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Abstract

Let S be a standard Sturmian word that is a fixed point of a non-trivial homomorphism. Associated to the infinite word S is a unique irrational number β with $0<\beta<1$. We prove that the standard Sturmian word S contains no fractional power with exponent greater than Ω and that for any real number $\varepsilon>0$ it contains a fractional power with exponent greater than $\Omega-\varepsilon$; here Ω is a constant that depends on β. The constant Ω is given explicitly. Using these results we are able to give a short proof of Mignosi's theorem and give an exact evaluation of the maximal power that can occur in a standard Sturmian word.

Keywords: Sturmian words; Critical exponent; Lyndon–Schutzenberger theorem

1. Introduction

If $\alpha \in (0,1)$ is irrational, then the *infinite characteristic word* associated with α, denoted by f_α, is the word $w_1w_2w_3\cdots$ where $w_n = p_\alpha(n)$ with

$$p_\alpha(n) := \lfloor (n+1)\alpha \rfloor - \lfloor n\alpha \rfloor$$

for n a positive integer. For example, if $\alpha = (\sqrt{5}-1)/2$, then f_α is the infinite Fibonacci word over the alphabet $\{0,1\}$:

$$f_\alpha = 10110101101\cdots$$

Characteristic words are also called *standard Sturmian* words, and may be viewed geometrically. For the remaining part of the paper we shall write Sturmian word instead of the longer standard Sturmian word. Let β be a positive irrational number and consider the ray $y=\beta x$ where x is non-negative. This ray lies in the positive quadrant of $\mathbb{R}^2$.

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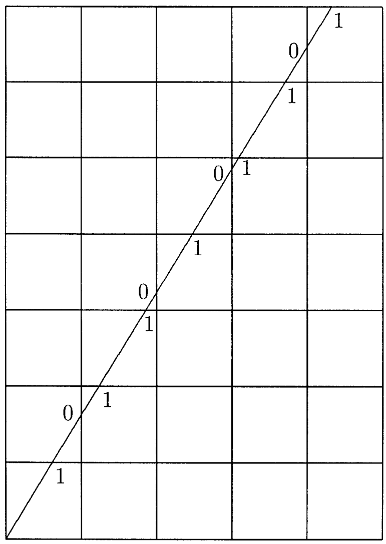

Fig. 1. The Fibonacci word.

By overlaying the quadrant with a uniform grid we can construct the infinite word S_β. Label the intersections of the ray and the grid according to the following rules. If the grid line crossed is vertical, label the intersection with a 0, and if the grid line crossed is horizontal, label the intersection with a 1. Now if $a_1, a_2, a_3, \ldots$ are the labels, read from left to right, then the infinite word $a_1a_2a_3\cdots$ is a Sturmian word. We denote this word by S_β. For example if we let $\beta = (\sqrt{5}+1)/2$, then the line construction would give us the Fibonacci word f_α (see Fig. 1).

There is a simple connection between the characteristic words f_α and the words S_β. Let α and β be irrational numbers with $0<\alpha<1$ and $\beta>0$. Then $f_\alpha = S_\beta$ if, and only if,

$$\alpha = \frac{1}{1+(1/\beta)}.$$

In 1993 Crisp et al. [4] proved that

$$S_\beta = \prod_{k=0}^{\infty} 1^{p(k)}0,$$

where β is irrational, $\beta>0$, and $p(k) = p_\beta(k) = \lfloor (k+1)\beta \rfloor - \lfloor k\beta \rfloor$.

On looking at the infinite Fibonacci word f_α, it can be seen that not only does the word 101 occur in f_α, but so does $(101)^2$. One may then ask what is the largest non-negative integer k such that $(101)^k$ occurs in f_α? More generally, given a subword u of a Sturmian word S_β what is the largest non-negative integer k such that u^k is a subword of S_β?

Similar questions were first looked at by Axel Thue [13, 14] in 1906 and 1912. In 1983 Karhumäki [6] showed that the infinite Fibonacci word is 4th power free. In 1989 Mignosi [9] proved a generalization of Karhumäki's result. He proved that the Sturmian word S_β is kth power free for some non-negative integer k if, and only if, β has a continued fraction expansion with bounded partial quotients. He did not however, determine what the smallest such k would be.

There is also the notion of a fractional power of a subword, which we define in Section 2. In 1992 Mignosi and Pirillo [10] showed that the infinite Fibonacci word contains no fractional power with exponent greater than $2+\varphi$, and for any $\varepsilon>0$ contains a fractional power with exponent greater than $2+\varphi-\varepsilon$. Here $\varphi=(\sqrt{5}+1)/2$, the golden ratio. Their proof relies on the properties of the finite Fibonacci words f_n, for $n\geqslant 0$. The finite Fibonacci words may be defined as follows. Let $f_0=0$ and $f_1=1$ and for $n\geqslant 2$ let $f_n=f_{n-1}f_{n-2}$. One immediately notices that f_n is a subword of the infinite Fibonacci word and in fact each f_n, for $n\geqslant 1$, is a prefix of the infinite Fibonacci word.

The essential property needed in the proof of Mignosi and Pirillo is the following theorem due to Séébold [12]. Let v be a non-empty word. If v^2 is a subword of the infinite Fibonacci word then $v=wz$ with $zw=f_n$ for some non-negative integer n and words z and w, $|w|>0$.

In this paper, we extend Mignosi's and Pirillo's result to all Sturmian words S_β, $0<\beta<1$ with $\beta=[0;b_0,\overline{b_1,\ldots,b_m}]$, where $b_0,b_1,\ldots,b_m$ are positive integers and $b_m\geqslant b_0$. Using this result we are able to give a short proof of Mignosi's theorem and give an exact evaluation of the maximal power that can occur in a Sturmian word. For a list of references on Sturmian words and history of the subject see Brown [3].

2. Preliminaries

In this section we give some of the definitions needed in this paper and state the known results that are used throughout. For any notation not explicitly defined, refer to Lothaire [8].

Let Σ be a finite alphabet, and let Σ^* be the free monoid generated by Σ. Denote the identity of Σ^*, the empty word, by ε; and let $\Sigma^+=\Sigma^*\backslash\{\varepsilon\}$. The elements of Σ^* are called words and the length of a given word $u\in\Sigma^*$ is denoted by $|u|$. A word u is a prefix (suffix) of the word v if there exists a word w such that $v=uw$ $(v=wu)$. If u is a prefix of v then we write $u\lhd v$, and if u is a suffix of v then write $u\rhd v$. If there exist words $x,y\in\Sigma^*$ such that $u=xvy$ for $u,v\in\Sigma^*$ then we say that v is a subword (factor) of u, written $v\,|\,u$. The reversal of a word $w\in\Sigma^+, w=w_0w_1\ldots w_n$ is $w_nw_{n-1}\ldots w_1w_0$, written as w^R; we also define $\varepsilon^R=\varepsilon$.

Now for a given word $w\in\Sigma^+$ we define w_i, for $i=1,\ldots,|w|$, to be the ith letter of w (counting from the left). Likewise we define w_{-i}, for $i=1,\ldots,|w|$, to be the ith letter of w (counting from the right). Let $u\in\Sigma^*$, $v\in\Sigma^+$. Define $|u|_v$ to be the number of times v occurs in the word u. We define $c(w)$ for $w\in\Sigma^+$ with $|w|\geqslant 2$ to be $w_0w_1\ldots w_{|w|-2}w_{|w|}w_{|w|-1}$, i.e. the word w with the last two letters interchanged.

