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Generalized enrichment of categories ☆

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Abstract

We define the phrase 'category enriched in an **fc**-multicategory' and explore some examples. An **fc**-multicategory is a very general kind of two-dimensional structure, special cases of which are double categories, bicategories, monoidal categories and ordinary multicategories. Enrichment in an **fc**-multicategory extends the (more or less well-known) theories of enrichment in a monoidal category, in a bicategory, and in a multicategory. Moreover, **fc**-multicategories provide a natural setting for the bimodules construction, traditionally performed on suitably cocomplete bicategories. Although this paper is elementary and self-contained, we also explain why, from one point of view, **fc**-multicategories are the *natural* structures in which to enrich categories. © 2001 Elsevier Science B.V. All rights reserved.

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A general question in category theory is: given some kind of categorical structure, what might it be enriched in? For instance, suppose we take braided monoidal categories. Then the question asks: what kind of thing must $\mathscr{V}$ be if we are to speak sensibly of $\mathscr{V}$-enriched braided monoidal categories? (The usual answer is that $\mathscr{V}$ must be a symmetric monoidal category.)

In another paper, [7], I have given an answer to the general question for a certain family of categorical structures (generalized multicategories). In particular, this theory gives an answer to the question 'what kind of structure $\mathscr{V}$ can a category be enriched in'? The answer is: an '**fc**-multicategory'.

Of course, the traditional answer to this question is that $\mathscr{V}$ is a monoidal category. But there is also a notion of a category enriched in a bicategory (see Walters [15]). And generalizing in a different direction, it is easy to see how one might speak

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of a category enriched in an ordinary multicategory ('change tensors to commas'). An **fc**-multicategory is, in fact, a very general kind of two-dimensional categorical structure, encompassing monoidal categories, bicategories, multicategories and double categories. The theory of categories enriched in an **fc**-multicategory extends all of the aforementioned theories of enrichment.

So from the point of view of [7], **fc**-multicategories are the natural structures in which to enrich a category. In this work, however, we do not assume any knowledge of [7] or of generalized multicategories. Instead, we define **fc**-multicategory in an elementary fashion (Section 1) and then define what a category enriched in an **fc**-multicategory is (Section 2). Along the way we see how enrichment in an **fc**-multicategory extends the previously-mentioned theories of enrichment, and look at various examples.

fc-multicategories also provide a natural setting for the bimodules construction (Section 3), traditionally carried out on bicategories satisfying certain cocompleteness conditions. At the level of **fc**-multicategories, the construction is both more general and free of technical restrictions. We show, in particular, that a category enriched in an **fc**-multicategory $\mathscr{V}$ naturally gives rise to a category enriched in the **fc**-multicategory **Bim**($\mathscr{V}$) of bimodules in $\mathscr{V}$. This result is functorial (that is, a $\mathscr{V}$-enriched functor gives rise to a **Bim**($\mathscr{V}$)-enriched functor), a statement which only holds if we work with **fc**-multicategories rather than bicategories.

1. fc-multicategories

In a moment, an explicit and elementary definition of **fc**-multicategory will be given. But first it might be helpful to look briefly at the wider context in which this definition sits: the theory of 'generalized multicategories'. The reader is reassured that no knowledge of this wider context is required in order to understand the rest of the paper.

Given a monad T on a category $\mathscr{E}$, both having certain convenient properties, there is a category of *T-multicategories*. A T-multicategory C consists of a diagram

$$T(C_0) \xleftarrow{\text{dom}} C_1 \xrightarrow{\text{cod}} C_0$$

in $\mathscr{E}$ (a *T-graph*) together with functions defining 'composition' and 'identity'; the full details can be found in Burroni [3] or Leinster ([6] or [8]). Thus when T is the identity monad on $\mathscr{E} = \mathbf{Set}$, a T-multicategory is simply a category. When T is the free-monoid monad on $\mathscr{E} = \mathbf{Set}$, a T-multicategory is a multicategory in the original sense of Lambek [5]. When T is the free (strict) ∞-category monad on the category $\mathscr{E}$ of globular sets ('∞-graphs'), a T-multicategory C with $C_0 = 1$ is a higher operad in the sense of Batanin [1]. The example which concerns us here is when T is the free category monad **fc** on the category $\mathscr{E}$ of directed graphs. A T-multicategory is then an **fc**-multicategory in the sense of the following explicit definition.

Definition 1. An **fc**-*multicategory* consists of

- a class of *objects* $x, x', \ldots$,
- for each pair (x, x') of objects, a class of *vertical* 1-*cells*

$$\begin{array}{c} x \\ \downarrow \\ x', \end{array}$$

  denoted $f, f', \ldots$,
- for each pair (x, x') of objects, a class of *horizontal* 1-*cells* $x \to x'$, denoted $m, m', \ldots$,
- for each $n \geqslant 0$, objects $x_0, \ldots, x_n, x, x'$, vertical 1-cells f, f', and horizontal 1-cells $m_1, \ldots, m_n, m$, a class of 2-*cells*

$$\begin{array}{ccccccc} x_0 & \xrightarrow{m_1} & x_1 & \xrightarrow{m_2} & \cdots & \xrightarrow{m_n} & x_n \\ {\scriptstyle f}\downarrow & & & \Downarrow & & & \downarrow{\scriptstyle f'} \\ x & & & \xrightarrow[m]{\qquad\qquad\qquad} & & & x', \end{array} \tag{1}$$

  denoted $\theta, \theta', \ldots$,
- *composition* and *identity* functions making the objects and vertical 1-cells into a category,
- a *composition* function for 2-cells, as in the picture

