



ELSEVIER

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

On sums of squares of primes II

Glyn Harman^a, Angel Kumchev^{b,*}^a Department of Mathematics, Royal Holloway University of London, Egham, Surrey TW20 0EX, UK^b Department of Mathematics, Towson University, 7800 York Road, Towson, MD 21252, USA

ARTICLE INFO

Article history:

Received 16 October 2009

Available online 15 May 2010

Communicated by Robert C. Vaughan

MSC:

11P32

Keywords:

Squares of primes

Exponential sums

Sieve methods

ABSTRACT

In this paper we continue our study, begun in G. Harman and A.V. Kumchev (2006) [10], of the exceptional set of integers, not restricted by elementary congruence conditions, which cannot be represented as sums of three or four squares of primes. We correct a serious oversight in our first paper, but make further progress on the exponential sums estimates needed, together with an embellishment of the previous sieve technique employed. This leads to an improvement in our bounds for the maximal size of the exceptional sets.

© 2010 Elsevier Inc. All rights reserved.

1. Introduction

As in [10] we write

$$\mathcal{A}_3 = \{n \in \mathbb{N} : n \equiv 3 \pmod{24}, n \not\equiv 0 \pmod{5}\},$$

$$\mathcal{A}_4 = \{n \in \mathbb{N} : n \equiv 4 \pmod{24}\}.$$

We further put

$$E_j(N) = |\{n \in \mathcal{A}_j : n \leq N, n \neq p_1^2 + \cdots + p_j^2, \text{ for any primes } p_u\}|, \quad j = 3, 4.$$

Our purpose in writing this article is to correct an error in our previous discussion of upper bounds for these sets and also to introduce further refinements to the method which lead to superior results.

* Corresponding author.

E-mail addresses: G.Harman@rhul.ac.uk (G. Harman), akumchev@towson.edu (A. Kumchev).

Although the improvement in the exponent is relatively small (the crucial change is from $1/7$ to $3/20$) the modifications to the method have independent interest and may have further applications—we state one such result below as Theorem 3.

It is conjectured that every sufficiently large integer in \mathcal{A}_j can be represented as the sum of j squares of primes, and so $E_j(N) = O(1)$. The expected main terms from an application of the Circle Method lead one to the following hypothetical asymptotic formulae:

$$\sum_{p_1^2+p_2^2+p_3^2=n} (\log p_1)(\log p_2)(\log p_3) \sim \frac{\pi}{4} \mathfrak{S}_3(n)n^{1/2} \tag{1.1}$$

and

$$\sum_{p_1^2+\dots+p_4^2=n} (\log p_1) \dots (\log p_4) \sim \frac{\pi^2}{16} \mathfrak{S}_4(n)n, \tag{1.2}$$

where $\mathfrak{S}_j(n) > 0$ for all large $n \in \mathcal{A}_j$. In 1938 Hua [12] proved a general result on representing almost all numbers in suitable residue classes as the sum of two squares of primes and the k -th power of a prime, from which it follows that almost all $n \in \mathcal{A}_3$ are representable as sums of three squares of primes. Of course, we then immediately obtain that almost all $n \in \mathcal{A}_4$ are representable as sums of four squares of primes. The subsequent history of this problem is documented in [10] (charting the developments in [2,13,14,16,18–20,22,24]), culminating in the authors' demonstration that

$$E_3(N) \ll N^{6/7+\epsilon} \quad \text{and} \quad E_4(N) \ll N^{5/14+\epsilon}.$$

Unfortunately there was a serious oversight in our proofs. To be precise, the display (4.16) in [10] which gives an estimate on average for the singular series, namely

$$\sum_{N/2 < n \leq N} \left| \mathfrak{S}_3(n, Q) - 8 \prod_{2 < p \leq Q} (1 + \mathfrak{s}(p, n)) \right| \ll N^{1+\epsilon/2} Q^{-1/2},$$

is not good enough for the stated result to follow. We would like to thank Claus Bauer and Hongze Li who both independently alerted the authors to this error. We correct this in Section 3.5 here. However, we can now move beyond what seemed a difficult barrier with the previous exponents, which arose as $1 - \sigma$ and $\frac{1}{2} - \sigma$ with $\sigma = 1/7$. The following results show that we can now increase σ to $3/20$.

Theorem 1. *Let $\epsilon > 0$ be given. Then for all large N we have*

$$E_3(N) \ll N^{17/20+\epsilon}. \tag{1.3}$$

Theorem 2. *Let $\epsilon > 0$ be given. Then for all large N we have*

$$E_4(N) \ll N^{7/20+\epsilon}. \tag{1.4}$$

Combining the new ideas in the present work with [15] we obtain the following.

Theorem 3. *Let $E(N)$ represent the cardinality of the set*

$$\{n \leq N: n \equiv 1 \text{ or } 3 \pmod{6}, n \neq p_1 + p_2^2 + p_3^2\}.$$

Then, for every $\epsilon > 0$,

$$E(N) \ll N^{7/20+\epsilon}. \tag{1.5}$$

2. The method

We shall only prove Theorem 1; the straightforward modifications needed for Theorem 2 follow as in [10], and for Theorem 3 as in [15]. It suffices to estimate the number of exceptional integers n in the set $\mathcal{B} = \mathcal{A}_3 \cap (\frac{1}{2}N, N]$ where N will be our main parameter, which we assume to be “sufficiently large”. We write

$$P = N^{1/2}, \quad L = \log P, \quad \mathcal{J} = \left[\frac{1}{3}P, \frac{2}{3}P \right).$$

We use c to denote an absolute constant, not necessarily the same at each occurrence. In the following, σ will be a parameter in the range $\frac{1}{7} \leq \sigma \leq \frac{3}{20}$, and our method will show that $E_3(N) \ll N^{1-\sigma+\epsilon}$. Here, as elsewhere in the following, ϵ is an arbitrary small positive real.

We wish to represent integers n in the form $m_1^2 + m_2^2 + m_3^2$ where each m_j is restricted to prime values. In our previous paper we sieved only one of the variables, say m_3 . In our current work we will sieve two variables, albeit in a rather asymmetric way. To be precise, let $\rho_1(m)$ be the characteristic function of the set of primes. Suppose that, for suitable non-negative functions $\rho_j(m)$, $2 \leq j \leq 5$, we have

$$\rho_2(m) \geq \rho_1(m) = \rho_3(m) - \rho_4(m) + \rho_5(m).$$

Then

$$\sum_{\substack{m_1^2+m_2^2+m_3^2=n \\ m_j \in \mathcal{J}}} \rho_1(m_1)\rho_1(m_2)\rho_1(m_3) \geq S_1 - S_2,$$

where

$$S_1 = \sum_{\substack{m_1^2+m_2^2+m_3^2=n \\ m_j \in \mathcal{J}}} \rho_1(m_1)\rho_1(m_2)\rho_3(m_3),$$

$$S_2 = \sum_{\substack{m_1^2+m_2^2+m_3^2=n \\ m_j \in \mathcal{J}}} \rho_1(m_1)\rho_2(m_2)\rho_4(m_3).$$

The circle method then gives

$$\sum_{\substack{m_1^2+m_2^2+m_3^2=n \\ m_j \in \mathcal{J}}} \rho_j(m_1)\rho_k(m_2)\rho_\ell(m_3) = \int_0^1 f_j(\alpha)f_k(\alpha)f_\ell(\alpha)e(-\alpha n) d\alpha, \tag{2.1}$$

where we write $e(x) = \exp(2\pi ix)$ and, for $1 \leq j \leq 4$,

$$f_j(\alpha) = \sum_{m \in \mathcal{J}} \rho_j(m)e(\alpha m^2). \tag{2.2}$$

Here we will want the ρ_j , $2 \leq j \leq 4$, to satisfy

$$\sum_{m \leq X} \rho_j(m) = C_j X L^{-1} (1 + o(1)) \tag{2.3}$$

for $P^{1/2} \leq X \leq P$, where

$$C_3 - C_2 C_4 > 0. \tag{2.4}$$

It then remains to establish that

$$\int_0^1 f_1(\alpha) f_k(\alpha) f_\ell(\alpha) e(-\alpha n) d\alpha = K_n C_k C_\ell \Pi(n, Q) P L^{-3} (1 + o(1))$$

for the same value K_n in the two cases $k = 1, \ell = 3, k = 2, \ell = 4$ where $C_1 = 1$, with at most $E_3(N)$ exceptions up to N . Here $\Pi(n, Q)$ is an approximation to $\mathfrak{S}(n)$ which we define later and which satisfies $\Pi(n, Q) \gg L^{-3}$. When we state the main term more explicitly it will be clear that $1 \ll K_n \ll 1$ with absolute constants. The properties of the ρ_j necessary to achieve this will be introduced when relevant. In particular, it should be noted that we require ρ_2 and ρ_3 to satisfy the most stringent conditions.

Our application of the circle method has the same format as our previous work; see [23] for a general introduction. The main contribution to the right side of (2.1) comes from the *major arcs* which we denote by \mathfrak{M} and are defined as follows. Let $Q = P^{2\sigma-3\epsilon}$ and write (shifting $[0, 1)$ by $\omega = Q P^{-2+\epsilon}$ which does not change (2.1))

$$\mathfrak{M} = [\omega, 1 + \omega) \cap \bigcup_{1 \leq q \leq Q} \bigcup_{(a,q)=1} \left[\frac{a}{q} - \frac{\omega}{q}, \frac{a}{q} + \frac{\omega}{q} \right). \tag{2.5}$$

The *minor arcs* \mathfrak{m} are then given by $\mathfrak{m} = [\omega, 1 + \omega) \setminus \mathfrak{M}$.

For technical reasons, it is convenient to modify $f_j(\alpha)$, $j \geq 2$, on the major arcs to remove interference between possible prime divisors of m (when $\rho(m) < 0$) and approximation denominators. We introduce a function $\theta(m, \alpha)$ which is 1 except when there exist integers a and q such that

$$|q\alpha - a| < \omega, \quad (a, q) = 1, \quad q \leq Q, \quad (m, q) \geq P^\sigma,$$

in which case $\theta(m, \alpha) = 0$. Write

$$g_j(\alpha) = \sum_{m \in \mathfrak{J}} \rho_j(m) \theta(m, \alpha) e(\alpha m^2).$$

We note that $g_j(\alpha) = f_j(\alpha)$ for $\alpha \in \mathfrak{m}$ and that

$$f_j(\alpha) - g_j(\alpha) \ll P^{1-\sigma} \tag{2.6}$$

for all α .

3. The major arcs

The major arc contributions to S_1 and S_2 are dominated by the integrals

$$\int_{\mathfrak{M}} f_1(\alpha)^2 g_3(\alpha) e(-\alpha n) d\alpha \quad \text{and} \quad \int_{\mathfrak{M}} f_1(\alpha) g_2(\alpha) g_4(\alpha) e(-\alpha n) d\alpha,$$

respectively. In this section, we evaluate the latter integral. The evaluation of the former can be carried out in a similar fashion and is, in fact, less technical.

As in [10], we suppose that ρ_j , $j = 2, 3, 4$, have asymptotic properties similar to those of ρ_1 . To be precise, we assume that ρ_j satisfy the following two hypotheses:

- (i) Let $A, B > 0$ be fixed, let χ be a non-principal character modulo q , $q \leq L^B$, and let \mathcal{J}' be a subinterval of \mathcal{J} . Then

$$\sum_{m \in \mathcal{J}'} \rho_j(m) \chi(m) \ll PL^{-A}. \tag{3.1}$$

- (ii) Let $A > 0$ be fixed and let \mathcal{J}' be a subinterval of \mathcal{J} . There exists a smooth function δ_j on \mathcal{J} such that

$$\sum_{m \in \mathcal{J}'} \rho_j(m) = \sum_{m \in \mathcal{J}'} \delta_j(m) + O(PL^{-A}). \tag{3.2}$$

Of course, by the Siegel–Walfisz theorem, these hypotheses hold also for $\rho_1(m)$ with $\delta_1(m) = (\log m)^{-1}$. We note that (3.2) gives

$$\int_{\mathcal{J}} \delta_j(u) du = C_j \frac{P}{3L} (1 + o(1)).$$

Furthermore, we assume that:

- (iii) $\rho_j(m) = 0$ if m has a prime divisor $p < Z = P^{1-6\sigma}$.

For $j = 1, \dots, 4$, we define functions $f_j^*(\alpha)$ on \mathfrak{M} by setting

$$f_j^*(\alpha) = \frac{S(\chi_0, a)}{\phi(q)} \sum_{m \in \mathcal{J}} \delta_j(m) e((\alpha - a/q)m^2) \quad \text{if } \alpha \in \mathfrak{M}(q, a).$$

Here χ_0 is the principal character modulo q and

$$S(\chi, a) = \sum_{h=1}^q \bar{\chi}(h) e_q(ah^2).$$

We now proceed to estimate the integral

$$\int_{\mathfrak{M}} (f_1(\alpha) g_2(\alpha) g_4(\alpha) - f_1^*(\alpha) f_2^*(\alpha) f_4^*(\alpha)) e(-\alpha n) d\alpha, \tag{3.3}$$

which we think of as the error of approximation of the contribution from \mathfrak{M} by the expected main term. For our purposes, it suffices to show that this quantity is $O(PL^{-A})$ for any fixed $A > 0$, for example.