For $i=0,\dots,|w|$, $w_i\in\Sigma^+$ we define $r_i(w)=w_1w_2\dots w_{|w|-i}$, and define $l_i(w)$ to be the word such that $r_i(w)l_i(w)=w$. Two words $u,v\in\Sigma^+$ are said to be conjugate if there exists words $x,y\in\Sigma^*$ such that $u=xy$ and $v=yx$; we write $u\sim v$.

The set of right infinite words over a finite alphabet Σ is denoted by Σ^ω, and a word in Σ^ω is called an ω-word. Let w be a word and define $\Lambda(w)$ to be the set of letters in w. Let $w\in\Sigma^+$ with $|w|=n$. A positive integer i is a period of w if $w_{m+i}=w_m$ for $0<m\leqslant n-i$. If $r\in\mathbb{Q}, r\geqslant 1$, then we say that $w=z^r$ if, z is a prefix of w, $|z|$ is a period of w, and $r=|w|/|z|$. We say that a given ω-word w is *critical* if there exists a real number $\Omega>1$, called the *critical exponent* of ω, such that the follows conditions hold:

(i) *if* $u\in\Sigma^+, t\geqslant 1$ *and* $u^t \mid w$ *then* $t\leqslant\Omega$; *and*
(ii) *for all* $\varepsilon>0$ *there exists a* $v\in\Sigma^+$ *and a rational number* $r>\Omega-\varepsilon$ *such that* $v^r\mid w$.

Thus the result of Mignosi and Pirillo can be restated as: the infinite Fibonacci word is critical and has a critical exponent of $2+\varphi$, where $\varphi=(\sqrt{5}+1)/2$.

A useful result will be the Lyndon–Schützenberger theorem: let x,y be words, then $xy=yx$ if, and only if, there exists a word z and positive integers k and l such that $x=z^k$ and $y=z^l$.

Let $w\in\Sigma^+\cup\Sigma^\omega$; then for $n\geqslant 1$, let $\rho_w(n)$ denote the number of factors of w of length n. The function $\rho_w(n)$ is called the subword complexity of the word w. With this definition we have the following result (see [11]).

Proposition 1 (Morse and Hedlund [11]). *Let* β *be a positive irrational and* S_β *its corresponding Sturmian word. Then* $\rho_{S_\beta}(n)=n+1$.

Let Π be a binary alphabet. Let $\mathrm{Em}(\Pi)$ be the monoid obtained by composing endomorphisms on $\Pi^+\cup\Pi^\omega$. For the rest of the paper assume that $\Sigma=\{0,1\}$. Let $\Gamma(\Sigma)$ be the submonoid of $\mathrm{Em}(\Sigma)$ generated by X,L and R, where

$$\begin{array}{lll} X:\Sigma^*\to\Sigma^* & L:\Sigma^*\to\Sigma^* & R:\Sigma^*\to\Sigma^* \\ 0\mapsto 1 & 0\mapsto 0 & 0\mapsto 0 \\ 1\mapsto 0 & 1\mapsto 01 & 1\mapsto 10 \end{array}$$

Within the submonoid $\Gamma(\Sigma)$ we have the following relations:

$$(1)\ \ X^2=I,\qquad (2)\ \ LR=RL,\qquad (3)\ \ \text{for all } k\geqslant 0\ \ RXR^kL=LXL^kR,$$

where I is the identity homomorphism.

We note here that if $\mathscr{F}$ is the infinite Fibonacci word and $T\in\Gamma(\Sigma)$ with $T=XL$ then $T(\mathscr{F})=\mathscr{F}$. That is, $\mathscr{F}$ is a fixed point of the homomorphism T. This raises the question: given a Sturmian word S_β, can one find a homomorphism $T\in\Gamma(\Sigma)$ such that S_β is a fixed point of T? This question was partially answered in 1991 by Brown [2] and completely answered in 1993 by Crisp et al. [4].

Proposition 2 (Crisp et al. [4]). (a) *Let $\beta>1$ be an irrational. The Sturmian word S_β is invariant under some non-trivial homomorphism T if, and only if, β has a continued fraction expansion of the form $[b_0;\overline{b_1,\dots,b_m}]$ where $b_m\geqslant b_0\geqslant 1$. Further, if that is the case and m is minimal, then T must be a power of*

$$XL^{b_0}XL^{b_1}\cdots XL^{b_{m-1}}XL^{b_m-b_0}X$$

and all such homomorphisms leave S_β fixed.

(b) *Let $0<\beta<1$ be irrational. The Sturmian word S_β is invariant under some non-trivial homomorphism T if, and only if, β has a continued fraction expansion of the form $[0;b_0,\overline{b_1,\dots,b_m}]$ where $b_m\geqslant b_0\geqslant 1$. Further, if that is the case and m is minimal then T must be a power of*

$$L^{b_0}XL^{b_1}X\cdots L^{b_{m-1}}XL^{b_m-b_0}$$

and all such homomorphisms leave S_β fixed.

3. Factors of Sturmian words

As noted in Section 1, the essential property used in determining the critical exponent of the infinite Fibonacci word is a theorem due to Séébold. We would like to have an analogous result regarding the Sturmian words S_β, with $0<\beta<1$ an irrational number such that $\beta=[0;b_0,\overline{b_1,\dots,b_m}]$, where $b_0,b_1,\dots,b_m$ are positive integers and $b_m\geqslant b_0$. To do this we will first have to determine which words are analogous to the finite Fibonacci words in the Sturmian word S_β. Using the observation that the infinite Fibonacci word $\mathscr{F}$ is the fixed point of $T=XL$ and that $T^n(0)=f_n$ for $n\geqslant 0$ we make the following definitions.

Let β be an irrational number such that $0<\beta<1$ and $\beta=[0;b_0,\overline{b_1,b_2,\dots,b_m}]$ with the b_i positive integers for $i=0,\dots,m$ and $b_m\geqslant b_0$. By Proposition 2, S_β is the fixed point of the following homomorphism:

$$T=L^{b_0}XL^{b_1}X\cdots L^{b_{m-1}}XL^{b_m-b_0}.$$

If the period m of the continued fraction for β is minimal we will call T the fundamental homomorphism of S_β. It is clear that $T\in\Gamma(\Sigma)$. Now for $t=0,1,\dots,m-1$ let

$$F_t=L^{b_0}XL^{b_1}X\cdots L^{b_t}X$$

with $F_{-1}=I$. For $t=0,1,\dots,m-2$ let

$$E_t=L^{b_{t+1}}X\cdots L^{b_{m-1}}XL^{b_m-b_0}$$

with $E_{m-1}=L^{b_m-b_0}$ and $E_{-1}=I$. Note that $T=F_tE_t$.

We will show that the finite words of the form $T^nF_t(0)$ for $n\geqslant 0$ and $t=0,1,\dots,m-1$ are subwords of S_β analogous to the finite Fibonacci words view as subwords of the infinite Fibonacci word. We now examine some properties of these words.

Lemma 3. *With T and F_t defined as above, we have the following*:
(a) $T(0)=F_{m-1}(0)$;
(b) $T(1)=F_{m-1}(0)^{b_m-b_0}F_{m-1}(1)$;
(c) $F_t(1)=F_{t-1}(0)$ *for* $t=0,\dots,m-1$; *and*
(d) $F_t(0)=F_{t-1}(0)^{b_t}F_{t-1}(1)$ *for* $t=0,\dots,m-1$.