$$\begin{array}{ccccccccccccccc} \bullet & \xrightarrow{m_1^1} & \cdots & \xrightarrow{m_1^{r_1}} & \bullet & \xrightarrow{m_2^1} & \cdots & \xrightarrow{m_2^{r_2}} & \bullet \ \cdots \ \bullet & \xrightarrow{m_n^1} & \cdots & \xrightarrow{m_n^{r_n}} & \bullet \\ {\scriptstyle f_0}\downarrow & & \Downarrow\theta_1 & & \downarrow & & \Downarrow\theta_2 & & \downarrow\ \cdots\ \downarrow & & \Downarrow\theta_n & & \downarrow{\scriptstyle f_n} \\ \bullet & & \xrightarrow[m_1]{} & & \bullet & & \xrightarrow[m_2]{} & & \bullet \ \cdots \ \bullet & & \xrightarrow[m_n]{} & & \bullet \\ {\scriptstyle f}\downarrow & & & & & & \Downarrow\theta & & & & & & \downarrow{\scriptstyle f'} \\ \bullet & & & & & & \xrightarrow[m]{\qquad\qquad} & & & & & & \bullet \end{array}$$

$$\longmapsto$$

$$\begin{array}{ccccc} \bullet & \xrightarrow{m_1^1} & \cdots & \xrightarrow{m_n^{r_n}} & \bullet \\ {\scriptstyle f\circ f_0}\downarrow & & \Downarrow\theta\circ(\theta_1,\theta_2,\ldots,\theta_n) & & \downarrow{\scriptstyle f'\circ f_n} \\ \bullet & & \xrightarrow[m]{\qquad\qquad} & & \bullet \end{array}$$

  ($n \geqslant 0$, $r_i \geqslant 0$), where the $\bullet$'s represent objects,
- an *identity* function

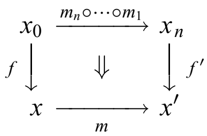
$$x \xrightarrow{m} x' \quad \longmapsto \quad \begin{array}{ccc} x & \xrightarrow{m} & x' \\ {\scriptstyle 1_x}\downarrow & \Downarrow 1_m & \downarrow{\scriptstyle 1_{x'}} \\ x & \xrightarrow[m]{} & x'. \end{array}$$

The 2-cell composition and identities are required to obey associativity and identity laws.

The associativity and identity laws ensure that any diagram of pasted-together 2-cells with a rectangular boundary has a well-defined composite.

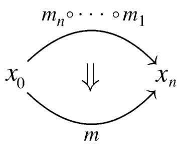
Examples

(1) Any double category gives an **fc**-multicategory, in which a 2-cell as in diagram (1) is a 2-cell



in the double category. If the double category is called $\mathscr{D}$ then we also call the resulting **fc**-multicategory $\mathscr{D}$, and we use the same convention for bicategories (next example).

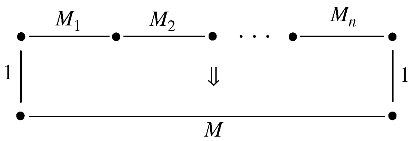
(2) Any bicategory gives an **fc**-multicategory in which the only vertical 1-cells are identity maps, and a 2-cell as in diagram (1) is a 2-cell



in the bicategory (with $x = x_0$ and $x' = x_n$).

Here $m_n \circ \cdots \circ m_1$ denotes some n-fold composite of the 1-cells $m_n, \ldots, m_1$ in the bicategory. For the sake of argument let us decide to associate to the left, so that $m_4 \circ m_3 \circ m_2 \circ m_1$ means $((m_4 \circ m_3) \circ m_2) \circ m_1$. A different choice of bracketing would only affect the resulting **fc**-multicategory up to isomorphism (in the obvious sense).

(3) Any monoidal category $\mathscr{M}$ gives rise to an **fc**-multicategory $\Sigma\mathscr{M}$ (the *suspension* of $\mathscr{M}$) in which there is one object and one vertical 1-cell, and a 2-cell

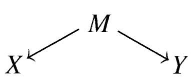


(2)

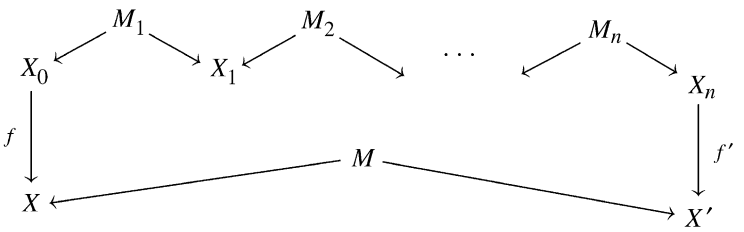
is a morphism $M_n \otimes \cdots \otimes M_1 \to M$ in $\mathscr{M}$. This is a special case of Example (2).

(4) Similarly, any ordinary multicategory $\mathscr{M}$ gives an **fc**-multicategory $\Sigma\mathscr{M}$: there is one object, one vertical 1-cell, and a 2-cell as in diagram (2) is a map $M_1, \ldots, M_n \to M$ in $\mathscr{M}$.

(5) We define an **fc**-multicategory **Span**. Objects are sets, vertical 1-cells are functions, a horizontal 1-cell $X \to Y$ is a diagram

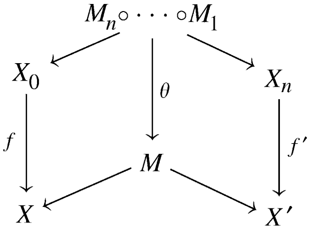


of sets and functions, and a 2-cell inside



(3)

is a function θ making



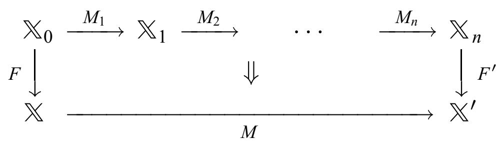
commute. Here $M_n \circ \cdots \circ M_1$ is the limit of the top row of diagram (3), an iterated pullback. Composition is defined in the obvious way.

