2^j(x + 2\pi k)))_{k \in \mathbb{Z}} : j > 0\}$ for $x \in T$. Then $(S(V_0))_{\|x\|} = F(x)$ for a.e. $x \in T$. In particular, if $V_0$ is shift-invariant, then $V_{0\|x\|} = F(x)$ for a.e. $x \in T$.

We now give our characterization of those Riesz wavelets which are associated with MRAs in the following main theorem in this section. We prove the equivalence of (a) and (b) in this section. This equivalence will be used in the proof of Theorem 3.5. Other equivalences will then be proved at the end of Section 3 by use of Theorems 2.8 and 3.5.

THEOREM 2.6. Suppose $\psi$ is a Riesz wavelet. Under the same notations as in Lemma 2.5, the following assertions are equivalent:

(a) $\psi$ is associated with an MRA;
(b) $V_0$ is shift-invariant and $\dim F(x) = 1$ for a.e. $x \in T$;
(c) $V_0$ is shift-invariant and $\dim F(x) \geq 1$ for a.e. $x \in T$;
(d) $V_0$ is shift-invariant and $\sum_{j \geq 1} \sum_{k \in \mathbb{Z}} |\hat{\psi}(2^j(x + 2\pi k))|^2 > 0$ for a.e. $x \in T$.

Proof. We prove the equivalence of (a) and (b) here, and the equivalences of other conditions are given at the end of Section 3.
Proof. (a) ⇒ (b): This follows from Lemma 2.5 and Proposition 2.4, since the association of \( \psi \) with an MRA implies that \( V_0 = S(\{ \psi \}) \) for some \( \varphi \in L^2(\mathbb{R}) \) and that the family of the translates of \( \varphi \) is a Riesz basis of \( V_0 \). Thus, \( V_0 \) is shift invariant and \( \mathbb{F}(x) = (S(\{ \psi \}))_{k \in \mathbb{Z}} = \text{span}[\{ \hat{\varphi}(x + 2\pi k) \}_{k \in \mathbb{Z}}] \) is one-dimensional for a.e. \( x \in \mathbb{T} \).

(b) ⇒ (a): This follows from Lemma 2.5 and Proposition 2.3.  

We show that the shift-invariance of \( V_0 \) and the condition

\[
\sum_{j \geq 1} \sum_{k \in \mathbb{Z}} | \hat{\psi}(2^j(x + 2\pi k)) |^2 > 0
\]

for a.e. \( x \in \mathbb{T} \) are independent in Section 4.

Recall that if \( \psi \) is a Riesz wavelet, then it has a dual Riesz basis \( \{ \psi_{jk} : j, k \in \mathbb{Z} \} \) [10, 12]. More precisely, let \( Sf := \sum_{j,k \in \mathbb{Z}} \langle f, \psi_{jk} \rangle \psi_{jk}, f \in L^2(\mathbb{R}) \), be the frame operator with respect to the Riesz bases \( \{ \psi_{jk} : j, k \in \mathbb{Z} \} \). Then \( \{ \psi_{jk} : j, k \in \mathbb{Z} \} \) is the dual Riesz basis of \( \{ \psi_{jk} : j, k \in \mathbb{Z} \} \) satisfying \( \langle \psi_{jk} \rangle, \langle \psi_{lm} \rangle = \delta_{jl} \delta_{km} \) [10, 12]. It is well known that not all dual Riesz bases of Riesz wavelets are wavelet Riesz bases. That is, there exists a Riesz wavelet whose dual Riesz basis is not of the form \( \{ D^j T_k \hat{\psi} : j, k \in \mathbb{Z} \} \) for any \( \hat{\psi} \in L^2(\mathbb{R}) \) [7 (Sect. 1.4), 8]. We now characterize those Riesz wavelets whose dual Riesz bases are also Riesz wavelets. This characterization will later be used to prove the remaining parts of Theorem 2.6. First we need a lemma.