A difficulty arises upon reducing σ below $1/7$ —the function $\theta(m, \alpha)$ no longer covers the interference between all possible prime divisors of m (when $\rho(m) < 0$) and the major arc denominators. To be precise, we need a new argument for the range from Z to P^σ . To deal with this, for an integer q , we write \mathcal{S}_q for the set of primes p in the range $Z \leq p < P^\sigma$ that divide q . In particular, \mathcal{S}_0 is simply the set of primes p with $Z \leq p < P^\sigma$. We also write $\mathcal{S}'_q = \mathcal{S}_q \cup \{1\}$. Since $Z^2 > P^\sigma$, under hypothesis (iii), we have

$$g_j(\alpha) = \sum_{l \in \mathcal{S}'_q} g_{j,l}(\alpha) = g_{j,1}(\alpha) + \sum_{p \in \mathcal{S}_q} g_{j,p}(\alpha),$$

where for $\alpha \in \mathfrak{M}(q, a)$ and $l \in \mathcal{S}'_q$,

$$g_{j,l}(\alpha) = \sum_{\substack{m \in \mathcal{J} \\ (m,q)=l}} \rho_j(m)\theta(m, \alpha)e(\alpha m^2).$$

Similarly to (4.1) in [10], when $\alpha \in \mathfrak{M}(q, a)$ and $l \in \mathcal{S}'_q$, we have

$$g_{j,l}(\alpha) = \frac{1}{\phi(q_l)} \sum_{\chi \bmod q_l} S(\chi, al) \sum_{lm \in \mathcal{J}} \rho_j(lm)\chi(m)e(\beta l^2 m^2), \tag{3.4}$$

where $q_l = q/l$ and $\beta = \alpha - a/q$. If χ is a character and l a natural number, we now define

$$W_{j,l}(\chi, \beta) = \sum_{lm \in \mathcal{J}} (\rho_j(lm)\chi(m) - D_l(\chi)\delta_j(lm))e(\beta l^2 m^2),$$

where $D_l(\chi) = 1$ when $l = 1$ and χ is principal and $D_l(\chi) = 0$ otherwise. By (3.4) above and (4.1) in [10],

$$\Delta_1(\alpha) = f_1(\alpha) - f_1^*(\alpha) = \frac{1}{\phi(q)} \sum_{\chi \bmod q} S(\chi, a)W_{1,1}(\chi, \alpha - a/q), \tag{3.5}$$

$$\Delta_j(\alpha) = g_{j,1}(\alpha) - f_j^*(\alpha) = \frac{1}{\phi(q)} \sum_{\chi \bmod q} S(\chi, a)W_{j,1}(\chi, \alpha - a/q), \tag{3.6}$$

$$g_{j,p}(\alpha) = \frac{1}{\phi(q_p)} \sum_{\chi \bmod q_p} S(\chi, ap)W_{j,p}(\chi, \alpha - a/q). \tag{3.7}$$

Using (3.5)–(3.7), we can express the integral (3.3) as the linear combination of seventeen quantities of the form

$$\int_{\mathfrak{M}} \Delta_1^b(\alpha)\Delta_2^b(\alpha)\Delta_4^b(\alpha)e(-\alpha n) d\alpha,$$

with $\Delta_1^b(\alpha)$ one of $f_1^*(\alpha)$ or $\Delta_1(\alpha)$ and $\Delta_j^b(\alpha)$, $j = 2, 4$, one of

$$f_j^*(\alpha), \quad \Delta_j(\alpha) \quad \text{or} \quad \sum_{p \in \mathcal{S}_q} g_{j,p}(\alpha).$$

To be more precise, each of the eighteen possible combinations occurs with the exception of $f_1^*(\alpha)f_2^*(\alpha)f_3^*(\alpha)$ which we later show to give the main term.

We shall restrict our attention here to the two most troublesome combinations:

$$I_1 = \int_{\mathfrak{M}} \Delta_1(\alpha)\Delta_2(\alpha)\Delta_4(\alpha)e(-\alpha n) d\alpha, \tag{3.8}$$

$$I_2 = \sum_{p_1, p_2 \in \mathcal{S}_0} \int_{\mathfrak{M}_p} \Delta_1(\alpha)g_{2,p_1}(\alpha)g_{4,p_2}(\alpha)e(-\alpha n) d\alpha, \tag{3.9}$$

where \mathfrak{M}_p denotes the subset of \mathfrak{M} consisting of the major arcs $\mathfrak{M}(q, a)$, with q divisible by p_1 and p_2 . However, before we estimate I_1 and I_2 , we need to establish some lemmas.

3.1. Bounds for averages of $W_{j,l}(\chi, \beta)$

At this point, we need to make a hypothesis about the structure of the sieve weights ρ_j . Henceforth, we write

$$\psi(m, z) = \begin{cases} 1 & \text{if } p \mid m \Rightarrow p \geq z, \\ 0 & \text{otherwise.} \end{cases} \tag{3.10}$$

We also extend $\psi(m, z)$ to all real $m > 0$ by setting $\psi(m, z) = 0$ when m is not an integer. Our construction will yield coefficients ρ_j that are linear combinations of convolutions of the form

$$\sum_{r \sim R} \sum_{s \sim S} \xi_r \eta_s \psi(rs, z) \psi(m/rs, z), \tag{3.11}$$

where $|\xi_r| \leq \tau(r)^c$ and $|\eta_s| \leq \tau(s)^c$. In our applications the value of z will often depend on certain variables. To help set up the necessary hypotheses for our auxiliary results we therefore write $z(r, s)$ for a positive real-valued function, which in practice will either be fixed, or take the value p for some prime divisor of r or s ; see Section 5 for the specific cases of interest. We also put

$$Y = p^{1-5\sigma}, \quad V = p^{2\sigma}, \quad W = p^{1-4\sigma}. \tag{3.12}$$

We now require that ρ_j satisfies the following additional hypothesis:

(iv) ρ_j can be expressed as a linear combination of $O(L^c)$ convolutions of the form (3.11), where

$$1 \leq R \leq V, \quad 1 \leq S \leq W, \quad Z \leq z(r, s) \leq p^{8/35}. \tag{3.13}$$

For the remainder of Section 3.1, we suppress the index j and write $W_l(\chi, \beta)$ for $W_{j,l}(\chi, \beta)$, ρ for ρ_j , etc.

Lemma 1. Let α, β be reals with $0 < \alpha < \beta$, let n, g be positive integers, and let (A_q) be a sequence of positive reals such that

$$\sum_{\substack{q \sim Q \\ d|q}} A_q \leq B_1 + d^{-1} B_2.$$

Then

$$\sum_{q \sim Q} (n, [q, g])^\alpha [q, g]^{-\beta} A_q \ll (n, g)^\alpha g^{-\beta+\epsilon} (B_1 + Q^{-\beta'} B_2),$$

where $\beta' = \min(\beta - \alpha, 1)$. Furthermore, if $ghQ \geq n^\delta$ for some $\delta > 0$, then

$$\sum_{q \sim Q} (n, [q, g])^\alpha [q, g]^{-\beta} A_q \ll (n, g)^\alpha g^{-\beta+\epsilon} (B_1 + Q^{-\beta''} B_2),$$

where $\beta'' = \min(\beta, 1)$.

Proof. These inequalities can be established by a slight generalization of the arguments leading to (5.21) and (5.23) in [17]. In particular, see (5.20) and (5.22) in [17]. \square

Lemma 2. Suppose that ρ is a convolution of the form (3.11) and Φ is a complex-valued function defined on \mathcal{J} . Suppose also that the parameters R and S and the function $z(r, s)$ satisfy

$$\max(R, S) \leq P^{11/20}, \quad z(r, s) \min(R, S) \leq P^{11/20}, \quad z(r, s) \leq P^{8/35}. \tag{3.14}$$

Then the sum

$$\sum_{m \in \mathcal{J}} \rho(m) \Phi(m)$$

can be expressed as a linear combination of $O(L^\epsilon)$ sums of the form

$$\sum_{r \sim R_1} \sum_{s \sim S_1} \sum_{rsk \in \mathcal{J}} \xi_r^* \eta_s^* \zeta_k \Phi(rsk), \tag{3.15}$$

where $|\xi_r^*| \leq \tau(r)^\epsilon$, $|\eta_s^*| \leq \tau(s)^\epsilon$, $\max(R_1, S_1) \leq P^{11/20}$, and either $\zeta_k = 1$ for all k , or $|\zeta_k| \leq \tau(k)^\epsilon$ and $R_1 S_1 \geq P^{27/35}$.

Proof. This can be established similarly to Lemma 5.4 in [13], which contains (essentially) the case $\Phi(m) = \chi(m)e(\beta m^2)$. The second and third conditions in (3.14) can serve as a replacement for the hypothesis $z \leq P^{23/140}$ in [13]. \square

The above result covers ρ_2 and ρ_3 , while the following lemma covers additional sums that arise in ρ_4 .

Lemma 3. Let $W \leq R \leq P^{1/2}$. Then the sum

$$\sum_{m \in \mathcal{J}} \sum_{p \sim R} \psi(m/p, p) \tag{3.16}$$

can be expressed as a linear combination of $O(L^c)$ sums of the form (3.15) where the parameters satisfy the same conditions as in Lemma 2. The same conclusion is also reached for the sum

$$\sum_{m \in \mathcal{J}} \sum_{\substack{pr > W, qr > Y \\ Z < r < q < Y \\ p < V}} \psi(m/(pqr), r). \tag{3.17}$$

Proof. We begin with the sum (3.16) which clearly detects products of two primes since $p > (m/p)^{1/2}$ here. If $P^{9/20} \leq R \leq P^{1/2}$ the result is immediate with the variable k identically equal to 1. Otherwise, let $u = P^{9/40}R^{-1/2}$. We apply Heath–Brown’s generalized Vaughan Identity to the variable m/r as given by [9, Lemma 2.8] (note that $12Nu$ there should read $N^{1/2}u$). This gives Type II sums with one range of size $P^{9/20}/R$ to $P^{1/3}$, and Type I sums where the variable with an “unknown” weight has size $\leq (P/R)^{1/2}u$. These sums are quickly shown to have the required properties.

For the sum in (3.17) we immediately have a sum of the correct form when $r < P^{8/35}$ or $pqr < P^{11/20}$. For the remainder of the sum we note that $P^{8/35} > (P/pqr)^{1/2}$ and so $\psi(m/pqr, r)$ detects primes only. Again we can apply Heath–Brown’s identity and obtain the required result.

For either of the above sums we could have used the Alternative Sieve technique we employ later, but the appeal to Heath–Brown’s identity is quicker. \square

Lemma 4. Let $l \in \mathcal{S}_0 \cup \{1\}$ and $g, n, D \in \mathbb{N}$. Suppose that ρ is either a convolution of the form (3.11) that satisfies (3.13), or one of (3.16), (3.17). Suppose also that G, Δ are reals such that $\Delta l D G^{1+\epsilon} \leq P^{-31/20}$ and that $\mathcal{H}(D, G)$ is a set of characters $\chi = \xi \psi$, where ξ is a character modulo D and ψ a primitive character modulo q , with $q \leq G$ and $(q, D) = 1$. Then

$$\sum_{\chi \in \mathcal{H}(D, G)} w(q) \left(\int_{-\Delta}^{\Delta} |W_l(\chi, \beta)|^2 d\beta \right)^{1/2} \ll l^{-1} w(1) g^\epsilon L^c, \tag{3.18}$$

where $w(q) = (n, [q, g])^{1/2} [q, g]^{-1+\epsilon}$.

Proof. We can use Lemma 1 to deduce (3.18) from the inequality

$$\sum_{\substack{\chi \in \mathcal{H}(D, G) \\ d|q}} \left(\int_{-\Delta}^{\Delta} |W_l(\chi, \beta)|^2 d\beta \right)^{1/2} \ll l^{-1} (1 + d^{-1} l D G^2 \Delta P^{31/20}) L^c. \tag{3.19}$$

We first consider the case $l = 1$. If ρ is of the form (3.11), it follows from the hypotheses (3.13) that R, S and $z(r, s)$ satisfy the hypotheses (3.14) of Lemma 2. By Lemma 3 we obtain the required conclusions for the convolutions (3.16) and (3.17) too. Thus, we may assume that $W_1(\chi, \beta)$ is given by (3.15) with $\Phi(m) = \chi(m)e(\beta m^2)$ (with the appropriate adjustment when χ is principal). Then the

argument of Lemma 4.3 in [4] shows that the left side of (3.19) is bounded above by

$$\Delta P L T^{-1} \sum_{\substack{\chi \in \mathcal{H}(D, G) \\ d|q}} \int_{-T}^T |F(it, \chi)| dt + d^{-1} G^2 \Delta P^{1/2}, \tag{3.20}$$

where $\Delta P^2 \leq T \leq P^{10}$ and $F(it, \chi)$ is the Dirichlet polynomial

$$F(it, \chi) = \sum_{r \sim R_1} \sum_{s \sim S_1} \sum_{k \asymp P/(R_1 S_1)} \xi_r^* \eta_s^* \zeta_k \chi(rsk)(rsk)^{-it}.$$

We can apply Theorem 2.1 in [4] to the sum in (3.20) to obtain (3.19) with $l = 1$.