Proof. We have

(a) $$\begin{aligned} T(0) &= L^{b_0}XL^{b_1}X\cdots L^{b_{m-1}}XL^{b_m-b_0}(0) \\ &= L^{b_0}XL^{b_1}X\cdots L^{b_{m-1}}X(0) \\ &= F_{m-1}(0); \end{aligned}$$

(b) $$\begin{aligned} T(1) &= L^{b_0}XL^{b_1}X\cdots L^{b_{m-1}}XL^{b_m-b_0}(1) \\ &= L^{b_0}XL^{b_1}X\cdots L^{b_{m-1}}X(0^{b_m-b_0}1) \\ &= F_{m-1}(0)^{b_m-b_0}F_{m-1}(1); \end{aligned}$$

(c) $$\begin{aligned} F_t(1) &= L^{b_0}XL^{b_1}X\cdots L^{b_t}X(1) \\ &= L^{b_0}X\cdots L^{b_{t-1}}X(0) \qquad \text{for } t=1,\dots,m-1 \\ &= F_{t-1}(0); \end{aligned}$$

with the result clearly true when $t=0$. Finally,

(d) $$\begin{aligned} F_t(0) &= L^{b_0}XL^{b_1}X\cdots L^{b_t}X(0) \\ &= L^{b_0}X\cdots L^{b_{t-1}}X(0^{b_t}1) \quad \text{for } t=1,\dots,m-1 \\ &= F_{t-1}(0)^{b_t}F_{t-1}(1). \end{aligned}$$

with the result clearly true when $t=0$. □

Lemma 4. *Let* $D = L^{a_1}XL^{a_2}X\cdots L^{a_n}XL^b$ *where* $a_1,a_2,\dots,a_n$ *are positive integers and* b *is a non-negative integer. Then*
(a) *if* $n\equiv 0\,(\mathrm{mod}\,2)$ *then* $l_2(D(0))=10$ *and* $l_2(D(1))=01$;
(b) *if* $n\equiv 1\,(\mathrm{mod}\,2)$ *then* $l_2(D(0))=01$, *and*
  (i) *if* $b=0$ *and* $n=1$ *then* $D(1)=0$;
  (ii) *if* $b=0$, $n\geqslant 2$ *then* $l_2(D(1))=10$.

Proof. Let $D=L^{a_1}XL^{a_2}X\cdots L^{a_n}XL^b$. Now if $n=1$ and $b=0$ then $D=L^{a_1}X$ and so $D(1)=0$ and $l_2(D(0))=01$. Assume then that if $b=0$ then $n\geqslant 2$ and if $b\neq 0$ then $n\geqslant 1$. With this we see that $|D(1)|>1$ and $|D(0)|>1$, so both $l_2(D(0))$ and $l_2(D(1))$ are defined. Consider the homomorphisms given by $D_1=L^{c_1}X$ and $D_2=L^{c_1}XL^{c_2}X$ where c_1 and c_2 are positive integers. Let $w\in\Sigma^*$ with $l_2(w)=01$. Then $l_2(D_1(w))=10$ and $l_2(D_2(w))=01$. Similarly, if $w\in\Sigma^*$ with $l_2(w)=10$ then $l_2(D_1(w))=01$ and $l_2(D_2(w))=10$. Thus if $b=0$ then
(a) if $n\equiv 0\,(\mathrm{mod}\,2)$, then $l_2(D(0))=10$ and $l_2(D(1))=01$;

(b) if $n \equiv 1 \pmod 2$, then $l_2(D(0)) = 01$, and
 (i) if $b = 0$, $n = 1$, then $D(1) = 0$;
 (ii) if $b = 0$ and $n \geqslant 2$, then $l_2(D(1)) = 10$.

Now by observing that $L^b(0) = 0$ and $L^b(1) = 0^b1$ our result follows. □

Corollary 5. *Let $D = L^{a_1}XL^{a_2}X \cdots L^{a_m}XL^b$ where $a_1, a_2, \ldots, a_m$ are positive integers and b is a non-negative integer. Then if $n \geqslant 2$,*
(a) *if $m \equiv 0 \pmod 2$, then $l_2(D^n(1)) = 01$;*
(b) *if $m \equiv 1 \pmod 2$, and $n \equiv 0 \pmod 2$, then $l_2(D^n(0)) = 10$ and $l_2(D^n(1)) = 01$;*
(c) *if $m \equiv 1 \pmod 2$, and $n \equiv 1 \pmod 2$, then $l_2(D^n(0)) = 01$ and $l_2(D^n(1)) = 10$.*

The finite Fibonacci words f_n have the following important properties. The first belongs to folklore (see for example [1, 5] or [7]): $f_n f_{n-1} = c(f_{n-1} f_n)$. The second is due to de Luca [5]: $r_2(f_n f_{n-1})$ is a palindrome, and each word f_n is primitive; that is, $f_n \neq v^k$ for any $v \in \Sigma^+$ with k a positive integer greater than 2. We prove the following generalizations of these facts. Let $\Pi = \{a, b\}$, $a \neq b$, be a binary alphabet. Let $S : \Pi \to \Sigma$ be the homomorphism given by $S(a) = 0$ and $S(b) = 1$. Let $\Delta(\Pi)$ be the submonoid of $\mathrm{Em}(\Pi)$ generated by XS and LS.

Theorem 6. *Let Π be a finite alphabet with $x, y \in \Pi^*$. Then $xy = c(yx)$ with $(xy)_{-1} \neq (xy)_{-2}$ if, and only if, $|\Lambda(xy)| = 2$ and there exists a $D \in \Delta(\Lambda(xy))$ such that $D((xy)_{-1}) = x$ and $D((xy)_{-2}) = y$. Moreover, if $xy = c(yx)$ with $(xy)_{-1} \neq (xy)_{-2}$ then $r_2(xy) = r_2(yx)$ is a palindrome.*

Proof. Let Π be a finite alphabet. Let $x, y \in \Pi^*$ such that $xy = c(yx)$ with $(xy)_{-1} \neq (xy)_{-2}$. Without loss of generality, we may assume that $|\Pi| \geqslant 2$ with $(xy)_{-1} = 0$ and $(xy)_{-2} = 1$. We note that one cannot have $x = \varepsilon$ or $y = \varepsilon$. Consider when $|x| = 1$, then $x = 1$ and $y = 1^k0$ for some $k \geqslant 0$. So let $D = XL^k \in \Delta(\Sigma)$, then $D(0) = x$ and $D(1) = y$. Moreover $\Lambda(xy) = \Sigma$ and $r_2(xy) = r_2(yx)$ is a palindrome. Similarly for $|y| = 1$. Assume now that $|x| \geqslant 2$ and $|y| \geqslant 2$. Assume that for $w_1, w_2 \in \Pi^*$ with $w_1w_2 = c(w_2w_1)$, $(w_1w_2)_{-1} \neq (w_1w_2)_{-2}$ and $|w_1| + |w_2| < |x| + |y|$ that there exists a $D \in \Delta(\Sigma)$ such that $D(0) = w_1$ and $D(1) = w_2$. Moreover that $\Lambda(w_1w_2) = \Sigma$ and $r_2(w_1w_2) = r_2(w_2w_1)$ is a palindrome.

Since $xy = c(yx)$ there exist $x_1, y_1 \in \Pi^*$ such that $x = x_101$ and $y = y_110$. Thus $x_101y_1 = y_110x_1$. If $x_1 = \varepsilon$ then $y_1 = (01)^k0$ for some integer $k \geqslant 0$. Hence $x = 01$ and $y = (01)^{k+1}0$. So let $D = LXL^{k+1}$ then $D \in \Delta(\Sigma)$ and $D(0) = x$ with $D(1) = y$. Moreover, $\Lambda(xy) = \Sigma$ and $r_2(xy) = r_2(yx)$ is a palindrome. Similarly for $y = \varepsilon$.

