JOURNAL OF FUNCTIONAL ANALYSIS **160**, 337–365 (1998)
ARTICLE NO. FU983335

Geometry and Curvature of Diffeomorphism Groups with H^1 Metric and Mean Hydrodynamics

Steve Shkoller

Center for Nonlinear Studies, MS-B258, Los Alamos, New Mexico 87545; and CDS, California Institute of Technology, 107-81, Pasadena, California 91125
E-mail: shkoller@cds.caltech.edu

Received June 16, 1998; accepted July 10, 1998

$V=(1-\alpha^2\Delta)\,U$. In this model, the momentum V is transported by the velocity U, with the effect that nonlinear interaction between modes corresponding to length scales smaller than α is negligible. We generalize this equation to the setting of an n-dimensional compact Riemannian manifold. The resulting equation is the Euler–Poincaré equation associated with the geodesic flow of the H^1 right invariant metric on $\mathscr{D}^s_\mu$, the group of volume preserving Hilbert diffeomorphisms of class H^s. We prove that the geodesic spray is continuously differentiable from $T\mathscr{D}^s_\mu(M)$ into $TT\mathscr{D}^s_\mu(M)$ so that a standard Picard iteration argument proves existence and uniqueness on a finite time interval. Our goal in this paper is to establish the foundations for Lagrangian stability analysis following Arnold (*Ann. Inst. Grenoble* **16** (1966), 319–361). To do so, we use submanifold geometry, and prove that the *weak* curvature tensor of the right invariant H^1 metric on $\mathscr{D}^s_\mu$ is a bounded trilinear map in the H^s topology, from which it follows that solutions to Jacobi's equation exist. Using such solutions, we are able to study the infinitesimal stability behavior of geodesics. © 1998 Academic Press

Key Words: geodesics; Hilbert diffeomorphism groups; Lagrangian stability.

1. INTRODUCTION

1.1. *Background*

The Lagrangian formalism for the hydrodynamics of incompressible ideal fluids considers geodesic motion on $\mathscr{D}^s_\mu(M)$, the group of all volume preserving Hilbert diffeomorphisms of the fluid container M of class H^s. Arnold [A] and Ebin and Marsden [EM] showed that if $\eta(t)$ is a smooth geodesic of the weak L^2 right invariant metric in $\mathscr{D}^s_\mu(M)$, and if

0022-1236/98 $25.00

$U(t)=\dot{\eta}(t)\circ\eta(t)^{-1}$, then the Eulerian velocity $U(t)$ is a solution of the Euler equations

$$\begin{aligned}&\partial_t U(t)+\nabla_{U(t)}U(t)=-\text{grad } p(t)\\&\text{div } U(t)=0, \qquad U(0)=U_0,\end{aligned} \tag{1.1}$$

where $p(t)$ is the pressure function completely determined by $U(t)$.

The Lagrangian stability of the solutions to (1.1) is obtained by studying the behavior of nearby geodesics. A flow $\eta(t)$ is stable if all geodesics in $\mathscr{D}^s_\mu(M)$ with sufficiently close initial conditions at $t=0$ remain close for all $t\geqslant 0$. Thus, one must study the curvature of $\mathscr{D}^s_\mu(M)$ as this enters the linearization of the equations of geodesic flow. The study of the curvature of the volume preserving diffeomorphism group with weak L^2 right invariant metric was initiated by Arnold in [A]. Therein, he computed a formula for the sectional curvature at the identity of a group with one-side invariant metric in terms of the coadjoint and adjoint action, and used this formula to show that the sectional curvature of the volume preserving diffeomorphisms of the flat torus is negative in "many" directions. Using this computation, Arnold was able to demonstrate that for an idealized model of the earth's atmosphere, deviations of fluid particles with nearby initial conditions grow by a factor of 10^5 in two months, making longterm dynamical weather forecast nearly impossible. See the book by Arnold and Khesin [AK1] (as well as [AK2]) for a detailed account.

This work initiated a detailed study of the geometry of the volume preserving diffeomorphism group with L^2 right invariant metric. Ebin and Marsden [EM] provided the differentiable structure for the diffeomorphism groups of Sobolev class and established the functional-analytic foundations of study (see also [E]). Lukatskii [L1, L2, L3] gave detailed explicit computations of the curvature of the measure-preserving diffeomorphism group on the torus. Misiolek [M1, M2] and Bao, Lafontaine, and Ratiu [BLR] used submanifold geometry to compute the sectional curvature of $\mathscr{D}^s_\mu(M)$ for arbitrary manifolds M. Shnirelman [S1, S2] has studied the Riemannian distance on $\mathscr{D}_\mu$ induced by the L^2 metric, and obtained bounds on the diameter of $\mathscr{D}_\mu$. Again, see [AK1] for a comprehensive account of all of these developments.

1.2. *Motivation for the H^1 Metric*

Our interest is in developing the geometry of the volume preserving diffeomorphism group with weak H^1 right invariant metric and studying the properties of its curvature operator. We are motivated by the recently developed models of Holm, Marsden, and Ratiu [HMR1], [HMR2] for the mean hydrodynamic motion of incompressible ideal fluids in Euclidean space. Their basic idea was to obtain a model which averages over small scale fluctuations of order α using an additive decomposition of a given vector field into its mean and oscillatory components. We generalize this procedure to diffeomorphism groups of Riemannian manifolds where mappings are "decomposed" as opposed to vector fields. We shall give a detailed report of this in [HKMRS] for manifolds M with boundary. Herein, we merely outline the basic construction to motivate our study. To do so, we shall need some notation.

Let $\alpha \mapsto \sigma^\alpha \in C^\infty([0,1], M)$. If $U \in C^\infty(TM)$, then $U \circ \sigma \in C^\infty(TM|_{\text{Image}(\sigma)})$. U is said to be parallel along σ if $\nabla_{\sigma'} U = 0$, where $\sigma' = (d/d\alpha)|_0 \sigma^\alpha$. We set $\alpha \mapsto P_\alpha$ to be the unique solution of $\nabla_{\dot\sigma} P_\alpha = 0$, $P_0 = \mathrm{Id}_{T_{\sigma(0)}M}$. P_α is a linear isomorphism between $T_{\sigma(0)}M$ and $T_{\sigma(\alpha)}M$, and is called the parallel transport along σ up to time α.

We consider a geodesic curve in $\mathscr{D}^s_\mu(M)$ and decompose it into its mean $\eta(t)$ and its small scale fluctuations $\zeta^\alpha(t)$ about the mean. The curve $\eta^\alpha(t) = \zeta^\alpha \circ \eta(t)$ describes the motion of the fluid and is defined such that $\eta^0(t) = \eta(t)$. We assume that $\eta' := (d/d\alpha)|_0 \eta^\alpha$ has mean zero, and we Taylor expand $P_\alpha^{-1}(U \circ \eta^\alpha)$ about $\alpha = 0$, where P_α is the parallel transport along the curve $\alpha \mapsto \eta^\alpha(x)$. We use the fact that $P_\alpha^{-1} \nabla_{\eta'} U = (d/d\alpha)[P_\alpha^{-1} U(\eta^\alpha)]$, to obtain $P_\alpha^{-1} U \circ \eta^\alpha = U \circ \eta + \alpha \nabla U \cdot \eta' + O(\alpha^2)$. Substitution of this Taylor expansion into the kinetic energy followed by a computation of its mean gives $\frac{1}{2}\int_M [\langle U, U\rangle + \alpha^2 \langle \nabla U, \nabla U\rangle]\,\mu + O(\alpha^4)$, where μ is the volume form on M and where, for simplicity, we set $\overline{\eta' \otimes \eta'} = \mathrm{Id}$. This is not essential as the term $\langle \overline{\eta' \otimes \eta'}\, \nabla U, \nabla U\rangle$ may also be used to define the H^1 metric at the identity.

The resulting Euler–Poincaré equation for the H^1 metric provides a new model for the mean motion of incompressible ideal fluids given by[1]

$$
\begin{gathered}
\dot V(t) + \nabla_{U(t)} V(t) + \alpha^2 [\nabla U(t)]^t \cdot \Delta U(t) = -\mathrm{grad}\, p(t) \\
V = (1 + \alpha^2 \Delta)\, U, \\
\mathrm{div}\, U = 0, \qquad U(0) = U_0.
\end{gathered}
\tag{1.2}
$$

We call this equation the Euler-α equation or the averaged Euler equation. Unlike the Euler equation (1.1) which conserves the L^2 kinetic energy $\|u\|_{L^2}$, this model conserves the H^1 "kinetic" energy $\|u\|_{H^1}$. Geodesic motion of the α-H^1 right invariant metric on the volume preserving diffeomorphism group has the following effect on solutions U of (1.2): nonlinear interaction among modes corresponding to scales smaller than α is regularized by the inversion of the elliptic operator $(1 - \alpha^2 \Delta)$, so that the behavior of the solution at small scales is controlled by nonlinear dispersion instead of viscous dissipation, and an H^1 conservation law is preserved. Dissipation may then be added to (1.2) to obtain a Navier–Stokes-α model

[1] In this expression, Δ is the Laplace–de Rham operator. This equation may also be written as $\dot V + \mathscr{L}_U V = -d\eta$ for 1-forms V.

(see [FHT] for the proof of global existence of the Navier–Stokes-α model in three dimensions as well as bounds on the dimension of the global attractor).

1.3. *Outline*

The goal of this paper is to develop the foundations for the Lagrangian stability analysis of equation (1.2). For our analysis, we shall set $\alpha = 1$. Volume preserving diffeomorphism groups on Riemannian manifolds equipped with the H^1 right invariant metric have not previously been studied, so we begin by developing the fundamental geometric structures.

After computing the unique Riemannian covariant derivative of the H^1 right invariant metric on the diffeomorphism group $\mathscr{D}^s(M)$, M a compact Riemannian manifold, we use the Hodge theorem to induce the H^1 Riemannian covariant derivative on $\mathscr{D}^s_\mu(M)$. This, in turn, provides the geodesic spray $\mathscr{S}$: $T\mathscr{D}^s_\mu(M) \to TT\mathscr{D}^s_\mu(M)$ which, just as in the case of the Euler equations, is continuously differentiable for $s > (n/2) + 1$. A standard Picard iteration argument may then be used to establish the existence and uniqueness of (1.2) on a finite time interval. In the case that the compact manifold M has a boundary, there are two very interesting subgroups of $\mathscr{D}^s_\mu(M)$ on which the geodesic flow of the right invariant H^1 metric is also C^1. In [HKMRS], we shall define these subgroups which take into account two different kinds of boundary conditions that may be imposed on the Euler-α equations.

Having this result, we proceed to study the curvature of the right invariant H^1 connection. We follow Misiolek [M1] and use basic submanifold geometry, in particular the Gauss equation, to define the curvature on the volume preserving diffeomorphism group, thought of as a weak submanifold (and subgroup) in the weak H^1 topology of the full diffeomorphism group. We are able to prove that this weak curvature tensor is a bounded trilinear map in the H^s topology on M for $s > (n/2) + 2$, and hence that solutions to the Jacobi equation exist. We note that due to the weak metric, the boundedness of the curvature of the H^1 connection cannot be immediately inferred from the regularity of the geodesic spray.

Next, we show that, just as for the Euler equations, pressure constant flows in directions with negative sectional curvature of the full diffeomorphism group, imply that the sectional curvature of the volume preserving subgroup is negative, and hence that such flows are Lagrangian unstable, and do not possess conjugate points.

We remark, that even if M is a flat manifold such as the flat torus $\mathbb{T}^n$, the volume preserving diffeomorphism group $\mathscr{D}^s_\mu(\mathbb{T}^n)$ is not flat. In fact, even the curvature of the right invariant H^1 metric on $\mathscr{D}^s(\mathbb{T}^n)$ does not vanish. Note that this is in contrast with the curvature of the right invariant L^2 metric on $\mathscr{D}^s(\mathbb{T}^n)$ which does vanish.

The paper is structured as follows. In Section 2, we describe the functional analytic setting of the geometry of the diffeomorphism group with H^1 metric. In Section 3, we define the covariant derivative of the H^1 metric and prove the local well-posedness of the geodesic equations of this H^1 metric on the volume preserving diffeomorphism group. In Section 4, we define the curvature of the H^1 metric on $\mathscr{D}^s_\mu(M)$, prove that it is bounded in the strong H^s topology, and establish existence and uniqueness results for the Jacobi equation. Finally, in Section 5, we describe the Lagrangian instability of the Euler-α equations.