Suppose now that $l = p \in S_0$. We note that this case cannot occur for (3.16). The following argument is for the case where ρ is of the form (3.11), but it can easily be adapted for (3.17). We may assume that $z(r, s) \leq P^\sigma$, since $W_p(\chi, \beta)$ is otherwise an empty sum. The left side of (3.19) equals

$$p^{-1} \sum_{\substack{\chi \in \mathcal{H}(D, G) \\ d|q}} \left(\int_{-\Delta_p}^{\Delta_p} |W_p(\chi, \beta p^{-2})|^2 d\beta \right)^{1/2}, \quad \Delta_p = p^2 \Delta. \tag{3.21}$$

The sum $W_p(\chi, \beta p^{-2})$ splits into three subsums: a subsum where $r = pr'$; a subsum where $p \nmid r$ and $s = ps'$; and a subsum where $p \nmid rs$ and $k = pk'$. Each of these three subsums can be represented in the form $W_p^*(\chi, \beta)$, where

$$W_p^*(\chi, \beta) = \sum_{r \sim R'} \sum_{s \sim S'} \sum_{k \sim P/(prs)} \xi_r' \eta_s' \psi(rsk, z) \chi(rsk) e(\beta(rsk)^2), \tag{3.22}$$

with $R' \leq R, S' \leq S, |\xi_r'| \leq \tau(r)^c$ and $|\eta_s'| \leq \tau(s)^c$. We have

$$\begin{aligned} \max(R', S') &\leq W \leq (P/p)^{11/20} && \text{since } 1 - 4\sigma \leq \frac{11}{20}(1 - \sigma); \\ z \min(R', S') &\leq P^\sigma V \leq (P/p)^{11/20} && \text{since } 3\sigma \leq \frac{11}{20}(1 - \sigma); \\ z &\leq P^\sigma \leq (P/p)^{8/35} && \text{since } \sigma \leq \frac{8}{35}(1 - \sigma). \end{aligned}$$

We can therefore apply Lemma 2 to decompose $W_p^*(\chi, \beta)$ into sums of the form (3.15) with $\Phi(m) = \chi(m)e(\beta m^2)$ and Pp^{-1} in place of P . By Theorem 2.1 in [4], the quantity (3.21) with $W_p^*(\chi, \beta)$ in place of $W_p(\chi, \beta p^{-2})$ is bounded by

$$p^{-1} (1 + d^{-1} D G^2 \Delta P^{31/20} p^{9/20}) L^c,$$

whence (3.19) with $l = p$ follows. \square

3.2. Bounds for complete exponential sums

Given characters χ_1, χ_2, χ_3 modulo q and a vector $\mathbf{b} \in \mathbb{Z}^4$, we define

$$B(q, \mathbf{b}; \chi_1, \chi_2, \chi_3) = \frac{1}{\phi(q)^3} \sum_{\substack{1 \leq a \leq q \\ (a,q)=1}} S(\chi_1, ab_1)S(\chi_2, ab_2)S(\chi_3, ab_3)e_q(-ab_4). \tag{3.23}$$

It is not difficult to express $B(q, \mathbf{b}; \chi_1, \chi_2, \chi_3)$ as a linear combination of Gauss sums

$$\tau_a(\chi) = \sum_{1 \leq h \leq q} \chi(h)e_q(ah).$$

Indeed, by the orthogonality of the characters modulo q ,

$$S(\chi, a) = \sum_{\xi^2 = \bar{\chi}} \tau_a(\xi),$$

where the summation is over the characters ξ modulo q with $\xi^2 = \bar{\chi}$. Thus,

$$B(q, \mathbf{b}; \chi_1, \chi_2, \chi_3) = \frac{1}{\phi(q)^3} \sum_{\substack{\xi_1, \xi_2, \xi_3 \\ \xi_j^2 = \bar{\chi}_j}} \tau_{b_1}(\xi_1)\tau_{b_2}(\xi_2)\tau_{b_3}(\xi_3)\overline{\tau_{b_4}(\xi_1\xi_2\xi_3)}. \tag{3.24}$$

We also note that $B(q)$ is multiplicative as a function of q in the following sense: if $q = q_1q_2$, $(q_1, q_2) = 1$, and $\chi_j = \chi_{j,1}\chi_{j,2}$ with $\chi_{j,i}$ a character modulo q_i , then

$$B(q, \mathbf{b}; \chi_1, \chi_2, \chi_3) = B(q_1, \mathbf{b}; \chi_{1,1}, \chi_{2,1}, \chi_{3,1})B(q_2, \mathbf{b}; \chi_{1,2}, \chi_{2,2}, \chi_{3,2}). \tag{3.25}$$

The proofs of the above properties are similar to those of parts (a) and (d) of Lemma 2.5 in [3]. We now record upper bounds for $|B(q, \mathbf{b}; \chi_1, \chi_2, \chi_3)|$ for several special choices of \mathbf{b} .

Case 1. $\mathbf{b} = \mathbf{b}_1 = (1, 1, 1, n)$. By virtue of (3.25), it suffices to consider the case when $q = p^e$ for some prime p . We deal with the case of an odd prime p . The case $q = 2^e$, $e \geq 3$, can be dealt with in a similar fashion, and when $q = 2$ or 4 , we may use the trivial bound. Using the bound (see Lemma 3.1 in [17])

$$|\tau_a(\chi)| \leq (a, q)^{1/2}q^{1/2},$$

we deduce immediately from (3.24) that

$$B(p^e, \mathbf{b}_1; \chi_1, \chi_2, \chi_3) \leq 8(n, p^e)^{1/2}p^{3-e}\phi(p)^{-3}. \tag{3.26}$$

In the special case when χ_1, χ_2 and χ_3 are all principal, we can improve on this. We pause at this stage to write

$$s(q, n) = B(q, \mathbf{b}_1; \chi_0, \chi_0, \chi_0) = \sum_{\substack{1 \leq a \leq q \\ (a,q)=1}} \frac{S(\chi_0, a)^3}{\phi(q)^3} e_q(-an).$$

We also write, for future reference,

$$\mathfrak{S}_3(n, Q) = \sum_{q \leq Q} \mathfrak{s}(q, n).$$

We then have the following result.

Lemma 5. For all $q \geq 2$ and $n \geq 1$ we have the two estimates:

$$|\mathfrak{s}(q, n)| \leq \tau(q)^3 \frac{q^2}{\phi(q)^3}, \tag{3.27}$$

and

$$|\mathfrak{s}(q, n)| \leq \tau((q, n))^2 \frac{(\log \log q)^{10}}{q}. \tag{3.28}$$

Moreover, if $p^2 \mid q$ with $p > 2$, or if $16 \mid q$, we have $\mathfrak{s}(q, n) = 0$.

Proof. The cases $q = 2^j$ can be quickly checked. It then suffices to consider the case $q = p > 2$, since the factors $\tau_1(\xi_j)$ in (3.24) vanish when $q = p^e$, $e \geq 2$. When $q = p$ and χ_j is principal, each ξ_j is either principal or a Legendre symbol. If some ξ_j is principal, we have $|\tau_1(\xi_j)| = 1$. We also note that if ξ is non-principal, then

$$|\tau_n(\xi)| = \begin{cases} 0 & \text{if } p \mid n, \\ p^{1/2} & \text{otherwise.} \end{cases}$$

On the other hand, if ξ is principal, then

$$|\tau_n(\xi)| = \begin{cases} p - 1 & \text{if } p \mid n, \\ 1 & \text{otherwise.} \end{cases}$$

When $p \mid n$, we deduce that the modulus of the sum on the right-hand side of (3.24) is

$$\leq \begin{cases} p - 1 & \text{if each } \xi_j \text{ is principal,} \\ 3p(p - 1) & \text{if exactly one } \xi_j \text{ is principal,} \\ 0 & \text{otherwise.} \end{cases}$$

When $p \nmid n$, the modulus of the sum on the right-hand side of (3.24) is

$$\leq \begin{cases} 1 & \text{if each } \xi_j \text{ is principal,} \\ p^2 & \text{if no } \xi_j \text{ is principal,} \\ 3p & \text{otherwise.} \end{cases}$$

We thus have

$$|\mathfrak{s}(p, n)| \leq \begin{cases} p^2 \phi(p)^{-3} (1 + 7/p) & \text{if } p \nmid n, \\ 3p^2 \phi(p)^{-3} & \text{if } p \mid n. \end{cases}$$

The bounds (3.27) and (3.28) quickly follow. \square

We note, for future reference, that the main contribution in the case $p \nmid n$ can be explicitly calculated, namely

$$s(p, n) = \left(\frac{-n}{p}\right) \frac{p^2}{(p-1)^3} + \gamma(p, n), \tag{3.29}$$

where $|\gamma(p, n)| \leq 7p/(p-1)^3$.

Suppose now that χ_j has conductor q_j , and let $q_0 = [q_1, q_2, q_3]$. By (3.25)–(3.27),

$$B(q, \mathbf{b}_1; \chi_1, \chi_2, \chi_3) \ll (n, q_0)^{1/2} q^{-1} \tau(q)^c, \tag{3.30}$$

whence

$$\sum_{\substack{q \leq Q \\ q_0 | q}} B(q, \mathbf{b}_1; \chi_1 \chi_0, \chi_2 \chi_0, \chi_3 \chi_0) \ll (n, q_0)^{1/2} q_0^{-1+\epsilon} L^c. \tag{3.31}$$

Here χ_0 denotes the principal character modulo q .

Case 2. $\mathbf{b} = \mathbf{b}_D = (1, p_1^2, p_2^2, n)$, where p_1, p_2 are distinct odd primes and $D = p_1 p_2$. When $(q, D) = 1$, similarly to (3.30), we have

$$B(q, \mathbf{b}_D; \chi_1, \chi_2, \chi_3) \ll (n, q_0)^{1/2} q^{-1} \tau(q)^c,$$

where $q_0 = [q_1, q_2, q_3]$, q_j being the conductor of χ_j . When $q = p_1$, the factor $\tau_{p_1^2}(\xi_2)$ in (3.24) vanishes unless ξ_2 is principal, in which case that factor equals $\phi(p_1)$. Hence,

$$\begin{aligned} |B(p_1, \mathbf{b}_D; \chi_1, \chi_2, \chi_3)| &\leq \frac{1}{\phi(p_1)^2} \sum_{\substack{\xi_1, \xi_3 \\ \xi_j^2 = \bar{\chi}_j}} |\tau_1(\xi_1) \tau_{p_2^2}(\xi_3) \tau_n(\xi_1 \xi_3)| \\ &\leq 4(n, p_1)^{1/2} p_1^{3/2} \phi(p_1)^{-2}. \end{aligned}$$

Similarly,

$$|B(p_2, \mathbf{b}_D; \chi_1, \chi_2, \chi_3)| \leq 4(n, p_2)^{1/2} p_2^{3/2} \phi(p_2)^{-2}.$$

Now, let $q = p_1 p_2 r$, where $(r, p_1 p_2) = 1$, and suppose that χ_j has conductor q_j . We deduce that

$$B(q, \mathbf{b}_D; \chi_1, \chi_2, \chi_3) \ll \sqrt{D(n, D)(n, r_0)} q^{-1} \tau(q)^c, \tag{3.32}$$

where $r_0 = ([q_1, q_2, q_3], r)$.

Case 3. $\mathbf{b} = \mathbf{b}_p = (1, p^2, p^2, n)$. When $q = p^e$, $e \leq 2$, the factors $\tau_{p^2}(\xi_j)$, $j = 2, 3$, in (3.24) vanish unless ξ_j is principal. Hence,

$$|B(p^e, \mathbf{b}_p; \chi_1, \chi_2, \chi_3)| \leq \frac{1}{\phi(p^e)} \sum_{\xi_1^2 = \bar{\chi}_1} |\tau_1(\xi_1) \tau_n(\xi_1)| \leq 2p\phi(p)^{-1},$$

on noting that $|\tau_n(\xi_1)| \leq p^{e/2}$ when ξ_1 is non-principal and $|\tau_1(\xi_1)| \leq 1$ when ξ_1 is principal. Suppose now that $q = p^e r$, with $e \leq 2$ and $(r, p) = 1$, and that χ_j has conductor q_j . Then, similarly to (3.32), we have

$$B(q, \mathbf{b}_p; \chi_1, \chi_2, \chi_3) \ll p^e (n, r_0)^{1/2} q^{-1} \tau(q)^c, \tag{3.33}$$

where $r_0 = ([q_1, q_2, q_3], r)$. We also remark that when $e = 2$, the left side of (3.33) vanishes unless $p^2 \mid q_1$ and $(p, q_2 q_3) = 1$, in which case $r_0 = [q_1 p^{-2}, q_2, q_3]$.

