Available at WWW.MATHEMATICSWEB.ORG
POWERED BY SCIENCE DIRECT

ACADEMIC PRESS

Advances in Applied Mathematics 30 (2003) 219–227

ADVANCES IN Applied Mathematics

www.elsevier.com/locate/aam

Witt's formula for restricted Lie algebras

V.M. Petrogradsky [*,1]

Faculty of Mathematics, Ulyanovsk State University, Lev Tolstoy 42, Ulyanovsk, 432700, Russia

Received 15 July 2001; accepted 20 December 2001

Abstract

Suppose that L is the free Lie algebra generated by k elements. This is a homogeneous algebra and the dimensions of its homogeneous components are given by Witt's formula $\dim L_n = \frac{1}{n}\sum_{a|n}\mu(a)k^{n/a}$, where $\mu(n)$ is the Möbius function. Recently the author discovered generating functions for the whole of the free Lie superalgebra. These series yield some new dimension formulas. Now we present generating functions for the free Lie p-algebras. As a corollary, we suggest dimension formulas similar to Witt's formula.

1. Introduction: Witt's formula for free Lie superalgebras

Suppose that W is a superalgebra generated by sets $X_+ = \{x_1, \dots, x_m\}$, $X_- = \{x_{m+1}, \dots, x_{m+k}\}$ and which is multihomogeneous with respect to these sets. For elements of W we introduce the *multidegree* $\alpha = (\alpha_1, \dots, \alpha_{m+k}) \in \mathbb{N}_0^{m+k}$, the *degree* $|\alpha| = \alpha_1 + \cdots + \alpha_{m+k}$, and the numbers $|\alpha_+| = \alpha_1 + \cdots + \alpha_m$, $|\alpha_-| = \alpha_{m+1} + \cdots + \alpha_{m+k}$, where $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. By $a|\alpha$ we denote that a divides all components α_i. One has components for the gradation by the multidegree W_α, by the degree W_n, and by the *superdegree* W_{ij}, where the latter space consists of elements of degree i with respect to $x_1, \dots, x_m$, and degree j with respect to $x_{m+1}, \dots, x_{m+k}$. Let $W_n = W_{n,+} \oplus W_{n,-}$ denote the decomposition into the even and odd components, one also defines the *superdimension* as $\operatorname{sdim} W_n = \dim W_{n,+} - \dim W_{n,-}$. For a multihomogeneous subspace $V \subset W$ we define the following *Hilbert–Poincaré series*

[*] Current address: Math. Institut, Heinrich-Heine Univ., Universitätsstr. 1, D-40225 Düsseldorf, Germany.
E-mail addresses: petrogra@math.uni-duesseldorf.de, petrogradsky@hotbox.ru.
[1] Adv. Appl. Math., FPSAC'01 in Arizona; Supported by grants RFBR 99-01-00233, 01-01-00728.

0196-8858/03/$ – see front matter
doi:10.1016/S0196-8858(02)00533-X

$$V = \bigoplus_{\alpha\in\mathbb{N}_0^{m+k}} V_\alpha, \qquad \mathcal{H}(V,t_1,\dots,t_{m+k}) = \sum_{\alpha\in\mathbb{N}_0^{m+k}} \dim V_\alpha\, t_1^{\alpha_1}\cdots t_{m+k}^{\alpha_{m+k}},$$

$$V = \bigoplus_{i,j=0}^{\infty} V_{ij}, \qquad \mathcal{H}(V,t_+,t_-) = \sum_{i,j=0}^{\infty} \dim V_{ij}\, t_+^i t_-^j,$$

$$V = \bigoplus_{n=0}^{\infty} V_n, \qquad \mathcal{H}(V,t) = \sum_{n=0}^{\infty} \dim V_n\, t^n = \mathcal{H}(V,t_+,t_-)|_{t_+=t_-=t}.$$