**Lemma 2.7.** Suppose that \( (\psi, \tilde{\psi}) \) is a pair of biorthogonal wavelets such that \( \{ \psi_{jk} : j, k \in \mathbb{Z} \} \) and \( \{ \psi_{lm} : l, m \in \mathbb{Z} \} \) are dual Riesz bases of \( L^2(\mathbb{R}) \). Define, for each \( j \in \mathbb{Z}, W_j := \text{span}[\psi_{jk} : k \in \mathbb{Z}], \tilde{W}_j := \text{span}[\tilde{\psi}_{jk} : k \in \mathbb{Z}], V_j := +_{k < j} W_k, \tilde{V}_j := +_{k < j} \tilde{W}_k, Z_j := +_{k \geq j} W_k, \) and \( \tilde{Z}_j := +_{k \geq j} \tilde{W}_k. \) Then \( L^2(\mathbb{R}) = V_0 + Z_0 = \tilde{V}_0 + \tilde{Z}_0. \)

**Proof.** Since \( L^2(\mathbb{R}) = +_{k \in \mathbb{Z}} W_k, L^2(\mathbb{R}) = V_0 + Z_0 \). Since \( \{ \psi_{jk} : j, k \in \mathbb{Z} \} \) and \( \{ \psi_{lm} : l, m \in \mathbb{Z} \} \) are biorthogonal, \( \tilde{V}_0 \perp Z_0 \). We show that \( V_0 \perp Z_0 = L^2(\mathbb{R}) \). Suppose \( f \perp V_0 \perp Z_0 \). Since \( \{ \psi_{jk} : j, k \in \mathbb{Z} \} \) is a Riesz basis for \( L^2(\mathbb{R}) \), there exists \( (a_{jk})_{j,k \in \mathbb{Z}} \) such that \( f = \sum_{j,k \in \mathbb{Z}} a_{jk} \psi_{jk}. f \perp V_0 \) implies that \( a_{jk} = 0 \) for each \( j < 0, k \in \mathbb{Z} \). Hence \( f = \sum_{j \geq 0, k \in \mathbb{Z}} a_{jk} \psi_{jk} \in Z_0, \) Then \( f = 0 \) because \( f \perp Z_0 \).  

The following theorem gives a characterization in order that a Riesz wavelet has its dual wavelet.

**Theorem 2.8.** Suppose that \( \psi \) is a Riesz wavelet, and let \( S \) be the frame operator with respect to the Riesz basis \( \{ \psi_{jk} : j, k \in \mathbb{Z} \} \). Then the following assertions are equivalent:

(a) There exists \( \tilde{\psi} \in L^2(\mathbb{R}) \) such that \( \tilde{\psi}_{jk} : j, k \in \mathbb{Z} \) is the dual Riesz basis of \( \psi_{jk} ; \)

(b) \( V_0 := +_{j \geq 0} W_j \) is shift-invariant;

(c) \( S^{-1} T_k \psi = T_k S^{-1} \psi \) for each \( k \in \mathbb{Z} \).

**Proof.** (a) ⇒ (b): Suppose (a) holds. Then by Lemma 2.7 and by the duality we have that \( L^2(\mathbb{R}) = V_0 + +_{j \geq 0} \tilde{W}_j, \) where \( \tilde{W}_j = \text{span}[D^j T_k \tilde{\psi} : k \in \mathbb{Z}] \). Notice that by the commutation relation \( T_j D^j = D^j T_j, \) \( \tilde{W}_j \) is shift-invariant for \( j \geq 0 \) and so is \( +_{j \geq 0} \tilde{W}_j \). Therefore, \( V_0 \) is shift-invariant by Lemma 2.4.

(b) ⇒ (c): Fix \( k \in \mathbb{Z}. \) Since \( \{ \psi_{il} : i, l \in \mathbb{Z} \} \) is a Riesz basis of \( L^2(\mathbb{R}) \), it suffices to show that \( \langle S^{-1} T_k \psi - T_k S^{-1} \psi, \psi_{il} \rangle = 0 \) for each \( i, l \in \mathbb{Z} \). Notice that \( \langle S^{-1} D^j T_k \psi, D^j T_l \psi \rangle =
Let us introduce a few definitions and lemmas. The remaining part of our paper is influenced by the ideas in [1, 2, 18]. Let \( X \) be a closed subspace of a Hilbert space \( H \). We define the angle \( R(X, Y) \) for \( X, Y \) closed. Suppose that \( \{\psi_{ij}\} \) is a Riesz wavelet that is associated with an MRA, then it has the dual Riesz basis of \( \hat{\psi}_{il} \). Hence, \( \langle \psi_{ij}, \hat{\psi}_{il} \rangle = \delta_{ij} \delta_{kl} \). Hence \( \langle S^{-1}T_k \psi - T_k S^{-1} \psi, \psi_{ij} \rangle = \delta_{ij} \delta_{kl} - \langle T_k S^{-1} \psi, \psi_{ij} \rangle \).