3.3. Estimation of I_1

We can rewrite I_1 as the multiple sum

$$\sum_{q \leq Q} \sum_{\chi_1 \bmod q} \sum_{\chi_2 \bmod q} \sum_{\chi_3 \bmod q} B(q, \mathbf{b}_1; \chi_1, \chi_2, \chi_3) J(q, n; \chi_1, \chi_2, \chi_3), \tag{3.34}$$

where $B(q, \mathbf{b}_1; \chi_1, \chi_2, \chi_3)$ is defined by (3.23) with $\mathbf{b} = \mathbf{b}_1 = (1, 1, 1, n)$ and

$$J(q, n; \chi_1, \chi_2, \chi_3) = \int_{-\omega/q}^{\omega/q} W_1(\chi_1, \beta) W_{2,1}(\chi_2, \beta) W_{4,1}(\chi_3, \beta) e(-\beta n) d\beta.$$

We now pass to primitive characters in (3.34). In general, if $\chi \bmod q$, $q \leq Q$, is induced by a primitive character $\chi^* \bmod r$, $r \mid q$, we have

$$W_1(\chi, \beta) = W_1(\chi^*, \beta) \tag{3.35}$$

and

$$W_{j,1}(\chi, \beta) \ll |W_{j,1}(\chi^*, \beta)| + \sum_{\substack{p \in \mathcal{S}_q \\ p \nmid r}} |W_{j,p}(\chi^*, \beta)| + E(q, r), \tag{3.36}$$

where $E(q, r)$ denotes the number of integers $m \in \mathcal{J}$ with $(m, r) = 1$, $\psi(m, Z) = 1$ and $(m, q) \geq P^\sigma$. Since $Z^3 > Q$, if an integer m is counted in $E(q, r)$, then (m, q) is either a prime $p \geq P^\sigma$ or the product $p_1 p_2$ of two distinct primes $p_1, p_2 \geq Z$. Now, given a character χ modulo r , we define

$$W_0(\chi) = \max_{|\beta| \leq \omega/r} |W_1(\chi, \beta)|, \quad W_{j,l}(\chi) = \left(\int_{-\omega/lr}^{\omega/lr} |W_{j,l}(\chi, \beta)|^2 d\beta \right)^{1/2},$$

$$W_j(\chi) = W_{j,1}(\chi), \quad W_j^\sharp(\chi) = \sum_{p \in \mathcal{S}_0} W_{j,p}(\chi).$$

Let χ_j^* denote the primitive character modulo q_j , $q_j \mid q$, inducing χ_j . By (3.35) and (3.36),

$$J(q, n; \chi_1, \chi_2, \chi_3) \ll \sum_{1 \leq i \leq 9} J_i(q; \chi_1^*, \chi_2^*, \chi_3^*),$$

where each $J_i(q; \chi_1^*, \chi_2^*, \chi_3^*)$ is a product of the form $W_0(\chi_1^*)W_2^b(\chi_2^*)W_4^b(\chi_3^*)$, with $W_j^b(\chi)$ one of the following:

$$W_j(\chi), \quad W_j^\sharp(\chi), \quad (\omega/q)^{1/2}E(q, q_j).$$

Suppose first that $J_i(q; \chi_1^*, \chi_2^*, \chi_3^*)$ is one of the four products involving only $W_j(\chi)$ and $W_j^\sharp(\chi)$. We note that in this case $J_i(q; \chi_1^*, \chi_2^*, \chi_3^*)$ depends only on the characters and not on q . Thus, its contribution to the final bound for (3.34) is bounded above by

$$\sum_{q_1, \chi_1}^* \sum_{q_2, \chi_2}^* \sum_{q_3, \chi_3}^* J_i(\chi_1, \chi_2, \chi_3) B_1(\chi_1, \chi_2, \chi_3), \tag{3.37}$$

where \sum_{q_j, χ_j}^* denotes a summation over the primitive characters of moduli $q_j \leq Q$, and $B_1(\chi_1, \chi_2, \chi_3)$ is the sum in (3.31). Hence, by (3.31), the sum (3.37) is bounded by

$$L^c \sum_{q_1, \chi_1}^* \sum_{q_2, \chi_2}^* \sum_{q_3, \chi_3}^* (n, q_0)^{1/2} q_0^{-1+\epsilon} J_i(\chi_1, \chi_2, \chi_3), \tag{3.38}$$

where $q_0 = [q_1, q_2, q_3]$. The four such sums can be estimated in a similar fashion, so we present only the details of the estimation of

$$\sum_{q_1, \chi_1}^* \sum_{q_2, \chi_2}^* \sum_{q_3, \chi_3}^* (n, q_0)^{1/2} q_0^{-1+\epsilon} W_0(\chi_1) W_2^\sharp(\chi_2) W_4(\chi_3).$$

Since ρ_4 satisfies hypothesis (iv), Lemma 4 with $l = 1$ gives

$$\sum_{q_3, \chi_3}^* (n, q_0)^{1/2} q_0^{-1+\epsilon} W_4(\chi_3) \ll (n, \tilde{q}_0)^{1/2} \tilde{q}_0^{-1+2\epsilon} L^c,$$

where $\tilde{q}_0 = [q_1, q_2]$. Furthermore, since ρ_2 satisfies hypothesis (iv), Lemma 4 with $l = p$, $p \in \mathcal{S}_0$, gives

$$\sum_{q_2, \chi_2}^* (n, \tilde{q}_0)^{1/2} \tilde{q}_0^{-1+2\epsilon} W_2^\sharp(\chi_2) \ll (n, q_1)^{1/2} q_1^{-1+3\epsilon} L^c. \tag{3.39}$$

Finally, by Lemma 2.3 in [21],

$$\sum_{q_1, \chi_1}^* (q_1, n)^{1/2} q_1^{-1+3\epsilon} W_0(\chi) \ll PL^{-A} \tag{3.40}$$

for any fixed $A > 0$.

Next, we estimate the contribution to (3.34) from a product $J_i(q; \chi_1^*, \chi_2^*, \chi_3^*)$ where at least one of the factors $W_j^b(\chi)$ is of the form $(\omega/q)^{1/2}E(q, q_j)$. Let us consider, for example, the contribution from the product

$$W_0(\chi_1^*)W_2(\chi_2^*)(\omega/q)^{1/2}E(q, q_3).$$

By (3.30), this contribution does not exceed

$$\omega^{1/2} \sum_{q_1, \chi_1}^* \sum_{q_2, \chi_2}^* \sum_{q_3, \chi_3}^* W_0(\chi_1) W_2(\chi_2)(n, q_0)^{1/2} \sum_{\substack{q \leq Q \\ q_0 | q}} E(q, q_3) q^{-3/2+\epsilon}, \tag{3.41}$$

where $q_0 = [q_1, q_2, q_3]$. Let \mathcal{D} denote the set of integers $d \leq Q$ that are either a prime $p \geq P^\sigma$ or a product $p_1 p_2$ of two primes $p_1, p_2 \geq Z$. The innermost sum in (3.41) is bounded by

$$\sum_{d \in \mathcal{D}} \frac{P}{d} \sum_{\substack{q \leq Q \\ [q_0, d] | q}} q^{-3/2+\epsilon} \ll P \sum_{\substack{d \in \mathcal{D} \\ d q_3 \leq Q}} d^{-1} [q_0, d]^{-3/2+\epsilon} = \Sigma(q_3), \quad \text{say.}$$

Put $\tilde{q}_0 = [q_1, q_2]$. Summing this bound over q_3 , we find that

$$\begin{aligned} \sum_{q_3, \chi_3}^* (n, q_0)^{1/2} \Sigma(q_3) &\ll P \sum_{d \in \mathcal{D}} \frac{1}{d} \sum_{q_3 \leq Q/d} \frac{(n, q_0)^{1/2} q_3}{[q_0, d]^{3/2-\epsilon}} \\ &\ll Q^{1/2} P \sum_{d \in \mathcal{D}} \frac{1}{d^2} \sum_{q_3 \leq Q/d} \frac{(n, q_0)^{1/2}}{[q_0, d]^{1-\epsilon}} \\ &\ll Q^{1/2} P \sum_{d \in \mathcal{D}} \frac{(\tilde{q}_0, d)^{1-\epsilon}}{d^{3-\epsilon}} \sum_{q_3 \leq Q/d} (n, q_0)^{1/2} q_0^{-1+\epsilon} \\ &\ll Q^{1/2+\epsilon} P (n, \tilde{q}_0)^{1/2} \tilde{q}_0^{-1+2\epsilon} \sum_{d \in \mathcal{D}} \frac{(\tilde{q}_0, d)^{1-\epsilon}}{d^3} \\ &\ll Q^{1/2+\epsilon} P^{1-2\sigma} (n, \tilde{q}_0)^{1/2} \tilde{q}_0^{-1+2\epsilon}, \end{aligned} \tag{3.42}$$

where we have used that $q_3 [q_0, d]^{-1/2} \leq Q^{1/2} d^{-1}$ and $(q_0, d) = (\tilde{q}_0, d)$. Combining (3.42), the variant of (3.39) for $W_2(\chi_2)$, and (3.40), we conclude that the quantity (3.41) does not exceed

$$\omega^{1/2} p^{2-2\sigma+\epsilon} Q^{1/2} \ll Q P^{1-2\sigma+2\epsilon} \ll P^{1-\epsilon}.$$

This completes the estimation of I_1 .

3.4. Estimation of I_2

We first consider the part of I_2 where $p_1 \neq p_2$. For every such pair of primes $\mathbf{p} = (p_1, p_2)$, the integral over $\mathfrak{M}_{\mathbf{p}}$ equals

$$\sum_{\substack{q \leq Q \\ D | q}} \sum_{\chi_1 \bmod q} \sum_{\chi_2 \bmod q/p_1} \sum_{\chi_3 \bmod q/p_2} B(q, \mathbf{b}_D; \chi_1, \chi_2, \chi_3) J(q, n, D; \chi_1, \chi_2, \chi_3), \tag{3.43}$$

where $D = p_1 p_2$,

$$B(q, \mathbf{b}_D; \chi_1, \chi_2, \chi_3) = \frac{\phi(D)}{\phi(q)^3} \sum_{\substack{1 \leq a \leq q \\ (a,q)=1}} S(\chi_1, a)S(\chi_2, ap_1)S(\chi_3, ap_2)e_q(-an),$$

$$J(q, n, D; \chi_1, \chi_2, \chi_3) = \int_{-\omega/q}^{\omega/q} W_1(\chi_1, \beta)W_{2,p_1}(\chi_2, \beta)W_{4,p_2}(\chi_3, \beta)e(-\beta n) d\beta.$$

When $p \mid q$ and χ is a character modulo q_p , we have

$$S(\chi, ap) = \begin{cases} \phi(p)^{-1}S(\chi \chi_0, ap^2) & \text{if } p^2 \nmid q, \\ p^{-1}S(\chi \chi_0, ap^2) & \text{if } p^2 \mid q, \end{cases} \tag{3.44}$$

where $\chi \chi_0$ is the character modulo q induced by χ . Therefore, $B(q, \mathbf{b}_D; \chi_1, \chi_2, \chi_3)$ is, in fact, the sum (3.23) with $\mathbf{b} = \mathbf{b}_D = (1, p_1^2, p_2^2, n)$. Similarly to (3.36), we have

$$W_{j,p}(\chi, \beta) \ll |W_{j,p}(\chi^*, \beta)| + E(q, r) \tag{3.45}$$

whenever χ is character modulo q_p induced by a character χ^* modulo r . Using (3.35), (3.45) and (3.32), we reduce the estimation of (3.43) to the estimation of four sums of the form

$$\sqrt{D(n, D)} \sum_{q_1, \chi_1}^* \sum_{q_2, \chi_2}^* \sum_{q_3, \chi_3}^* (n, r_0)^{1/2} W_0(\chi_1) \sum_{\substack{q \leq Q \\ q_0 \mid q}} W_{2,p_1}^b(\chi_2)W_{4,p_2}^b(\chi_3)q^{-1+\epsilon}, \tag{3.46}$$

where $q_0 = [q_1, p_1q_2, p_2q_3]$, $r_0 = q_0/D$, $W_{2,p_1}^b(\chi_2)$ represents either $W_{2,p_1}(\chi_2)$ or $(\omega/q)^{1/2}E(q, q_2)$, and $W_{4,p_2}^b(\chi_3)$ is defined similarly to $W_{2,p_1}^b(\chi_2)$. The contribution from the sum involving the factor $W_{2,p_1}(\chi_2)W_{4,p_2}(\chi_3)$ is bounded by

$$Q^\epsilon \sqrt{\frac{(n, D)}{D}} \sum_{q_1, \chi_1}^* \sum_{q_2, \chi_2}^* \sum_{\substack{q_3, \chi_3 \\ q_0 \leq Q}}^* (n, r_0)^{1/2} r_0^{-1} W_0(\chi_1)W_{2,p_1}(\chi_2)W_{4,p_2}(\chi_3). \tag{3.47}$$

The condition $[q_1, p_1q_2, p_2q_3] \leq Q$ implies that each character χ_3 is either primitive with a modulus $r_3 \leq QD^{-1}$, or the product of such a character and a (primitive) character modulo p_1 . Thus, the sum over q_3 and χ_3 splits into two sums of the form appearing in Lemma 4: one over $\chi_3 \in \mathcal{H}(1, QD^{-1})$ and one over $\chi_3 \in \mathcal{H}(p_1, QD^{-1})$. Similarly, the sum over q_2 and χ_2 splits into sums over $\chi_2 \in \mathcal{H}(1, QD^{-1})$ and $\chi_2 \in \mathcal{H}(p_2, QD^{-1})$, and the sum over q_1 and χ_1 splits into four sums over the sets $\mathcal{H}(d, QD^{-1})$, $d \mid D$. Observe that $r_0 = [r_1, r_2, r_3]$, where the r_j 's are the moduli of the primitive characters with moduli $\leq QD^{-1}$. Hence, Lemma 4 with $l = p_1$, $D = l \in \{1, p_1\}$ and $\Delta = \omega/(p_2r_3l)$ gives

$$\sum_{\chi_3 \in \mathcal{H}(l, QD^{-1})} (n, r_0)^{1/2} r_0^{-1} W_{4,p_2}(\chi_3) \ll p_2^{-1}(n, [r_1, r_2])^{1/2} [r_1, r_2]^{-1+\epsilon} L^c.$$

Another application of Lemma 4 to the sum over χ_2 and an application of a variant of Lemma 4.1 in [4] (a combination of that lemma with Lemma 1 above) to the sum over χ_1 show that the sum (3.47) is $\ll (n, D)^{1/2} D^{-3/2} Q P^{1+\epsilon}$.