The generating functions for the polynomial rings and the free associative algebras are well known. Let $W = K[X]$, $X = \{x_1,\dots,x_m\}$, be the polynomial ring and $W = K\langle X\rangle$, $X = \{x_1,\dots,x_m\}$ the free associative algebra. Then

$$\mathcal{H}\big(K[X],t_1,\dots,t_m\big) = \frac{1}{(1-t_1)\cdots(1-t_m)},$$
$$\mathcal{H}\big(K\langle X\rangle,t_1,\dots,t_m\big) = \frac{1}{1-(t_1+\cdots+t_m)}. \tag{1}$$

These formulas have many applications. In particular, they are applied to study invariants [4,5,10]. In [11,12] the author found the generating functions for the free Lie (super)algebras. It seems that these formulas were not known before.

Theorem 1.1 [12]. *Let $L = L(X)$, $X = X_+ \cup X_-$ be the free Lie superalgebra where $X_+ = \{x_1,\dots,x_m\}$, $X_- = \{x_{m+1},\dots,x_{m+k}\}$ are the even and odd generators, respectively. Then*

$$\mathcal{H}(L,t_1,\dots,t_m,t_{m+1},\dots,t_{m+k}) = -\sum_{a=1}^{\infty} \frac{\mu(a)}{a} \ln\big(1-t_1^a-\cdots-t_m^a+(-t_{m+1})^a+\cdots+(-t_{m+k})^a\big),$$

$$\mathcal{H}(L,t_+,t_-) = -\sum_{a=1}^{\infty} \frac{\mu(a)}{a} \ln\big(1-mt_+^a+k(-t_-)^a\big),$$

$$\mathcal{H}(L,t) = -\sum_{a=1}^{\infty} \frac{\mu(a)}{a} \ln\big(1-\big(m-(-1)^a k\big)t^a\big).$$

In the case of free Lie (super) algebras there are formulas for dimensions of homogeneous and multihomogeneous components [1–3], see also [14]. But these formulas include two cases, they correspond to the fact that a basis of the homogeneous component either consists only of regular words or it also contains squares of regular odd words. We suggest new formulas without two cases (of course, they give the same values). See also a more general approach in generality of gradings by some semigroups [13].

Corollary 1.1 [12].

$$\dim L_n = \frac{1}{n}\sum_{a|n}\mu(a)\bigl(m-(-1)^a k\bigr)^{n/a},$$

$$\dim L_{n,+} = \frac{1}{n}\sum_{a|n}\mu(a)\frac{(m-(-1)^a k)^{n/a}+(m-k)^{n/a}}{2},$$

$$\dim L_{n,-} = \frac{1}{n}\sum_{a|n}\mu(a)\frac{(m-(-1)^a k)^{n/a}-(m-k)^{n/a}}{2},$$

$$\operatorname{sdim} L_n = \frac{1}{n}\sum_{a|n}\mu(a)(m-k)^{n/a}, \tag{2}$$

$$\dim L_\alpha = \frac{(-1)^{|\alpha_-|}}{|\alpha|}\sum_{a|\alpha}\mu(a)\frac{(|\alpha|/a)!\,(-1)^{|\alpha_-|/a}}{(\alpha_1/a)!\cdots(\alpha_{m+k}/a)!}. \tag{3}$$

The formula for the superdimensions (2) was obtained in [7,8]. Also, (3) was known for the case $X = X_-$ [9].