2. FUNCTIONAL–ANALYTIC SETTING

2.1. *Preliminaries*

Let $(M, \langle \cdot, \cdot \rangle)$ be a compact oriented Riemannian n dimensional manifold without boundary and define $\mathscr{D}^s(M)$ to be the set of all bijective maps $\eta: M \to M$ such that η and η^{-1} are of Sobolev class H^s. For $s > (n/2)+1$, $\mathscr{D}^s(M)$ is a C^∞ infinite dimensional Hilbert manifold which, about each η, is locally diffeomorphic to the Hilbert space $H^s_\eta(TM) := \{X \in H^s(M, TM): \pi \circ X = \eta\}$ where $\pi: TM \to M$. The condition $s > (n/2)+1$ ensures that $\mathscr{D}^s(M) \subset H^s(M, M)$ is open (see [MEF], Proposition 2.3.1).

A local chart is given by $\omega_{\exp}: H^s_\eta(TM) \to \mathscr{D}^s(M)$, $\omega_{\exp}(X) = \exp \circ X$, where exp is the Riemannian exponential map of $\langle \cdot, \cdot \rangle$. The manifold $\mathscr{D}^s(M)$ is a topological group with composition being the group operation. The ω-lemma asserts that for each $\eta \in \mathscr{D}^s(M)$, right composition $\alpha_\eta: \mathscr{D}^s(M) \to \mathscr{D}^s(M)$ is C^∞, while for all $\eta \in \mathscr{D}^{s+r}(M)$, left composition $\omega_\eta: \mathscr{D}^s(M) \to \mathscr{D}^s(M)$ is C^r.

2.2. *Weak L^2 Structure*

The weak L^2 right invariant Riemannian metric on $\mathscr{D}^s(M)$ is given by

$$\langle X_\eta, Y_\eta \rangle_0 = \int_M \langle X_\eta(x), Y_\eta(x) \rangle_{\eta(x)}\, \mu(x), \tag{2.1}$$

where $\eta \in \mathscr{D}^s(M)$, $X_\eta, Y_\eta \in T_\eta \mathscr{D}^s(M)$, and $\langle \cdot, \cdot \rangle$ and μ are the Riemannian metric and volume element on M. We let ∇ be the Levi–Civita covariant derivative of $\langle \cdot, \cdot \rangle$ on M, and $K: T^2M \to TM$ the induced connector.

Remark 2.1. Associated to the unique Riemannian connector K of the metric $\langle \cdot, \cdot \rangle$ on M are unique local connection 1-forms which can also be used to define ∇. Let us denote by $\mathscr{V}$ the model space of TM. By definition, there exists an open cover $\{\mathscr{O}_a\}$ of M and functions $\{\psi_a\}$ defined on $\mathscr{O}_a$ such that for all $x \in \mathscr{O}_a$, $\psi_a(x): \mathscr{V} \to T_xM$ is an isomorphism and the map $x \mapsto \psi_a(x)\,\xi$ from $\mathscr{O}_a$ to TM is smooth for all $\xi \in \mathscr{V}$. If $U \in C^\infty(TM)$ and

$V \in T\mathcal{O}_a$, then $U(x) = \psi_a(x)\,\xi(x)$ where $\xi(x) = \psi_a(x)^{-1}\,U(x) \in \mathcal{V}$ for all $x \in \mathcal{O}_a$, and ∇ on TM necessarily has the form $\nabla_V U = \psi_a(x)[T\xi \cdot V + \mathcal{A}^a\langle V\rangle\,\xi(x)]$, where the local connection 1-forms $\mathcal{A}^a$ are defined by $\mathcal{A}^a\langle V\rangle\,\xi := \psi_a(x)^{-1}\,\nabla_V[\psi_a(x)\,\xi]$ for all $\xi \in \mathcal{V}$.

It is a fact that the unique Levi–Civita L^2 covariant derivative ∇^0 of $\langle\cdot,\cdot\rangle_0$ is given pointwise by ∇ (see [EM]); namely, if $X, Y \in C^\infty(T\mathcal{D}^s(M))$, then

$$\nabla^0_X Y = K \circ (TY \cdot X). \tag{2.2}$$

Furthermore, ∇^0 is right invariant. For $X_\eta, Y_\eta \in T_\eta\mathcal{D}^s(M)$, let X, Y be their C^∞ extensions to vector fields on $\mathcal{D}^s(M)$. Let $t \mapsto \eta_t$ be a smooth curve in $\mathcal{D}^s(M)$ such that $\eta_0 = \eta$ and $(d/dt)|_0\,\eta_t = X_\eta$. Then

$$\begin{aligned}\nabla^0_X Y(\eta) &= \frac{d}{dt}\Big|_0 Y(\eta_t) + \Gamma_\eta(X_\eta, Y_\eta)\\ &= \frac{d}{dt}\Big|_0 Y(\eta_t \circ \eta^{-1}) \circ \eta + (\nabla_{X_\eta \circ \eta^{-1}} Y_\eta \circ \eta^{-1}) \circ \eta,\end{aligned}$$

where $\Gamma_\eta \colon T_\eta\mathcal{D}^s(M) \times T_\eta\mathcal{D}^s(M) \to T_\eta\mathcal{D}^s(M)$ is the Christoffel map. Namely, for fixed $\eta \in \mathcal{D}^s(M)$, let $(\mathcal{O}_a, \psi_a)$ be a local frame (or trivialization) for the bundle

$$\mathcal{E}_\eta = \bigcup_{x \in M} T_{\eta(x)} M \downarrow \eta(M)$$

modeled on $\mathcal{W}$. Then for each $x \in \mathcal{O}_a$, $\psi_a(x) \colon \mathcal{W} \to T_{\eta(x)}M$ is an isomorphism. Letting $\xi(x) = \psi_a(x)^{-1}\,Y_\eta(x)$, for each $x \in \mathcal{O}_a$, the Christoffel map is given by $\Gamma_\eta(X_\eta, Y_\eta)(x) = \psi_a(x)[\mathcal{A}^a(\eta(x))\langle X_\eta(x)\rangle\,\xi(x)]$. The covariant derivative ∇ on $\mathcal{E}_\eta$ is given by the operator $\nabla \colon C^\infty(\mathcal{E}_\eta) \times \mathcal{E}_\eta \to C^\infty(\mathcal{E}_\eta)$, or for $X_{\eta(x)}$, $Y_{\eta(x)}$ elements of the fiber $\mathcal{E}_{\eta(x)}$ over $\eta(x)$, $\nabla_{X_{\eta(x)}} Y_{\eta(x)} \in \mathcal{E}_{\eta(x)}$. It is clear that this is equivalent to $\nabla_{(Y_\eta \circ \eta^{-1})}(X_\eta \circ \eta^{-1}) \circ \eta$ using the symbol ∇ here to denote the covariant derivative on M (or TM). We shall use the symbol ∇ to denote the covariant derivative on both TM and $\mathcal{E}_\eta$, as the context will be clear.

We may also consider M as the base manifold, in which case we define the pull-back bundle $\eta^*(TM) = \bigcup_{x \in M} T_{\eta(x)} M \downarrow M$. The covariant derivative on this bundle is the operator $\nabla \colon C^\infty(\mathcal{E}_\eta) \times TM \to C^\infty(\mathcal{E}_\eta)$. In this setting, we differentiate a vector $Y_{\eta(x)}$ in the direction of a vector in TM, and this vector is often obtained by the push-forward of a vector $X_{\eta(x)} \in T_{\eta(x)}M$ by η^{-1}. For example, $\nabla_{T\eta^{-1}(\eta(x))\,X_\eta(x)} Y_{\eta(x)} \in T_{\eta(x)}\,M$. It is often convenient for computations to take this equivalent point of view.

2.3. *The Laplacian*

Letting $\Delta = d\delta + \delta d$ denote the Laplace–de Rham operator,[2] we define the H^s metric as follows. Let $X, Y \in T_e\mathscr{D}^s(M)$ and set

$$\langle X, Y \rangle_s = \int_M \langle X(x), (1 + \Delta^s)\, Y(x) \rangle\, \mu(x). \tag{2.3}$$

Extending $\langle \cdot, \cdot \rangle_s$ to $\mathscr{D}^s(M)$ by right invariance gives a smooth invariant metric on $\mathscr{D}^s(M)$. We shall be particularly interested in the metric $\langle \cdot, \cdot \rangle_1$.

In order to obtain formulas for the unique Levi–Civita covariant derivative of $\langle \cdot, \cdot \rangle_1$, it is convenient to express the metric (2.3) in terms of the rough Laplacian $\hat{\Delta} = \mathrm{Tr}\, \nabla\nabla$. We will need the relationship between the rough Laplacian and the Laplace–de Rham operator so that we may express (2.3) in terms of $\hat{\Delta}$. Let ∇^* denote the L^2 formal adjoint of ∇ so that for any $X \in C^\infty(TM)$ and $S, T \in C^\infty(E)$, E a vector bundle over M, $\langle \nabla_X^* S(x), T(x) \rangle_0 = \langle S(x), \nabla_X T(x) \rangle_0$. Then $\nabla_X^* = -\nabla + \mathrm{div}\, X$. To see this, note that

$$\langle \nabla_X^*, S, T \rangle_0 = \int \langle S, \nabla_X T \rangle\, \mu = \int X \langle S, T \rangle\, \mu - \langle \nabla_X S, T \rangle_0$$

$$= \int \langle S, T \rangle\, \mathrm{div}\, X \mu - \langle \nabla_X S, T \rangle_0.$$

If $\mathrm{div}\, X = 0$, then $\nabla_X^* = -\nabla_X$ which we shall often make use of.

Next, let $\tau \in C^\infty(T^*M \otimes TM)$, let $\{e_i\}$ be a local orthonormal frame on M, and let $\sigma \in C^\infty(TM)$ with support in the domain of definition of the local frame $\{e_i\}$. Then

$$\langle \nabla^* \tau, \sigma \rangle_0 = \langle \tau, \nabla \sigma \rangle_0 = \langle \tau \langle e_i \rangle, \nabla_{e_i} \sigma \rangle_0 = \langle \nabla_{e_i}^* (\tau \langle e_i \rangle), \sigma \rangle_0.$$

We may choose the frame $\{e_i\}$, so that locally $\nabla e_i = 0$ and hence $\mathrm{div}\, e_i = 0$. Then

$$\nabla^* \tau = \nabla_{e_i}^* \tau \langle e_i \rangle = -\nabla_{e_i} (\tau \langle e_i \rangle) = -(\nabla_{e_i} \tau) \langle e_i \rangle = -\nabla \tau(e_i, e_i),$$

where the last equality follows from our choice of frame, since $\nabla_{e_i}(\tau \langle e_i \rangle) = (\nabla_{e_i} \tau) \langle e_i \rangle = \nabla \tau \langle e_i, e_i \rangle$. Hence $\nabla^* \tau = -\nabla \tau(e_i, e_i)$, and since $\nabla X \in C^\infty(T^*M \otimes TM)$, we have that

$$\hat{\Delta} = -\nabla^* \nabla.$$

[2] We identify vector fields and 1-forms on M.

With the notation established, we write Bochner's formula relating $\hat{\Delta}$ with Δ on 1-forms as

$$\Delta\alpha = \hat{\Delta}\alpha + \alpha\langle Ric\langle \cdot \rangle\rangle, \tag{2.4}$$

where $Ric\langle X\rangle := R(e_i, X)\, e_i$, R being the curvature of ∇ on M (see, for example, [R]). Because the Ricci tensor is a self-adjoint operator with respect to the metric on TM, for $X \in C^\infty(TM)$, we have that

$$\Delta X = \nabla^*\nabla X + Ric\langle X\rangle.$$

2.4. *Weak H^1 Metric*

Using (2.3), the H^1 metric at the identity may be reexpressed as

$$\begin{aligned}\langle X, Y\rangle_1 &= \langle X, (1 + Ric)\ Y\rangle_{L^2} + \langle X, \nabla^*\nabla Y\rangle_{L^2} \\ &= \langle X, (1 + Ric)\ Y\rangle_{L^2} + \langle \nabla X, \nabla Y\rangle_{L^2}\end{aligned} \tag{2.5}$$

for all $X, Y \in T_e\mathscr{D}^s_\mu(M)$. The metric (2.5) extends smoothly by right translation in the following way. Let $X_\eta, Y_\eta \in T_\eta\mathscr{D}^s_\mu(M)$. Then

$$\begin{aligned}\langle X_\eta, Y_\eta\rangle_1 = \int_M &\langle X_\eta(x), Y_\eta(x) + Ric\langle Y_\eta \circ \eta^{-1}\rangle \circ \eta(x)\rangle_{\eta(x)} \\ &+ \langle \nabla(X_\eta \circ \eta^{-1}) \circ \eta(x), \nabla(Y_\eta \circ \eta^{-1}) \circ \eta(x)\rangle_{\eta(x)}\, \mu.\end{aligned} \tag{2.6}$$

From the implicit function theorem, the set of all volume preserving H^s diffeomorphisms of M, $\mathscr{D}^s_\mu(M) := \{\eta \in \mathscr{D}^s(M) : \eta^*(\mu) = \mu\}$, is a submanifold of $\mathscr{D}^s(M)$ with the induced right invariant H^1 Riemannian metric, as well as a subgroup. For each $\eta \in \mathscr{D}^s_\mu(M)$, the metric (2.6) defines a smooth orthogonal projection $P_\eta : T_\eta\mathscr{D}^s(M) \to T_\eta\mathscr{D}^s_\mu(M)$ defined by

$$P_\eta(X) = (P_e(X \circ \eta^{-1})) \circ \eta, \quad X \in T_\eta\mathscr{D}^s(M),$$

where P_e is the H^1 orthogonal projection onto the 1-forms $\{\alpha \in H^s : \alpha \in \ker \delta\}$ in the Hodge decomposition

$$H^s(T^*M) = \ker \delta \oplus_{H^1} dH^{s+1}(M). \tag{2.7}$$

See [Mor] for a detailed proof of the Hodge decomposition.