For \( i \geq 0 \),
\[
\langle T_k S^{-1} \psi, D^j T_l \psi \rangle = \langle S^{-1} \psi, T_{-k} D^j T_l \psi \rangle = \langle S^{-1} \psi, D^j T_{l-2k} \psi \rangle = \delta_{0l} \delta_{0j} \delta_{kl}.
\]

For \( i < 0 \), \( \psi_{ij} \in V_0 \). Since \( V_0 \) is shift-invariant, \( T_{-k} \psi_{ij} \in V_0 = \{ \hat{\psi}_{l} \} \). Notice that from the biorthogonality relations we have that \( S^{-1} \psi = \sum_{j} W_j \psi_{ij} \). Hence \( \langle T_k S^{-1} \psi, \psi_{ij} \rangle = \langle S^{-1} \psi, T_{-k} \psi_{ij} \rangle = 0 \). This shows that the left-hand side of (2.2) is 0 for each \( i, l \in \mathbb{Z} \). Therefore, \( S^{-1}T_k \psi = T_k S^{-1} \psi \).

As a corollary we have:

**Corollary 2.9.** If \( \psi \) is a Riesz wavelet that is associated with an MRA, then it has a dual Riesz wavelet.

**Proof.** This follows from Theorems 2.6 and 2.8. \( \blacksquare \)

### 3. BIORTHOGONAL MRA WAVELETS

In this section, we give several equivalent conditions so that a pair of biorthogonal wavelets be associated with biorthogonal MRAs. Before we state and prove our main result let us introduce a few definitions and lemmas. The remaining part of our paper is influenced by the ideas in [1, 2, 18]. Let \( X \) and \( Y \) be two closed subspaces of a Hilbert space \( H \). We define the angle \( R(X, Y) \) between \( X \) and \( Y \) via \( R(X, Y) := \inf \| P_X (y) \| / \| y \| : y \in Y \setminus \{0\} \) \), where \( P_X \) denotes the orthogonal projection onto \( X \). For the geometric meaning and the properties of this concept we refer to [1, 18]. The following lemma is due to Unser and Aldroubi [18].

**Lemma 3.1** [18]. For \( f, g \in L^2(\mathbb{R}) \) let \( X := \text{span}\{T_k f : k \in \mathbb{Z}\} \) and \( Y := \text{span}\{T_k g : k \in \mathbb{Z}\} \). Suppose that \( \{T_k f : k \in \mathbb{Z}\} \) and \( \{T_k g : k \in \mathbb{Z}\} \) are Riesz bases for \( X \) and \( Y \), respectively. Then
\[
R(X, Y) = R(Y, X) = \text{ess-inf}_{x, y} \left\{ \frac{| \sum_{k \in \mathbb{Z}} \tilde{f}(x + 2\pi k) \tilde{g}(x + 2\pi k) |}{( \sum_{k \in \mathbb{Z}} | f(x + 2\pi k) |^2 )^{1/2} ( \sum_{k \in \mathbb{Z}} | g(x + 2\pi k) |^2 )^{1/2}} \right\}.
\]

The following lemma is well known. See [9, Sect. 2.3], for example.

**Lemma 3.2.** If \( H = X + Y \), then there exists a bounded operator \( E \), called the projection from \( H \) onto \( X \), such that \( E^2 = E \), \( Ex = x \) for each \( x \in X \), ran\( E = X \), and ker\( E = Y \).
LEMMA 3.3. Let $U$, $V$, $\tilde{V}$ be closed subspaces of a Hilbert space $H$ satisfying $H = V + U = \tilde{V} \oplus U$. Then $R(\tilde{V}, V) > 0$.