Span is an example of a 'weak double category', which is just like a double category except that horizontal 1-cell composition only obeys associativity and identity axioms up to coherent isomorphism.

It is rather idiosyncratic to name this **fc**-multicategory after its horizontal 1-cells: usually one names a categorical structure after its objects (e.g. **Group**, **Set**). However, we do not want to confuse the **fc**-multicategory **Span** of sets with the mere category **Set** of sets, so we will stick to this convention.

Notice, incidentally, that **Set** is the category formed by the objects and vertical 1-cells of **Span**, and that the **fc**-multicategory Σ**Set** arising from the monoidal category $(\mathbf{Set}, \times, 1)$ is the 'full' sub-**fc**-multicategory of **Span** whose only object is 1.

(6) There is an **fc**-multicategory **Prof**, in which the category formed by the objects and vertical 1-cells is the usual category of (small) categories and functors. Horizontal 1-cells are profunctors (bimodules): that is, a horizontal 1-cell $\mathbb{X} \to \mathbb{X}'$ is a functor $\mathbb{X}^{\mathrm{op}} \times \mathbb{X}' \to \mathbf{Set}$. A 2-cell



consists of a function

$$M_n(x_{n-1}, x_n) \times \cdots \times M_1(x_0, x_1) \to M(F(x_0), F'(x_n))$$

for each $x_0 \in \mathbb{X}_0, \ldots, x_n \in \mathbb{X}_n$, such that this family of functions is natural in the x_i's. So if the functors F and F' are identities then this is a morphism of profunctors $M_n \otimes \cdots \otimes M_1 \to M$.

(7) In a similar spirit, **Bimod** is the following **fc**-multicategory:
- objects are rings (with identity, not necessarily commutative),
- vertical 1-cells are ring homomorphisms,
- a horizontal 1-cell $R \to S$ is an (S,R)-bimodule,
- a 2-cell

$$\begin{array}{ccccccc} R_0 & \xrightarrow{M_1} & R_1 & \xrightarrow{M_2} & \cdots & \xrightarrow{M_n} & R_n \\ {\scriptstyle f}\downarrow & & & \Downarrow\theta & & & \downarrow{\scriptstyle f'} \\ R & & & \xrightarrow[M]{\hspace{6em}} & & & R' \end{array}$$

is a function $\theta: M_n \times \cdots \times M_1 \to M$ which preserves addition in each component separately (is 'multi-additive') and satisfies the equations

$$\begin{aligned} \theta(r_n \cdot m_n, m_{n-1}, \ldots) &= f'(r_n) \cdot \theta(m_n, m_{n-1}, \ldots) \\ \theta(m_n \cdot r_{n-1}, m_{n-1}, \ldots) &= \theta(m_n, r_{n-1} \cdot m_{n-1}, \ldots) \\ &\vdots \\ \theta(\ldots, m_2 \cdot r_1, m_1) &= \theta(\ldots, m_2, r_1 \cdot m_1) \\ \theta(\ldots, m_2, m_1 \cdot r_0) &= \theta(\ldots, m_2, m_1) \cdot f(r_0), \end{aligned}$$

- composition and identities are defined in the evident way.

(8) If we remove all the additive structure involved in **Bimod** then we obtain an **fc**-multicategory **Action**; alternatively, **Action** is the 'full' sub-**fc**-multicategory of **Prof** in which the only objects allowed are 1-object categories. Thus the objects of **Action** are monoids, the vertical 1-cells are monoid homomorphisms, a horizontal 1-cell $R \to S$ is a set with commuting left S-action and right R-action, and 2-cells are defined as in Example (7).

2. Enrichment

The purpose of this paper is to explore in an elementary way the concept of a category enriched in an **fc**-multicategory. But just as the elementary definition of **fc**-multicategory (Definition 1) is plucked out of a much larger theory (as explained in the introduction to Section 1), so too is the definition of category enriched in an **fc**-multicategory. There is a whole theory [7] of enrichment for generalized multicategories, of which the present work is just the most simple case. This wider theory runs as follows.

Any T-multicategory has an underlying T-graph, as explained above, and so there is a forgetful functor

$$T\text{-}\mathbf{Multicat} \to T\text{-}\mathbf{Graph}.$$

Under mild conditions on $\mathscr{E}$ and T, this functor has a left adjoint. We thus obtain a monad T' on the category $\mathscr{E}' = T$-**Graph**. We can then speak of T'-multicategories, and if $\mathscr{V}$ is a T'-multicategory one can make a definition of *$\mathscr{V}$-enriched T-multicategory*. So: we can speak of a T-multicategory enriched in a T'-multicategory.

The most simple case is the identity monad T on $\mathscr{E} = \mathbf{Set}$. Then T-multicategories are categories, T' is the free category monad **fc** on $\mathscr{E}' = \mathbf{Graph}$, and T'-multicategories are **fc**-multicategories. So the general theory gives a concept of category enriched in an **fc**-multicategory. The main part of this section is a direct description of this concept.

The next most simple case is the free monoid monad T on $\mathscr{E} = \mathbf{Set}$, and here there are two especially interesting examples of enriched T-multicategories. Firstly, it turns out that any symmetric monoidal category $\mathscr{S}$ gives rise to a T'-multicategory $\mathscr{V}$, and a one-object $\mathscr{V}$-enriched T-multicategory is then exactly what topologists call a (non-symmetric) operad in $\mathscr{S}$ (see e.g. [9]). Secondly, there is a certain naturally-arising T'-multicategory $\mathscr{V}$ such that $\mathscr{V}$-enriched T-multicategories are the structures called 'relaxed multicategories' by Borcherds in his definition of vertex algebras over a vertex group [2,10,11], and called 'pseudo-monoidal categories' by Soibelman in his work on quantum affine algebras [12,13].