Remark 2.2. We remark here that it is essential to use the Laplace–de Rham operator in defining the metric (2.6) in order for the Hodge decomposition to hold. Using the rough Laplacian instead to define the H^1 metric would not provide an orthogonal decomposition in the H^1 topology of divergence-free vector fields and gradients of functions, unless the manifold M is either flat or Einstein, as can be seen from (2.4).

3. H^1 COVARIANT DERIVATIVE AND ITS GEODESIC FLOW

3.1 *Weak H^1 Riemannian Connection*

Next, we compute the Riemannian covariant derivative on $\mathscr{D}^s(M)$ of the H^1 right invariant metric restricted to vectors tangent to $\mathscr{D}^s_\mu(M)$. Using the Hodge decomposition, we define the induced covariant derivative $\tilde{\nabla}^1$ on $\mathscr{D}^s_\mu(M)$. We then prove the local well-posedness of the geodesic equations of $\tilde{\nabla}^1$.

THEOREM 3.1. *The unique Levi–Civita covariant derivative ∇^1 of $\langle\cdot,\cdot\rangle_1$ restricted to vector fields in $T\mathscr{D}^s_\mu(M)$ is given by*

$$\nabla^1_X Y = \nabla^0_X Y + A(X, Y) + B(X, Y) + C(X, Y), \tag{3.1}$$

where for any $\eta \in \mathscr{D}^s_\mu(M)$,

$$\begin{aligned}
A_\eta(X_\eta, Y_\eta) = \tfrac{1}{2}(1 + Ric_\eta - \hat{\Delta}_\eta)^{-1}\, [\nabla^*\{&\nabla X_\eta[T\eta]^{-1}\,\nabla Y_\eta[T\eta]^{-1}\,[T\eta]^{-1^t}\\
&+ \nabla Y_\eta[T\eta]^{-1}\,\nabla X_\eta[T\eta]^{-1}\,[T\eta]^{-1^t}\\
&+ (\nabla X_\eta[T\eta]^{-1})(\nabla Y_\eta[T\eta]^{-1})^t\,[T\eta]^{-1^t}\\
&+ (\nabla Y_\eta[T\eta]^{-1})(\nabla X_\eta[T\eta]^{-1})^t\,[T\eta]^{-1^t}\\
&- (\nabla X_\eta[T\eta]^{-1})^t\,(\nabla Y_\eta[T\eta]^{-1})[T\eta]^{-1^t}\\
&- (\nabla Y_\eta[T\eta]^{-1})^t\,(\nabla X_\eta[T\eta]^{-1})[T\eta]^{-1^t}\}\,],\\
B_\eta(X_\eta, Y_\eta) = \tfrac{1}{2}(1 + Ric_\eta - \hat{\Delta}_\eta)^{-1}\,\{&-\mathrm{Tr}[R(\nabla X_\eta T\eta^{-1}\langle\cdot\rangle, Y_\eta)\,\cdot\\
&+ R(\nabla Y_\eta T\eta^{-1}\langle\cdot\rangle, X_\eta)\,\cdot + R(X_\eta,\cdot)\,\nabla Y_\eta T\eta^{-1}\langle\cdot\rangle\\
&+ R(Y_\eta,\cdot)\,\nabla X_\eta T\eta^{-1}\langle\cdot\rangle]\\
&+ \nabla^*[R(X_\eta, T\eta^{-1^t})\,Y_\eta + R(Y_\eta, T\eta^{-1^t})\,X_\eta]\},\\
C_\eta(X_\eta, Y_\eta) = (1 + Ric_\eta - \hat{\Delta}_\eta)^{-1}\,[&(\nabla_{X_\eta} Ric)\langle Y_\eta\rangle + (\nabla_{Y_\eta} Ric)\langle X_\eta\rangle\\
&- \tfrac{1}{2}[\langle(\nabla Ric\langle\cdot\rangle\langle X_\eta\rangle, Y_\eta\rangle^\# + \langle(\nabla Ric\langle\cdot\rangle\langle Y_\eta\rangle, X_\eta\rangle^\#]\\
&- Ric_\eta\langle[X_\eta, Y_\eta]\rangle],
\end{aligned} \tag{3.2}$$

where X_η, $Y_\eta \in T_\eta\mathscr{D}^s_\mu(M)$,

$$Ric_\eta\langle X_\eta\rangle = Ric\langle X_\eta \circ \eta^{-1}\rangle \circ \eta$$

is the right-translated Ricci tensor,

$$\hat{\Delta}_\eta = -\nabla^*[\nabla(\cdot)(T\eta)^{-1}\,(T\eta)^{-1^t}],$$

and $(\cdot)^{\#}$ *is the operator mapping* 1*-forms to vector fields through the given metric on* M.

Proof. Formula (3.1) is obtained by a lengthy computation using (2.6) and the fundamental theorem of Riemannian geometry which associates to every strong metric, a unique Levi–Civita covariant derivative. Although $\langle \cdot, \cdot \rangle_1$ is a weak metric, ∇^1 is still uniquely defined by virtue of the existence of a C^1 geodesic spray restricted to tangent vectors on $\mathscr{D}^s_\mu(M)$ (see Theorem 3.3). ∎

Remark 3.1. Note that for $X_\eta \in H^s_\eta(TM)$, the operators $[T\eta]^{-1}$, $[T\eta]^{-1^t}$, and ∇X_η induce the following pointwise operators

$$[T\eta(x)]^{-1} : T_{\eta(x)}M \to T_x M,$$

$$[T\eta(x)]^{-1^t} : T_x M \to T_{\eta(x)}M,$$

$$(\nabla X_\eta)(x) : T_x M \to T_{\eta(x)}M.$$

Remark 3.2. Since $[T\eta]^{-1}[T\eta]^{-1^t}$ is positive symmetric, the spectrum of $-\hat{\Delta}_\eta$, $\sigma(-\hat{\Delta}_\eta)$, is positive. We can ensure that $0 \notin \sigma(1 + Ric_\eta - \hat{\Delta})$ by requiring that M have nonnegative Ricci curvature or in the case that M has negative Ricci curvature, by insisting that $|-\sigma(Ric_\eta)| \leqslant 1$. More generally, we require $\mathrm{Ker}(1 + Ric_\eta - \hat{\Delta})$ to be either empty or unique for all $x \in M$, $\eta \in \mathscr{D}^s_\mu(M)$. In the case that the kernel is not empty, we shall restrict our phase space to the orthogonal complement of $\mathrm{Ker}(1 + Ric_\eta - \hat{\Delta}_\eta)$ but this may only occur on manifolds M with negative Ricci curvature (this is essentially Bochner's theorem).

Now, on $H^{s+1}(M)$, $\Delta = d\delta = -\mathrm{div}\ \mathrm{grad}$, so an explicit formula for P_e: $T_e\mathscr{D}^s(M) \to T_e\mathscr{D}^s_\mu(M)$ is obtained as follows. Suppose that $V \in H^s(TM)$, and let $p \in H^{s+1}(M)$ solve $\Delta p = \mathrm{div}\ V$. Then

$$P_e(V) = V - \mathrm{grad}\ \Delta^{-1}\ \mathrm{div}\ V.$$

We shall denote the orthogonal projection onto $dH^{s+1}(M)$ by

$$Q_e(V) = \mathrm{grad}\ \Delta^{-1}\ \mathrm{div}\ V. \tag{3.3}$$

$\mathscr{D}^s_\mu(M)$ thus becomes a weak Riemannian submanifold of $\mathscr{D}^s(M)$ with the metric (2.6), and the induced covariant derivative

$$\tilde{\nabla}^1 = P \circ \nabla^1$$

is inherited from $\mathscr{D}^s(M)$.

3.2. *Geodesic Flow of* $\tilde{\nabla}^1$

THEOREM 3.2. *If $\eta(t)$ is a geodesic of $\tilde{\nabla}^1$, then $U(t)=\dot{\eta}\circ\eta^{-1}(t)$ is a vector field on M which satisfies the mean motion equations of an ideal fluid,*

$$\begin{aligned}\partial_t U(t)+(1+\varDelta)^{-1}\,[\nabla_{U(t)}(1+\varDelta)\,U(t)+\langle\nabla U(t)\langle\cdot\rangle,\varDelta U(t)\rangle^{\#}]\\ =-\text{grad}\;p(t)\end{aligned} \tag{3.4}$$

$$\text{div}\;U(t)=0,\qquad U(0)=U_0,$$

where $p(t)$ is the pressure function which is determined from $V(t)$.

Proof. Together with the Hodge decomposition (2.7), a straightforward computation of the coadjoint action ad* of $\mathscr{D}^s_\mu(M)$ given by

$$\begin{aligned}\langle \text{ad}^*_V W, U\rangle_1 &= \langle \text{ad}_V U, W\rangle_1,\\ \text{ad}_U V = -[U,V], &\qquad U,V,W\in T_e\mathscr{D}^s_\mu(M)\end{aligned} \tag{3.5}$$

shows that (3.4) is simply

$$\dot{U}(t) = -P_e\circ\text{ad}^*_{U(t)}\,U(t),$$

the Euler–Poincaré equation for the induced H^1 metric on $\mathscr{D}^s_\mu(M)$. ▮

Remark 3.3. Notice that the Euler–Poincaré equation (3.4) is expressed in terms of the Laplace–de Rham operator $\varDelta$. In terms of the rough Laplacian $\hat{\varDelta}$,

$$P_e\circ\text{ad}^*_U U=P_e\circ(1+Ric-\hat{\varDelta})^{-1}\,[\nabla_U(1+Ric-\hat{\varDelta})\,U-\nabla U^t\cdot[Ric+\hat{\varDelta}]\,U].$$

We shall need the following lemmas, the first of which is similar to Lemma 2 of Appendix A in [EM].

LEMMA 3.1. *Let $\hat{\varDelta}_{(\cdot)}\colon \bigcup_{\eta\in\mathscr{D}^s_\mu(M)} H^s_\eta(TM)\downarrow\mathscr{D}^s_\mu(M)\to\bigcup_{\eta\in\mathscr{D}^s_\mu(M)} H^{s-2}_\eta(TM)\downarrow\mathscr{D}^s_\mu(M)$ be given by*

$$\hat{\varDelta}_\eta=-\nabla^*[\nabla(\cdot)(T\eta)^{-1}\,(T\eta)^{-1^t}]$$

and the identity on $\mathscr{D}^s_\mu(M)$. Then $\hat{\varDelta}_{(\cdot)}$ is a C^1 bundle map.