So assume that x and y are non-empty words. Without loss of generality, we may assume that $x_1 \lhd y_1$. Then there exists a word $y_2 \in \Pi^+$ such that $y_1 = x_1y_2$. Whence $01x_1y_2 = y_210x_1$. This implies that $0 \lhd y_2$, and so there exists a word $y_3 \in \Pi^*$ such that $y_2 = 0y_3$. Thus $1x_10y_3 = y_310x_1$. Here we have two possibilities, either $y_3 = \varepsilon$ or $y_3 \neq \varepsilon$.

If $y_3 = \varepsilon$ then $x_1 0 = 0 x_1$, and so there exists a non-negative integer k such that $x_1 = 0^k$ and $y_1 = 0^{k+1}$. This give $x = 0^{k+1}1$ and $y = 0^{k+1}10$. So let $D = L^{k+1}XL$. Then $D \in \Delta(\Sigma)$ and $D(0) = x$ with $D(1) = y$. Moreover, $\Lambda(xy) = \Sigma$ and $r_2(xy) = r_2(yx)$ is a palindrome.

If $y_3 \neq \varepsilon$ then there exists a word $y_4 \in \Pi^*$ such that $y_3 = 1y_4$. Thus $x_1 01 y_4 = y_4 10 x_1$. Now let $w_1 = x_1 01$ and $w_2 = y_4 10$. Then $w_1 w_2 = c(w_2 w_1)$ with $(w_1 w_2)_{-1} \neq (w_1 w_2)_{-2}$ and $|w_1| + |w_2| < |x| + |y|$. So by our induction hypothesis there exists a $D \in \Delta(\Sigma)$ such that $D(0) = w_1$ and $D(1) = w_2$. Moreover, $\Lambda(w_1 w_2) = \Sigma$ and $r_2(w_1 w_2) = r_2(w_2 w_1)$ is a palindrome. Let $D_1 = DL$ then $D_1(0) = D(0) = x_1 01 = x$ and $D_1(1) = D(01) = w_1 w_2 = y_1 10 = y$. Moreover, since $r_2(w_1)$ and $r_2(w_2)$ are also palindromes we have that $r_2(xy) = r_2(yx)$ is a palindrome with $\Lambda(xy) = \Sigma$.

Assume now that $\Pi = \Sigma$ and that $D \in \Delta(\Sigma)$. Then there exist positive integers $a_1, a_2, \ldots, a_n$ and non-negative integers i and j such that

$$D = X^i L^{a_1} X L^{a_2} X \cdots L^{a_n} X L^j.$$

The proof now follows easily by induction on n. □

Lemma 7. *Let Π be a binary alphabet, $P \in \Gamma(\Pi)$, $w \in \Pi^*$. Then $P(w)$ is primitive if, and only if, w is primitive.*

Proof. Assume without loss of generality, that $\Pi = \Sigma$. We clearly have that $X(w)$ is primitive if, and only if, w is primitive. We show that $L(w)$ is primitive if, and only if, w is primitive. We first note that if

$$g = \prod_{i=1}^{p} 0^{a_i}(01)^{b_i},$$

for some non-negative integers $a_i, b_i, i = 1, \ldots, p$, then $L(\bar{g}) = g$ where

$$\bar{g} = \prod_{i=1}^{p} 0^{a_i} 1^{b_i},$$

with $\bar{g}$ being unique up to juxtaposition of the empty word.

Assume that $L(w)$ is primitive but w is not primitive. Then $w = v^k$ for some $v \in \Sigma^*$ and some integer $k \geqslant 2$. Thus $L(w) = L(v^k) = L(v)^k$, which is a contradiction. Now assume that w is primitive but $L(w)$ is not primitive. Then $L(w) = v^k$ for some $v \in \Sigma^*$ and some integer $k \geqslant 2$. Now $1 \not\trianglelefteq L(w)$, hence $1 \not\trianglelefteq v$, and since $v \mid L(w)$ we must have

$$v = \prod_{i=1}^{q} 0^{c_i}(01)^{d_i},$$

for some non-negative integers $c_i, d_i, i = 1, \ldots, q$ because

$$L(w) = \prod_{i=1}^{p} 0^{a_i}(01)^{b_i}$$

for some non-negative integers $a_i, b_i, i = 1, \ldots, p$. Whence from the comment above, $L(\bar{v}) = v$, where

$$\bar{v} = \prod_{i=1}^{q} 0^{c_i} 1^{d_i}.$$

Thus $w = \bar{v}^k$, which is a contradiction. Similarly, we have that $R(w)$ is primitive if, and only if, w is primitive. The result now follows. □

Lastly, we would like to know the lengths of the words $T^n F_t(0)$. If $w \in \Sigma^+$, let $\sigma(w) = (m, n)$ where $m = |w|_0$ and $n = |w|_1$. Then if $w \in \Sigma^+$ with $\sigma(w) = (m, n)$ it is easy to see that

(1) $\sigma(X(w)) = (n, m)$;
(2) for $k \in \mathbb{Z}^+$, $\sigma(L^k(w)) = (m + kn, n) = \sigma(R^k(w))$; and
(3) $|w| = m + n$.

Using this notation we prove the following result.

Lemma 8. *Let $D = L^{a_1} X L^{a_2} X \cdots L^{a_n} X$ with $a_1, a_2, \ldots, a_n$ positive integers and $n \geqslant 2$. If*

$$[a_n; a_{n-1}, \ldots, a_1] = \frac{P_0}{Q_0}, \quad (P_0, Q_0) = 1$$

and

$$[a_n; a_{n-1}, \ldots, a_2] = \frac{P}{Q_1}, \quad (P, Q_1) = 1,$$

then

$$|T(0)|_0 = P_0, \qquad |T(0)|_1 = P, \qquad |T(1)|_0 = Q_0, \qquad |T(1)|_1 = Q_1.$$

If $D = L^{a_1} X$ then

$$|T(0)|_0 = a_1, \qquad |T(0)|_1 = 1, \qquad |T(1)|_0 = 1, \qquad |T(1)|_1 = 0.$$

Proof. The proof is by induction. The result is clear for $n = 2$. We now proceed by induction on n. Let $a_1, a_2, \ldots, a_{n+1}$ be positive integers. Assume that

$$[a_{n+1}; a_n, \ldots, a_2] = \frac{P_0'}{Q_0'}, \quad (P_0', Q_0') = 1$$

and

$$[a_{n+1}; a_n, \ldots, a_3] = \frac{P_1'}{Q_1'}, \quad (P_1', Q_1') = 1.$$

Then if $w_1 = L^{a_2} X L^{a_3} X \cdots L^{a_{n+1}} X(0)$ and $w_2 = L^{a_2} X L^{a_3} X \cdots L^{a_{n+1}} X(1)$, we have by our inductive assumption that $\sigma(w_1) = (P_0', P_1')$ and $\sigma(w_2) = (Q_0', Q_1')$. Thus,

$$\begin{aligned}\sigma(L^{a_1} X L^{a_2} X \cdots L^{a_{n+1}} X(0)) &= \sigma(L^{a_1}(w_1)) \\ &= (P_1' + a_1 P_0', P_0'),\end{aligned}$$

and

$$\sigma(L^{a_1}XL^{a_2}X\cdots L^{a_{n+1}}X(1)) = \sigma(L^{a_1}X(w_2))$$
$$= (Q_1' + a_1Q_0', Q_0').$$