In [12] these formulas were proved directly. In [13] we consider another approach and a more general situation of the generating set $X = \bigcup_{\alpha\in\Gamma} X_\alpha$ being graded by a semigroup Γ, and instead of series we study characters for the whole of a superalgebra and for multihomogeneous components. As a particular case we consider the semigroup $\Gamma = \mathbb{N}^{m+k}\setminus\{0\}$ and monoid $\overline{\Gamma} = \mathbb{N}^{m+k}$. Denote $e_i = (\ldots,0,1,0,\ldots)$, $i = 1,\ldots,m+k$, with 1 on the ith place. We set $\varepsilon(e_1) = \cdots = \varepsilon(e_m) = 1$, $\varepsilon(e_{m+1}) = \cdots = \varepsilon(e_{m+k}) = -1$. We obtain the homomorphism $\varepsilon : \overline{\Gamma} \to \{\pm 1\}$ and the decomposition into the even and odd components $\Gamma = \Gamma_+ \cup \Gamma_-$, $\Gamma_+ = \{\alpha \in \Gamma \mid \varepsilon(\alpha) = 1\}$, $\Gamma_- = \{\alpha \in \Gamma \mid \varepsilon(\alpha) = -1\}$. We consider Lie superalgebras L that are decomposed into the even and odd components as follows $L = L_+ \oplus L_-$, $L_+ = \bigoplus_{\alpha\in\Gamma_+} L_\alpha$, $L_- = \bigoplus_{\alpha\in\Gamma_-} L_\alpha$. We denote $\mathbb{Q}[\![\mathbf{t}]\!] = \mathbb{Q}[\![t_1,\ldots,t_{m+k}]\!]$ and let $\mathbb{Q}[\![\mathbf{t}]\!]_0$ be the series without a constant term. An arbitrary series from $\mathbb{Q}[\![\mathbf{t}]\!]$ we write as $\phi(\mathbf{t})$. Suppose that $V = \bigoplus_{\alpha\in\overline{\Gamma}} V_\alpha$ is a $\overline{\Gamma}$-graded vector space, then we also denote $\mathcal{H}(V,\mathbf{t}) = \mathcal{H}(V,t_1,\ldots,t_{m+k})$.

We introduce two important operators as follows.

$$\mathcal{E}(\phi)(\mathbf{t}) = \exp\left(\sum_{b=1}^{\infty}\frac{1}{b}\phi\Big|_{(t_i=\varepsilon(t_i)^{b+1}t_i^b)}\right), \quad \phi(\mathbf{t}) \in \mathbb{Q}[\![\mathbf{t}]\!]_0,$$

$$\mathcal{L}(\phi)(\mathbf{t}) = \sum_{a=1}^{\infty}\frac{\mu(a)}{a}\ln\bigl(\phi|_{(t_i=\varepsilon(t_i)^{a+1}t_i^a)}\bigr), \quad \phi(\mathbf{t}) \in 1+\mathbb{Q}[\![\mathbf{t}]\!]_0.$$

The application of these operators is based on the following fact.

Theorem 1.2 [13]. *Let $L=\bigoplus_{\alpha\in\Gamma} L_\alpha$ be a Γ-graded Lie superalgebra, $\Gamma=\mathbb{N}^{m+k}\setminus\{0\}$. Then the Hilbert–Poincaré series for L and its universal enveloping algebra $U(L)$ are connected as follows*:

(1) $\mathcal{H}(U(L),\mathbf{t})=\mathcal{E}(\mathcal{H}(L,\mathbf{t}))$;
(2) $\mathcal{H}(L,\mathbf{t})=\mathcal{L}(\mathcal{H}(U(L),\mathbf{t}))$.

Let $L=L(X)$ be the free Lie superalgebra generated by $X=X_+\cup X_-$, where $X_+=\{x_1,\dots,x_m\}$ and $X_-=\{x_{m+1},\dots,x_{m+k}\}$ are the even and odd generators. It is well known that $U(L)\cong K\langle X\rangle$ [2]. Now using (1) and the second formula one can easily derive the formula for $\mathcal{H}(L,\mathbf{t})$ given by Theorem 1.1.