The general definition of enriched T-multicategory is very simple. Take a monad T on a category $\mathscr{E}$, and let T' be the free T-multicategory monad, as above. Given an object A of $\mathscr{E}$, we can form $I(A)$ (with I for 'indiscrete'), the unique T-multicategory with graph

$$T(A) \xleftarrow{\mathrm{pr}_1} T(A) \times A \xrightarrow{\mathrm{pr}_2} A.$$

Then $I(A)$ is a T'-algebra, say $h : T'(I(A)) \to I(A)$. Arising from this is a T'-multicategory $M(I(A))$, the unique such with graph

$$T'(I(A)) \xleftarrow{1} T'(I(A)) \xrightarrow{h} I(A).$$

For a fixed T'-multicategory $\mathscr{V}$, a *$\mathscr{V}$-enriched T-multicategory* is defined as an object C_0 of $\mathscr{E}$ together with a map $T'(I(C_0)) \to \mathscr{V}$ of T'-multicategories. Maps between $\mathscr{V}$-enriched T-multicategories are also defined in a simple way, thus giving a category.

In the case concerning us, $\mathscr{E} = \mathbf{Set}$ and $T = id$, the definition of enriched (T-multi)-category is therefore as follows. Given a set A, we obtain the indiscrete category $I(A)$ on A. In the **fc**-multicategory $M(I(A))$, an object is an element of A, the only vertical 1-cells are identities, there is one horizontal 1-cell $a \to b$ for each $a, b \in A$, and for each $a_0, \dots, a_n \in A$ there is precisely one 2-cell of the form

$$\begin{array}{ccccccc} a_0 & \longrightarrow & a_1 & \longrightarrow & \cdots & \longrightarrow & a_n \\ {\scriptstyle 1}\downarrow & & & \Downarrow & & & \downarrow{\scriptstyle 1} \\ a_0 & & & \longrightarrow & & & a_n. \end{array}$$

Composition and identities are uniquely determined. A category enriched in an **fc**-multicategory $\mathscr{V}$ consists of a set C_0 together with a map from the **fc**-multicategory $M(I(C_0))$ to $\mathscr{V}$. This definition is plainly equivalent to Definition 2 below.

That concludes the sketch of the theory of enriched generalized multicategories, and we now return to the elementary account.

Fix an **fc**-multicategory $\mathscr{V}$.

Definition 2. A *category enriched in* $\mathscr{V}$, or $\mathscr{V}$*-enriched category*, C, consists of

- a class C_0 (of 'objects'),
- for each $a \in C_0$, an object $C[a]$ of $\mathscr{V}$,
- for each $a, b \in C_0$, a horizontal 1-cell $C[a] \xrightarrow{C[a,b]} C[b]$ in $\mathscr{V}$,
- for each $a, b, c \in C_0$, a 'composition' 2-cell

$$\begin{array}{ccccc} C[a] & \xrightarrow{C[a,b]} & C[b] & \xrightarrow{C[b,c]} & C[c] \\ {\scriptstyle f}\downarrow & & \Downarrow comp_{a,b,c} & & \downarrow{\scriptstyle 1} \\ C[a] & & \xrightarrow[C[a,c]]{\qquad\qquad} & & C[c], \end{array}$$

- for each $a \in C_0$, an 'identity' 2-cell

$$\begin{array}{ccc} C[a] & =\!=\!=\!= & C[a] \\ {\scriptstyle 1}\downarrow & \Downarrow ids_a & \downarrow{\scriptstyle 1} \\ C[a] & \xrightarrow[C[a,a]]{} & C[a] \end{array}$$

(where the equality sign along the top denotes a string of 0 horizontal 1-cells), such that *comp* and *ids* satisfy associativity and identity axioms.

To the reader used to enrichment in a monoidal category, the only unfamiliar piece of data in this definition is the family of objects $C[a]$. To the reader used to enrichment in bicategories even this will be familiar; indeed, since the vertical 1-cells are not used in any significant way, our definition looks very much like the definition of category enriched in a bicategory (see [15]). This lack of use of the vertical 1-cells might seem to weigh against the claim that **fc**-multicategories are, in some sense, the natural structures in which to enrich categories. However, the vertical 1-cells *are* used in the definition of $\mathscr{V}$-enriched functor, which is given next. This makes the theory of enrichment in an **fc**-multicategory run more smoothly (sometimes, at least) than that of enrichment in a bicategory, as we shall see towards the end of Section 3.

Definition 3. Let C and D be $\mathscr{V}$-enriched categories. A $\mathscr{V}$*-enriched functor* $F : C \to D$ consists of

- a function $F : C_0 \to D_0$,
- for each $a \in C_0$, a vertical 1-cell

$$\begin{array}{c} C[a] \\ \downarrow{\scriptstyle F_a} \\ D[F(a)]\,, \end{array}$$

- for each $a, b \in C_0$, a 2-cell

$$\begin{array}{ccc} C[a] & \xrightarrow{C[a,b]} & C[b] \\ {\scriptstyle F_a}\downarrow & \Downarrow F_{ab} & \downarrow{\scriptstyle F_b} \\ D[F(a)] & \xrightarrow[D[F(a),F(b)]]{} & D[F(b)] \end{array},$$

such that the F_{ab}'s commute with the composition and identity 2-cells in C and D, in an evident sense.

With the obvious notion of composition of $\mathscr{V}$-enriched functors, we obtain a category $\mathscr{V}$-**Cat** of $\mathscr{V}$-enriched categories and functors.