As to the estimation of the remaining three sums of the form (3.46), we note that the condition $q_0 \mid q$ implies that $E(q, q_j) < PD^{-1}$ and $q_j \leq QD^{-1}$. Hence, the contribution from the product involving $E(q, q_2)E(q, q_3)$, for example, is bounded above by

$$\begin{aligned} & \omega P^2 \sqrt{\frac{(n, D)}{D^3}} \sum_{q_1, \chi_1}^* W_0(\chi_1) \sum_{q_2, q_3 \leq Q D^{-1}} q_2 q_3 (n, r_0)^{1/2} q_0^{-2+\epsilon} \\ & \ll \omega Q P^{2+\epsilon} \sqrt{\frac{(n, D)}{D^9}} \sum_{l|D} \sum_{\chi \in \mathcal{H}(l, Q D^{-1})} (n, r)^{1/2} r^{-1+3\epsilon} W_0(\chi). \end{aligned}$$

Another appeal to the variant of Lemma 4.1 in [4] used above shows that the last expression is

$$\ll \omega Q P^{3+2\epsilon} \sqrt{\frac{(n, D)}{D^9}} \ll (n, D)^{1/2} D^{-3/2} Q^2 P^{1+3\epsilon} Z^{-6}.$$

Therefore, the total contribution to I_2 from pairs (p_1, p_2) of distinct primes is

$$\ll P^{1+3\epsilon} \sum_{p_1, p_2 \in S_0} (n, p_1 p_2)^{1/2} (p_1 p_2)^{-3/2} \ll P^{1+3\epsilon} Z^{-1}.$$

Finally, let $p \in S_0$ and $\mathbf{p} = (p, p)$. Then the integral over $\mathfrak{M}_{\mathbf{p}}$ appearing in I_2 can be expressed as

$$\sum_{\substack{q \leq Q \\ p|q}} \sum_{\chi_1 \bmod q} \sum_{\chi_2 \bmod q_p} \sum_{\chi_3 \bmod q_p} B(q, \mathbf{b}_p; \chi_1, \chi_2, \chi_3) J(q, n, p; \chi_1, \chi_2, \chi_3), \tag{3.48}$$

where $q_p = qp^{-1}$,

$$\begin{aligned} B(q, \mathbf{b}_p; \chi_1, \chi_2, \chi_3) &= \frac{1}{\phi(q)\phi(q_p)^2} \sum_{\substack{1 \leq a \leq q \\ (a, q) = 1}} S(\chi_1, a) S(\chi_2, ap) S(\chi_3, ap) e_q(-an), \\ J(q, n, p; \chi_1, \chi_2, \chi_3) &= \int_{-\omega/q}^{\omega/q} W_1(\chi_1, \beta) W_{2,p}(\chi_2, \beta) W_{4,p}(\chi_3, \beta) e(-\beta n) d\beta. \end{aligned}$$

By (3.44), $B(q, \mathbf{b}_p; \chi_1, \chi_2, \chi_3)$ is the sum (3.23) with $\mathbf{b} = \mathbf{b}_p = (1, p^2, p^2, n)$. Hence, by (3.35), (3.45) and (3.33), the contribution to (3.48) from moduli q divisible by p but not by p^2 does not exceed the linear combination of four sums of the form

$$p \sum_{q_1, \chi_1}^* \sum_{q_2, \chi_2}^* \sum_{q_3, \chi_3}^* (n, r_0)^{1/2} W_0(\chi_1) \sum_{\substack{q \leq Q \\ q_0|q, p^2 \nmid q}} W_{2,p}^b(\chi_2) W_{4,p}^b(\chi_3) q^{-1+\epsilon}, \tag{3.49}$$

where $q_0 = [q_1, pq_2, pq_3]$, $r_0 = q_0 p^{-1}$, and $W_{j,p}^b(\chi)$ has the same meaning as in (3.46). The sum (3.49) involving the product $W_{2,p}(\chi_2) W_{4,p}(\chi_3)$ is bounded by

$$Q^\epsilon \sum_{q_1, \chi_1}^* \sum_{q_2, \chi_2}^* \sum_{\substack{q_3, \chi_3 \\ q_0 \leq Q, p^2 \nmid q_0}}^* (n, r_0)^{1/2} r_0^{-1} W_0(\chi_1) W_{2,p}(\chi_2) W_{4,p}(\chi_3).$$

The conditions $q_0 \leq Q$ and $p^2 \nmid q$ imply that $q_2, q_3 \leq Q p^{-1}$ and that $q_1 = r_1$ or $q_1 = pr_1$, where $(p, r_1) = 1$. Furthermore, $r_0 = [r_1, q_2, q_3]$. Thus, we can again use Lemma 4 and a variant of Lemma 4.1

in [4] to show that the last sum is $\ll p^{-2}P^{1+\epsilon}$. The sums (3.49) involving factors $(\omega/q)E(q, q_j)$ satisfy the same bound. For example, one of those does not exceed

$$\begin{aligned} &\omega^{1/2}P \sum_{q_1, \chi_1}^* \sum_{q_2, \chi_2}^* W_0(\chi_1)W_{2,p}(\chi_2) \sum_{p_1 \geq Z} \frac{1}{p_1} \sum_{q_3 \leq Q/(pp_1)} (n, r_0)^{1/2}q_3 \sum_{\substack{q \leq Q \\ s|q}} q^{-3/2+\epsilon} \\ &\ll \omega^{1/2}P \sum_{q_1, \chi_1}^* \sum_{\substack{q_2, \chi_2 \\ [q_1, pq_2] \leq Q}}^* W_0(\chi_1)W_{2,p}(\chi_2) \sum_{p_1 \geq Z} \frac{Q^{1/2}}{pp_1^2} \sum_{q_3 \leq Q/(pp_1)} (n, r_0)^{1/2}s^{-1+\epsilon}, \end{aligned}$$

where $s = [q_1, pq_2, pp_1q_3]$. Another application of Lemma 4 and of the same variant of Lemma 4.1 in [4] as before show that the last sum is $\ll p^{-2}P^{1+\epsilon}QZ^{-3}$. Therefore, the total contribution to (3.48) from moduli not divisible by p^2 is $\ll p^{-2}P^{1+\epsilon}$.

Finally, we consider the contribution to (3.48) from moduli q divisible by p^2 . For such moduli, the term $E(q, r)$ in (3.45) is superfluous. Thus, by (3.33) and the remark following it, this contribution is bounded by

$$p^2Q^\epsilon \sum_{q_1, \chi_1}^* \sum_{q_2, \chi_2}^* \sum_{\substack{q_3, \chi_3 \\ q_0 \leq Q}}^* (n, r_0)^{1/2}q_0^{-1+\epsilon} W_0(\chi_1)W_{2,p}(\chi_2)W_{4,p}(\chi_3),$$

where $q_0 = [q_1, pq_2, pq_3]$, $r_0 = q_0p^{-2}$ and the moduli q_1, q_2, q_3 satisfy the conditions $p^2 \mid q_1, (p, q_2q_3) = 1$. We note that these conditions imply that $q_2, q_3 \leq Qp^{-2}$ and that $\chi_1 \in \mathcal{H}(p^2, Qp^{-2})$. Thus, once again, we can use Lemma 4 and a variant of Lemma 4.1 in [4] to show that the last sum does not exceed $\ll p^{-2}P^{1+\epsilon}$. Therefore, the total contribution to I_2 from pairs (p, p) , with $p \in \mathcal{S}_0$ is

$$\ll P^{1+\epsilon} \sum_{p \in \mathcal{S}_0} p^{-2} \ll P^{1+\epsilon} Z^{-1}.$$

We have thus shown that the integral in (3.3) is $O(PL^{-A})$. We then have

$$\int_{\mathfrak{M}} f_1^*(\alpha)f_2^*(\alpha)f_4^*(\alpha)e(-\alpha n) d\alpha = \sum_{q \leq Q} s(q, n)I(q, n),$$

where

$$I(q, n) = \int_{-\omega/q}^{\omega/q} f_1^*(\beta)f_2^*(\beta)f_4^*(\beta)e(-\beta n) d\beta.$$

We can then use standard major arc techniques to show that, for $|\beta| < \omega/q$, we have

$$\begin{aligned} f_j^*(\beta) &= \sum_{\sqrt{u} \in \mathcal{J}} \frac{\delta_j(\sqrt{u})}{2\sqrt{u}} e(\beta u) + O(P\omega/q) \\ &\ll \frac{P}{p^2|\beta| + 1} + O(P\omega/q). \end{aligned}$$

The error arising from any terms involving $O(P\omega/q)$ will be smaller than the other errors which arise. We then complete the integral over $[-\omega/q, \omega/q]$ to an integral over $[-1/2, 1/2]$ incurring an error bounded by a constant times

$$\sum_{q \leq Q} |\mathfrak{s}(q, n)| \frac{1}{P^3} \frac{q^2}{\omega^2} \ll \sum_{q \leq Q} \tau(q)^3 \frac{q^2}{\phi(q)^3} q^2 Q^{-2} P^{1-2\epsilon} \ll P^{1-\epsilon}$$

using (3.27). This shows the main term to be

$$\sum_{q \leq Q} \mathfrak{s}(q, n) \sum_{\substack{\sqrt{m_j} \in \mathcal{J} \\ m_1+m_2+m_3=n}} \frac{\delta_1(\sqrt{m_1})\delta_2(\sqrt{m_2})\delta_4(\sqrt{m_3})}{8\sqrt{m_1 m_2 m_3}}.$$

Clearly we can write the main term as

$$\mathfrak{S}_3(n, Q) C_2 C_4 P L^{-3} K_n (1 + o(1)),$$

where $1 \ll K_n \ll 1$ with absolute constants. As indicated earlier, similar but simpler working leads to an analogous result for

$$\int_{\mathfrak{M}} f_1(\alpha)^2 g_3(\alpha) e(-\alpha n) d\alpha,$$

with a main term $\mathfrak{S}_3(n, Q) C_3 P L^{-3} K_n (1 + o(1))$. Thus we obtain

$$\int_{\mathfrak{M}} (f_1^2(\alpha) g_3(\alpha) - f_1(\alpha) g_2(\alpha) g_4(\alpha)) d\alpha = \mathfrak{S}_3(n, Q) (C_3 - C_2 C_4) K_n P L^{-3} (1 + o(1)) + O(P L^{-A}).$$

3.5. The singular series

Our goal in this section is to prove the following result.

Lemma 6. Write

$$G(\alpha) = f_1(\alpha) (f_1(\alpha) g_3(\alpha) - g_2(\alpha) g_4(\alpha)).$$

Then, for all but $O(N^{1-\sigma+\epsilon})$ integers $n \in \mathcal{B}$, we have

$$\int_{\mathfrak{M}} G(\alpha) e(-\alpha n) d\alpha \gg (C_3 - C_2 C_4) P L^{-6}. \tag{3.50}$$

Write

$$\Pi(n, Q) = \begin{cases} 8 \prod_{3 \leq p \leq Q} (1 + \mathfrak{s}(p, n)) & \text{if } n \in \mathcal{A}_3, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 6 will thus follow from our previous work once we demonstrate the following.

Lemma 7. For all but $O(N^{1+\epsilon} Q^{-1})$ integers in \mathcal{B} we have

$$\mathfrak{S}(n, Q) = \Pi(n, Q) + O(\exp(-(\log L)^{1+\epsilon})). \tag{3.51}$$

Remark. The reader will note in the proof that the value Q can be taken as large as $N^{1/5}$ in this part of argument.