2. Witt's formula for free restricted Lie algebras

Let L be a Lie algebra over a field K of characteristic $p>0$. We use standard notation $\operatorname{ad}x : L\to L$, $\operatorname{ad}x(y)=[x,y]$, $x,y\in L$. Recall that L is called the *restricted Lie algebra* or *Lie p-algebra*, [6] if it additionally affords a unary operation $x\mapsto x^{[p]}$, $x\in L$, satisfying

(1) $(\lambda x)^{[p]}=\lambda^p x^{[p]}$, when $\lambda\in K$, $x\in L$;
(2) $\operatorname{ad}(x^{[p]})=(\operatorname{ad}x)^p$;
(3) $(x+y)^{[p]}=x^{[p]}+y^{[p]}+\sum_{i=1}^{p-1}s_i(x,y)$; $x,y\in L$, where $is_i(x,y)$ is the coefficient of t^{i-1} in the polynomial $\operatorname{ad}(tx+y)^{p-1}(x)\in L[t]$.

This notion is motivated by the following construction. Let A be an associative algebra over K, $\operatorname{char}K=p>0$. The algebra A is turned into a Lie algebra by the new operation $[x,y]=xy-yx$, where $x,y\in A$. Then the mapping $x\mapsto x^p$, $x\in A$ satisfies these three properties.

Suppose that L is a restricted Lie algebra. Let J be the ideal in the universal enveloping algebra $U(L)$ generated by all elements $x^{[p]}-x^p$, $x\in L$. Then $u(L)=U(L)/J$ is called the *restricted enveloping algebra*. In this algebra the formal operation $x^{[p]}$ coincides with the pth power x^p for all $x\in L$. Let $\{a_i \mid i\in I\}$ be a linearly ordered basis for L. By an analogue of Poincaré–Birkhoff–Witt's theorem [6], the restricted enveloping algebra has the following canonical basis:

$$u(L)=\bigl\langle a_{q_1}^{n_1}\cdots a_{q_s}^{n_s}\bigm| q_1<\cdots<q_s,\ 0\leqslant n_i<p;\ s\geqslant 0\bigr\rangle_K. \tag{4}$$

Recall that a function of a natural argument $\alpha(n)$ is called *multiplicative* if $\alpha(nm)=\alpha(n)\alpha(m)$ for any coprime n,m. We introduce the following two functions:

$$1_p(n)=\begin{cases}1, & (p,n)=1,\\ 1-p, & (p,n)=p,\end{cases}$$

$$\mu_p(n)=\begin{cases}\mu(n), & (p,n)=1,\\ \mu(m)\bigl(p^s-p^{s-1}\bigr), & n=mp^s,\ (p,m)=1,\ s\geqslant 1.\end{cases}$$

It is easy to check that these functions are multiplicative. One may view $\mu_p(n)$ as a deformation of the Möbius function.

By analogy with ordinary Lie algebras, one defines the free restricted Lie algebra L [1]. The goal of this paper is to find generating functions and dimension formulas for L. These formulas are completely analogous to those for the free Lie superalgebras.

Theorem 2.1. *Let $L = L_p(X)$ be the free Lie p-algebra generated by $X = \{x_1, \dots, x_m\}$. Then*

$$\mathcal{H}(L, t_1, \dots, t_m) = -\sum_{a=1}^{\infty} \frac{\mu_p(a)}{a} \ln\bigl(1 - t_1^a - \cdots - t_m^a\bigr),$$

$$\mathcal{H}(L, t) = -\sum_{a=1}^{\infty} \frac{\mu_p(a)}{a} \ln(1 - mt^a).$$

Corollary 2.1. *Let $L = L_p(X)$ be the free Lie p-algebra generated by $X = \{x_1, \dots, x_m\}$. Then*

$$\dim L_n = \frac{1}{n} \sum_{a|n} \mu_p(a) m^{n/a},$$

$$\dim L_\alpha = \frac{1}{|\alpha|} \sum_{a|\alpha_i} \mu_p(a) \frac{(|\alpha|/a)!}{(\alpha_1/a)! \cdots (\alpha_m/a)!}.$$