Hence $|D(0)|_0 = P_1' + a_1P_0'$, $|D(0)|_1 = P_0'$, $|D(1)|_0 = Q_1' + a_1Q_0'$ and $|D(1)|_1 = Q_0'$, where $D = L^{a_1}XL^{a_2}X\cdots L^{a_{n+1}}X$. Now

$$[a_{n+1}; a_n, \ldots, a_2, a_1] = \frac{a_1P_0' + P_1'}{a_1Q_0' + Q_1'}$$

and

$$[a_{n+1}; a_n, \ldots, a_2] = \frac{P_0'}{Q_0'}$$

and so the result follows. □

4. Factorizations of S_β

We are now in a position to look at how the words $T^nF_t(0)$ factor in S_β. We have already seen one factorization of S_β due to Crisp et al. [4]. That is

$$S_\beta = \prod_{k=0}^{\infty} 1^{p(k)}0,$$

where β is irrational, $\beta > 0$, and $p(k) = \lfloor (k+1)\beta \rfloor - \lfloor k\beta \rfloor$. This factorization has the disadvantage that when β is irrational with $0<\beta<1$, $p(k)$ is zero most of the time. Thus we do not get a clear indication on how S_β factors. To fix this problem we have the following modification.

Lemma 9. *Let $0<\beta<1$ be an irrational number. Then if $q(k) = p_{\beta^{-1}}(k)$ we have*

$$S_\beta = \prod_{k=0}^{\infty} 0^{q(k)}1.$$

Proof. Assume that

$$S_\beta = \prod_{k=0}^{\infty} 0^{h(k)}1$$

for some non-negative integer valued function $h(k)$. Now $X(S_\beta) = S_{\beta^{-1}}$; thus

$$S_{\beta^{-1}} = \prod_{k=0}^{\infty} 1^{h(k)}0,$$

but from above we have that

$$S_{\beta^{-1}} = \prod_{k=0}^{\infty} 1^{q(k)}0,$$

thus $h(k) = q(k)$ for $k \geqslant 0$. □

We now take advantage of the fact that S_β is the fixed point of the homomorphism T. That is, since $T(S_\beta)=S_\beta$, we have by Lemma 9 that for $n\geqslant 0$

$$S_\beta = T^n(S_\beta) = T^n\left(\prod_{k=0}^{\infty} 0^{q(k)}1\right) = \prod_{k=0}^{\infty} T^n(0)^{q(k)}T^n(1).$$

So we begin to get an idea how the factors of S_β occur. The following theorem and corollary show how the factors $T^nF_t(0)$ occur in the word S_β.

Theorem 10. *Let* $0<\beta<1$ *with* $\beta=[0;b_0,\overline{b_1,\ldots,b_m}]$, *where* $b_0,\ldots,b_m$ *are positive integers and* $b_m\geqslant b_0$. *Then*

$$S_\beta = \prod_{k=0}^{\infty} F_t(0)^{s_t(k)}F_t(1),$$

where $s_t(k)=\lfloor (k+1)\beta_t^{-1}\rfloor - \lfloor k\beta_t^{-1}\rfloor$, *and*

$$\beta_t=[0;b_{t+1},b_{t+2},\ldots,b_m,\overline{b_1,b_2,\ldots,b_m}]$$

for $t=-1,0,1,\ldots,m-1$.

Proof. Let T be the fundamental homomorphism of S_β. We have by Lemma 9 that $X(S_\beta)=S_{\beta^{-1}}$ and for integers $k\geqslant 0$, $L^k(S_\beta)=S_{(k+\beta^{-1})^{-1}}$. Hence for $t=0,1,\ldots,m-2$ we have

$$E_t(S_\beta)=L^{b_{t+1}}X\cdots L^{b_{m-1}}XL^{b_m-b_0}(S_\beta)=S_\gamma,$$

where

$$\begin{aligned}\gamma&=[0;b_{t+1},b_{t+2},\ldots,b_{m-1},b_m-b_0,0,b_0,\overline{b_1,\ldots,b_m}]\\&=[0;b_{t+1},b_{t+2},\ldots,b_m,\overline{b_1,b_2,\ldots,b_m}].\end{aligned}$$

Whence

$$S_\beta=T(S_\beta)=F_tE_t(S_\beta)=F_t(S_{\beta_t})=\prod_{k=0}^{\infty} F_t(0)^{s_t(k)}F_t(1),$$

where $s_t(k)=\lfloor (k+1)\beta_t^{-1}\rfloor-\lfloor k\beta_t^{-1}\rfloor$ and $\beta_t=[0;b_{t+1},b_{t+2},\ldots,b_m,\overline{b_1,\ldots,b_m}]$. The result is also true when $t=-1$ and $t=m-1$. □

Corollary 11. *Let* $0<\beta<1$ *with* $\beta=[0;b_0,\overline{b_1,\ldots,b_m}]$, *where* $b_0,\ldots,b_m$ *are positive integers and* $b_m\geqslant b_0$. *Let* T *be the fundamental homomorphism of* S_β. *Then for* $n\geqslant 0$

$$S_\beta = \prod_{k=0}^{\infty} (T^nF_t(0))^{s_t(k)}T^nF_t(1),$$

where $s_t(k)=\lfloor (k+1)\beta_t^{-1}\rfloor - \lfloor k\beta_t^{-1}\rfloor$ *and* $\beta_t=[0;b_{t+1},b_{t+2},\ldots,b_m,\overline{b_1,\ldots,b_m}]$ *for* $t=-1,0,1,\ldots,m-1$.

Let Π be an alphabet (finite or infinite) and $w \in \Pi^\omega$ with $w = w_1 w_2 w_3 \ldots$. Now let $w(n.k) = w_n w_{n+1} \cdots w_{n+k-1}$ for positive integers n and k. For finite subwords u and v of w we define

$$\mathrm{dist}_w(u,v) = \min_{\substack{n \neq m \\ w(n,|u|)=u \\ w(m,|v|)=v}} |n-m|.$$

Now if u is a finite subword of w such that $\mathrm{dist}_w(u,u) > |u|$ then we call u a stationary factor of w. The following lemmas will be useful in studying stationary factors, and factors of S_β.

Lemma 12. *Let $w \in \Sigma^+$. Then $\rho_{w^2}(|w|) = |w|$ if, and only if, w is primitive.*

Proof. It is clear that $\rho_{w^2}(|w|) \leqslant |w|$ for all $w \in \Sigma^+$. Assume that w is primitive yet $\rho_{w^2}(|w|) < |w|$. Then there exist $u_1, u_2, v_1, v_2, x \in \Sigma^+$ such that $w^2 = u_1 x v_1 = u_2 x v_2$ with $x \sim w$ and $|u_1| \neq |u_2|$. Assume without loss of generality, that $u_1 \rhd u_2$ and $v_2 \rhd v_1$. Then there exist words $u_3, v_3 \in \Sigma^+$ such that $u_2 = u_1 u_3$ and $v_1 = v_3 v_2$. Now $|x v_3| = |u_3 x|$, so $|v_3| = |u_3|$, hence $v_3 = u_3$. Thus $u_3 x = x u_3$ and so x is not primitive by the Lyndon–Schützenberger theorem. Whence w is not primitive, which is a contradiction. The other direction is easy. □

Lemma 13. *Let $0 < \beta < 1$ with $\beta = [0; b_0, \overline{b_1, \ldots, b_m}]$, where the $b_0, \ldots, b_m$ are positive integers and $b_m \geqslant b_0$. Let T be the fundamental homomorphism of S_β. Then if $x = (T^n F_t(0))_{-2}$ we have*

$$\{s \in \Sigma^* \colon s \mid S_\beta, |s| = |T^n F_t(0)|\}$$
$$= \{u \in \Sigma^* \colon u \sim T^n F_t(0)\} \cup \{x r_2(T^n F_t(0)) x\}$$

for $n \geqslant 0$ and $t = 0, 1, \ldots, m-1$.