Proof. The proof of this can be found in [1, Theorem 3.2]. Since the proof therein is slightly abstract, we give an elementary one. Suppose that $R(\tilde{V}, V) = 0$. Then there exists $\{f_n : n \in \mathbb{N}\} \subset V$ such that $\|f_n\| = 1$ and $P_{\tilde{V}}f_n \to 0$ as $n \to \infty$. Since $H = \tilde{V} \oplus U$, there exists $\{h_n : n \in \mathbb{N}\} \subset U$ such that $f_n = P_{\tilde{V}}f_n + h_n$. Then $V + U \ni f_n - h_n = P_{\tilde{V}}f_n \to 0$ as $n \to \infty$. Let $E$ be the bounded operator in Lemma 3.2 whose range is $V$ and whose kernel is $U$. Then $1 = \|f_n\| = \|E(f_n - h_n)\| \leq \|E\| \|f_n - h_n\| \to 0$ as $n \to \infty$. This contradiction shows that $R(\tilde{V}, V) > 0$. \hfill $\blacksquare$

The following lemma is Lemma 5.47 of [19] in which Auscher’s reproducing lemma ([3, Proposition 5.2]) plays a key role.

LEMMA 3.4. Suppose that $(\psi, \tilde{\psi})$ is a pair of biorthogonal wavelets such that $\{\psi_{jk} : j, k \in \mathbb{Z}\}$ and $\{\tilde{\psi}_{lm} : l, m \in \mathbb{Z}\}$ are dual Riesz bases of $L^2(\mathbb{R})$. Let $\Psi_j(x) := (\tilde{\psi}(2^j(x + 2\pi k)))_{k \in \mathbb{Z}}$, $\tilde{\Psi}_j(x) := (\psi(2^j(x + 2\pi k)))_{k \in \mathbb{Z}}$, $\mathcal{F}(x) := \overline{\text{span}}(\Psi_j(x) : j \in \mathbb{N})$, and $\tilde{\mathcal{F}}(x) := \overline{\text{span}}(\tilde{\Psi}_j(x) : j \in \mathbb{N})$. Then for a.e. $x \in \mathbb{T}$

$$\dim \mathcal{F}(x) = \dim \tilde{\mathcal{F}}(x) = \sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}} \tilde{\psi}(2^j(x + 2\pi k))\tilde{\psi}(2^j(x + \pi k)).$$

We now state and prove our main result in this section. The following theorem gives a series of equivalent conditions so that a pair $(\psi, \tilde{\psi})$ of biorthogonal wavelets be associated with biorthogonal MRAs. As we mentioned in the Introduction, our result shows that one of the two conditions in Wang’s characterization (Condition (b) in Section 1) is redundant [19].

THEOREM 3.5. Suppose that $(\psi, \tilde{\psi})$ is a pair of biorthogonal wavelets such that $\{\psi_{jk} : j, k \in \mathbb{Z}\}$ and $\{\tilde{\psi}_{lm} : l, m \in \mathbb{Z}\}$ are dual Riesz bases of $L^2(\mathbb{R})$. Let $\Psi_j(x) := (\tilde{\psi}(2^j(x + 2\pi k)))_{k \in \mathbb{Z}}$, $\tilde{\Psi}_j(x) := (\psi(2^j(x + 2\pi k)))_{k \in \mathbb{Z}}$, $\mathcal{F}(x) := \overline{\text{span}}(\Psi_j(x) : j \in \mathbb{N})$, and $\tilde{\mathcal{F}}(x) := \overline{\text{span}}(\tilde{\Psi}_j(x) : j \in \mathbb{N})$. Then the following statements are equivalent:

1. $\dim \mathcal{F}(x) = \dim \tilde{\mathcal{F}}(x) = 1$ for a.e. $x \in \mathbb{T}$;
2. Either $\dim \mathcal{F}(x) = 1$ or $\dim \tilde{\mathcal{F}}(x) = 1$ for a.e. $x \in \mathbb{T}$;
3. Both $\psi$ and $\tilde{\psi}$ are associated with MRAs;
4. Either $\psi$ or $\tilde{\psi}$ is associated with an MRA;
5. $(\psi, \tilde{\psi})$ is associated with biorthogonal MRAs;
6. $\sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}} \tilde{\psi}(2^j(x + 2\pi k))\psi(2^j(x + 2\pi k)) = 1$ for a.e. $x \in \mathbb{T}$;
7. $\sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}} \tilde{\psi}(2^j(x + 2\pi k))\psi(2^j(x + 2\pi k)) \neq 0$ for a.e. $x \in \mathbb{T}$;
8. $\sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}} |\tilde{\psi}(2^j(x + 2\pi k))|^2 > 0$ and $\sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}} |\psi(2^j(x + 2\pi k))|^2 > 0$ for a.e. $x \in \mathbb{T}$;
9. Either $\sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}} |\tilde{\psi}(2^j(x + 2\pi k))|^2 > 0$ or $\sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}} |\psi(2^j(x + 2\pi k))|^2 > 0$ for a.e. $x \in \mathbb{T}$.