Examples

(1) Let $\mathscr{M}$ be a monoidal category and consider a category C enriched in the **fc**-multicategory $\Sigma\mathscr{M}$ (defined in Example 1(3)). There is only one possible choice for the $C[a]$'s, so the data for C consists of the set C_0, the objects $C[a,b]$ of $\mathscr{M}$, and the maps

$$C[b,c] \otimes C[a,b] \to C[a,c], \qquad I \to C[a,a].$$

Thus it turns out that a category enriched in $\Sigma\mathscr{M}$ is just a category enriched (in the usual sense) in $\mathscr{M}$. The same goes for enriched functors, so $(\Sigma\mathscr{M})$-**Cat** is isomorphic to the usual category of $\mathscr{M}$-enriched categories and functors.

(2) Let $\mathscr{M}$ be an (ordinary) multicategory. There is an obvious notion of category enriched in $\mathscr{M}$: that is, a set C_0 together with an object $C[a,b]$ of $\mathscr{M}$ for each $a, b \in C_0$ and arrows

$$C[a,b], C[b,c] \to C[a,c], \qquad \cdot \to C[a,a]$$

(where $\cdot$ is the empty sequence), obeying suitable axioms. This is precisely the same thing as a category enriched in $\Sigma\mathscr{M}$.

(3) If $\mathscr{B}$ is a bicategory then our $\mathscr{B}$-**Cat** is isomorphic to the category of $\mathscr{B}$-enriched categories defined in Walters [15].

(4) Fix a topological space A. Then there is a bicategory $\Pi_2 A$, the *homotopy bicategory* of A, in which an object is a point of A, a 1-cell is a path in A, and a 2-cell is a homotopy class of path homotopies in A. For any 1-cell $\gamma : a \to b$ there is an associated 1-cell $\gamma^* : b \to a$ (that is, γ run backwards), and there are canonical 2-cells $1_b \to \gamma \circ \gamma^*$ and $\gamma^* \circ \gamma \to 1_a$.
Now suppose that A is non-empty and path-connected, and make a choice of a basepoint a_0 and for each $a \in A$ a path $\gamma_a : a_0 \to a$. Then we obtain a category C enriched in $\Pi_2 A$, as follows:
   - C_0 is the underlying set of A,
   - $C[a] = a$,
   - $C[a,b] = \gamma_b \circ \gamma_a^*$ (a path from a to b),

- composition $C[b,c] \circ C[a,b] \to C[a,c]$ is the 2-cell

$$(\gamma_c \circ \gamma_b^*) \circ (\gamma_b \circ \gamma_a^*) \to \gamma_c \circ \gamma_a^*$$

coming from the canonical 2-cell $\gamma_b^* \circ \gamma_b \to 1_{a_0}$,
- the identity 2-cell $1_a \to C[a,a]$ is the canonical 2-cell $1_a \to \gamma_a \circ \gamma_a^*$.

(5) In the previous example, the bicategory $\Pi_2 A$ can be replaced by any bicategory $\mathscr{B}$ in which the underlying directed graph of objects and 1-cells is (non-empty and) connected, and every 1-cell has a left adjoint. (I thank the referee for alerting me to this.)

(6) **Span**-**Cat** is equivalent to the comma category (ob $\downarrow$ **Set**), where **ob** : **Cat** $\to$ **Set** is the objects functor. This means that a category enriched in **Span** consists of a category D, a set I, and a function $\mathrm{ob}(D) \to I$. To see why this is true, recall that a category C enriched in **Span** consists of
- a set C_0,
- for each $i \in C_0$, a set $C[i]$,
- for each $i,j \in C_0$, a span

$$C[i] \xleftarrow{s_{ij}} C[i,j] \xrightarrow{t_{ij}} C[j],$$

- composition functions $C[j,k] \times_{C[j]} C[i,j] \to C[i,k]$,
- identity functions $C[i] \to C[i,i]$,

all satisfying axioms. We can construct from C a category D with object-set $\coprod_{i \in C_0} C[i]$, arrow-set $\coprod_{i,j \in C_0} C[i,j]$, source and target maps given by the s_{ij}'s and t_{ij}'s, and composition and identity operations coming from those in C. By taking $I = C_0$ and the projection function $\mathrm{ob}(D) \to I$, we now have an object of (ob $\downarrow$ **Set**). A similar analysis of **Span**-enriched functors can be carried out, and we end up with a functor

$$\mathbf{Span}\text{-}\mathbf{Cat} \to (\mathrm{ob} \downarrow \mathbf{Set}).$$

It is easy to see that this functor is an equivalence.

Let us briefly consider enriched categories with only one object. In the classical case of enrichment in a monoidal category $\mathscr{M}$, the category of one-object $\mathscr{M}$-enriched categories is the category **Mon**($\mathscr{M}$) of monoids in $\mathscr{M}$. For an arbitrary **fc**-multicategory $\mathscr{V}$, we therefore define **Mon**($\mathscr{V}$) to be the full subcategory of $\mathscr{V}$-**Cat** whose objects are $\mathscr{V}$-enriched categories C with $|C_0| = 1$. Definitions 2 and 3 yield an explicit description of **Mon**($\mathscr{V}$).

Examples

(1) If $\mathscr{M}$ is a monoidal category then **Mon**($\Sigma\mathscr{M}$) is the category of monoids in $\mathscr{M}$.

(2) If $\mathscr{M}$ is a multicategory then an object of **Mon**($\Sigma\mathscr{M}$) consists of an object M of $\mathscr{M}$ together with maps

$$M, M \to M, \qquad \cdot \to M$$

satisfying associativity and identity laws—in other words, a 'monoid in $\mathscr{M}$'. A monoid in $\mathscr{M}$ is also the same thing as a multicategory map $\mathbf{1} \to \mathscr{M}$, where $\mathbf{1}$ is the terminal multicategory.