Proof of Lemma 7. In the following we can assume that whenever the variable q appears it has no square odd factor exceeding 1 and is not divisible by 16. We write

$$\Psi(r, z) = \begin{cases} 1 & \text{if } p \mid r \Rightarrow p \leq z, \\ 0 & \text{otherwise.} \end{cases}$$

Let R be a parameter exceeding Q to be determined later. We begin by writing

$$\begin{aligned} \Pi(n, Q) - \mathfrak{S}(n, Q) &= \sum_{q>Q} s(q, n) \Psi(q, Q) \\ &= \Sigma_1(n) + \Sigma_2(n), \end{aligned}$$

where $Q < q \leq R$ in $\Sigma_1(n)$ and $q > R$ in $\Sigma_2(n)$. We now use (3.28) to obtain

$$\Sigma_2(n) \ll \sum_{d \mid n} \mu^2(d) \tau^2(d) \sum_{\substack{q>R \\ d \mid q}} \Psi(q, Q) \frac{(\log \log q)^{10}}{q}.$$

From our restriction on q we note that $\Psi(q, Q)$ vanishes when $q \geq \exp(2Q)$. Hence

$$\Sigma_2(n) \ll L^{10} \sum_{d \mid n} \frac{\tau^2(d)}{d} \sum_{q>R/d} \frac{\Psi(q, Q)}{q}.$$

We now choose R to satisfy

$$\log R = L(\log L)^{1+2\epsilon}.$$

Then, using standard bounds on the number of integers up to 2^j having all their prime factors $\leq Q$ from [11], we have

$$\begin{aligned} \Sigma_2(n) &\ll L^{14} \sum_{q>R/n} \frac{\Psi(q, Q)}{q} \ll L^{14} \sum_{2^j>R/n} 2^{-j} \sum_{q \leq 2^j} \Psi(q, Q) \\ &\ll L^{14} \sum_{2^j>R/n} \exp\left(-\frac{\log 2^j}{\log Q}\right) \ll \exp(-(\log L)^{1+\epsilon}). \end{aligned}$$

We thus have

$$\Pi(n, Q) - \mathfrak{S}(n, Q) = \Sigma_1(n) + O(\exp(-(\log L)^{1+\epsilon})).$$

Recalling (3.29) we define $\gamma(p, n)$ for $p > 2$ by

$$\gamma(p, n) = \mathfrak{s}(p, n) - \left(\frac{-n}{p}\right) \frac{p^2}{(p-1)^3},$$

and extend this definition to obtain a multiplicative function $\gamma(q, n)$ defined on odd square-free q . We have

$$\sum_{Q < q \leq R} \Psi(q, Q) \mathfrak{s}(q, n) \ll \left| \sum_{Q' < q \leq R'} \theta_q \mathfrak{s}(q, n) \right|,$$

where $\theta_q \in \{0, 1\}$ and $\theta_q = 0$ unless q is odd and square-free. Also $2^j(Q', R') = (Q, R)$ for some $j \in \{0, 1, 2, 3\}$. For the values of q of relevance we have

$$\mathfrak{s}(q, n) = \sum_{d|q} \frac{d^2}{\phi(d)^3} \left(\frac{-n}{d}\right) \gamma(q/d, n).$$

It then suffices to estimate

$$\Sigma_3(n) = \sum_{Q < dq \leq R} \theta_q \theta_d \frac{q^2}{\phi(q)^3} \left(\frac{-n}{q}\right) \gamma(d, n).$$

We write $\Sigma_4(n)$ to be the subsum of this expression with $d > N^\epsilon$. Then

$$\Sigma_4(n) \ll N^{-\epsilon} \sum_{dq} d \theta_q \theta_d \frac{q^2}{\phi(q)^3} \left(\frac{-n}{d}\right) |\gamma(d, n)|.$$

It then follows that

$$\Sigma_4(n) \ll N^{-\epsilon} \prod_{3 \leq p \leq Q} \left(1 + \frac{p^2}{(p-1)^3}\right) \prod_{3 \leq p \leq Q} (1 + p |\gamma(p, n)|).$$

From our earlier work we know that

$$|\gamma(p, n)| \leq \begin{cases} 3p^2(p-1)^{-3} & \text{if } p | n, \\ 7p^{-2} & \text{if } p \nmid n. \end{cases}$$

Hence,

$$\begin{aligned} \Sigma_4(n) &\ll N^{-\epsilon} \tau(n)^2 \prod_{p \leq Q} \left(1 + \frac{1}{p}\right) \prod_{p \leq Q} \left(1 + \frac{7}{p}\right) \\ &\ll N^{-\epsilon} \tau(n)^2 L^8 \ll N^{-\epsilon/2}. \end{aligned}$$

It therefore remains to bound

$$\Sigma_5(n) = \sum_{d \leq N^\epsilon} |B(d, n)| \left| \sum_{Q < qd \leq R} \theta_q \frac{q^2}{\phi(q)^3} \left(\frac{-n}{d}\right) \right|.$$

Let $\Sigma_5(n, G)$ denote the part of the inner sum with $G < q \leq 2G$. So

$$\Sigma_5(n) \ll L \sum_{d \leq N^\epsilon} \frac{\tau((d, n))^2}{d} \sum_G |\Sigma_5(G, n)|.$$

Write

$$\mathcal{E}(G) = \{n \in \mathcal{A}_3 : n \leq N, |\Sigma_5(G, n)| \geq N^{-\epsilon/3}\}.$$

We will obtain two different bounds for the cardinality of this set: one to cover the range $G \leq N/Q$, the other for the remaining values of G .

First consider the sum

$$\begin{aligned} \sum_{n \leq N} |\Sigma_5(G, n)|^2 &= \sum_{n \leq N} \left| \sum_{q \sim G} \theta_q \frac{q^2}{\phi(q)^3} \left(\frac{-n}{q}\right) \right|^2 \\ &= \sum_{q_1 \sim G} \theta_{q_1} \theta_{q_2} \frac{(q_1 q_2)^2}{\phi(q_1)^3 \phi(q_2)^3} \sum_{n \leq N} \left(\frac{-n}{q_1 q_2}\right). \end{aligned}$$

The terms with $q_1 = q_2$ can only be estimated trivially. They give a contribution

$$\leq \sum_{q \sim G} \frac{q^4}{\phi(q)^6} N \ll \frac{N}{G} L.$$

On the other hand, if $q_1 \neq q_2$ then $(\frac{-n}{q_1 q_2})$ is a non-principal character modulo $q_1 q_2$. Hence, these terms contribute

$$\ll \sum_{q_j \sim G} (q_1 q_2)^{\epsilon-1/2} \ll G^{1+\epsilon}.$$

We thus have

$$\sum_{n \leq N} |\Sigma_5(G, n)|^2 \ll (N/Q + G) N^\epsilon.$$

It follows that

$$|\mathcal{E}(G)| \ll N^{1+\epsilon}/Q$$

if $Q \leq G \leq N/Q$.

Now we use the method of [14] to estimate

$$\sum_{n \leq N} |\Sigma_5(G, n)|.$$

In the following the parameter m will satisfy $2 \leq m \ll (\log L)^c$. From [14, Lemma 6.5] we have

$$\begin{aligned} \sum_{n \leq N} |\Sigma_5(G, n)| &\ll (N + G^{\frac{1}{m}} N^{\frac{1}{2}}) (m \log N + 1)^{\frac{m^2-1}{2m}} \left(\sum_{q \sim G} \left(\frac{q^2}{\phi(q)^3} \right)^{\frac{2m}{2m-1}} \right)^{\frac{2m-1}{2m}} \\ &\ll (N + G^{\frac{1}{m}} N^{\frac{1}{2}}) L^m (\log \log G)^3 \left(\sum_{q \sim G} q^{-\frac{2m}{2m-1}} \right)^{\frac{2m-1}{2m}} \\ &\ll (NG^{-\frac{1}{2m}} + G^{\frac{1}{2m}} N^{\frac{1}{2}}) L^{m+1} \ll (NG^{-\frac{1}{2m}} + G^{\frac{1}{2m}} N^{\frac{1}{2}}) N^\epsilon. \end{aligned}$$

Hence, we obtain

$$|\mathcal{E}(G)| \ll N^{1+\delta} G^{-\frac{1}{2m}} + G^{\frac{1}{2m}} N^{\frac{1}{2}+\epsilon} \ll N^{1+\epsilon} / Q,$$

provided that $Q^{2m} \leq G \leq N^m Q^{-2m}$. Thus, so long as $Q \leq N^{1/5}$, this covers the whole range from Q^2 to R .

We can now combine our two bounds to obtain

$$|\mathcal{E}(G)| \ll N^{1+\epsilon} Q^{-1}$$

for each of the $O(L^2)$ choices for G in the range $Q \leq G \leq R$. This gives

$$\sum_G |\Sigma_5(G, n)| \ll L^2 N^{-\epsilon/3}$$

for all but $O(N^{1+\delta} Q^{-1})$ integers $n \in \mathcal{B}$. It follows that

$$\Sigma_1(n) \ll N^{-\epsilon/4}$$

for all but $O(N^{1+\epsilon} Q^{-1})$ integers in $n \in \mathcal{B}$, which completes the proof. \square

4. Minor arc estimates for $f_j(\alpha)$

One limit to the exponent saving that we can make in our theorems comes from the best bounds we can obtain for one of the $f_j(\alpha)$ on the minor arcs. The best result at the moment for $f_1(\alpha)$ would lead to only $\sigma = 1/8$ (see [5,13]). The combination of a sieve method and bilinear exponential sum estimates in [10] led to $\sigma = 1/7$. We now describe how to sharpen the method.

In [10], our estimates for bilinear exponential sums were based partially on a general result of the second author: Lemma 5.6 in [13]. However, that result is no longer sufficient when $\sigma \geq 1/7$. Our first lemma is a variant of Lemma 5.6 in [13] that can be applied in the present context. The reader will recognize the major arc arguments from the previous section and note that the complications from primes in \mathcal{S}_q continue to be a nuisance.

Lemma 8. *Suppose that α is real and that a, q are integers with*

$$|q\alpha - a| \leq Q^2 P^{-2}, \quad 1 \leq a \leq q \leq Q^2, \quad (a, q) = 1,$$

and let ξ_r and η_s be complex numbers, with $|\xi_r| \leq 1$ and $|\eta_s| \leq 1$. Suppose also that R, S are reals, which together with $z = z(r, s)$ satisfy

$$1 \leq R \leq V, \quad 1 \leq S \leq W, \quad Z \leq z(r, s) \leq P^{8/35}.$$

Then the exponential sum

$$g(\alpha) = \sum_{r \sim R} \sum_{s \sim S} \sum_{x \sim P/(rs)} \xi_r \eta_s \psi(rsx, z) e(\alpha(rsx)^2)$$

satisfies the inequality

$$g(\alpha) \ll P^{1+\epsilon} (q + P^2|q\alpha - a|)^{-1/2} + P^{1-\sigma+\epsilon}.$$

Proof. We will initially treat $g(\alpha)$ like the sum $g_j(\alpha)$ studied in the previous section. We write

$$g(\alpha) = h_1(\alpha) + \sum_{p \in S_q} h_p(\alpha) + h^*(\alpha), \tag{4.1}$$

where $h_l(\alpha)$ denotes the subsum of $g(\alpha)$ with $(rsx, q) = l$, and $h^*(\alpha)$ denotes the subsum of $g(\alpha)$ where $(rsx, q) \geq P^\sigma$. We note that $h_d(\alpha)$ is a subsum of $h^*(\alpha)$ unless d is a prime p in the range $Z \leq p < P^\sigma$. Of course, if terms involving p arise we must have $q \geq p \geq Z$. The reader will note that for $q < z(r, s)$ some of the more awkward terms in the following do not occur. We estimate $h^*(\alpha)$ trivially:

$$h^*(\alpha) \ll \sum_{\substack{d|q \\ d \geq P^\sigma}} \sum_{\substack{k \sim P \\ d|k}} k^{\epsilon/2} \ll P^{1+\epsilon/2} \sum_{\substack{d|q \\ d \geq P^\sigma}} d^{-1} \ll P^{1-\sigma+\epsilon}. \tag{4.2}$$

Now, let $l = 1$ or $l = p \in S_q$. We write $q = lq_0$ and $\beta = \alpha - a/q$. We can then use the arguments of the previous section to write $h_l(\alpha)$ in the form

$$\frac{1}{\phi(q_0)} \sum_{\chi \bmod q_0} S(\chi, al) \sum_{r \sim R} \sum_{s \sim S} \sum_{\substack{x \sim P/(rs) \\ l|rsx}} \xi_r \eta_s \psi(rsx, z) \chi(rsx/l) e(\beta(rsx)^2).$$

We can then estimate this as the sum of at most three terms of the form

$$\frac{1}{\phi(q_0)} \sum_{\chi \bmod q_0} |S(\chi, al)| h_l(\beta, \chi)$$

with

$$h_l(\beta, \chi) = |W_l^*(\chi, \beta l^2)|,$$

the sum defined by (3.22) with $R' \leq R, S' \leq S$ as before (and so these parameters satisfy the same inequalities as given in Lemma 3). When $l = 1$, we deduce that

$$h_1(\alpha) \ll q^{-1/2+\epsilon} \sum_{\chi \bmod q} h_1(\beta, \chi). \tag{4.3}$$

For $l = p$ we must estimate sums of the form

$$\frac{1}{\phi(q_0)} \sum_{\chi \bmod q_0} |S(\chi, ap)| h_p(\beta, \chi). \tag{4.4}$$

Let $q = p^e q_1$, where $e \geq 1$ and $(p, q_1) = 1$. We also write

$$q_2 = \max(qp^{-e}, qp^{-2}) = \begin{cases} q_1 & \text{if } e = 1, \\ qp^{-2} & \text{if } e \geq 2. \end{cases}$$

We consider two cases.