Proof. From Corollary 11 we see that $(T^n F_t(0))^2 \mid S_\beta$ for $n \geqslant 0$ and $t = 0, \ldots, m-1$. Thus if $u \sim T^n F_t(0)$ then $u \mid S_\beta$. By Lemma 7 we know $T^n F_t(0)$ is primitive, thus by Lemma 13 $\rho_{(T^n F_t(0))^2}(|T^n F_t(0)|) = |T^n F_t(0)|$. We show that $x r_2(T^n F_t(0)) x \mid S_\beta$. By Corollary 11 $T^n F_t(1) T^n F_t(0) \mid S_\beta$, and by Corollary 5 $l_2(T^n F_t(1)) = c(l_2(T^n F_t(0)))$, thus $x r_2(T^n F_t(0)) x$ is a factor of S_β. Now by Theorem 1, $\rho_{S_\beta}(|T^n F_t(0)|) = |T^n F_t(0)| + 1$, and hence the proof is complete. □

Lemma 14. *Let $0 < \beta < 1$ with $\beta = [0; b_0, \overline{b_1, \ldots, b_m}]$, where $b_0, b_1, \ldots, b_m$ are positive integers and $b_m \geqslant b_0$. Let T be the fundamental homomorphism of S_β. The factors 1 and $x r_2(T^n F_t(0)) x$ where $x = (T^n F_t(0))_{-2}$ are stationary in S_β, for $n \geqslant 0$ and $t = 0, \ldots, m-1$.*

Proof. Since 11 is not a subword of S_β, 1 is clearly a stationary factor of S_β. Let t be an integer with $0 \leqslant t \leqslant m-1$. Since $r_2(T^n F_t(0)) xx \dagger S_\beta$, any occurrence of the word $xT^nF_t(0))x$ in S_β corresponds to an occurrence of the word $xT^nF_t(0)$. Now by Corollary 11 we have that

$$S_\beta = \prod_{i=0}^{\infty}(T^nF_t(0))^{s_t(k)}T^nF_t(1),$$

where $s_t(k) = \lfloor (k+1)\beta_t^{-1} \rfloor - \lfloor k\beta_t^{-1} \rfloor$; with $\beta_t = [0; b_{t+1}, \ldots, b_m, \overline{b_1, \ldots, b_m}]$. Thus, since $T^nF_t(0)$ is a primitive word by Lemma 7, by the Lyndon–Schützenberger theorem it follows that any occurrence of $xr_r(T^nF_T(0))x$ must occur in the subword $T^nF_t(0)T^n F_t(1)T^nF_t(0)$. As

$$T^nF_t(1)T^nF_t(0) = T^nF_t(0)c(T^nF_t(1)),$$

it follows from the previous argument that the word $xr_r(T^nF_T(0))x$ must occur in the word $T^nF_t(1)T^nF_t(0)$, and only occurs once. The result now follows. □

5. The main theorems

We are now in a position to prove a generalization of Séébold's theorem, and thus determine the critical exponent of S_β under the condition that S_β is the fixed point of a homomorphism.

Theorem 15. *Let* $0<\beta<1$ *with* $\beta = [0; b_0, \overline{b_1, \ldots, b_m}]$, *where* $b_0, \ldots, b_m$ *are positive integers and* $b_m \geqslant b_0$. *Let* T *be the fundamental homomorphism of* S_β *and* $k = 2 + \max_{1\leqslant i\leqslant m} b_i$. *If* $u \in \Sigma^+$ *with* $u^k \mid S_\beta$, *then* $u \sim T^nF_t(0)$ *for some* $n \geqslant 0$ *and some* $0 \leqslant t \leqslant m-1$.

Proof. Let $u \in \Sigma^+$ such that $u^k \mid S_\beta$ where $k = 2 + \max_{1\leqslant i\leqslant m} b_i$. Assume that $u \not\sim T^n F_t(0)$ for any integer $n \geqslant 0$ and integer t with $-1 \leqslant t \leqslant m-1$. We first note that $|u| \neq |T^nF_t(0)|$ for any integer $n \geqslant 0$ and integer t with $-1 \leqslant t \leqslant m-1$. For if $|u| = |T^n F_t(0)|$ then by Lemmas 13 and 14, u would be a stationary factor of S_β, which contradicts $k \geqslant 3$. Now by Lemma 3, for an integer $n \geqslant 1$, $T^nF_{-1}(0) = T^{n-1}F_{m-1}(0)$. Thus there exists an $n \geqslant 0$ and an integer t with $0 \leqslant t \leqslant m-1$ (with $n \geqslant 1$ if $t = 0$) such that

$$|T^nF_{t-1}(0)| < |u| < |T^nF_t(0)|. \tag{1}$$

Hence $u = vw$ where $|v| = |T^nF_{t-1}(0)|$. Now from (1) and Lemma 3

$$\begin{aligned} 0 < |w| &< |T^nF_t(0)| - |T^nF_{t-1}(0)| \\ &= (b_t - 1)|T^nF_{t-1}(0)| + |T^nF_{t-1}(1)|. \end{aligned} \tag{2}$$

By Lemma 13 we must have either $v \sim T^nF_{t-1}(0)$ or $v = xr_2(T^nF_{t-1}(0))x$ where $x = (T^nF_{t-1}(0))_{-2}$. We consider the two cases separately. Let us first assume that $v = xr_2 (T^nF_{t-1}(0))x$. Let $y = (T^nF_{t-1}(0))_{-1}$. Since v is stationary, we need to determine

$\lambda = \mathrm{dist}_{S_\beta}(v,v) - |v|$. By Lemma 14, $\lambda > 0$. Now by Corollary 11 we have that

$$S_\beta = \prod_{i=0}^{\infty} (T^n F_{t-1}(0))^{s_{t-1}(i)} T^n F_{t-1}(1), \tag{3}$$

where $s_{t-1}(i) = \lfloor (i+1)\beta_{t-1}^{-1} \rfloor - \lfloor i\beta_{t-1}^{-1} \rfloor$ and

$$\beta_{t-1} = [0; b_t, b_{t+1}, \ldots, b_m, \overline{b_1, \ldots, b_m}].$$

Consider when $1 \leqslant t \leqslant m-1$ with $n \geqslant 0$. Then $|T^n F_{t-1}(0)| > |T^n F_{t-1}(1)|$. Now $v \mid S_\beta$ only where $d := T^n F_{t-1}(1) T^n F_{t-1}(0) \mid S_\beta$. From (3), two consecutive occurrences of d in S_β are separated by $(T^n F_{t-1}(0))^{s_{t-1}(i)-1}$. Now

$$\begin{aligned} &T^n F_{t-1}(0) T^n F_{t-1}(1) (T^n F_{t-1}(0))^{s_{t-1}(i)} T^n F_{t-1}(1) T^n F_{t-1}(0) \\ &\quad = T^n F_{t-1}(1) r_2(T^n F_{t-1}(0)) y x r_2(T^n F_{t-1}(0)) x y (T^n F_{t-1}(0))^{s_{t-1}(i)-2} T^n F_{t-1}(1) \\ &\qquad r_2(T^n F_{t-1}(0)) y x r_2(T^n F_{t-1}(0)) x y. \end{aligned}$$

Thus,

$$\begin{aligned} \lambda &= |y (T^n F_{t-1}(0))^{\min(s_{t-1}(i)-2)} T^n F_{t-1}(1) r_2(T^n F_{t-1}(0)) y| \\ &= |(T^n F_{t-1}(0))^{\min(s_{t-1}(i)-1)} T^n F_{t-1}(1)| \\ &= (b_t - 1)|T^n F_{t-1}(0)| + |T^n F_{t-1}(1)|, \end{aligned}$$

and since $|w| \geqslant \lambda$ we get a contradiction of (2).