It is possible to prove these formulas directly. Denote $\mathbb{Q}[\![\mathbf{t}]\!] = \mathbb{Q}[\![t_1, \dots, t_m]\!]$. Suppose that $L(X)$ is the free Lie algebra generated by X. One has the natural embedding $L(X) \subset U(L(X)) \cong K\langle X\rangle$. Let $L_p(X) = \langle v^{p^n} \mid v \in L, n \geqslant 0\rangle_K$, then $L_p(X)$ is the free restricted Lie algebra generated by X and the p-operation $v \mapsto v^p$, $v \in L$; moreover $u(L_p(X)) \cong K\langle X\rangle$ [2]. Also, one has the following construction of a basis for $L_p(X)$. Suppose that $\{w_\alpha \mid \alpha \in \Lambda\}$ is a homogeneous basis for $L(X)$, then $\{w_\alpha^{[p^n]} \mid \alpha \in \Lambda, n \geqslant 0\}$ is a basis for the free p-algebra $L_p(X)$ [2]. This observation leads to the formula

$$\mathcal{H}\bigl(L_p(X), \mathbf{t}\bigr) = \sum_{n=0}^{\infty} \mathcal{H}\bigl(L(X), \mathbf{t}\bigr)\Big|_{\left(t_i = t_i^{p^n}\right)}.$$

Now it is possible to use Theorem 1.1 and Corollary 1.1.

But let us consider another method. We introduce a dilatation on the formal power series

$$^{[b]} : \mathbb{Q}[\![\mathbf{t}]\!] \to \mathbb{Q}[\![\mathbf{t}]\!], \; (\phi)^{[b]}(\mathbf{t}) = \phi|_{(t_i = t_i^b)}, \quad b \in \mathbb{N}.$$

Remark that to study the case of Lie superalgebras we needed an operator of a *twisted dilatation* [13]. The following properties are easily checked.

Lemma 2.1. (1) $f^{[1]} = f$.
(2) *The twisted dilatation* $f \mapsto f^{[b]}$ *is an endomorphism of the algebra* $\mathbb{Q}[\![\mathbf{t}]\!]$.
(3) $(f^{[a]})^{[b]} = (f^{[b]})^{[a]} = f^{[ab]}$, *where* $f \in \mathbb{Q}[\![\mathbf{t}]\!]$, $a, b \in \mathbb{N}$.

We suggest a modification of operators $\mathcal{E}$, $\mathcal{L}$ as follows:

$$\mathcal{E}_p(\phi)(\mathbf{t}) = \exp\left(\sum_{b=1}^{\infty} \frac{1_p(b)}{b} \phi(\mathbf{t})^{[b]}\right), \quad \phi(\mathbf{t}) \in \mathbb{Q}[\![\mathbf{t}]\!]_0,$$

$$\mathcal{L}_p(\phi)(\mathbf{t}) = \sum_{a=1}^{\infty} \frac{\mu_p(a)}{a} \ln\bigl(\phi(\mathbf{t})^{[a]}\bigr), \quad \phi(\mathbf{t}) \in 1 + \mathbb{Q}[\![\mathbf{t}]\!]_0.$$

Lemma 2.2. *The mappings* $\mathcal{E}_p : \mathbb{Q}[\![\mathbf{t}]\!]_0 \to 1 + \mathbb{Q}[\![\mathbf{t}]\!]_0$ *and* $\mathcal{L}_p : 1 + \mathbb{Q}[\![\mathbf{t}]\!]_0 \to \mathbb{Q}[\![\mathbf{t}]\!]_0$ *are well-defined and mutually inverse.*

Proof. We denote $\mathbf{t}^\alpha = t_1^{\alpha_1} \cdots t_m^{\alpha_m}$, for $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$. By definition of these operators, we obtain only finitely many terms corresponding to each $\mathbf{t}^\alpha$. Therefore, these operators are well-defined.