Proof. As in the proof of (a) $\Rightarrow$ (b) of Theorem 2.8 we note that both $V_0 := \mathbb{H}_{j<0} W_j$ and $\tilde{V}_0 := \mathbb{H}_{j<0} \tilde{W}_j$ are shift-invariant.

Lemma 4.16, Chap. 3] in the biorthogonal wavelet setting.

Moreover, for a.e. $x$ there exist positive constants $A$ and $B$ such that $A \leq |\sum_{k \in \mathbb{Z}} \hat{\psi}(x+2\pi k)\hat{\psi}(x+2\pi k)| \leq B$ for a.e. $x \in \mathbb{T}$ by Lemma 3.1 and Proposition 2.1. Define $\varphi^\#$ by

$$\hat{\varphi}^\#(x) := \frac{\hat{\psi}(x)}{\sum_{k \in \mathbb{Z}} \hat{\psi}(x+2\pi k)\hat{\psi}(x+2\pi k)}$$

for a.e. $x \in \mathbb{R}$. Then it is easy to see that $\tilde{V}_j = \text{span}\{\varphi^\#_{jk} : k \in \mathbb{Z}\}$ for $j \in \mathbb{Z}$ [10, Sect. 5.3.1]. Moreover, $\{T_k \varphi^\# : k \in \mathbb{Z}\}$ is a Riesz basis for $\tilde{V}_0$ by Proposition 2.1. Hence $\hat{\psi}$ is also associated with the MRA $\{\tilde{V}_j\}_{j \in \mathbb{Z}}$ with a scaling function $\varphi^\#$. A direct calculation shows that $\sum_{k \in \mathbb{Z}} \hat{\psi}(x+2\pi k)\hat{\psi^\#}(x+2\pi k) = 1$ for a.e. $x \in \mathbb{T}$. This shows that $\langle T_k \varphi, T_l \varphi^\# \rangle = \delta_{kl}$, $k, l \in \mathbb{Z}$.

(5) $\Rightarrow$ (4): Obvious.

(5) $\Rightarrow$ (6): This follows from the implication (5) $\Rightarrow$ (4), the equivalence of Conditions (a) and (b) in Theorem 2.6 and Lemma 3.4.

(6) $\Rightarrow$ (7) $\Rightarrow$ (8) $\Rightarrow$ (9): Trivial.

(9) $\Rightarrow$ (8) $\Rightarrow$ (7): These follow by Lemma 3.4.

(7) $\Rightarrow$ (6): Lemma 3.4 and (7) imply

$$\sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}} \hat{\psi}(2^j(x+2\pi k))\overline{\hat{\psi}(2^j(x+2\pi k))} \in \mathbb{N}$$

for a.e. $x \in \mathbb{T}$. Since $\langle T_k \psi, T_l \tilde{\psi} \rangle = \delta_{kl}$, $k, l \in \mathbb{Z}$, we have $\sum_{k \in \mathbb{Z}} \hat{\psi}(x+2\pi k) \times \overline{\hat{\psi}(x+2\pi k)} = 1$ for a.e. $x \in \mathbb{T}$. We now imitate the calculations in the proof of [13, Lemma 4.16, Chap. 3] in the biorthogonal wavelet setting.

$$\int_0^{2\pi} \sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}} \hat{\psi}(2^j(x+2\pi k))\overline{\hat{\psi}(2^j(x+2\pi k))} \, dx$$

$$= \sum_{k \in \mathbb{Z}} \int_{2\pi k}^{2\pi(k+1)} \sum_{j=1}^{\infty} \hat{\psi}(2^jx)\overline{\hat{\psi}(2^jx)} \, dx$$

$$= \sum_{j=1}^{\infty} \int_{\mathbb{R}} \hat{\psi}(2^jx)\overline{\hat{\psi}(2^jx)} \, dx$$

$$= \sum_{j=1}^{\infty} \frac{1}{2^j} \int_{\mathbb{R}} \hat{\psi}(x)\overline{\hat{\psi}(x)} \, dx$$
\[
\int_{\mathbb{R}} \hat{\psi}(x) \overline{\hat{\psi}}(x) \, dx = \int_{0}^{2\pi} \sum_{k \in \mathbb{Z}} \hat{\psi}(x + 2\pi k) \overline{\hat{\psi}}(x + 2\pi k) \, dx = \int_{0}^{2\pi} 1 \, dx = 2\pi.
\]

Hence,
\[
\sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}} \hat{\psi}(2^j (x + 2\pi k)) \overline{\hat{\psi}}(2^j (x + 2\pi k)) = 1
\]
for a.e. \(x \in T\).