We now consider the case when $t = 0$ and $n \geqslant 1$. Here we look at two cases: (i) when $b_0 = b_m$ and (ii) when $b_0 < b_m$. If $b_0 = b_m$ then $T^n F_{-1} = T^n$ and $|T^n(0)| > |T^n(1)|$, and the argument follows as above. Now if $b_0 < b_m$ then $|T^n(0)| < |T^n(1)|$. Here we use that for $n \geqslant 1$, $T^n F_{-1}(0) = T^{n-1} F_{m-1}(0)$ and that $|T^{n-1} F_{m-1}(0)| > |T^{n-1} F_{m-1}(1)|$. By Corollary 11 we have that

$$S_\beta = \prod_{i=0}^{\infty} (T^{n-1} F_{m-1}(0))^{s_{m-1}(i)} T^{n-1} F_{m-1}(1), \tag{4}$$

where $s_{m-1}(i) = \lfloor (i+1)\beta_{m-1}^{-1} \rfloor - \lfloor i\beta_{m-1}^{-1} \rfloor$ and $\beta_{m-1} = [0; b_m, \overline{b_1, b_2, \ldots, b_m}]$. Hence from the previous argument we see that by Lemma 3

$$\begin{aligned} |w| \geqslant \lambda &= (b_m - 1)|T^{n-1} F_{m-1}(0)| + |T^{n-1} F_{m-1}(1)| \\ &= (b_0 - 1)|T^{n-1} F_{m-1}(0)| + (b_m - b_0)|T^{n-1} F_{m-1}(0)| + |T^{n-1} F_{m-1}(0)| \\ &= (b_0 - 1)|T^n F_{-1}(0)| + |T^n F_{-1}(1)|, \end{aligned}$$

which contradicts (2).

Assume now that $v \sim T^n F_{t-1}(0)$; in fact, we may assume, without loss of generality, that $v = T^n F_{t-1}(0)$. Consider when $1 \leqslant t \leqslant m-1$. Consider the word u_1 which occurs in S_β and with respect to the factorization of Corollary 11 is represented by

$$u_1 = v^{s_{t-1}(i)-j} T^n F_{t-1}(1) v^l \tag{5}$$

with $0 \leqslant j \leqslant s_{t-1}(i) - 1$ and $0 \leqslant l \leqslant s_{t-1}(i+1) + 1$. Assume that $u_1^3 \mid S_\beta$ with the representation by (5) of u_1 representing the first occurrence of u_1 in u_1^3. This implies that

$$\begin{aligned} u_1^3 &= v^{s_{t-1}(i)-j} T^n F_{t-1}(1) v^l v^{s_{t-1}(i)-j} T^n F_{t-1}(1) v^l v^{s_{t-1}(i)-j} T^n F_{t-1}(1) v^l \\ &= v^{s_{t-1}(i)-j} T^n F_{t-1}(1) v^{s_{t-1}(i)+l-j} T^n F_{t-1}(1) v^{s_{t-1}(i)+l-j} T^n F_{t-1}(1) v^l \\ &\lhd v^{s_{t-1}(i)-j} T^n F_{t-1}(1) v^{s_{t-1}(i+1)} T^n F_{t-1}(1) v^{s_{t-1}(i+2)} T^n F_{t-1}(1) v^{s_{t-1}(i+3)} \end{aligned}$$

which gives us that

$$\begin{aligned} & v^{s_{t-1}(i)+l-j} T^n F_{t-1}(1) v^{s_{t-1}(i)+l-j} T^n F_{t-1}(1) v^l \\ & \quad \lhd v^{s_{t-1}(i+1)} T^n F_{t-1}(1) v^{s_{t-1}(i+2)} T^n F_{t-1}(1) v^{s_{t-1}(i+3)}. \end{aligned} \tag{6}$$

Now $m(i)+l-j \geqslant 1$, for $k \geqslant 0$, $s_{t-1}(k) \geqslant 1$ and $|v| > |T^n F_{t-1}(1)|$. Thus since $T^n F_{t-1}(1) \lhd v^2$ and $T^n F_{t-1}(1) v \neq v T^n F_{t-1}(1)$, we must have that $s_{t-1}(i)+l-j = m(i+1)$. Hence $l-j \in \{-1, 0, 1\}$. Now if $l-j \in \{0, 1\}$ then

$$\begin{aligned} |u_1| - |T^n F_{t-1}(0)| &= |v^{s_{t-1}(i)+l-j-1} T^n F_{t-1}(1)| \\ &\geqslant \min(s_{t-1}(i) + l - j - 1)|T^n F_{t-1}(0)| + |T^n F_{t-1}(1)| \\ &= (b_t - 1)|T^n F_{t-1}(0)| + |T^n F_{t-1}(1)|. \end{aligned}$$

Consider when $l-j = -1$. Again from (6) we either have $s_{t-1}(i) - 1 = s_{t-1}(i+2)$ or $s_{t-1}(i) = s_{t-1}(i+2)$. If $s_{t-1}(i) - 1 = s_{t-1}(i+2)$ then $s_{t-1}(i+1) = s_{t-1}(i) - 1 = s_{t-1}(i+2)$ which is a contradiction. If $s_{t-1}(i) = s_{t-1}(i+2)$ then $s_{t-1}(i) = b_t + 1$, hence

$$\begin{aligned} |u_1| - |T^n F_{t-1}(0)| &= (s_{t-1}(i) - 2)|T^n F_{t-1}(0)| + |T^n F_{t-1}(1)| \\ &= (b_t - 1)|T^n F_{t-1}(0)| + |T^n F_{t-1}(1)|. \end{aligned}$$

Now we have that $u^3 \mid S_\beta$ so $vwvwvw \mid S_\beta$. We look at the possible positions within S_β where the word $vwvwvw$ occurs. Thus we must look at the possible positions of v within S_β. Now we have that

$$S_\beta = \prod_{i=0}^{\infty} v^{s_{t-1}(i)} T^n F_{t-1}(1),$$

and so since v is primitive, by the Lyndon–Schützenberger theorem, we have determined most of the possible positions of v in S_β. There only remains the possibility that v occurs somewhere in the factor $v T^n F_{t-1}(1) v$, other than as a prefix or suffix. We have that $T^n F_{t-1}(1) v = v c(T^n F_{t-1}(1))$, and we claim that this is the only other possible position for v. Now $|c(T^n F_{t-1}(1))| < |v|$, thus if another position of v was possible then $v \mid v c(T^n F_{t-1}(1)))$. Then by Corollary 5 and the Lyndon–Schützenberger theorem $v = x^m$ for some integer $m \geqslant 2$ with $x \in \Sigma^+$, which contradicts that v is primitive.