First, let us prove the property $\sum_{ab=n} 1_p(b)\mu_p(a) = 0$ for all $n > 1$, that resembles the property of the Möbius function $\sum_{ab=n} \mu(a) = 0$, $n > 1$. Indeed, let us decompose $n = n'p^k$, $(p, n') = 1$ and $a = a'p^\alpha$, $(p, a') = 1$; $b = b'p^\beta$, $(p, b') = 1$. In case $k = 0$ the statement follows from the property of the Möbius function. Consider the case $k \geqslant 1$.

$$\begin{aligned}
\sum_{ab=n} 1_p(b)\mu_p(a) &= \sum_{a'b'=n'} 1_p(b')\mu_p(a') \sum_{\alpha+\beta=k} 1_p\bigl(p^\beta\bigr)\mu_p(p^\alpha) \\
&= \Bigl(\sum_{a'b'=n'} \mu(a')\Bigr)\Bigl(1\bigl(p^k - p^{k-1}\bigr) + (1-p)\bigl(p^{k-1} - p^{k-2}\bigr) + \cdots \\
&\qquad + (1-p)(p-1) + (1-p)1\Bigr) \\
&= \Bigl(\sum_{a'b'=n'} \mu(a')\Bigr)\Bigl(1\bigl(p^k - p^{k-1}\bigr) + (1-p)p^{k-1}\Bigr) = 0.
\end{aligned}$$

Let $f = f(\mathbf{t}) \in \mathbb{Q}[\![\mathbf{t}]\!]_0$; then

$$\begin{aligned}
\mathcal{L}_p(\mathcal{E}_p f) &= \mathcal{L}_p\left(\exp\left(\sum_{b=1}^{\infty} \frac{1_p(b)}{b} f^{[b]}\right)\right) = \sum_{a=1}^{\infty} \frac{\mu_p(a)}{a} \sum_{b=1}^{\infty} \frac{1_p(b)}{b} \bigl(f^{[b]}\bigr)^{[a]} \\
&= \sum_{a=1}^{\infty}\sum_{b=1}^{\infty} \frac{f^{[ba]}}{ba} 1_p(b)\mu_p(a) = \sum_{n=1}^{\infty} \frac{f^{[n]}}{n} \sum_{ab=n} 1_p(b)\mu_p(a) = f^{[1]} = f.
\end{aligned}$$

The property $\mathcal{E}_p(\mathcal{L}_p f) = f$, $f = f(\mathbf{t}) \in 1 + \mathbb{Q}[\![\mathbf{t}]\!]_0$ is proved in the same way. $\square$

Now suppose that L is an $\mathbb{N}^m \setminus \{0\}$-graded restricted Lie algebra, here we additionally suppose that for all $x \in L_\alpha$, $\alpha \in \mathbb{N}^m \setminus \{0\}$, we have $x^{[p]} \in L_{p\alpha}$.

Theorem 2.2. *Let $L = \bigoplus_{\alpha\in\Gamma} L_\alpha$ be a Γ-graded restricted Lie algebra, $\Gamma = \mathbb{N}^m \setminus \{0\}$. Then the Hilbert–Poincaré series for L and its restricted enveloping algebra $u(L)$ are connected as follows*:

(1) $\mathcal{H}(u(L), \mathbf{t}) = \mathcal{E}_p(\mathcal{H}(L, \mathbf{t}))$;
(2) $\mathcal{H}(L, \mathbf{t}) = \mathcal{L}_p(\mathcal{H}(u(L), \mathbf{t}))$.

Proof. Let $\{a_i \mid i \in I\}$ be an ordered homogeneous basis for L. Using canonical basis for restricted enveloping algebra (4), we obtain

