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NOTE

On the Distortion Theorem for Quasiconformal Mappings with Fixed Boundary Values[1]

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For a planar domain Ω with at least three boundary points and ρ_Ω the hyperbolic metric of Ω with constant curvature -1, G. J. Martin poses a problem that asks, if f is a K-quasiconformal self-homeomorphism of Ω with boundary values given by the identity mapping, whether $\rho_\Omega(z, f(z)) \leq \log K$ holds for $z, f(z) \in \Omega$. In this note, we give a negative answer to this question. © 2001 Academic Press

Key Words: quasiconformal mappings; distortion theorem; Teichmüller theorem.

1. INTRODUCTION

It is well known that if (a_1, a_2, a_3, a_4) and (b_1, b_2, b_3, b_4) are two ordered quadruples of distinct complex numbers, then there exists a conformal mapping of the whole extended plane which takes a_k into b_k if and only if the cross-ratios are equal. If they are not equal, it is natural to ask for what K does there exist a K-quasiconformal mapping which transforms

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one quadruple into another? The Teichmüller distortion theorem plays an important role in answering this question (see [1]). Gehring and Martin [2] pointed out a connection between two classical theorems: Schottky's theorem and the distortion theorem for planar quasiconformal mappings. Recently, Martin [3] found that the best possible estimates could be obtained either by using the sharp form of Schottky's theorem or the sharp form of the distortion theorem for quasiconformal mappings. A sense preserving homeomorphism f of a domain $\Omega \subset C$ is a K-quasiconformal mapping, $1 \le K < \infty$, if f is an L^2-solution of the Beltrami equation $\frac{\partial f(z)}{\partial \bar{z}} = \mu(z)\frac{\partial f(z)}{\partial z}$, where $\mu(z)$ is a Borel measurable function in Ω with $\| \mu(z)\|_\infty \le \frac{K-1}{K+1} < 1$. Let $C(z_1, \dots, z_n)$ be the extended plane minus the points $z_1, \dots, z_n$. If $n \ge 3$, let us represent the universal covering surface of $C(z_1, \dots, z_n)$ by the upper half plane $\mathrm{Im}(\tau) > 0$, and let $z = \lambda(\tau)$ be an analytic covering. The hyperbolic density $\rho(z)$ in the domain $C(z_1, \dots, z_n)$ is defined by

$$\rho(z) = \frac{|f'(z)|}{\mathrm{Im}(f(z))}, \tag{1}$$

where f is any local inverse for λ. Then $\rho(z)$ is independent of both λ and $f(z)$, and $\rho(z)$ satisfies the differential equation

$$\Delta \log \rho(z) = \rho^2(z), \tag{2}$$

which is known as $\rho(z)$ with constant Gaussian curvature -1. The hyperbolic distance, $\rho(z, w)$, in the domain $C(z_1, \dots, z_n)$ is defined by

$$\rho(z,w) = \inf_\gamma \int_\gamma \rho(z)|dz|, \tag{3}$$

where the infimum is taken over all rectifiable curves γ connecting z and w in $C(z_1, \dots, z_n)$. For a planar domain with at least three boundary points, corresponding to the Teichmüller distortion theorem (see [3, 4]), Martin proved in [3] the following

THEOREM A. *Let Ω be a planar domain with at least three boundary points and let $\rho_\Omega(z, w)$ be the hyperbolic metric of Ω with constant curvature -1. Suppose $z, w \in \Omega$ and*

$$\rho_\Omega(z,w) \le \log K. \tag{4}$$

Then there is a K-quasiconformal self-homeomorphism $f(z)$ of Ω such that

(1) $f(\zeta) = \zeta$ *for all* $\zeta \in \partial\Omega$,

(2) $f(z) = w$.

Comparing this with the Teichmüller distortion theorem, Martin posed the following converse problem in [3]:

Problem. Let Ω be a planar domain with at least three boundary points and let ρ_Ω be the hyperbolic metric of Ω with constant curvature -1. Suppose $f(z)$ is a K-quasiconformal self-homeomorphism of Ω such that

(1) $f(\zeta) = \zeta$ for all $\zeta \in \partial\Omega$,
(2) $f(z) = w$ for $z, \quad w \in \Omega$.

Does $\rho_\Omega(z, f(z)) \le \log K$ still hold?

2. MAIN RESULT AND ITS PROOF

In this note, we will solve this problem by proving that it does not hold in general. Our result can be stated as follows:

THEOREM. *There are planar simply connected domains Ω with at least two boundary points, and $\rho_\Omega(z,w)$ is the hyperbolic metric of Ω with constant curvature -1. For every $1 < K < \infty$, there is a K-quasiconformal self-homeomorphism $f_\Omega(z)$ of Ω such that*

(1) $f_\Omega(\zeta) = \zeta$ *for all* $\zeta \in \partial\Omega$,
(2) $f_\Omega(z) = w$, *for some* $z, w \in \Omega$,
(3) $\rho_\Omega(z, f_\Omega(z)) > \log K$.

Proof of Theorem. Let Ω be a planar simply connected domain with at least two boundary points. According to the Riemann mapping theorem, for a point $z_0 \in \Omega$, there exists a conformal mapping in Ω onto the disk $D = \{w \mid |w| < 1\}$, such that $f(z_0) = 0$, $f'(z_0) > 0$. We obtain that the hyperbolic density of Ω with constant Gaussian curvature -1 is given by $\rho_\Omega(z) = 2|f'(z)|/(1 - |f(z)|^2)$. We choose a $1/(1-\beta)$-quasiconformal mapping $\phi(w)$ of D as

$$\phi(w) = w - (1 - |w|)\,\beta, \qquad 0 < \beta < 1, \quad w \in D,$$

and let $F(z) = f^{-1} \circ \phi \circ f(z)$. Thus, $F(w)$ is a $1/(1-\beta)$-quasiconformal mapping of Ω onto itself with the following properties:

(1) $F(\zeta) = \zeta$, for all $\zeta \in \partial\Omega$,

(2) the complex dilatation of $F(z)$ is given as $\mu_F(z) = \mu_\phi(f(z))\frac{\overline{f_z(z)}}{f_z(z)}$,

(3) $\|\mu_F\|_\infty = \frac{\beta}{2-\beta}$.

Thus $F(z)$ is a $1/(1-\beta)$-quasiconformal mapping of Ω onto itself which keeps the boundary points $\partial\Omega$ fixed. Since $\phi(0) = -\beta$, we have $F(z_0) = f^{-1}(-\beta)$; the hyperbolic distant between z_0 and $f^{-1}(-\beta)$ in Ω is given by

$$\rho_\Omega(z_0, F(z_0)) = \inf \int_\gamma \frac{2|f'(z)|}{1-|f(z)|^2}|dz|, \tag{5}$$

where the infimum is taken over all rectifiable curves γ connecting z_0 and $f^{-1}(-\beta)$ in Ω.

We obtain that

$$\rho_\Omega(z_0, F(z_0)) = \rho_D(0, -\beta) = \log\frac{1+\beta}{1-\beta} > \log\frac{1}{1-\beta}.$$

Because $K = \frac{1}{1-\beta}$ and $0 < \beta < 1$, K can range from $1 < K < \infty$. Therefore the proof of the theorem is completed.

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