Proof. Let $H^{s-1}_\eta(T^*M\otimes TM)=H^{s-1}(\bigcup_{x\in M}(T^*_{\eta(x)}M\otimes T_{\eta(x)}M)\downarrow M)$, and let

$$f(\eta)=\nabla(\cdot)(T\eta)^{-1}\,(T\eta)^{-1^t}.$$

We first show that f is a C^1 section of the bundle

$$\bigcup_{\eta \in \mathscr{D}^s_\mu(M)} \mathrm{Hom}(H^s_\eta(TM), H^{s-1}_\eta(T^*M \otimes TM)) \downarrow \mathscr{D}^s_\mu(M).$$

Continuity of f is clear. We compute its derivative. With $V \in H^s_\eta(TM)$, the ω-lemma asserts that

$$\begin{aligned} Df(\eta)\langle V\rangle = {} & \nabla(\cdot)[T\eta]^{-1}(\nabla V)[T\eta]^{-1}[T\eta]^{-1^t} \\ & - \nabla(\cdot)[T\eta]^{-1}[T\eta]^{-1^t}(\nabla V)^t[T\eta]^{-1^t}. \end{aligned}$$

Now,

$$\begin{aligned} & \|Df(\eta)\|_{\mathscr{L}(H^s_\eta(TM), \mathrm{Hom}(H^s_\eta(TM), H^{s-1}_\eta(T^*M \otimes TM)))} \\ & \quad = \sup_{V \in H^s_\eta(TM),\, \|V\|_s = 1} \|Df(\eta)\langle V\rangle\|_{\mathrm{Hom}(H^s_\eta(TM), H^{s-1}_\eta(T^*M \otimes TM)))} \\ & \quad = \sup_{V \in H^s_\eta(TM),\, \|V\|_s = 1} \; \sup_{W \in H^s_\eta(TM),\, \|W\|_s = 1} \|(Df(\eta)\langle V\rangle)\langle W\rangle\|_{H^{s-1}_\eta(T^*M \otimes TM)} \\ & \quad \leqslant C(\|T\eta\|_{s-1}, \|[T\eta]^{-1}\|_{s-1} < \infty, \end{aligned}$$

where the last two inequalities are due to the ω-lemma and the fact that $[T\eta]^{-1} \in H^{s-1}$ whenever $\eta \in H^s$, again by the ω-lemma. Let $\mathscr{O} \subset \mathscr{D}^s_\mu(M)$ be a be neighborhood of some η. Locally $\hat{\Delta}_{(\cdot)}$ acts on $\mathscr{O} \otimes \mathscr{F}$, for a trivialization $\{\psi(\eta)\}_{\eta \in \mathscr{O}}$ such that $\psi(\eta)$: $H^s_\eta(TM) \to \mathscr{F}$ isomorphically.

Computing the supremum of

$$\|Df(\eta)\|_{\mathscr{L}(H^s_\eta(TM), \mathrm{Hom}(H^s_\eta(TM), H^{s-1}_\eta(T^*M \otimes TM)))}$$

over all $\eta \in \mathscr{O}$ defines the C^1 topology. Since we may bound the supremum, we have proven that f is C^1. Now thinking of $\nabla(\cdot)[T\eta]^{-1}[T\eta]^{-1^t}$ as a map on $\mathscr{F}$, it is smooth by the ω-lemma. To see this, it suffices to consider the fiber over the identity e, where the operator is linear and hence a smooth bundle map.

The operator ∇^* acts fiberwise, and is linear, hence smooth as a bundle map. This proves that $\hat{\Delta}_{(\cdot)}$ is a C^1 bundle map, which proves the lemma. ∎

Remark 3.4. Although we shall only need the C^1 regularity, it seems likely that by considering higher order derivatives of $\nabla(\cdot)[T\eta]^{-1}[T\eta]^{-1^t}$, thought of as a bundle map, we could obtain the C^k regularity of $\hat{\Delta}_{(\cdot)}$ for any nonnegative integer k.

LEMMA 3.2. *The operator* $(1+Ric_{(\cdot)}-\hat{\Delta}_{(\cdot)})^{-1}\colon \bigcup_{\eta\in\mathscr{D}^s_\mu(M)} H^{s-2}_\eta(TM)\downarrow \mathscr{D}^s_\mu(M)\to \bigcup_{\eta\in\mathscr{D}^s_\mu(M)} H^s_\eta(TM)\downarrow\mathscr{D}^s_\mu(M)$ *is a* C^1 *bundle map.*

Proof. By the smoothness of right translation, the map $\eta\mapsto Ric_\eta$ is smooth. Thus, $(1+Ric_{(\cdot)}-\hat{\Delta}_{(\cdot)})$ is C^1 (using Lemma 3.1) and by assumption has trivial kernel and closed range, hence is a C^1 bijection. By the inverse function theorem, a C^1 bijective bundle map covering the identity has a C^1 inverse. ∎

For the following theorem, recall that $TT\mathscr{D}^s_\mu(M)$ is identified with H^s maps $\mathscr{Y}\colon M\to TTM$ covering some $X_\eta\in T_\eta\mathscr{D}^s_\mu(M)$.

THEOREM 3.3. *For* $s>(n/2)+1$, *there exists a neighborhood of* $e\in\mathscr{D}^s_\mu(M)$ *and an* $\varepsilon>0$ *such that for any* $V\in T_e\mathscr{D}^s_\mu(M)$ *with* $\|V\|_s<\varepsilon$, *there exists a unique geodesic* $\dot\eta\in C^1((-2,2), T\mathscr{D}^s_\mu(M))$ *satisfying*

$$\tilde{\nabla}^1_{\dot\eta}\dot\eta=0,\quad \eta(0)=e,\quad \dot\eta(0)=V,$$

with smooth dependence on V.

Proof. Let $\eta(t)$ be a curve in $\mathscr{D}^s_\mu(M)$. Using the formula for the induced covariant derivative of the H^1 metric (3.1) on $\mathscr{D}^s_\mu(M)$ or by a computation of the first variation of the energy (see [HKMRS] for the detailed computation)

$$\mathscr{E}(\eta)=\tfrac{1}{2}\int_{\mathbb{R}}\langle\dot\eta(t),\dot\eta(t)\rangle_1\,dt, \tag{3.6}$$

we find that

$$\begin{aligned}P_\eta\circ\nabla_{\dot\eta}\dot\eta&=P_\eta\circ(1+Ric_\eta-\hat{\Delta}_\eta)^{-1}\,[\nabla^*[\{-(\nabla\dot\eta[T\eta]^{-1})^t\,(\nabla\dot\eta[T\eta]^{-1})\\&\quad+\nabla\dot\eta[T\eta]^{-1}\,\nabla\dot\eta[T\eta]^{-1}+(\nabla\dot\eta[T\eta]^{-1})(\nabla\dot\eta[T\eta]^{-1})^t\}\,[T\eta]^{-1^t}]\\&\quad+(\nabla_{\dot\eta}Ric)\langle\dot\eta\rangle-\tfrac{1}{2}\langle\nabla Ric\langle\cdot\rangle\langle\dot\eta\rangle,\dot\eta\rangle^\#-\{\mathrm{Tr}[R(\nabla\dot\eta T\eta^{-1}\langle\cdot\rangle,\dot\eta)\cdot\\&\quad+R(\dot\eta,\cdot)\,\nabla\dot\eta T\eta^{-1}\langle\cdot\rangle]+\nabla^*\{R(\dot\eta,T\eta^{-1^t})\,\dot\eta\}\,] \qquad (3.7)\\&:=P_\eta\circ F_\eta(\dot\eta).\end{aligned}$$

Using the notation of Remark 2.1, we let $(\mathscr{O}_a,\psi_a)$ be a trivialization of $\mathscr{E}_\eta$ and set $\dot\eta(x)=\psi_a(x)\,\xi(x)$. For all $x\in\mathscr{O}_a$, we express $\nabla_{\dot\eta(x)}\dot\eta(x)$ by $\nabla_{\dot\eta}\dot\eta(x)=\psi_a(x)[\dot\xi+(\mathscr{A}^a\circ\eta)(x)\langle\dot\eta\rangle\xi(x)]$. Let $\tilde{F}_\eta$ be the localization of F_η in $(\mathscr{O}_a,\psi_a)$. Then, in this trivialization, we may write (3.7) in the form of a geodesic spray $\mathscr{S}\colon T\mathscr{D}^s_\mu(M)\to TT\mathscr{D}^s_\mu(M)$. We have, locally, that

$$\mathscr{S}_\eta(\dot\eta)=\frac{d}{dt}(\eta,\psi_a^{-1}\dot\eta)=(\xi,Q_n\psi_a\dot\xi-P_\eta[\psi_a(\mathscr{A}^a\circ\eta)\langle\psi_a\xi\rangle\,\xi-\psi_a\tilde{F}_\eta]).$$

We show that $\mathscr{S}_\eta$ is a quadratic form. Clearly, F_η is quadratic; as for the term $Q_\eta \psi_a \dot{\xi}$, we note that

$$\dot{\xi} = \psi_a^{-1}[(\psi_a \xi \circ \eta^{-1} + \nabla_{\psi_a \xi \circ \eta^{-1}}(\psi_a \xi \circ \eta^{-1})) \circ \eta],$$

and since $\operatorname{div}(\psi_a \xi \circ \eta^{-1}) = 0$, $Q_e(\psi_a \dot{\xi} \circ \eta^{-1}) \circ \eta = Q_e[T(\psi_a \xi \circ \eta^{-1}) \cdot (\psi_a \xi \circ \eta^{-1})] \circ \eta$, so that

$$\begin{aligned} Q_e(\psi_a \dot{\xi} \circ \eta^{-1}) \circ \eta + Q_\eta[\psi_a(\mathscr{A}^a \circ \eta)\langle \psi_a \xi \rangle\, \xi] \\ = Q_e[\nabla_{\psi_a \xi \circ \eta^{-1}}(\psi_a \xi \circ \eta^{-1}] \circ \eta \\ = \operatorname{grad} \Delta^{-1}[\operatorname{Ric}(\psi_a \xi \circ \eta^{-1}, \psi_a \xi \circ \eta^{-1}) \\ + \operatorname{Tr}(\nabla(\psi_a \xi \circ \eta^{-1}) \cdot \nabla(\psi_a \xi \circ \eta^{-1}))] \circ \eta, \end{aligned}$$

where $\operatorname{Ric}(V, W) = Ric\langle V \rangle W$. This shows that S_η is quadratic in ξ.

The projection P_η is a smooth bundle map. Namely, P: $T\mathscr{D}^s(M) \downarrow \mathscr{D}^s_\mu(M) \to T\mathscr{D}^s_\mu(M)$ is C^∞. (To prove this one need only replace the L^2 orthogonal projection onto the harmonic forms by the H^1 orthogonal projection onto harmonic forms in Lemma 4 of Appendix A in [EM].)

The map $x \mapsto (\mathscr{A}^a \circ \eta)(x) \in C^\infty(\mathscr{O}_a, [T^*_{\eta(x)} M]^2 \otimes T_{\eta(x)} M)$ since the local connection 1-forms and right translation are both smooth maps. Since $\psi_a(x)$ is an isomorphism, $\psi_a[(\mathscr{A}^a \circ \eta)\langle \cdot \rangle(\cdot)]$: $(H^s_\eta)^2 \to H^s_\eta$ smoothly.

By Lemma 3.2, $(1 + Ric_{(\cdot)} - \hat{\Delta}_{(\cdot)})^{-1}$ is a C^1 bundle map. Since R and Ric are fiberwise multilinear maps, it follows from the smoothness of right translation that all terms involving the curvature are smooth bundle maps. Letting $U = \dot{\eta} \circ \eta^{-1}$, we need only prove that the terms $[-(\nabla U)^t (\nabla U) + (\nabla U)(\nabla U) + (\nabla U)(\nabla U)^t][T\eta]^{-1^t}$ are C^1 bundle maps. The argument for this is identical to that of Lemma 3.1.

We have shown that $\mathscr{S}$: $T\mathscr{D}^s_\mu(M) \to TT\mathscr{D}^s_\mu(M)$ is a C^1 bundle map. A standard Picard iteration argument for ordinary differential equations in a Banach space then proves the existence of a unique C^1 flow (see [La], Theorem 1.11), and this proves the theorem. ∎

Together with Theorem 3.2, we have proven the local well-posedness of the Cauchy problem for the Euler-α equations (3.4) on M. This implies the following facts.[3]

Corollary 3.1. *Let $\eta \in \mathscr{D}^s_\mu(M)$ be in a sufficiently small neighborhood of e. Then, there exists a vector field V on M such that* $\exp_e(V) = \eta$. *In other words, the Euler-α flow with initial condition V reaches η in time* 1.

[3] We would like to thank the referee for pointing these out and suggesting their inclusion in this paper.

As another corollary, we immediately have the H^1 analog of Theorem 12.1 of [EM].

COROLLARY 3.2. *For $s > (n/2)+1$, let $\eta(t)$ be a geodesic of the right invariant H^1 metric on $\mathscr{D}^s_\mu(M)$. If $\eta(0) \in \mathscr{D}^{s+k}_\mu(M)$ and $\dot\eta(0) \in T_{\eta(0)}\mathscr{D}^{s+k}_\mu(M)$ for $0 \leqslant k \leqslant \infty$, then $\eta(t)$ is H^{s+k} on M for all t for which $\eta(t)$ was defined in $\mathscr{D}^s_\mu(M)$.*

The proof of this theorem exactly follows the proof of Theorem 12.1 of [EM] once we have the regularity properties of the exponential map. As noted in [EM] for the case of the Euler equations, this has the important consequence that the time of existence of a geodesic does not depend on s, so that a geodesic with C^∞ initial conditions is a curve in

$$\mathscr{D}_\mu(M) = \bigcap_{s > n/2} \mathscr{D}^s_\mu(M),$$

where $\mathscr{D}_\mu(M)$ is the ILH (inverse limit Hilbert) Lie group of C^∞ diffeomorphisms.