$$\begin{aligned}
\mathcal{H}\bigl(u(L), \mathbf{t}\bigr) &= \prod_{\alpha\in\Gamma} \bigl(1 + \mathbf{t}^\alpha + \mathbf{t}^{2\alpha} + \cdots + \mathbf{t}^{(p-1)\alpha}\bigr)^{\dim L_\alpha} = \prod_{\alpha\in\Gamma} \left(\frac{1-\mathbf{t}^{p\alpha}}{1-\mathbf{t}^\alpha}\right)^{\dim L_\alpha} \\
&= \exp\left(\sum_{\alpha\in\Gamma} \dim L_\alpha \bigl(-\ln(1-\mathbf{t}^\alpha) + \ln(1-\mathbf{t}^{p\alpha})\bigr)\right) \\
&= \exp\left(\sum_{\alpha\in\Gamma} \dim L_\alpha \left(\sum_{b=1}^{\infty} \frac{\mathbf{t}^{b\alpha}}{b} - \sum_{b=1}^{\infty} \frac{\mathbf{t}^{pb\alpha}}{b}\right)\right) \\
&= \exp\left(\sum_{\alpha\in\Gamma} \dim L_\alpha \sum_{b=1}^{\infty} \mathbf{t}^{b\alpha} \frac{1_p(b)}{b}\right) = \exp\left(\sum_{b=1}^{\infty} \frac{1_p(b)}{b} \sum_{\alpha\in\Gamma} \dim L_\alpha \mathbf{t}^{b\alpha}\right) \\
&= \exp\left(\sum_{b=1}^{\infty} \frac{1_p(b)}{b} \mathcal{H}(L, \mathbf{t})^{[b]}\right) = \mathcal{E}_p\bigl(\mathcal{H}(L, \mathbf{t})\bigr).
\end{aligned}$$

The another relation follows by Lemma 2.2:

$$\mathcal{H}(L, \mathbf{t}) = (\mathcal{L}_p\mathcal{E}_p)\mathcal{H}(L, \mathbf{t}) = \mathcal{L}_p\bigl(\mathcal{E}_p\mathcal{H}(L, \mathbf{t})\bigr) = \mathcal{L}_p\bigl(\mathcal{H}\bigl(u(L), \mathbf{t}\bigr)\bigr). \qquad \square$$

Proof of Theorem 2.1. Let $L = L_p(X)$ be the free restricted Lie algebra generated by $X = \{x_1, \ldots, x_m\}$. It is well known that $u(L) \cong K\langle X\rangle$ [2]. Using (1) and the second formula of Theorem 2.2, we immediately derive the formula for $\mathcal{H}(L, \mathbf{t})$:

$$\begin{aligned}
\mathcal{H}(L, \mathbf{t}) &= \mathcal{L}_p\bigl(\mathcal{H}\bigl(u(L), \mathbf{t}\bigr)\bigr) = \mathcal{L}_p\bigl(\mathcal{H}\bigl(K\langle X\rangle, \mathbf{t}\bigr)\bigr) = \mathcal{L}_p \frac{1}{1 - t_1 - \cdots - t_m} \\
&= -\sum_{a=1}^{\infty} \frac{\mu_p(a)}{a} \ln\bigl(1 - t_1^a - \cdots - t_m^a\bigr). \qquad \square
\end{aligned}$$

Proof of Corollary 2.1. Let us expand the formula for $\mathcal{H}(L, t)$ given by Theorem 2.1:

$$\begin{aligned}\mathcal{H}(L,t) &= -\sum_{a=1}^{\infty}\frac{\mu_p(a)}{a}\ln(1-mt^a)=\sum_{a=1}^{\infty}\frac{\mu_p(a)}{a}\sum_{s=1}^{\infty}\frac{m^s}{s}t^{as}\\ &= \sum_{a=1}^{\infty}\sum_{s=1}^{\infty}\frac{t^{as}}{as}\mu_p(a)m^s=\sum_{n=1}^{\infty}\frac{t^n}{n}\sum_{a|n}\mu_p(a)m^{n/a}.\end{aligned}$$