Case 1. $e = 1$. Then $(ap, q_0) = 1$ and $q_0 = q_2$, and we have (similarly to the case $l = 1$)

$$h_p(\alpha) \ll q_2^{-1/2+\epsilon} \sum_{\chi \bmod q_2} h_p(\beta, \chi). \tag{4.5}$$

Case 2. $e \geq 2$. Then the exponential sum $S(\chi, ap)$ vanishes when the conductor of χ is divisible by p^{e-1} . Otherwise, we have $|S(\chi, ap)| \ll pq_2^{1/2+\epsilon}$. Hence,

$$h_p(\alpha) \ll q_2^{-1/2+\epsilon} \sum_{\chi \bmod q_2} h_p(\beta, \chi \chi_0), \tag{4.6}$$

where χ_0 is the principal character modulo p . When $e \geq 3$, we have $\chi \chi_0 = \chi$, so (4.6) turns into (4.5). When $e = 2$, we deduce from (4.6) that

$$h_p(\alpha) \ll q_2^{-1/2+\epsilon} \sum_{\chi \bmod q_2} h_p(\beta, \chi) + \Delta,$$

where

$$\Delta = q_2^{-1/2+\epsilon} p^{\epsilon/4} \sum_{\chi \bmod q_2} \left| \sum_{n \sim P/p^2} \theta_n \chi(n) \right|,$$

with coefficients θ_n satisfying $|\theta_n| \leq 1$. By Cauchy’s inequality and the orthogonality of the characters modulo q_2 , we obtain

$$\begin{aligned} \Delta^2 &\ll p^{2\epsilon} \sum_{\chi \bmod q_2} \left| \sum_{n \sim P/p^2} \theta_n \chi(n) \right|^2 \ll p^{2\epsilon} \sum_{\substack{m, n \sim Pp^{-2} \\ m \equiv n \pmod{q_2}}} \phi(q_2) \\ &\ll p^{2+2\epsilon} p^{-4} (1 + pq_2 P^{-1}) \ll p^{2+2\epsilon} p^{-4} \ll p^{2-2\sigma+\epsilon}. \end{aligned}$$

Hence,

$$h_p(\alpha) \ll q_2^{-1/2+\epsilon} \sum_{\chi \bmod q_2} h_p(\beta, \chi) + p^{1-\sigma+\epsilon}. \tag{4.7}$$

Combining (4.1), (4.2), (4.3), (4.5) and (4.7), we get

$$g(\alpha) \ll q^{-1/2+\epsilon} \sum_{\chi \bmod q} h_1(\beta, \chi) + q_2^{-1/2+\epsilon} \sum_{\chi \bmod q_2} h_p(\beta, \chi) + P^{1-\sigma+\epsilon}. \tag{4.8}$$

By the argument of Lemma 5.1 in [13],

$$\sum_{\chi \bmod q} h_1(\beta, \chi) \ll \sum'_{\chi \bmod q} h_1(\beta, \chi) + h_1(\beta, \chi^0) + \sum_{\substack{rsx \sim P \\ (rsx, q) > 1}} \psi(rsx, z), \tag{4.9}$$

where χ^0 is the trivial character and \sum' denotes summation over the non-principal characters modulo q . Since the last sum on the right side of (4.9) vanishes when $q < Z$, we have

$$\sum_{\substack{rsx \sim P \\ (rsx, q) > 1}} \psi(rsx, z) \ll \sum_{\substack{d|q \\ d \geq Z}} \sum_{\substack{k \sim P \\ d|k}} k^{\epsilon/4} \ll P^{1+\epsilon/3} Z^{-1} \ll q^{1/2-\epsilon} P^{1+\epsilon} Z^{-3/2}. \tag{4.10}$$

We now note that Lemma 5.4 in [13] remains true if one replaces the hypothesis $z \leq P^{23/140}$ of that lemma by the hypotheses

$$z \leq P^{8/35}, \quad z \min(R, S) \leq P^{11/20}.$$

Thus, the first two terms on the right side of (4.9) are bounded above by

$$P^{1+\epsilon/2} (1 + P^2|\beta|)^{-1/2} + qP^{11/20+\epsilon/2} (1 + P^2|\beta|)^{1/2},$$

provided that

$$z(r, s) \leq \min(P^{8/35}, P^{11/20}V^{-1}) = P^{8/35}.$$

We conclude that

$$q^{-1/2+\epsilon} \sum_{\chi \bmod q} |h_1(\beta, \chi)| \ll \frac{P^{1+\epsilon}}{(q + qP^2|\beta|)^{1/2}} + P^{11/20+\epsilon} Q + P^{1+\epsilon} Z^{-3/2}. \tag{4.11}$$

Before estimating the terms arising from $h_p(\alpha)$ we note that there is no contribution unless $z(r, s) \leq p \leq P^\sigma$. Hence, working in an analogous fashion to the above, we obtain

$$\begin{aligned} q_2^{-1/2+\epsilon} \sum_{\chi \bmod q_2} h_p(\beta, \chi) &\ll \frac{(P/p)^{1+\epsilon}}{(q_2 + q_2P^2|\beta|)^{1/2}} + (P/p)^{11/20+\epsilon} Q + P^{1+\epsilon} Z^{-2} \\ &\ll \frac{P^{1+\epsilon}}{(q + qP^2|\beta|)^{1/2}} + P^{11/20} Q + P^{1+\epsilon} Z^{-2}, \end{aligned} \tag{4.12}$$

provided that

$$W \leq (P/p)^{11/20}, \quad z(r, s) \leq \min((P/p)^{8/35}, (P/p)^{11/20}V^{-1}).$$

These follow from the inequalities

$$1 - 4\sigma \leq \frac{11}{20}(1 - \sigma), \quad 3\sigma \leq \frac{11}{20}(1 - \sigma).$$

The desired estimate follows from (4.8), (4.11) and (4.12). \square

Lemma 9. *Suppose that $\alpha \in \mathfrak{m}$ and that the function ρ_j in (2.2) satisfies hypotheses (iii) and (iv) together with:*

(v) $\rho_j(\mathfrak{m})$ is the linear combination of $O(L^\epsilon)$ bilinear sums of the form

$$\sum_{rs=\mathfrak{m}} \xi_r \eta_s, \tag{4.13}$$

where $|\xi_r| \leq \tau(r)^\epsilon$, $|\eta_s| \leq \tau(s)^\epsilon$, and either $V \leq r \leq W$, or $r \geq P^{3\sigma}$ and $\eta_r = 1$ for all r .

Then

$$f_j(\alpha) \ll P^{1-\sigma+2\epsilon}. \tag{4.14}$$

Proof. By Dirichlet’s theorem in Diophantine approximation, we can find integers a, q with

$$1 \leq q \leq (P/Q)^2, \quad (a, q) = 1, \quad |q\alpha - a| < (Q/P)^2.$$

Under the assumption of hypothesis (v), the arguments in Sections 8 and 9 of [8] (see (34) in particular) yield the bound

$$f_j(\alpha) \ll P^{1-\sigma+\epsilon} + P^{1+\epsilon} \left(\frac{1}{q} + \frac{q}{P^2} \right)^{1/4}.$$

This establishes (4.14) when $q \geq Q^2$. On the other hand, when $q \leq Q^2$ hypotheses (iii) and (iv) ensure that we can appeal to Lemma 8. This yields the bound

$$f_j(\alpha) \ll \frac{P^{1+\epsilon}}{(q + P^2|q\alpha - a|)^{1/2}} + P^{1-\sigma+\epsilon},$$

from which (4.14) follows on noting that for $\alpha \in \mathfrak{m}$ we have

$$q + P^2|q\alpha - a| > Q. \quad \square$$

5. The sieve method

We now show how functions ρ_j having properties (i)–(v) (when $j = 2, 3$) or (i)–(iv) ($j = 4$) above can be constructed using the sieve method originating in [6] and developed in [7,1] by modifying the construction used in [8]. Verification of hypotheses (iii) and (iv) is straightforward, so we shall concentrate on checking hypothesis (v). It is immediate that $\psi(m, Z)$ satisfies hypothesis (v) by Theorem 3.1 in [9]. Indeed we can actually obtain the same result for

$$\sum_{r \leq V} c_r \psi(m/r, Z), \tag{5.1}$$

where $p \mid r \Rightarrow p \geq Z$ if $c_r \neq 0$, and $|c_r| \ll 1$. We now state as a lemma a further refinement.

Lemma 10. *Suppose that $p \mid r \Rightarrow p \geq Z$ if either $c_r \neq 0$ or $b_r \neq 0$ and $|c_r|, |b_r| \leq 1$. Then*

$$\sum_{\substack{r \leq V \\ s \leq Y}} c_r b_s \psi(m/(rs), Z) \tag{5.2}$$

satisfies hypothesis (v).

Proof. We can reduce the case $rs \leq V$ to (5.1). The case $V \leq rs \leq W$ is immediately in the correct form. We may therefore suppose that $rs > W$. Let

$$\Pi = \prod_{p < Z} p.$$

Then

$$\sum_{r,s} c_r b_s \psi(m/(rs), Z) = \sum_{d \mid \Pi} \mu(d) \sum_{rsnd=m} c_r b_s.$$

We can then use the technique used in the proof of [9, Theorem 3.1] to decompose the sum into $O(L^2)$ sums of the requisite types. The basic idea is to take out the prime factors of d one by one until a suitable combination of factors lies in the range from V to W or the size of the “free variable” exceeds $P^{3\sigma}$. This is possible since each of the prime factors is bounded above by Z , so (also using $s \leq Y$)

$$rp_1 \dots p_u < V \Rightarrow rp_1 \dots p_{u+1} < P^{1-4\sigma} \quad \text{and} \quad rsp_1 \dots p_u < P^{1-3\sigma}.$$

This completes the proof. \square

Now $\psi(m, P^{1/2})$ is the characteristic function of the set of primes in J . So Buchstab’s identity gives

$$\rho_1(m) = \psi(m, P^{1/2}) = \psi(m, Z) - \sum_{Z \leq p < P^{1/2}} \psi(m/p, p). \tag{5.3}$$

We first construct $\rho_2(n)$, returning later to (5.3) for the lower bound. Clearly

$$\begin{aligned} \rho_1(n) &\leq \psi(m, Z) - \sum_{Z \leq p < Y} \psi(m/p, p) - \sum_{V \leq p < W} \psi(m/p, p) \\ &= \psi_1 - \psi_2 - \psi_3, \quad \text{say.} \end{aligned}$$

Now ψ_1 and ψ_3 satisfy hypothesis (v), and we apply Buchstab’s identity again to ψ_2 :

$$\psi_2 = \sum_{Z \leq p < Y} \psi(m/p, Z) - \sum_{Z \leq q < p < Y} \psi(m/(pq), q) = \psi_4 - \psi_5,$$

say. Again ψ_4 is in the required form and we can apply Buchstab one more time to ψ_5 to obtain:

$$\psi_5 = \sum_{Z \leq q < p < Y} \psi(m/(pq), Z) - \sum_{Z \leq r < q < p < Y} \psi(m/(pqr), r).$$

The first term on the right-hand side above is of the correct form, whereas the second term can be split into two parts: one which satisfies hypothesis (v), and the rest will be discarded since it is counted with a negative weight and we are seeking an upper bound. It follows that the value we obtain for C_2 , obtained by adding on to 1 various integrals corresponding to the discarded sums (compare [9, Chapter 6]), is

$$1 + \int_{1-5\sigma}^{2\sigma} w\left(\frac{1-\alpha}{\alpha}\right) \frac{d\alpha}{\alpha^2} + \int_{1-4\sigma}^{\frac{1}{2}} w\left(\frac{1-\alpha}{\alpha}\right) \frac{d\alpha}{\alpha^2} + \int_A w\left(\frac{1-\alpha-\beta-\gamma}{\gamma}\right) \frac{d\gamma}{\gamma^2} \frac{d\beta}{\beta} \frac{d\alpha}{\alpha}.$$

Here $w(u)$ is Buchstab’s function, defined as the continuous solution of

$$\begin{cases} (uw(u))' = w(u-1) & \text{if } u > 2, \\ w(u) = u^{-1} & \text{if } 1 < u \leq 2. \end{cases}$$

Also, A is the three-dimensional region given by:

$$1 - 6\sigma < \alpha < 1 - 5\sigma, \quad 1 - 6\sigma < \gamma < \beta < \alpha,$$

with the additional constraint that neither the sum of any pair of variables nor the sum of all three variables lies in the interval $[2\sigma, 1 - 4\sigma]$. Some simple calculations then yield $C_2 < 1.74$ when $\sigma = 3/20$.

We now begin the construction of our lower bound sieve function by breaking the final sum on the right-hand side of (5.3) into three parts:

$$\Psi_1 = \sum_{Z \leq p < V} \psi(m/p, p), \quad \Psi_2 = \sum_{V \leq p \leq W} \psi(m/p, p), \quad \Psi_3 = \sum_{W < p < P^{\frac{1}{2}}} \psi(m/p, p).$$

Hypothesis (v) is met for Ψ_2 . Since we cannot cast Ψ_3 into a form which satisfies (v) this term will contribute to $\rho_4(m)$. This contributes $\log(3/2) < 0.406$ to C_4 .