Remark 3.5. A computation of the first variation of (3.6) on the full diffeomorphism group shows that the geodesic spray has no derivative loss in this case as well and following our arguments is smooth. For example, on $\mathbb{S}^1$, with $\Delta := \eta_x^{-1}(\partial_x \eta_x^{-1} \partial_x)$ and for $\alpha > 0$, the principle part of the geodesic spray, for $s > 3/2$, is given by

$$\ddot\eta = (1-\alpha^2\Delta)^{-1}[(-2\dot\eta + \alpha^2 \Delta\dot\eta)\,\eta_x^{-1}\dot\eta_x]. \tag{3.8}$$

This gives the well-posedness of 1D Camassa–Holm for $s > 3/2$. It is clear that the nonlinear dispersion arising from the H^1 metric regularizes the shock formation of the Burger–Riemann equation into traveling peaked solutions (see [HMR1]). The fact that the Burger–Riemann equation which arises from the L^2 right invariant metric shocks, is a connected to the loss of smoothness of the spray, for in the $\alpha = 0$ limit, (3.8) is $\ddot\eta = -2\eta_x^{-1}\dot\eta_x\dot\eta$ which has derivative loss.

A similar but lengthier computation shows that for $s > n/2+2$, the geodesic spray has no derivative loss on the full diffeomorphism group in n dimensions, so that the covariant derivative ∇^1 can be uniquely defined for all vectors in $T\mathscr{D}^s(M)$.

4. CURVATURE OF THE H^1 METRIC

Because the Lie-theoretic computation of the sectional curvature is difficult to compute on manifolds M with nonvanishing curvature, we use basic submanifold geometry to estimate the curvature of the H^1 metric on $\mathscr{D}^s_\mu(M)$ for arbitrary smooth manifolds.

4.1. *Curvature of* ∇^1

We denote by R^0 the curvature of the L^2 metric ∇^0. Proposition 3.4 of [M1] states that R^0 is completely determined by R, the curvature of M, and is a bounded trilinear map in the H^s topology. Namely, for $X_\eta, Y_\eta, Z_\eta \in T_\eta \mathscr{D}^s(M)$ and using the right invariance of ∇^0, it is evident from formula (2.2) that R^0 may be expressed as

$$R^0(X_\eta, Y_\eta)\, Z_\eta = (R(X_\eta \circ \eta^{-1}, Y_\eta \circ \eta^{-1})\, Z_\eta \circ \eta^{-1}) \circ \eta.$$

It follows that R^0 is right invariant, and that

$$\|R^0_\eta(X_\eta, Y_\eta)\, Z_\eta\|_s \leqslant C\, \|X_\eta\|_s\, \|Y_\eta\|_s\, \|Z_\eta\|_s,$$

where C denotes any constant which may depend on s, η, and the derivatives of the metric $\langle \cdot, \cdot \rangle$ on M.

Now for each $\eta \in \mathscr{D}^s_\mu(M)$, the right-translated weak metric (2.5_1) splits $T_\eta \mathscr{D}^s(M)$ into the direct sum

$$T_\eta \mathscr{D}^s(M) = T_\eta \mathscr{D}^s_\mu(M) \oplus_{H^1} \nu_\eta \mathscr{D}^s_\mu(M),$$

where $\nu_\eta \mathscr{D}^s(M)$ is the H^1 orthogonal complement of $T_\eta \mathscr{D}^s_\mu(M)$ in $T_\eta \mathscr{D}^s(M)$. We now introduce the (weak) second fundamental form S of $\mathscr{D}^s_\mu(M)$ by assigning to each $\eta \in \mathscr{D}^s_\mu(M)$ a map

$$S_\eta \colon T_\eta \mathscr{D}^s_\mu(M) \times T_\eta \mathscr{D}^s_\mu(M) \to \nu_\eta \mathscr{D}^s_\mu(M).$$

Given $X_\eta, Y_\eta \in T_\eta \mathscr{D}^s_\mu(M)$, we extend them to C^∞ vector fields X, Y on $\mathscr{D}^s_\mu(M)$, and define

$$\begin{aligned} S_\eta(X_\eta, Y_\eta) &= Q_\eta(\nabla^1_X Y(\eta)), \\ &= Q_\eta(\nabla^0_X Y(\eta) + A_\eta(X_\eta, Y_\eta)) + B_\eta(X_\eta, Y_\eta) + C_\eta(X_\eta, Y_\eta)), \end{aligned} \tag{4.1}$$

where $\eta \in \mathscr{D}^s_\mu(M)$ and

$$Q_\eta(X_\eta) = (Q_e(X_\eta \circ \eta^{-1})) \circ \eta$$

can be computed explicitly from (3.3).

We next define the (weak) Riemannian curvature tensor R^1 of $\langle \cdot, \cdot \rangle_1$ on $\mathscr{D}^s(M)$. This is the trilinear map

$$R^1_\eta : T_\eta \mathscr{D}^s(M) \times T_\eta \mathscr{D}^s(M) \times T_\eta \mathscr{D}^s(M) \to T_\eta \mathscr{D}^s(M),$$
$$R^1_\eta(X_\eta, Y_\eta)\, Z_\eta = (\nabla^1_X \nabla^1_Y Z)_\eta - (\nabla^1_Y \nabla^1_X Z)_\eta - (\nabla^1_{[X, Y]} Z)_\eta,$$

where $\eta \in \mathscr{D}^s(M)$ and X, Y, Z are smooth extensions of vectors X_η, Y_η, Z_η to a neighborhood of η.

LEMMA 4.1. *For $\eta \in \mathscr{D}^s(M)$, B_η: $(H^s_\eta(TM))^2 \to H^{s+1}_\eta(TM)$ continuously.*

Proof. Let $X, Y, Z \in T_e\mathscr{D}^s(M)$. Since $s > (n/2)+1$, H^r is a multiplicative algebra for $r \geqslant s-1$; hence, it suffices to obtain the estimate at the identity e.

We use the fact that R^0 is a continuous trilinear map in the H^s topology, and estimate B_η using equation (3.2). For the terms $\mathrm{Tr}[R(\nabla \cdot X, Y)\cdot + R(\nabla \cdot Y, X)\cdot + R(X, \cdot)\nabla \cdot Y + R(Y, \cdot)\nabla \cdot X]$ we use the continuous embedding $H^{s-1}(TM) \hookrightarrow C^0(TM)$, while for the term $\nabla^*[R(X, \cdot)Y + R(Y, \cdot)X]$ we use that ∇^*: $H^s \to H^{s-1}$ is continuous. Since $(1-\hat{\Delta})^{-1}$ is a pseudodifferential operator of order -2, we obtain that

$$\|B(X, Y)\|_{s+1} \leqslant C\,\|X\|_s\,\|Y\|_s,$$

where the constant C may depend on R and s. ∎

The same argument shows that

COROLLARY 4.1. *For each $\eta \in \mathscr{D}^s(M)$, B_η: $H^s_\eta(TM) \times H^{s-1}_\eta(TM) \to H^s_\eta(TM)$ continuously.*

Similarly,

LEMMA 4.2. *For each $\eta \in \mathscr{D}^s(M)$, the following are bounded multilinear maps:*

(i) C_η: $(H^s_\eta(TM))^2 \to H^{s+1}_\eta(TM)$,

(ii) *for each* $X_\eta \in T_\eta\mathscr{D}^s(M)$, $\nabla^0_{X_\eta}$: $H^s_\eta(TM) \to H^{s-1}_\eta(TM)$,

(iii) A_η: $(H^s_\eta(TM))^2 \to H^s_\eta(TM)$.

Proof. Items (i) and (ii) are trivial, while for item (iii), we use that H^{s-1} is a Schauder ring. ∎

PROPOSITION 4.1. *Let M be a compact n dimensional manifold. For $s > (n/2)+2$, and $\eta \in \mathscr{D}^s_\mu(M)$, R^1_η: $(T_\eta\mathscr{D}^s_\mu(M))^3 \to T_\eta\mathscr{D}^s_\mu(M)$ is continuous in the H^s topology.*

Proof. For $\eta \in \mathscr{D}^s_\mu(M)$, let $X_\eta, Y_\eta, Z_\eta \in T_\eta\mathscr{D}^s_\mu(M)$, and let X, Y, Z be smooth extensions to a neighborhood of η. Let $D(X, Y) = A(X, Y) + B(X, Y) + C(X, Y)$. Then

$$\begin{aligned} R^1_\eta(X_\eta, Y_\eta)\, Z_\eta &= (\nabla^1_X \nabla^1_Y Z)(\eta) - (\nabla^1_Y \nabla^1_X Z)(\eta) - (\nabla^1_{[X,\,Y]} Z)(\eta) \\ &= R^0_\eta(X_\eta, Y_\eta)\, Z_\eta + D(X, \nabla^1_Y Z)(\eta) - D(Y, \nabla^1_X Z)(\eta) \\ &\quad + (\nabla^0_X D(Y, Z))(\eta) - (\nabla^0_Y D(X, Z))(\eta) \\ &\quad + D(X, D(Y, Z))(\eta) - D(Y, D(X, Z))(\eta) - D([X, Y], Z)(\eta). \end{aligned}$$

Since R^0 is a bounded trilinear map in the H^s topology, we must show that the remaining terms are bounded trilinear maps in H^s as well. These terms are of two types. Type *I* terms involve commutation between ∇^0 and D, while the type *II* terms involve commutation between the bilinear forms A, B, and C. From Lemmas 4.1 and 4.2 it is clear that the trilinear map formed by type *II* terms are bounded maps in the H^s topology; hence, we estimate type *I* terms.

We begin with type *I* terms which are the commutation of ∇^0 and B. Since for each $\eta \in \mathscr{D}^s_\mu(M)$, H^{s-2}_η is a Schauder ring, using the right invariance of $\|\cdot\|_s$ it suffices to obtain the continuity of the trilinear maps at the identity e. Using Lemma 4.1, it is clear that terms of the type $\nabla^0_X B(Y, Z)$ are continuous in H^s, while Corollary 4.1 gives the bound on the remaining terms involving B. Clearly, since C_η is as regularizing as B_η, by the same argument, we have that all type *I* terms involving the commutation of ∇^0 and C are continuous trilinear maps in H^s as well. The difficult type *I* terms to estimate are those involving the commutation of ∇^0 and A, since by part iii) of Lemma 4.2, it appears as though a derivative loss may occur in some of these terms.

In fact, such a derivative loss does not occur, and for the purpose of estimating these terms, it will suffice to replace A_e with

$$\bar{A}(X, Y) = \hat{\Delta}^{-1} \nabla^*(\nabla X \cdot \nabla Y)$$

for $X, Y \in T_e \mathscr{D}^s_\mu(M)$. The terms we must estimate are given by

$$\begin{aligned} &\nabla_Y \hat{\Delta}^{-1} \nabla^*(\nabla X \cdot \nabla Z) + \hat{\Delta}^{-1} \nabla^*(\nabla Y \cdot \nabla_X Z) + \hat{\Delta}^{-1} \nabla^*(\nabla Y \cdot \hat{\Delta}^{-1} \nabla^*(\nabla X \cdot \nabla Z)) \\ &\qquad - \nabla_X \hat{\Delta}^{-1} \nabla^*(\nabla Y \cdot \nabla Z) - \hat{\Delta}^{-1} \nabla^*(\nabla X \cdot \nabla_Y Z) \\ &\qquad - \hat{\Delta}^{-1} \nabla^*(\nabla X \cdot \hat{\Delta}^{-1} \nabla^*(\nabla Y \cdot \nabla Z)) \\ &\qquad - \hat{\Delta}^{-1} \nabla^*(\nabla [X, Y] \cdot \nabla Z). \end{aligned} \tag{4.2}$$

We shall need the following lemma which is Corollary 4.2 of [T].

LEMMA 4.3. *Let α and β be pseudodifferential operators with symbols of order m and n, respectively. Then the commutator $[\alpha, \beta]$ is a pseudo-differential operator with symbol of order $m + n - 1$.*

Using Lemma 4.3, $[\hat{\Delta}^{-1}\nabla^*, \nabla_Y]$ is a pseudodifferential operator of order -1, so that $[\hat{\Delta}^{-1}\nabla^*, \nabla_Y]: H^s \to H^{s+1}$ continuously. Hence, using the property of the Schauder ring, it is clear that

$$\| [\hat{\Delta}^{-1}\nabla^*, \nabla_Y](\nabla X \cdot \nabla Z) \|_s \leqslant C \|X\|_s \|Y\|_s \|Z\|_s,$$

where, in general, the constant C may depend on M and η. Similarly, we have the identical estimate for $[\hat{\Delta}^{-1}\nabla^*, \nabla_X](\nabla Y \cdot \nabla Z)$.