If the first occurrence of v in $vwvwvw$ in S_β occurs in $v^{s_{t-1}(i)}$ for some i, there must exist integers j and l such that $0 \leqslant j \leqslant s_{t-1}(i)-1$ and $0 \leqslant l \leqslant s_{t-1}(i+1)+1$ such that $vw = u_1$. But then

$$|w| \geqslant (b_t - 1)|T^n F_{t-1}(0)| + |T^n F_{t-1}(1)|$$

which is a contradiction. Thus, the first occurrence of v in $vwvwvw$ in S_β must occur in the factor $T^n F_{t-1}(1)v$. Hence, it easily follows that for some i

$$\begin{aligned}\lambda &\geqslant |c(T^n F_{t-1}(1))v^{s_{t-1}(i)-1}|\\ &\geqslant \min(s_{t-1}(i)-1)|T^n F_{t-1}(0)| + |T^n F_{t-1}(1)|\\ &= (b_t - 1)|T^n F_{t-1}(0)| + |T^n F_{t-1}(1)|\end{aligned}$$

which is again a contradiction.

Lastly, we must consider when $t=0$ with $n \geqslant 1$. Here we look at two cases: (i) when $b_0 = b_m$ and (ii) when $b_0 < b_m$. If $b_0 = b_m$ then $T^n F_{-1} = T^n$ and $|T^n(0)| > |T^n(1)|$, and the argument follows as before. If $b_0 < b_m$ then $|T^n(0)| < |T^n(1)|$. Here we use that for $n \geqslant 1$ $T^n F_{-1}(0) = T^{n-1} F_{m-1}(0)$ and that $|T^{n-1}F_{m-1}(0)| > |T^{n-1}F_{m-1}(1)|$. Thus, from above we have that

$$|w| \geqslant \lambda = (b_m - 1)|T^{n-1}F_{m-1}(0)| + |T^{n-1}F_{m-1}(1)|(b_0 - 1)|T^n F_{-1}(0)| + |T^n F_{-1}(1)|$$

which is a contradiction. The result now follows. □

The above theorem enables us to determine the critical exponent of S_β when S_β is the fixed point of a homomorphism.

Theorem 16. *Let $0 < \beta < 1$ with $\beta = [0; b_0, \overline{b_1, \dots, b_m}]$, where $b_0, \dots, b_m$ are positive integers and $b_m \geqslant b_0$. Then S_β has a critical exponent Ω, where*

$$\Omega = \max_{1 \leqslant t \leqslant m} [2 + b_t; b_{t-1}, \dots, b_1, \overline{b_m, \dots, b_1}].$$

Proof. We first show that given any $\varepsilon > 0$ there exists a $v \in \Sigma^+$ and an r such that $\Omega - \varepsilon < r < \Omega$ with $v^r \mid S_\beta$. Let T be the fundamental homomorphism of S_β. For $n \geqslant 1$ and t with $0 \leqslant t \leqslant m-1$, by Corollary 11 we have that

$$S_\beta = \prod_{k=0}^{\infty} T^n F_t(0)^{s_t(k)} T^n F_t(1),$$

where $s_t(k) = \lfloor (k+1)\beta_t^{-1} \rfloor - \lfloor k\beta_t^{-1} \rfloor$ and $\beta_t = [0; b_{t+1}, b_{t+2}, \dots, b_m, \overline{b_1, \dots, b_m}]$. Now by Theorem 6 we have $T^n F_t(1) T^n F_t(0) = T^n F_t(0) c(T^n F_t(1))$, and by Lemma 3, we have $r_2(T^n F_t(1)) \lhd T^n F_t(0)$. Thus $(T^n T_t(0))^s \mid S_\beta$, where

$$s = 2 + b_{t+1} + \frac{|T^n F_t(1)| - 2}{|T^n F_t(0)|}.$$

Now,

$$\begin{aligned} T^nF_t &= (L^{b_0}XL^{b_1}X\cdots L^{b_{m-1}}XL^{b_m-b_0})^n L^{b_0}XL^{b_1}X\cdots L^{b_t}X \\ &= L^{b_0}X(L^{b_1}X\cdots L^{b_m}X)^n L^{b_1}X\cdots L^{b_t}X, \end{aligned}$$

thus if

$$[b_t; b_{t-1}, \ldots b_1, \overline{b_m, \ldots, b_1}^n, b_0] = \frac{P_0}{Q_0} \quad (P_0, Q_0) = 1$$

and

$$[b_t; b_{t-1}, \ldots b_1, \overline{b_m, \ldots, b_1}^n] = \frac{P_1}{Q_1} \quad (P_1, Q_1) = 1$$

then by Lemma 8

$$\begin{aligned} \frac{|T^nF_t(1)| - 2}{|T^nF_t(0)|} &= \frac{Q_0 + Q_1 - 2}{P_0 + P_1} \\ &< [b_t; b_{t-1}, \ldots, b_1, \overline{b_m, \ldots, b_1}]^{-1} \\ &= [0; b_t, b_{t-1}, \ldots, b_1, \overline{b_m, \ldots, b_1}] \end{aligned}$$

and

$$\begin{aligned} \lim_{n\to\infty} \frac{|T^nF_t(1)| - 2}{|T^nF_t(0)|} &= \lim_{n\to\infty} \frac{Q_0 + Q_1 - 2}{P_0 + P_1} \\ &= [0; b_t, \ldots, b_1, \overline{b_m, \ldots, b_1}]. \end{aligned}$$

Hence there exists an $n \geqslant 1$ and a t with $0 \leqslant t \leqslant m-1$ such that $(T^nF_t(0))^s \mid S_\beta$ with $\Omega - \varepsilon < s < \Omega$.

We now show that if $u \in \Sigma^+$ and $u^r \mid S_\beta$, then $r < \Omega$. If $r \geqslant 2 + \max_{1 \leqslant i \leqslant m-1} b_i$, then by Theorem 15 we have $u \sim T^nF^t(0)$ for some $n \geqslant 0$ and some t with $0 \leqslant t \leqslant m-1$. Then $r < \Omega$, hence the result follows. □

As an example consider if $\Phi = (\sqrt{5} - 1)/2$, then $\Phi = [0; 1, 1, 1, \ldots]$ and so by Theorem 16 the Fibonacci infinite word $\mathscr{F} = S_\Phi$ has a critical exponent Ω where

$$\begin{aligned} \Omega &= 2 + [1; 1, 1, 1, \ldots] \\ &= 3 + \tfrac{1}{2}(\sqrt{5} - 1) \\ &\doteq 3.618034 \end{aligned}$$

as originally determined by Mignosi and Pirillo [10] in 1992. We now prove an explicit version of Mignosi's theorem.

Theorem 17. *Let $\beta > 0$ be an irrational number with $\beta = [b_0; b_1, b_2, \ldots]$. Then S_β is kth power-free for some integer k if, and only if, β has bounded partial quotients. Moreover, if β has bounded partial quotients then S_β is kth power-free but not $(k-1)$th power-free for $k = 3 + \max_{i \geqslant 0} b_i$.*

Proof. Let $\beta > 0$. Since $XS_\beta = S_{\beta^{-1}}$ we may assume that $0 < \beta < 1$ with $\beta = [0; \overline{b_0, b_1, b_2, \ldots}]$. Let $\beta_n = [0; \overline{b_0, b_1, b_2, \ldots, b_n}]$, then $\beta_n \to \beta$ as $n \to \infty$. By Theorem 16 S_{β_n} is k_nth power-free but contains $(k_n - 1)$th powers for

$$k_n = 3 + \max_{0 \leqslant i \leqslant n} b_i.$$

So there exist finite words $w_n \lhd S_{\beta_n}$ with $w_n \lhd S_\beta$ and $|w_n| \to \infty$ as $n \to \infty$ such that w_n is k_nth power-free but contains $(k_n - 1)$th powers. Since $k_n \geqslant k_{n-1}$ the result now follows. □

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