(3) If $\mathscr{B}$ is a bicategory then an object of **Mon**($\mathscr{B}$) is a monad in $\mathscr{B}$ in the sense of Street [15]: that is, it's an object X of $\mathscr{B}$ together with a 1-cell $t: X \to X$ and 2-cells $\mu: t \circ t \to t$, $\eta: 1 \to t$ satisfying the usual monad axioms. There are no maps $(X, t, \mu, \eta) \to (X', t', \mu', \eta')$ in **Mon**($\mathscr{B}$) unless $X = X'$, and in this case such a map consists of a 2-cell $t \to t'$ commuting with the μ's and η's. So **Mon**($\mathscr{B}$) is the category of monads and 'strict monad maps' in $\mathscr{B}$.

(4) Let $\mathscr{B}$ be a 2-category. Associated to $\mathscr{B}$ is not only the **fc**-multicategory $\mathscr{B}$ of the previous example—which we now call $\mathscr{V}$—but also two more **fc**-multicategories, $\mathscr{W}$ and $\mathscr{W}'$. Both $\mathscr{W}$ and $\mathscr{W}'$ are defined from double categories (see Example 1(1)), and in both cases an object is an object of $\mathscr{B}$, a vertical 1-cell is a 1-cell of $\mathscr{B}$, and a horizontal 1-cell is also a 1-cell of $\mathscr{B}$. In the case of $\mathscr{W}$, a 2-cell inside

$$\begin{array}{ccc} X & \xrightarrow{t} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X' & \xrightarrow[t']{} & Y' \end{array}$$

is a 2-cell $t' \circ f \to g \circ t$ in $\mathscr{B}$; in the case of $\mathscr{W}'$, it is a 2-cell $g \circ t \to t' \circ f$ in $\mathscr{B}$. Composition and identities are defined in the obvious way.

Since $\mathscr{V}$, $\mathscr{W}$ and $\mathscr{W}'$ are identical when we ignore the vertical 1-cells, the objects of **Mon**($\mathscr{W}$) and **Mon**($\mathscr{W}'$) are the same as the objects of **Mon**($\mathscr{V}$); that is, they are monads in $\mathscr{B}$. But by using $\mathscr{W}$ or $\mathscr{W}'$ we obtain a more flexible notion of a 'map of monads' than we did in Example (3): a map in **Mon**($\mathscr{W}$) is what Street called a *monad functor* in [14], and a map in **Mon**($\mathscr{W}'$) is a *monad opfunctor*.

3. Bimodules

Bimodules have traditionally been discussed in the context of bicategories. Thus given a bicategory $\mathscr{B}$, one constructs a new bicategory **Bim**($\mathscr{B}$) whose 1-cells are bimodules in $\mathscr{B}$ (see e.g. [4]). The drawback is that this is only possible when $\mathscr{B}$ has certain properties concerning the existence and behaviour of local reflexive coequalizers.

Here, we extend the **Bim** construction from bicategories to **fc**-multicategories, which allows us to drop the technical assumptions. In other words, we will construct an honest functor

$$\mathbf{Bim} : \mathbf{fc}\text{-}\mathbf{Multicat} \to \mathbf{fc}\text{-}\mathbf{Multicat}.$$

(**fc-Multicat** is the category of (small) **fc**-multicategories, with maps defined in the obvious way.)

I would like to be able to, but at present cannot, place the **Bim** construction in a more abstract setting: as it stands it is somewhat *ad hoc*. In particular, the definition does not appear to generalize to T-multicategories for arbitrary T.

The theories of bimodules and enrichment interact in the following way: given an **fc**-multicategory $\mathscr{V}$, there is a canonically-defined functor

$$\mathscr{V}\text{-}\mathbf{Cat} \to \mathbf{Bim}(\mathscr{V})\text{-}\mathbf{Cat}.$$

This is discussed at the end of the section, and provides lots of new examples of enriched categories.

We first have to define **Bim**. Let $\mathscr{V}$ be an **fc**-multicategory: then the **fc**-multicategory $\mathbf{Bim}(\mathscr{V})$ is defined as follows.

0-cells. A 0-cell of $\mathbf{Bim}(\mathscr{V})$ is an **fc**-multicategory map $1 \to \mathscr{V}$. That is, it is a 0-cell x of $\mathscr{V}$ together with a horizontal 1-cell $x \xrightarrow{t} x$ and 2-cells

$$\begin{array}{ccccc} x & \xrightarrow{t} & x & \xrightarrow{t} & x \\ {\scriptstyle 1}\downarrow & & \Downarrow \mu & & \downarrow{\scriptstyle 1} \\ x & & \xrightarrow[t]{} & & x \end{array} \qquad \begin{array}{ccc} x & = & x \\ {\scriptstyle 1}\downarrow & \Downarrow \eta & \downarrow{\scriptstyle 1} \\ x & \xrightarrow[t]{} & x \end{array}$$

satisfying the usual axioms for a monad, $\mu \circ (\mu, 1_t) = \mu \circ (1_t, \mu)$ and $\mu \circ (\eta, 1_t) = 1_t = \mu \circ (1_t, \eta)$.

Horizontal 1-cells. A horizontal 1-cell $(x,t,\eta,\mu) \to (x',t',\eta',\mu')$ consists of a horizontal 1-cell $x \xrightarrow{m} x'$ in $\mathscr{V}$ together with 2-cells

$$\begin{array}{ccccc} x & \xrightarrow{t} & x & \xrightarrow{m} & x' \\ {\scriptstyle 1}\downarrow & & \Downarrow \theta & & \downarrow{\scriptstyle 1} \\ x & & \xrightarrow[m]{} & & x' \end{array} \qquad \begin{array}{ccccc} x & \xrightarrow{m} & x' & \xrightarrow{t'} & x' \\ {\scriptstyle 1}\downarrow & & \Downarrow \theta' & & \downarrow{\scriptstyle 1} \\ x & & \xrightarrow[m]{} & & x' \end{array}$$

satisfying the usual module axioms $\theta \circ (\eta, 1_m) = 1_m$, $\theta \circ (\mu, 1_m) = \theta \circ (1_t, \theta)$, and dually for θ', and the 'commuting actions' axiom $\theta' \circ (\theta, 1_{t'}) = \theta \circ (1_t, \theta')$.