Next, we consider the endomorphism

$$\nabla_Y \nabla X \cdot \nabla Z + \nabla X \cdot \nabla_Y \nabla Z - \nabla_X \nabla Y \cdot \nabla Z - \nabla Y \cdot \nabla_X \nabla Z - \nabla \nabla_Y X + \nabla \nabla_X Y \cdot \nabla Z.$$

Again, using Lemma 4.3, $[\nabla_Y, \nabla]$ is order 1, so that

$$\| [\nabla_Y, \nabla] X \cdot \nabla Z \|_{s-1} \leqslant C \|X\|_s \|Y\|_s \|Z\|_s,$$

with the same estimate for $[\nabla_X, \nabla] Y \cdot \nabla Z$. After commutation, most of the terms in (4.2) cancel, and we are left to estimate

$$\hat{\Delta}^{-1}\nabla^*[\nabla X \cdot \nabla_Y \nabla Z - \nabla Y \cdot \nabla_X \nabla Z].$$

It suffices to estimate the first term. Now

$$\hat{\Delta}^{-1}\nabla^*[\nabla X \cdot \nabla_Y \nabla Z] = \hat{\Delta}^{-1}[(\nabla_Y \nabla Z)^t \cdot \hat{\Delta} X^t] + \hat{\Delta}^{-1}(\nabla^* \nabla_Y \nabla Z), \quad (4.3)$$

so the first term in the right-hand-side of (4.3) is clearly a continuous mapping in H^s. For the second term we use the identity on divergence-free vector fields given by

$$\operatorname{div} \nabla_X Y = \operatorname{Ric}(X, Y) + \operatorname{Tr}(\nabla X \cdot \nabla Y),$$

where $\operatorname{Ric}(X, Y) = \langle Ric\langle X \rangle, Y \rangle$. We obtain that

$$\nabla^* \nabla_Y \nabla Z = \operatorname{grad}[\operatorname{Ric}(Y, Z) + \operatorname{Tr}(\nabla Y \cdot \nabla Z)] + [\nabla^*, \nabla] \nabla_Y Z + \nabla^*[\nabla_Y, \nabla] Z.$$

Hence, using Lemma 4.3, $\nabla^* \nabla_Y \nabla Z: H^s \to H^{s-2}$ is continuous, so that

$$\| \hat{\Delta}^{-1}\nabla^*[\nabla X \cdot \nabla_Y \nabla Z - \nabla Y \cdot \nabla_X \nabla Z] \|_s \leqslant C \|X\|_s \|Y\|_s \|Z\|_s.$$

This completes the estimates on each term of $R_e^1(X, Y) Z$. Since we allow our constant to depend on η and since H^{s-2} is a multiplicative algebra, we have that for any $\eta \in \mathscr{D}^s(M)$,

$$\| R^1(X_\eta, Y_\eta) Z_\eta \|_s \leqslant C \|X_\eta\|_s \|Y_\eta\|_s \|Z_\eta\|_s,$$

where C denotes any constant which may depend on s, η, and derivatives of $\langle \cdot, \cdot \rangle$ on M. ▌

4.2. *Curvature of* $\tilde{\nabla}^1$

Next, we define the (weak) curvature $\tilde{R}^1$ of the induced metric $\langle \cdot, \cdot \rangle_1$ on $\mathscr{D}^s_\mu(M)$ as

$$\tilde{R}^1_\eta : T_\eta \mathscr{D}^s_\mu(M) \times T_\eta \mathscr{D}^s_\mu(M) \times T_\eta \mathscr{D}^s_\mu(M) \to T_\eta \mathscr{D}^s_\mu(M),$$
$$\tilde{R}^1_\eta(X_\eta, Y_\eta)\, Z_\eta = (\tilde{\nabla}^1_X \tilde{\nabla}^1_Y Z)_\eta - (\tilde{\nabla}^1_Y \tilde{\nabla}^1_X Z)_\eta - (\tilde{\nabla}^1_{[X, Y]} Z)_\eta,$$

where $\eta \in \mathscr{D}^s_\mu(M)$, and X, Y, Z are smooth extensions of X_η, Y_η, Z_η in a neighborhood of η.

In order to estimate $\tilde{R}^1$, we shall make use of the Gauss formula in submanifold geometry which relates the curvature of $\mathscr{D}^s(M)$ with the curvature of $\mathscr{D}^s_\mu(M)$ using the second fundamental form. Let X, Y, Z, and W be smooth vector fields on $\mathscr{D}^s_\mu(M)$. Then for any $\eta \in \mathscr{D}^s_\mu(M)$, we have

$$\langle \tilde{R}^1(X, Y)\, Z, W \rangle_1 = \langle R^1(X, Y)\, Z, W \rangle_1 + \langle S_\eta(Y, Z), S_\eta(X, W) \rangle_1 - \langle S_\eta(X, Z), S_\eta(Y, W) \rangle_1. \tag{4.4}$$

THEOREM 4.1. *The curvature $\tilde{R}^1$ of the induced H^1 metric on $\mathscr{D}^s_\mu(M)$ is a trilinear operator which is continuous in the H^s topology for $s > (n/2) + 2$.*

Proof. For the purpose of obtaining estimates on $\tilde{R}^1$ we shall use the equivalent H^s metric (under our assumptions) given at the identity for $X, Y \in T_e \mathscr{D}^s_\mu(M)$ by

$$\langle X, Y \rangle_s = \langle X, (1 - \hat{\Delta})^s\, Y \rangle_{L^2},$$

and then extended to $T\mathscr{D}^s_\mu(M)$ by right translation. This gives a smooth invariant metric on $\mathscr{D}^s_\mu(M)$ which induces a topology which is equivalent to the underlying topology of $\mathscr{D}^s_\mu(M)$.

We will estimate $\sup_{\|W\|_s = 1} \langle \tilde{R}^1(X, Y)\, Z, W \rangle_s$ using the Gauss formula (4.4). Let $X, Y, Z \in T_e \mathscr{D}^s_\mu(M)$, and let $W \in C^\infty(TM)$, $\operatorname{div} W = 0$. We have that

$$\begin{aligned}
&\langle \tilde{R}^1(X, Y)\, Z, (1 - \hat{\Delta})^s\, W \rangle_0 \\
&\quad = \langle R^1(X, Y)\, Z, (1 - \hat{\Delta})^s\, W \rangle_0 \\
&\qquad + \langle S_e(Y, Z), (1 - \hat{\Delta})\, S_e(X, (1 - \hat{\Delta})^{s-1}\, W) \rangle_0 \\
&\qquad - \langle S_e(X, Z), (1 - \hat{\Delta})\, S_e(Y, (1 - \hat{\Delta})^{s-1}\, W) \rangle_0.
\end{aligned} \tag{4.5}$$

Now, $S_e(X, Y) = Q_e(\nabla_X Y) + Q_e D(X, Y)$, where $D(X, Y) = A(X, Y) + B(X, Y) + C(X, Y)$, so

$$
\begin{aligned}
&\langle S_e(Y, Z), (1-\hat{\Delta})\, S_e(X, (1-\hat{\Delta})^{s-1} W)\rangle_0 \\
&\quad = \langle Q_e(\nabla_Y Z), (1-\hat{\Delta})\, Q_e \nabla_X (1-\hat{\Delta})^{s-1} W\rangle_0 \\
&\qquad + \langle Q_e(\nabla_Y Z), (1-\hat{\Delta})\, Q_e D(X, (1-\hat{\Delta})^{s-1} W)\rangle_0 \\
&\qquad + \langle Q_e D(Y, Z), (1-\hat{\Delta})\, Q_e(\nabla_X (1-\hat{\Delta})^{s-1} W)\rangle_0 \\
&\qquad + \langle Q_e D(Y, Z), (1-\hat{\Delta})\, Q_e D(X, (1-\hat{\Delta})^{s-1} W)\rangle_0. \qquad (4.6)
\end{aligned}
$$

For the first step, we will obtain the estimates for (4.6) in the case where D is just B. We begin by estimating the first term on the right-hand side of (4.6). Using the fact that Q_e is also an orthogonal projection in L^2, we have that

$$
\begin{aligned}
&\langle Q_e(\nabla_Y Z), (1-\hat{\Delta})\, Q_e \nabla_X (1-\hat{\Delta})^{s-1} W\rangle_0 \\
&\quad = -\langle (1-\hat{\Delta})^{(s-2)/2}\, \nabla_X Q_e (1-\hat{\Delta})\, Q_e \nabla_Y Z, (1-\hat{\Delta})^{s/2} W\rangle_0. \qquad (4.7)
\end{aligned}
$$

Using the identity for divergence-free vector fields

$$
\operatorname{div} \nabla_X Y = \operatorname{Ric}(X, Y) + \operatorname{Tr}(\nabla X \cdot \nabla Y),
$$

and choosing a smooth local orthonormal frame $\{e_i\}$ in which the rough Laplacian $\hat{\Delta} = \nabla_{e_i} \nabla_{e_i}$, we see that

$$
\begin{aligned}
Q_e \hat{\Delta} Q_e \nabla_Y Z = {} & \operatorname{grad} \hat{\Delta}^{-1} \operatorname{Ric}(e_i, \nabla_{e_i} \operatorname{grad} \hat{\Delta}^{-1} \operatorname{Ric}(Y, Z)) \\
& + \operatorname{grad} \hat{\Delta}^{-1} \operatorname{Tr}[\nabla e_i \cdot \nabla \nabla_{e_i} \operatorname{grad} \hat{\Delta}^{-1} \operatorname{Ric}(Y, Z)] \\
& + \operatorname{grad} \hat{\Delta}^{-1} \operatorname{Ric}(e_i, \nabla_{e_i} \operatorname{grad} \hat{\Delta}^{-1} \operatorname{Tr}[\nabla Y \cdot \nabla Z] \\
& + \operatorname{grad} \hat{\Delta}^{-1} \operatorname{Tr}[\nabla e_i \cdot \nabla \nabla_{e_i} \operatorname{grad} \hat{\Delta}^{-1} \operatorname{Tr}[\nabla Y \cdot \nabla Z]]. \qquad (4.8)
\end{aligned}
$$

We estimate the last term in (4.8) since it is least regular. We obtain

$$
\begin{aligned}
&\|\operatorname{grad} \hat{\Delta}^{-1} \operatorname{Tr}[\nabla e_i \cdot \nabla \nabla_{e_i} \operatorname{grad} \hat{\Delta}^{-1} \operatorname{Tr}[\nabla Y \cdot \nabla Z]]\|_{s-1} \\
&\quad \leqslant \|\operatorname{Tr}[\nabla e_i \cdot \nabla \nabla_{e_i} \operatorname{grad} \hat{\Delta}^{-1} \operatorname{Tr}[\nabla Y \cdot \nabla Z]]\|_{s-2} \\
&\quad \leqslant C\, \|\nabla_{e_i} \operatorname{grad} \hat{\Delta}^{-1} \operatorname{Tr}[\nabla Y \cdot \nabla Z]]\|_{s-1},
\end{aligned}
$$

where we used the fact that H^{s-2} is a multiplicative algebra, and the constant C may depend on e_i. Now

$$
\begin{aligned}
&\|\nabla_{e_i} \operatorname{grad} \hat{\Delta}^{-1} \operatorname{Tr}[\nabla Y \cdot \nabla Z]]\|_{s-1} \\
&\quad \leqslant \|\hat{\Delta}^{(s-1)/2} (\nabla \operatorname{grad} \hat{\Delta}^{-1} \operatorname{Tr}[\nabla Y \cdot \nabla Z]) \cdot e_i\|_0 \\
&\qquad + \|(\nabla \operatorname{grad} \hat{\Delta}^{-1} \operatorname{Tr}[\nabla Y \cdot \nabla Z]) \cdot \hat{\Delta}^{(s-1)/2} e_i\|_0 \\
&\quad \leqslant C\, \|\operatorname{Tr}[\nabla Y \cdot \nabla Z]\|_{s-1} \leqslant C\, \|Y\|_s\, \|Z\|_s.
\end{aligned}
$$

This shows that $\|Q_e(1-\hat{\Delta})\, Q_e\nabla_Y Z\|_{s-1} \leqslant C\,\|Y\|_s\,\|Z\|_s$, so that applying the Cauchy–Schwartz inequality to (4.7) we obtain

$$\begin{aligned}
&|\langle Q_e(\nabla_Y Z), (1-\hat{\Delta})\, Q_e\nabla_X(1-\hat{\Delta})^{s-1}\, W\rangle_0| \\
&\quad \leqslant C\,\|\nabla_X Q_e(1-\hat{\Delta})\, Q_e\nabla_Y Z\|_{s-2}\,\|W\|_s \\
&\quad \leqslant C\{\|\hat{\Delta}^{(s-2)/2}(\nabla Q_e(1-\hat{\Delta})\, Q_e\nabla_Y Z)\cdot X\|_0 \\
&\qquad\quad + \|\nabla Q_e(1-\hat{\Delta})\, Q_e\nabla_Y Z\cdot\hat{\Delta}^{(s-2)/2}X\|_0\}\,\|W\|_s \\
&\quad \leqslant C\{\|\nabla Q_e(1-\hat{\Delta})\, Q_e\nabla_Y Z\|_{s-2}\,\|X\|_\infty \\
&\qquad\quad + \|\nabla Q_e(1-\hat{\Delta})\, Q_e\nabla_Y Z\|_\infty\,\|X\|_{s-2}\}\,\|W\|_s \\
&\quad \leqslant C\,\|Q_e(1-\hat{\Delta})\, Q_e\nabla_Y Z\|_{s-1}\,\|X\|_s\,\|W\|_s \\
&\quad \leqslant C\,\|X\|_s\,\|Y\|_s\,\|Z\|_s\,\|W\|_s.
\end{aligned}$$

Since $B\colon H^s\times H^s\to H^{s+1}$ continuously, we have estimated the first and third terms on the right-hand side of (4.6).