We apply Buchstab’s identity again to Ψ_1 thus obtaining

$$\Psi_1 = \sum_{Z \leq p < V} \psi(m/p, Z) - \sum_{Z \leq q < p < V} \psi(m/(pq), q) = \Psi_4 - \Psi_5,$$

say. By (5.1) Ψ_4 satisfies (v). We split Ψ_5 into four sums $\Sigma_j, 1 \leq j \leq 4$ according to the sizes of p, q as follows:

- $j = 1: \quad V \leq pq \leq W;$
- $j = 2: \quad pq > W, \quad q > Y;$
- $j = 3: \quad pq > W, \quad q \leq Y;$
- $j = 4: \quad pq < V.$

Now Σ_1 automatically satisfies (v). We must discard the whole of Σ_2 and this is the main contribution to $\rho_5(n)$ leading to a “loss” at $\sigma = 3/20$:

$$\int_{1/4}^{3/10} \int_{1/4}^{\alpha} \frac{d\beta d\alpha}{\alpha\beta(1-\alpha-\beta)} < 0.037,$$

where we have noted that $w(u) = 1/u$ throughout the integration region. We can apply Buchstab’s identity again to Σ_3 , leading to

$$\sum_{pq > W, p < V} \psi(m/(pq), Z) - \sum_{pq > W, Z < r < q < Y} \psi(m/(pqr), r).$$

The first term here can be treated using Lemma 10. The second term can be split into three more sums depending on whether: $V \leq pr \leq W$, in which case (v) is automatically satisfied; $pr > W, qr > Y$ in which case we discard this portion which leads to another term in ρ_4 whose contribution to C_4 at $\sigma = 3/20$ is < 0.08 ; $qr < Y$ in which case we can decompose once more and the resulting sums all satisfy hypothesis (v) since

$$Z \leq s < r < q, \quad rq < Y \quad \Rightarrow \quad V \leq qrs \leq W$$

when $\sigma \leq 3/20$.

Finally, we can apply Buchstab’s identity again to Σ_4 to obtain

$$\sum_{\substack{Z \leq q < p < V \\ pq < V}} \psi(m/(pq), Z) - \sum_{\substack{Z \leq r < q < p < V \\ pq < V}} \psi(m/(pqr), r).$$

The first sum above satisfies hypothesis (v) by Lemma 10. We split the second sum into two sums, one with $V \leq pqr \leq W$ and one with $pqr > W$. The first sum immediately satisfies (v), while we can apply Buchstab’s identity to the second since $pq < V, r < q < V^{1/2} < Y$. This leads to a sum over four prime variables which can often be grouped into products lying between V and W . The rest of this sum leads to one last contribution to ρ_5 from a four-dimensional integral whose contribution at $\sigma = 3/20$ is < 0.0006 .

We can now gather all our results together to obtain

$$C_3 - C_2C_4 > (1 + C_4 - 0.038) - 1.74C_4 > 0.96 - 0.74 \times 0.49 = 0.5974,$$

as needed to complete the proof.

6. Proof of Theorem 1

The proof follows a very similar pattern to our previous paper, but we include all the details for completeness. There is one additional complication we must first deal with that did not arise in our earlier work. That is, we need a bound for

$$\int_0^1 |g_j(\alpha)|^4 d\alpha \quad \text{and not just for} \quad \int_0^1 |f_j(\alpha)|^4 d\alpha.$$

The bound $\ll P^{2+\epsilon}$ follows for the latter integral immediately from Hua’s lemma (Lemma 2.5 in [23]), but the result we require demands a little more work.

Lemma 11. *In the notation of previous sections, for $j = 2$ or 4 we have, for any $\epsilon > 0$,*

$$\int_0^1 |g_j(\alpha)|^4 d\alpha \ll P^{2+\epsilon}.$$

Proof. We recall the set $\mathcal{D} = \{m: P^\sigma \leq m \leq Q, p \mid m \Rightarrow p > Z\}$. Further, put $\mathcal{D}_q = \{d \in \mathcal{D}: (d, q) \geq P^\sigma\}$. We then have

$$\begin{aligned} \int_0^1 |g_j(\alpha)|^4 d\alpha &\ll \int_0^1 |f_j(\alpha)|^4 d\alpha + \int_0^1 |g_j(\alpha) - f_j(\alpha)|^4 d\alpha \\ &\ll P^{2+\epsilon} + \sum_{P^\sigma \leq q \leq Q} \sum_{(a,q)=1} I(a, q), \end{aligned}$$

where

$$I(a, q) = \int_{\mathfrak{M}(q,a)} \left| \sum_{d \in \mathcal{D}_q} \sum_{\substack{m \in \mathcal{J} \\ d \mid m}} \rho_j(m) e(\alpha m^2) \right|^4 d\alpha,$$

since it is only on these arcs that $\theta(m, \alpha) = 0$. By Hölder's inequality

$$I(a, q) \leq \left(\sum_{d \in \mathcal{D}_q} 1 \right)^3 \sum_{d \in \mathcal{D}_q} \int_{\mathfrak{M}(q,a)} \left| \sum_{\substack{m \in \mathcal{J} \\ d \mid m}} \rho_j(m) e(\alpha m^2) \right|^4 d\alpha.$$

Since each $d \in \mathcal{D}_q$ has at most two prime divisors, a simple change of integration variable and a swap in the order of summation give

$$\sum_{P^\sigma \leq q \leq Q} \sum_{(a,q)=1} I(a, q) \ll \sum_{d \in \mathcal{D}} \frac{1}{d^2} \sum_{\substack{q \leq Q \\ (d,q) > P^\sigma}} \sum_{(a,q)=1} \int_{\mathfrak{M}^*(a,d,q)} |\Sigma(\alpha, d)|^4 d\alpha,$$

where

$$\mathfrak{M}^*(a, d, q) = \left[\frac{ad^2}{q} - \frac{\omega d^2}{q}, \frac{ad^2}{q} + \frac{\omega d^2}{q} \right]$$

and

$$\Sigma(\alpha, d) = \sum_{md \in \mathcal{J}} \rho_j(md) e(\alpha m^2).$$

Since $\omega d^2/q < Q^{-2}$ the intervals $\mathfrak{M}^*(a_1, d, q)$, $\mathfrak{M}^*(a_2, d, q)$ overlap modulo 1 only if $a_1 d^2 \equiv a_2 d^2 \pmod{q}$. The number of overlaps is thus $\leq (q, d^2)$ as a runs through the reduced residues modulo q . Now, since $(q, d) \geq P^\sigma$ and $q < P^{2\sigma}$, we must have $(q, d^2) = (q, d)$. This value is either d or

otherwise the larger of the two prime divisors of d , which we denote by $p(d)$ if d is not a prime (and let $p(d)$ be zero if d is prime). Now, by Hua’s lemma

$$\int_{\mathfrak{M}^*(a,d,q)} |\Sigma(\alpha, d)|^4 d\alpha \leq \int_0^1 |\Sigma(\alpha, d)|^4 d\alpha \ll \left(\frac{P}{d}\right)^{2+\epsilon}.$$

Hence

$$\begin{aligned} \sum_{P^\sigma \leq q \leq Q} \sum_{(a,q)=1} I(a, q) &\ll P^{2+\epsilon} \sum_{d \in \mathcal{D}} \frac{1}{d^{4+\epsilon}} \left(\sum_{q' \leq Q/d} d + \sum_{q' \leq Q/p(d)} p(d) \right) \\ &\ll P^{2+\epsilon} Q \sum_{d \in \mathcal{D}} \frac{1}{d^4} \ll \frac{P^{2+\epsilon} Q}{P^{3\sigma}} < P^2. \end{aligned}$$

This completes the proof. \square

Let \mathfrak{J} be the set of integers $n \in \mathcal{B}$ for which (3.50) holds but which are not representable as sums of three squares of primes. We write $|\mathfrak{J}|$ for the cardinality of \mathfrak{J} and $Z(\alpha)$ for its generating function:

$$Z(\alpha) = \sum_{n \in \mathfrak{J}} e(-\alpha n).$$

Write

$$\begin{aligned} G^*(\alpha) &= f_1(\alpha)(f_1(\alpha)f_3(\alpha) - f_2(\alpha)f_4(\alpha)), \\ G(\alpha) &= f_1(\alpha)(f_1(\alpha)g_3(\alpha) - g_2(\alpha)g_4(\alpha)), \\ K(\alpha) &= G^*(\alpha) - G(\alpha) \\ &= f_1^2(f_3 - g_3) + f_1(f_2(f_4 - g_4) + g_4(f_2 - g_2)), \end{aligned}$$

where we have omitted the common variable α for all the functions on the last line in the interests of clarity. Then

$$\int_0^1 G^*(\alpha)Z(\alpha) d\alpha \leq 0$$

and

$$\int_{\mathfrak{M}} G(\alpha)Z(\alpha) d\alpha \gg |\mathfrak{J}|PL^{-6}.$$

Thus,

$$\left| \int_{\mathfrak{M}} G(\alpha)Z(\alpha) d\alpha + \int_0^1 K(\alpha)Z(\alpha) d\alpha \right| \gg |\mathfrak{J}|PL^{-6}.$$

Recalling Lemma 9 and (2.6), we deduce that

$$\begin{aligned} |\mathfrak{I}| &\ll L^6 P^{-1} \left(\int_m^1 |G(\alpha)Z(\alpha)| d\alpha + \int_0^1 |K(\alpha)Z(\alpha)| d\alpha \right) \\ &\ll P^{-3/20+\epsilon/2} \int_0^1 |h(\alpha)^2 Z(\alpha)| d\alpha, \end{aligned}$$

where $h(\alpha)$ is one of the $f_j(\alpha)$, $g_j(\alpha)$. Finally, using Cauchy's inequality, Parseval's identity and Hua's lemma (or Lemma 11 if $h(\alpha) = g_j(\alpha)$ for some j), we find that the last integral is bounded by

$$\left(\int_0^1 |Z(\alpha)|^2 d\alpha \right)^{1/2} \left(\int_0^1 |h(\alpha)|^4 d\alpha \right)^{1/2} \ll |\mathfrak{I}|^{1/2} P^{1+\epsilon/2},$$

and so

$$|\mathfrak{I}| \ll P^{17/10+2\epsilon} \ll N^{17/20+\epsilon}.$$

Combining this estimate with Lemma 6 then proves Theorem 1 as required.

References

- [1] R.C. Baker, G. Harman, J. Pintz, The exceptional set for Goldbach's problem in short intervals, in: Sieve Methods, Exponential Sums and Their Applications in Number Theory, in: London Math. Soc. Lecture Note Ser., vol. 237, Cambridge University Press, 1997, pp. 1–54.
- [2] C. Bauer, M.C. Liu, T. Zhan, On sums of three prime squares, *J. Number Theory* 85 (2000) 336–359.
- [3] S.K.K. Choi, A.V. Kumchev, Quadratic equations with five prime unknowns, *J. Number Theory* 107 (2004) 357–367.
- [4] S.K.K. Choi, A.V. Kumchev, Mean values of Dirichlet polynomials and applications to linear equations with prime variables, *Acta Arith.* 123 (2006) 125–142.
- [5] A. Ghosh, The distribution of αp^2 modulo 1, *Proc. Lond. Math. Soc.* (3) 42 (1981) 252–269.
- [6] G. Harman, On the distribution of αp modulo one, *J. Lond. Math. Soc.* (2) 27 (1983) 9–18.
- [7] G. Harman, On the distribution of αp modulo one II, *Proc. Lond. Math. Soc.* (3) 72 (1996) 241–260.
- [8] G. Harman, The values of ternary quadratic forms at prime arguments, *Mathematika* 51 (2004) 83–96.
- [9] G. Harman, *Prime Detecting Sieves*, Princeton University Press, 2007.
- [10] G. Harman, A.V. Kumchev, On sums of squares of primes, *Math. Proc. Cambridge Philos. Soc.* 140 (2006) 1–13.
- [11] A. Hildebrand, G. Tenenbaum, Integers without large prime factors, *J. Théor. Nombres Bordeaux* 5 (1993) 411–484.
- [12] L.K. Hua, Some results in additive prime number theory, *Quart. J. Math. Oxford* 9 (1938) 68–80.
- [13] A.V. Kumchev, On Weyl sums over primes and almost primes, *Michigan Math. J.* 54 (2006) 243–268.
- [14] M.C. Leung, M.C. Liu, On generalized quadratic equations in three prime variables, *Monatsh. Math.* 115 (1993) 133–169.
- [15] H.Z. Li, Sums of one prime and two prime squares, *Acta Arith.* 134 (2008) 1–9.
- [16] J.Y. Liu, On Lagrange's theorem with prime variables, *Quart. J. Math. Oxford* (2) 54 (2003) 453–462.
- [17] J.Y. Liu, K.M. Tsang, Small prime solutions of ternary linear equations, *Acta Arith.* 118 (2005) 79–100.
- [18] J.Y. Liu, T.D. Wooley, G. Yu, The quadratic Waring–Goldbach problem, *J. Number Theory* 107 (2004) 298–321.
- [19] J.Y. Liu, T. Zhan, Sums of five almost equal prime squares (II), *Sci. China* 41 (1998) 710–722.
- [20] J.Y. Liu, T. Zhan, Distribution of integers that are sums of three squares of primes, *Acta Arith.* 98 (2001) 207–228.
- [21] J.Y. Liu, T. Zhan, The exceptional set in Hua's theorem for three squares of primes, *Acta Math. Sinica (N.S.)* 21 (2005) 335–350.
- [22] W. Schwarz, Zur Darstellun von Zahlen durch Summen von Primzahlpotenzen, *J. Reine Angew. Math.* 206 (1961) 78–112.
- [23] R.C. Vaughan, *The Hardy–Littlewood Method*, second ed., Cambridge Tracts in Math., vol. 125, Cambridge University Press, 1997.
- [24] T.D. Wooley, Slim exceptional sets for sums of four squares, *Proc. Lond. Math. Soc.* (3) 85 (2002) 1–21.