Vertical 1-cells. A vertical 1-cell

$$\begin{array}{c} (x,t,\eta,\mu) \\ \downarrow \\ (\hat{x},\hat{t},\hat{\eta},\hat{\mu}) \end{array}$$

in $\mathbf{Bim}(\mathscr{V})$ is a vertical 1-cell

$$\begin{array}{c} x \\ \downarrow{\scriptstyle f} \\ \hat{x} \end{array}$$

in $\mathscr{V}$ together with a 2-cell

$$\begin{array}{ccc} x & \xrightarrow{t} & x \\ {\scriptstyle f}\downarrow & \Downarrow \omega & \downarrow{\scriptstyle f} \\ \hat{x} & \xrightarrow[\hat{t}]{} & \hat{x} \end{array}$$

satisfying $\omega \circ \mu = \hat{\mu} \circ (\omega, \omega)$ and a similar equation for units.

2-cells. A 2-cell

$$\begin{array}{ccccccc} t_0 & \xrightarrow{m_1} & t_1 & \xrightarrow{m_2} & \cdots & \xrightarrow{m_n} & t_n \\ {\scriptstyle f}\downarrow & & & \Downarrow & & & \downarrow{\scriptstyle f'} \\ t & & & \xrightarrow[m]{\qquad\qquad} & & & t' \end{array}$$

in $\mathbf{Bim}(\mathscr{V})$, where t stands for (x, t, η, μ), m for (m, θ, θ'), f for (f, ω), and so on, consists of a 2-cell

$$\begin{array}{ccccccc} x_0 & \xrightarrow{m_1} & x_1 & \xrightarrow{m_2} & \cdots & \xrightarrow{m_n} & x_n \\ {\scriptstyle f}\downarrow & & & \Downarrow\alpha & & & \downarrow{\scriptstyle f'} \\ x & & & \xrightarrow[m]{\qquad\qquad} & & & x' \end{array}$$

in $\mathscr{V}$, satisfying the 'external equivariance' axioms

$$\alpha \circ (\theta_1, 1_{m_2}, \ldots, 1_{m_n}) = \theta \circ (\omega, \alpha)$$
$$\alpha \circ (1_{m_1}, \ldots, 1_{m_{n-1}}, \theta'_n) = \theta' \circ (\alpha, \omega')$$

and the 'internal equivariance' axioms

$$\alpha \circ (1_{m_1}, \ldots, 1_{m_{i-2}}, \theta'_{i-1}, 1_{m_i}, 1_{m_{i+1}}, \ldots, 1_{m_n})$$
$$= \alpha \circ (1_{m_1}, \ldots, 1_{m_{i-2}}, 1_{m_{i-1}}, \theta_i, 1_{m_{i+1}}, \ldots, 1_{m_n})$$

for $2 \leqslant i \leqslant n$.

Composition and identities. For both 2-cells and vertical 1-cells in $\mathbf{Bim}(\mathscr{V})$, composition is defined directly from the composition in $\mathscr{V}$, and similarly identities.

Incidentally, the category formed by the objects and vertical 1-cells of $\mathbf{Bim}(\mathscr{V})$ is $\mathbf{Mon}(\mathscr{V})$, the category of monads in $\mathscr{V}$ defined earlier.

We have now defined an **fc**-multicategory $\mathbf{Bim}(\mathscr{V})$ for each **fc**-multicategory $\mathscr{V}$, and it is clear how to do the same thing for maps of **fc**-multicategories, so that we have a functor

$$\mathbf{Bim} : \mathbf{fc}\text{-}\mathbf{Multicat} \to \mathbf{fc}\text{-}\mathbf{Multicat}.$$

Again, we have been rather eccentric in naming the 'bimodules construction' after what it does to the horizontal 1-cells rather than the objects: perhaps we should call it the 'monads construction'. We are, however, following the traditional terminology.

Examples

(1) Let $\mathscr{B}$ be a bicategory satisfying the conditions on local reflexive coequalizers mentioned in the first paragraph of this section, so that it is possible to construct a bicategory $\mathbf{Bim}(\mathscr{B})$ in the traditional way. Let $\mathscr{V}$ be the **fc**-multicategory coming from $\mathscr{B}$. Then a 0-cell of $\mathbf{Bim}(\mathscr{V})$ is a monad in $\mathscr{B}$, a horizontal 1-cell $t \to t'$ is

a (t', t)-bimodule, and a 2-cell of the form

$$\begin{array}{ccccccc} t_0 & \xrightarrow{m_1} & t_1 & \xrightarrow{m_2} & \cdots & \xrightarrow{m_n} & t_n \\ {\scriptstyle 1}\big\downarrow & & & \Downarrow & & & \big\downarrow{\scriptstyle 1} \\ t_0 & & & \xrightarrow[m]{\hspace{6em}} & & & t_n \end{array}$$

is a map

$$m_n \otimes_{t_{n-1}} \cdots \otimes_{t_1} m_1 \to m$$

of (t_n, t_0)-bimodules, i.e. a 2-cell in **Bim**($\mathscr{B}$). So if we discard the non-identity 1-cells of **Bim**($\mathscr{V}$), then the resulting **fc**-multicategory is precisely the **fc**-multicategory associated with the bicategory **Bim**($\mathscr{B}$).

(2) **Bim**(**Span**) = **Prof**, where **Span** is the **fc**-multicategory of sets, functions, spans, etc, and **Prof** is the **fc**-multicategory of categories, functors, profunctors, etc. (Examples 1(5) and (6)).