As a corollary we give a generalization of [13, Corollary 3.15, Chap. 7].

**Corollary 3.6.** If \((\psi, \tilde{\psi})\) is a pair of biorthogonal wavelets such that either \(\psi\) or \(\tilde{\psi}\) has compact support, then \((\psi, \tilde{\psi})\) is associated with biorthogonal MRAs.

**Proof.** If \(\psi\) has compact support, then \(\hat{\psi}\) can be extended to be an entire function. Hence the zeros of \(\hat{\psi}\) is at most countable. The corollary follows by Condition (9) of Theorem 3.5.

We end this section with the completion of the proof of the main theorem of Section 2.

**Proof of the Remaining Equivalences in Theorem 2.6.**

(b) \(\Rightarrow\) (c): Trivial.
(c) \(\Rightarrow\) (d): Trivial.
(d) \(\Rightarrow\) (a): If \(V_0\) is shift-invariant, then by Theorem 2.8 there exists \(\tilde{\psi} \in L^2(\mathbb{R})\) such that \((\psi, \tilde{\psi})\) is a pair of biorthogonal wavelets. The second part of the statement (d) guarantees that Condition (9) of Theorem 3.5 is satisfied. Then (a) holds by Condition (3) of Theorem 3.5.

4. **EXAMPLES**

In this section, we show that the two conditions in Theorem 2.6(d) are independent by two examples. The first one provides an orthonormal wavelet not associated with an MRA (Theorem 2.6).

**Example 4.1.** There exists an orthonormal wavelet \(\psi\) such that \(\oplus_{j<0} W_j\) is shift-invariant but \(\sum_{j \geq 1} \sum_{k \in \mathbb{Z}} |\hat{\psi}(2^j (x + 2\pi k))|^2 = 0\) on a set of positive measure.

Consider Journé’s orthonormal wavelet \(\psi\) defined by \(\hat{\psi}(x) = \chi_K(x)\), where
\[
K = \left[ -\frac{32\pi}{7}, -4\pi \right] \cup \left[ -\pi, -\frac{4\pi}{7} \right] \cup \left[ \frac{4\pi}{7}, \pi \right] \cup \left[ 4\pi, \frac{32\pi}{7} \right].
\]

As in the proof of (a) \(\Rightarrow\) (b) of Theorem 2.8, we know that \(\oplus_{j<0} W_j\) is shift-invariant. But we can easily check that
\[
\sum_{j \geq 1} \sum_{k \in \mathbb{Z}} |\hat{\psi}(2^j (x + 2\pi k))|^2 = 0 \quad \text{for} \ 4\pi/7 \leq |x| < 6\pi/7.
\]
EXAMPLE 4.2. There exists a Riesz wavelet $\psi$ such that $\sum_{j \geq 1} \sum_{k \in \mathbb{Z}} |\hat{\psi}(2^j (x + 2\pi k))|^2 > 0$ for a.e. $x \in \mathbb{T}$ but $\oplus_{j < 0} W_j$ is not shift-invariant.

We define $\psi(x) = \chi_{(0,1/2-\epsilon)}(x) - \chi_{(1/2+\epsilon,1)}(x)$, $0 < \epsilon < 1/2$. We can check that

$$\hat{\psi}(x) = (2i/x) e^{-x^2/2} [\cos(x\epsilon) - \cos(x/2)] \neq 0$$

for a.e. $x \in \mathbb{R}$, and $\sum_{j \geq 1} \sum_{k \in \mathbb{Z}} |\hat{\psi}(2^j (x + 2\pi k))|^2 > 0$ a.e. $x \in \mathbb{T}$. But $\psi$ is known to be a Riesz wavelet which is not associated with an MRA [20, Example 3.2]. Hence $\oplus_{j < 0} W_j$ is not shift-invariant by Theorem 2.6(d).

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