Next we estimate the second term on the right-hand side of (4.6). We have that

$$\begin{aligned}
&B(X, (1-\hat{\Delta})^{s-1}\, W) \\
&\quad = \tfrac{1}{2}(1-\hat{\Delta})^{-1}\,\mathrm{Tr}[R(\,\cdot, \nabla.(1-\hat{\Delta})^{s-1}\, W)\, X \\
&\qquad + [R(\cdot, \nabla.X)(1-\hat{\Delta})^{s-1}\, W - \nabla.[R(X, \cdot)(1-\hat{\Delta})^{s-1}\, W \\
&\qquad + R((1-\hat{\Delta})^{s-1}\, W, \cdot)\, X]].
\end{aligned} \tag{4.9}$$

Let us begin our estimate with the first of the four terms in (4.9). Let

$$V = \tfrac{1}{2}(1-\hat{\Delta})^{-1}\, Q_e(1-\hat{\Delta})\, Q_e\nabla_Y Z,$$

which is of Sobolev class H^{s+1}. Then

$$\begin{aligned}
&\tfrac{1}{2}|\langle Q_e(\nabla_Y Z), (1-\hat{\Delta})\, Q_e(1-\hat{\Delta})^{-1}\,\mathrm{Tr}[R(e_i, \nabla_{e_i}(1-\hat{\Delta})^{s-1}\, W)\, X]\rangle_0| \\
&\quad = |\langle V, \mathrm{Tr}[R(e_i, \nabla_{e_i}(1-\hat{\Delta})^{s-1}\, W)\, X]\rangle_0\cdot| \\
&\quad = \bigg|\int_M \langle (1-\hat{\Delta})^{(s-2)/2}\,\{\langle\nabla_{e_i} V, R(e_i, X)\cdot\,\rangle^\# + \langle V, (\nabla_{e_i} R)(e_i, X)\cdot\,\rangle^\# \\
&\qquad + \langle V, R(\nabla_{e_i} e_i, X)\cdot\,\rangle^\# + \langle V, \mathrm{div}(e_i)\, R(e_i, X)\cdot\,\rangle^\# \\
&\qquad + \langle V, R(e_i, \nabla_{e_i} X)\cdot\,\rangle^\#\}, (1-\hat{\Delta})^{s/2}\, W\rangle\, dx\bigg|.
\end{aligned} \tag{4.10}$$

Now

$$\left|\int_M \langle (1-\hat{\Delta})^{(s-2)/2}\,\{\langle \nabla_{e_i} V, R(e_i, X)\cdot\rangle^{\#}\}, (1-\hat{\Delta})^{s/2}\, W\rangle\right.$$

$$= \left|\int_M \langle \mathrm{Tr}[\langle (1-\hat{\Delta})^{(s-2)/2}\,\nabla . V, R(\cdot, X)\rangle^{\#}\right.$$

$$\left. + \langle \nabla . V, (1-\hat{\Delta})^{(s-2)/2}\, R(\cdot, X)\rangle^{\#}], (1-\hat{\Delta})^{s/2}\, W\rangle\right|$$

$$\leqslant \|\mathrm{Tr}[\langle (1-\hat{\Delta})^{(s-2)/2}\,\nabla . V, R(\cdot, X)\rangle^{\#}$$

$$+ \langle \nabla . V, ((1-\hat{\Delta})^{(s-2)/2}\, R)(\cdot, X) + R(\cdot, (1-\hat{\Delta})^{(s-2)/2}\, X)\rangle^{\#}]\|_0\, \|W\|_s$$

$$\leqslant C\,[\,\|\nabla V\|_{s-2}\, \|R\|_\infty\, \|X\|_\infty + \|\nabla V\|_\infty\, \|(1-\hat{\Delta})^{(s-2)/2}\, R\|_\infty\, \|X\|_\infty$$

$$+ \|\nabla V\|_\infty\, \|R\|_\infty\, \|X\|_{s-2}]\, \|W\|_s$$

$$\leqslant C\,\|V\|_{s-1}\, \|X\|_s\, \|W\|_s \leqslant C\,\|Q_e(1-\hat{\Delta})\, Q_e \nabla_Y Z\|_{s-3}\, \|X\|_s\, \|W\|_s$$

$$\leqslant C\,\|X\|_s\, \|Y\|_s\, \|Z\|_s\, \|W\|_s,$$

where the constant C may depend on M, the derivatives of the metric $\langle \cdot, \cdot\rangle$ on M, and the local orthonormal frame. The remaining terms in (4.10) can be estimated in the same manner, so that

$$\tfrac{1}{2}\,|\langle Q_e(\nabla_Y Z), (1-\hat{\Delta})\, Q_e(1-\hat{\Delta})^{-1}\, \mathrm{Tr}[R(e_i, \nabla_{e_i}(1-\hat{\Delta})^{s-1}\, W)\, X]\rangle_0|$$

$$\leqslant C\,\|X\|_s\, \|Y\|_s\, \|Z\|_s\, \|W\|_s.$$

Using the same type of estimates, we may bound the remaining three terms in (4.9), so that the second term on the right-hand side of (4.6) with $D = B$ is majorized by $\|X\|_s\, \|Y\|_s\, \|Z\|_s\, \|W\|_s$. The fourth term on right-hand side of (4.6) with $D = B$ has more regularity than the second term, and thus has the same majorization.

Now, if we let $D = C$, we easily obtain the same estimates since C is as regularizing as B. For $D = A$, we must estimate the term

$$\langle Q_e \nabla_Y Z, (1-\hat{\Delta})\, Q_e(1-\hat{\Delta})^{-1}\, \nabla^*(\nabla X \cdot \nabla(1-\hat{\Delta})^{s/2}\, W)\rangle_0.$$

With similar estimates as above, we can bound this term by

$$C(\|(1-\hat{\Delta})^{(s-2)/2}\,\mathrm{grad\ div}\, X\|_0 \cdot \|\nabla(1-\hat{\Delta})^{-1}\, Q_e(1-\hat{\Delta})\, Q_e \nabla_Y Z\|_\infty$$

$$+ \|\mathrm{grad\ div}\, X\|_\infty \cdot \|(1-\hat{\Delta})^{(s-2)/2}\, \nabla(1-\hat{\Delta})^{-1}\, Q_e(1-\hat{\Delta})\, Q_e \nabla_Y Z\|_0$$

$$+ \|(1-\hat{\Delta})^{(s-2)/2}\, \nabla X\|_0 \cdot \|\nabla(\nabla(1-\hat{\Delta})^{-1}\, Q_e(1-\hat{\Delta})\, Q_e \nabla_Y Z)^t\|_\infty$$

$$+ \|\nabla X\|_\infty \cdot \|(1-\hat{\Delta})^{(s-2)/2}\, \nabla(1-\hat{\Delta})^{-1}\, Q_e(1-\hat{\Delta})\, Q_e \nabla_Y Z\|_0),$$

which is itself bounded by $C\,\|X\|_s\,\|Y\|_s\,\|Z\|_s\,\|W\|_s$. The estimates for the other terms involving A are similar.

Hence, we have estimated the second term on the right-hand side of (4.5), and by symmetry of the bound, the third term as well. Proposition 4.1 gives us the same majorization for the first term.

Since

$$\begin{aligned}
&\|\tilde{R}^1_e(X, Y)\, Z\|_s \\
&\qquad = \sup\{\langle \tilde{R}^1_e(X, Y)\, Z, W\rangle_s \colon W \in C^\infty(TM),\ \operatorname{div} W = 0,\ \|W\|_s < 1\} \\
&\qquad \leqslant C\,\|X\|_s\,\|Y\|_s\,\|Z\|_s,
\end{aligned}$$

where C depends on M and the derivatives of the metric on M, we have that $\tilde{R}^1_e$ is a bounded trilinear map on H^s.

Now the map $\eta \to P_\eta$ is continuously differentiable, and since right translation only introduces terms of the type $[T\eta]^{-1}$ and $[T\eta]^{-1^t}$, and as we have a multiplicative algebra, the general case follows. ∎

Remark 4.1. One might try to argue that the boundedness in H^s of $\tilde{R}^1$ follows immediately from the regularity of the geodesic spray, but this argument fails for the following reason. Let $\mathscr{U} \subset \mathscr{D}^s_\mu(M)$ be sufficiently small so as to allow a trivialization of $T\mathscr{D}^s_\mu(M)$, and let $\mathscr{A}^1$ be the local connection 1-form defining the H^1 covariant derivative $\tilde{\nabla}^1$. The fact that the geodesic spray of $\tilde{\nabla}^1$ is C^1 implies that $\mathscr{A}^1$ is a C^1 map as well. Now the curvature can be defined as $d\mathscr{A}^1 + \mathscr{A}^1 \wedge \mathscr{A}^1$, and it may seem that for all $\eta \in \mathscr{U}$, $d\mathscr{A}^1(\eta)$ is then necessarily a continuous operator from H^s into H^s. This is not the case, however, as the exterior derivative is defined in terms of the H^1-Frechet derivative, while the fact that $\mathscr{A}^1$ is C^1 is verified using the H^s-Frechet derivative. It is for this reason, that curvatures of strong metrics are trivially bounded operators in the strong topology of the manifold, while for weak metrics, one must verify any boundedness claims.

4.3. *Jacobi Equations*

We can now prove the existence of solutions to the Jacobi equation

$$\tilde{\nabla}^1_{\dot\eta}\tilde{\nabla}^1_{\dot\eta} Y + \tilde{R}^1_\eta(Y, \dot\eta)\,\dot\eta = 0 \tag{4.11}$$

along the geodesic $\eta(t)$ of the H^1-metric which solves the Euler-α equation (3.7) in Lagrangian coordinates. Note that (3.7) may equivalently be written as

$$\tilde{\nabla}^1_{\dot\eta}\dot\eta = 0 \tag{4.12}$$

for $\eta(t)$ a curve in $\mathscr{D}^s_\mu(M)$. The Jacobi equation (4.11) is the linearization of (4.12) along the geodesic.

THEOREM 4.2. *Let* $s > (n/2) + 2$ *and let* $Y_e, \dot{Y}_e \in T_e\mathscr{D}^s_\mu(M)$. *Then there exists a unique* H^s *vector field* $Y(t)$ *along* η *that is a solution to* (4.11) *with initial conditions* $Y(0) = Y_e$ *and* $\tilde{\nabla}^1_{\dot{\eta}} Y(0) = \dot{Y}_e$.

Proof. Let τ_t: $T_e\mathscr{D}^s_\mu(M) \to T_{\eta(t)}\mathscr{D}^s_\mu(M)$ be the parallel translation along η induced by $\tilde{\nabla}^1$. It is standard that τ_t is a linear isomorphism such that $[\tau_t, \tilde{\nabla}^1] = 0$, and $\tau_t^* \langle \cdot, \cdot \rangle_1 = \langle \cdot, \cdot \rangle_1$. We consider the curve in the algebra $V(t) = \tau_t^{-1} Y(t)$ where $(d/dt)\, V(t) = \tau_t^{-1} \tilde{\nabla}^1_{\eta(t)} Y(t)$, wherein the Jacobi equation takes the form

$$\frac{d^2}{dt^2} V(t) = -\tau_t^{-1} \tilde{R}^1_{\eta(t)}(\tau_t V(T), \dot{\eta}(t))\, \dot{\eta}(t).$$

By Theorem 4.1, $\tilde{R}^1$ is bounded in H^s, so existence and uniqueness immediately follow. ∎

5. STABILITY AND CURVATURE

In this section, we define the notion of Lagrangian linear stability (see [M1]).