On the other hand, $\mathcal{H}(L,t)=\sum_{n=1}^{\infty}\dim L_n t^n$, hence $\dim L_n=\frac{1}{n}\sum_{a|n}\mu_p(a)m^{n/a}$. The second formula follows by similar arguments:

$$\begin{aligned}\mathcal{H}(L,t_1,\dots,t_m) &= -\sum_{a=1}^{\infty}\frac{\mu_p(a)}{a}\ln\bigl(1-t_1^a-\cdots-t_m^a\bigr)\\ &= \sum_{a=1}^{\infty}\frac{\mu_p(a)}{a}\sum_{s=1}^{\infty}\frac{(t_1^a+\cdots+t_m^a)^s}{s}\\ &= \sum_{a=1}^{\infty}\frac{\mu_p(a)}{a}\sum_{s=1}^{\infty}\frac{1}{s}\sum_{|\beta|=s}\frac{|\beta|!}{\beta_1!\cdots\beta_m!}t_1^{a\beta_1}\cdots t_m^{a\beta_m}\\ &= \sum_{\alpha\in\mathbb{N}_0^m\setminus\{0\}}\frac{1}{|\alpha|}\sum_{a|\alpha_i}\mu_p(a)\frac{(|\alpha|/a)!}{(\alpha_1/a)!\cdots(\alpha_m/a)!}t_1^{\alpha_1}\cdots t_m^{\alpha_m}.\end{aligned}$$

Therefore,

$$\dim L_\alpha=\frac{1}{|\alpha|}\sum_{a|\alpha_i}\mu_p(a)\frac{(|\alpha|/a)!}{(\alpha_1/a)!\cdots(\alpha_m/a)!}. \qquad \square$$

Remark. One may consider the generality of *restricted Lie superalgebras* (see for definitions [2]). In this case analogous formulas are not so nice.

References

[1] Yu.A. Bahturin, Identical Relations in Lie Algebras, VNV Science Press, Utrecht, 1987.
[2] Yu.A. Bahturin, A.A. Mikhalev, V.M. Petrogradsky, M.V. Zaicev, Infinite Dimensional Lie Superalgebras, in: de Gruyter Exp. Math., Vol. 7, de Gruyter, Berlin, 1992.
[3] N. Bourbaki, Groupes et algèbres de Lie, Chapters 1–3, Hermann, Paris, 1968.
[4] N. Bourbaki, Groupes et algèbres de Lie, Chapters 4–6, Hermann, Paris, 1972.
[5] E. Formanek, Noncommutative Invariant Theory, Group actions on rings, in: Contemp. Math., Vol. 43, Amer. Math. Society, Providence, RI, 1985, 87–119.
[6] N. Jacobson, Lie Algebras, Interscience, New York, 1962.
[7] S.-J. Kang, Graded Lie superalgebras and the superdimension formula, J. Algebra 204 (2) (1998) 597–655.
[8] S.-J. Kang, Kim Myung-Hwan, Dimension formula for graded Lie algebras and its applications, Trans. Amer. Math. Soc. 351 (11) (1999) 4281–4336.
[9] A.I. Molev, L.M. Tsalenko, Representation of the symmetric group in the free Lie (super) algebra and in the space of harmonic polynomials, Funktsional. Anal. i Prilozhen. 20 (2) (1986) 76–77; English transl.: Funct. Anal. Appl. 20 (1986) 150–152.

[10] T. Molien, Über die Invarianten der linearen Substitutionsgruppen, Sitzungsber. König. Preuss. Akad. Wiss. (1897) 1152–1156.
[11] V.M. Petrogradsky, On invariants of actions of finite groups on free Lie algebras, Sibirsk. Mat. Zh. 41 (4) (2000) 915–925.
[12] V.M. Petrogradsky, On Witt's formula and invariants for free Lie superalgebras, in: Formal Power Series and Algebraic Combinatorics, Moscow, 2000, Springer-Verlag, Berlin, 2000, pp. 543–551.
[13] V.M. Petrogradsky, Characters and invariants for free Lie superalgebras, Algebra i Analiz 13 (1) (2001) 158–181; English transl.: St. Petersburg Math. J. 13 (2002) 107–122.
[14] R. Ree, Generalized Lie elements, Canad. J. Math. 12 (1960) 493–502.