(3) **Bim**(Σ**Ab**) = **Bimod** (Example 1(7)). Here **Ab** is regarded as a monoidal category under the usual tensor and Σ**Ab** is as in Example 1(3); or equivalently **Ab** is regarded as a multicategory with the usual multilinear maps, and Σ**Ab** is as in Example 1(4).

(4) Similarly, **Bim**(Σ**Set**) = **Action** (Example 1(8)), with cartesian product giving the monoidal category (or multicategory) structure on **Set**.

(5) It is possible to define an **fc**-multicategory **Span**($\mathscr{E}, T$), for any appropriate monad T on a category $\mathscr{E}$, and then **Bim**(**Span**($\mathscr{E}, T$)) is the **fc**-multicategory of T-multicategories and maps, profunctors, etc, between them. See [7] or [8] for details.

We now show how the bimodules construction produces new enriched categories from old.

Proposition 4. *For any* **fc**-*multicategory* $\mathscr{V}$, *there is a natural functor*

$$\tilde{(\;)} : \mathscr{V}\text{-}\mathbf{Cat} \to \mathbf{Bim}(\mathscr{V})\text{-}\mathbf{Cat},$$

preserving object-sets.

Proof. Take a $\mathscr{V}$-enriched category C. We must define a **Bim**($\mathscr{V}$)-enriched category $\tilde{C}$ with object-set C_0, and so, for instance, we must define an object $\tilde{C}[a]$ of **Bim**($\mathscr{V}$) for each $a \in C_0$. To do this we observe that $C[a]$ has a natural monad structure on it: that is, we put

$$\tilde{C}[a] = (C[a], C[a,a], ids_a, comp_{a,a,a}).$$

The rest of the construction is along similar lines; there is only one sensible way to proceed, and it is left to the reader. (An abstract account is in [8]). □

Examples

(1) Let C be a category enriched (in the usual sense) in the monoidal category **Ab** of abelian groups. Then the resulting **Bimod**-enriched category $\tilde{C}$ is as follows:

- $\tilde{C}_0$ is the set of objects of C,
- $\tilde{C}[a]$ is the ring $C[a,a]$, in which multiplication is given by composition in C,
- $\tilde{C}[a,b]$ is the abelian group $C[a,b]$ acted on by $\tilde{C}[a]=C[a,a]$ on the right and by $\tilde{C}[b]=C[b,b]$ on the left, both actions being by composition in C,
- composition and identities are as in C.

To illustrate the functoriality in the Proposition, take an **Ab**-enriched functor $F: C \to D$. This induces a **Bimod**-enriched functor $\tilde{F}: \tilde{C} \to \tilde{D}$ as follows:

- $\tilde{F}: C_0 \to D_0$ is the object-function of F,
- if $a \in C_0$ then $\tilde{F}_a$ is the ring homomorphism
$$\tilde{C}[a] = C[a,a] \to D[F(a),F(a)] = \tilde{D}[\tilde{F}(a)]$$
induced by F,
- if $a,b \in C_0$ then
$$\tilde{F}_{ab}: \tilde{C}[a,b] = C[a,b] \to D[F(a),F(b)] = \tilde{D}[\tilde{F}(a),\tilde{F}(b)]$$
is defined by the action of F on morphisms $a \to b$.

Note that in general, the ring homomorphism $\tilde{F}_a$ is not the identity; so the vertical 1-cells of **Bimod** get used in an essential way. This is the reason why the Proposition does not hold if we work throughout with bicategories rather than **fc**-multicategories: $\tilde{(\ )}$ is defined on objects of $\mathscr{V}$-**Cat**, but cannot sensibly be defined on morphisms.

(2) The non-additive version of (1) is that there is a canonical functor
$$\begin{aligned} \mathbf{Cat} &\to \mathbf{Action\text{-}Cat} \\ C &\mapsto \tilde{C} \end{aligned}$$
which exists because, for instance, the set of endomorphisms on an object of a category is naturally a monoid.

(3) In the previous example, part of the construction was to take $\tilde{C}[a]$ to be the monoid of all endomorphisms of a in C. However, we could just as well take only the automorphisms of a, and this would yield a different functor from **Cat** to **Action-Cat**.

(4) Applying the Proposition to $\mathscr{V} = \mathbf{Span}$ and recalling Example 2(6), we obtain a functor
$$(\mathrm{ob} \downarrow \mathbf{Set}) \to \mathbf{Prof\text{-}Cat}.$$
What this does on objects is as follows. Take a category D, a set I, and a function $\mathrm{ob}(D) \to I$. Then in the resulting **Prof**-enriched category E we have $E_0 = I$; $E[i]$ is the full subcategory of D whose objects are those lying over $i \in I$; and $E[i,j]$ is the profunctor
$$\begin{aligned} E[i]^{\mathrm{op}} \times E[j] &\to \mathbf{Set} \\ (d,d') &\mapsto D(d,d'). \end{aligned}$$
Composition and identities are defined as in D.

(5) To get more examples of **Prof**-enriched categories we can modify the previous example, taking $E[i]$ to be *any* subcategory of D whose objects are all in the fibre over i. Here are two specific instances (each with a vague flavour of topological quantum field theory about them). In the first, E_0 is the set $\mathbb{N}$ of natural numbers, $E[n]$ is the category of n-dimensional Hilbert spaces (= complex inner product spaces) and isometries, and $E[m,n]$ sends (H,H') to the set of all linear maps $H \to H'$. In the second, $E_0 = \mathbb{N}$ again, and we replace Hilbert spaces by differentiable manifolds, isometries by diffeomorphisms, and linear maps by differentiable maps.

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[1] E-print numbers refer to the archive at `http://arXiv.org`.