5.1. *Lagrangian Stability*

For $k \geq 1$, a fluid motion η is Lagrangian H^k (linearly) stable if every solution of the Jacobi equation (4.11) along η is bounded in the H^k norm.

THEOREM 5.1. *If* $\eta(t)$ *is a geodesic of* $\tilde{\nabla}^1$ *on* $\mathscr{D}^s_\mu(M)$ *whose pressure function* $p(t)$ *is constant for all* t *and if the sectional curvature of* R^1 *is nonpositive, then* η *is* H^k *Lagrangian unstable for* $k \geq 1$.

Proof. Let η solve $\tilde{\nabla}^1_{\dot{\eta}} \dot{\eta} = 0$ on $\mathscr{D}^s_\mu(M)$, and let $Y(t)$ be a nontrivial Jacobi field along η with $Y(0) = 0$, $\tilde{\nabla}^1_{\dot{\eta}} Y(0) = \dot{Y}_e$. If the sectional curvature of the plane spanned by $Y(t)$ and $\dot{\eta}$ is nonpositive for t, then η is H^k Lagrangian unstable for $k \geq 1$. This follows from Lemma 4.2 of [M1] by replacing the L^2 norm with the H^1 norm. Namely, for $t > 0$, let $Z = Y / \| Y \|_1$ and compute

$$\tilde{\nabla}^1_{\dot{\eta}} \tilde{\nabla}^1_{\dot{\eta}} Y = \frac{d^2}{dt^2}(\| Y \|_1)\, Z + 2\frac{d}{dt}(\| Y \|_1)\, \tilde{\nabla}^1_{\dot{\eta}} Z + \| Y \|_1\, \tilde{\nabla}^1_{\dot{\eta}} \tilde{\nabla}^1_{\dot{\eta}} Y.$$

Taking the inner product of $\tilde{\nabla}^1_{\dot{\eta}} \tilde{\nabla}^1_{\dot{\eta}} Y$ with Z, and noting that $\|Z\|_1 = 1$ and that Y solves (4.11), we obtain that

$$\frac{d^2}{dt^2}(\| Y \|_1) = [\, \|\tilde{\nabla}^1_{\dot{\eta}} Z\|_1^2 - \langle \tilde{R}^1(Z, \dot{\eta})\, \dot{\eta}, Z \rangle_1]\, \| Y \|_1.$$

Thus, $(d^2/dt^2)\,\|Y\|_1 \geqslant 0$, so that $\|Y\|_1 > ct$ for all $t>0$ and some positive constant c depending on $\dot{Y}_e$, which implies that $\|Y\|_k$ is unbounded for $k \geqslant 1$ by the compact embedding: $H^k \hookrightarrow H^1$.

Since η is a geodesic in $\mathscr{D}^s_\mu(M)$, Theorem 3.3 asserts that $U = \dot{\eta} \circ \eta^{-1}$ satisfies equation (3.4) on M. Thus, we have that

$$
\begin{aligned}
S_\eta(\dot{\eta}, \dot{\eta}) &= Q_\eta(\nabla^1_{\dot{\eta}} \dot{\eta}) \\
&= Q_e\{\partial_t U + (1+\Delta)^{-1}\,[\nabla_U (1+\Delta)\, U + \langle \nabla U \langle\,\cdot\,\rangle, \Delta U\rangle^{\#}]\} \circ \eta \\
&= -(\operatorname{grad} p) \circ \eta = 0,
\end{aligned}
$$

so η is a pressure constant geodesic of the right invariant H^1 metric on $\mathscr{D}^s_\mu(M)$ if and only if $S_\eta(\dot{\eta}, \dot{\eta}) = 0$.

From the Gauss equation (4.4),

$$
\langle \tilde{R}^1_\eta(X, \dot{\eta})\, \dot{\eta}, X\rangle_1 = \langle R^1(X, \dot{\eta})\, \dot{\eta}, X)_1 - \|S_\eta(\dot{\eta}, X)\|_1^2
$$

for any vector field $X(t)$ along the pressure constant geodesic η. Hence, $\langle \tilde{R}^1_\eta(X, \dot{\eta})\, \dot{\eta}, X\rangle_1$ is nonpositive whenever $\langle R^1(X, \dot{\eta})\, \dot{\eta}, X\rangle_1$ is nonpositive. ∎

Remark 5.1. Note that on the flat torus $\mathbb{T}^n$, the formula (3.1) simplifies to $\nabla^1_X Y = \nabla^0_X Y + A(X, Y)$, and since $R^0 = 0$, we have that for $X, Y, Z \in T_e \mathscr{D}^s_\mu(M)$,

$$
\begin{aligned}
R^1_e(X, Y)\, Z &= A_e(X, \nabla^1_Y Z) - A_e(Y, \nabla^1_X Z) + \nabla^0_X A_e(Y, Z) - \nabla^0_Y A_e(X, Z) \\
&\quad + A_e(X, A_e(Y, Z)) - A_e(Y, A_e(X, Z)) - A_e([X, Y], Z). \qquad (5.1)
\end{aligned}
$$

Choose a coordinate chart (U, x^i) on M. At the identity e,

$$
2A_e(X, Z) = (1-\Delta)^{-1}\,[\nabla^*(\nabla X \cdot \nabla Z + \nabla Z \cdot \nabla X)].
$$

Substitution of $(1-\Delta)^{-1}\,\nabla^*(\nabla X \cdot \nabla Z)$ into (5.1) yields

$$
\begin{aligned}
&\frac{\partial}{\partial x^j}\left[(1-\Delta)^{-1}\frac{\partial}{\partial x^l}\left(\frac{\partial Y^l}{\partial x^i}\frac{\partial Z^i}{\partial x^n}\right)\right] X^j - \frac{\partial}{\partial x^l}\left[(1-\Delta)^{-1}\frac{\partial}{\partial x^j}\left(\frac{\partial X^j}{\partial x^i}\frac{\partial Z^i}{\partial x^n}\right)\right] Y^l \\
&\quad + (1-\Delta)^{-1}\frac{\partial}{\partial x^j}\left[\frac{\partial X^j}{\partial x^i}\frac{\partial}{\partial x^n}\left(\frac{\partial Z^i}{\partial x^l}\, Y^l\right)\right] - (1-\Delta)^{-1}\frac{\partial}{\partial x^l}\left[\frac{\partial Y^l}{\partial x^i}\frac{\partial}{\partial x^n}\left(\frac{\partial Z_i}{\partial x^j}\, X^j\right)\right] \\
&\quad + (1-\Delta)^{-1}\frac{\partial}{\partial x^j}\left\{\frac{\partial Y^j}{\partial x^n}\frac{\partial}{\partial x^k}\left[(1-\Delta)^{-1}\frac{\partial}{\partial x^l}\left(\frac{\partial X^l}{\partial x^i}\frac{\partial Z^i}{\partial x^n}\right)\right]\right\} \\
&\quad - (1-\Delta)^{-1}\frac{\partial}{\partial x^l}\left\{\frac{\partial Y^l}{\partial x^n}\frac{\partial}{\partial x^k}\left[(1-\Delta)^{-1}\frac{\partial}{\partial x^j}\left(\frac{\partial X^j}{\partial x^i}\frac{\partial Z^i}{\partial x^n}\right)\right]\right\}.
\end{aligned}
$$

It is clear that R_e^1 vanishes when X, Y, Z have components of the form $e^{i\langle k,x\rangle}$. More interestingly, one may compute the sectional curvature $\langle R_e^1(X, Y)\, Y, X\rangle_1$ in the directions $X = \sin(\langle k, x\rangle)\, \partial/\partial x^1 + \cos(\langle m, x\rangle)\, \partial/\partial x^2$ and $Y = \cos(\langle k, x\rangle)\, \partial/\partial x^1 + \sin(\langle m, x\rangle)\, \partial/\partial x^2$. For example, when $X = (\sin(kx^1), 0)$ and $Y = (0, \cos(kx^2)$,

$$\langle R_e^1(X, Y)\, Y, X\rangle_1 = 0,$$

whereas if $X = (\sin(kx^1), 0)$ and $Y = (\cos(kx^1), 0)$, then

$$\langle R_e^1(X, Y)\, Y, X\rangle_1 < 0$$

for any choice of $k \neq 0$ (cf. [M3]). Recall that this computation of the curvature tensor of the full diffeomorphism group is restricted to divergence free vector fields, since we are ultimately only interested in the stability of the motion on the volume preserving subgroup.

If η is a geodesic in $\mathscr{D}_\mu^s(M)$, two points $\eta(t_1)$ and $\eta(t_2)$ are conjugate with respect to η if there exists a nonzero Jacobi field $Y(t)$ along η such that $Y(t_1) = Y(t_2) = 0$. Such Jacobi fields are thus stable perturbations of the initial flow.

Corollary 5.1. *Let η be a pressure constant geodesic in $\mathscr{D}_\mu^s(M)$. If the sectional curvature of R^1 is nonpositive, then there are no conjugate points along η.*

5.2. *Examples*

Example 5.1. A trivial example of a pressure constant geodesic in $\mathscr{D}_\mu(\mathbb{T}^2)$ is given by

$$\eta(t)(x^1, x^2) = (x^1 + h(x^2), x^2 + ct),$$

where c is a constant and h is a smooth periodic function. Let

$$\begin{aligned} G(\eta) = & -D(\dot\eta\circ\eta^{-1})^t\, D(\dot\eta\circ\eta^{-1})[T\eta]^{-1^t} + D(\dot\eta\circ\eta^{-1})\, D(\dot\eta\circ\eta^{-1})[T\eta]^{-1^t} \\ & + D(\dot\eta\circ\eta^{-1})\, D(\dot\eta\circ\eta^{-1})^t\, [T\eta]^{-1^t}. \end{aligned}$$

Then on $\mathbb{T}^n$, equation (4.12) simplifies to

$$\ddot\eta\circ\eta^{-1} - \operatorname{grad} \Delta^{-1} \operatorname{Tr}[D(\dot\eta\circ\eta^{-1})]^2 = (\mathrm{Id} - \operatorname{grad} \Delta^{-1} \operatorname{div})[(1-\hat\Delta_\eta)^{-1}\, G(\eta)],$$

and since $\dot\eta(x^1, x^2) = (0, c)$, then η is a geodesic.

EXAMPLE 5.2. Another example of a pressure constant geodesic in $\mathscr{D}_\mu(\mathbb{T}^2)$ is given by

$$\eta(t)(x^1, x^2) = (x^1 + th(x^2), x^2),$$

where again c is a constant and h is a smooth periodic function. In this case

$$\dot\eta \circ \eta^{-1}(y^1, y^2) = (h(y^2), 0),$$

and we must verify that

$$0 = P_e \circ \{\partial_t(\dot\eta \circ \eta^{-1}) + (1 - \hat\Delta)^{-1} [\nabla_{\dot\eta \circ \eta^{-1}}(1 - \hat\Delta)(\dot\eta \circ \eta^{-1}) - [\nabla\dot\eta \circ \eta^{-1}]^t \cdot \hat\Delta(\dot\eta \circ \eta^{-1})]\}. \tag{5.2}$$

Notice that for our choice of η, $(1-\hat\Delta)^{-1} [\nabla U]^t \cdot \hat\Delta U = \operatorname{grad} F$, for some $F \in C^\infty(M)$; hence, $P_e \circ (1-\hat\Delta)^{-1} [\nabla U]^t \cdot \hat\Delta U = 0$, so that (5.2) is simply

$$\partial_t(\dot\eta \circ \eta^{-1}) + (1-\hat\Delta)^{-1} \nabla_{\dot\eta \circ \eta^{-1}}(1-\hat\Delta)(\dot\eta \circ \eta^{-1}) = -\operatorname{grad} p. \tag{5.3}$$

But the left-hand side of (5.3) vanishes, so η is a pressure constant geodesic.

Remark 5.2. Theorem 5.1 and the remarks which follow its proof imply that the geodesic flows of the previous two examples with $h(x^2) = \sin(kx^2)$ are unstable to perturbations in the $\cos(kx^2)$ direction. Other such examples of unstable perturbations can be constructed.

ACKNOWLEDGMENTS

The author would like to thank Darryl Holm, Jerry Marsden, Gerard Misiołek, and Tudor Ratiu for the many stimulating discussions and helpful suggestions they have provided.

REFERENCES

[A] V. I. Arnold, Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluids parfaits, *Ann. Inst. Grenoble* **16** (1966), 319–361.

[AK1] V. I. Arnold and B. Khesin, "Topological Methods in Hydrodynamics," Springer-Verlag, New York, 1998.

[AK2] V. I. Arnold and B. Khesin, Topological methods in hydrodynamics, *in* "Annual Review of Fluid Mechanics," Vol. 24, pp. 145–166, Annual Reviews, Palo Alto, CA, 1992.

[BLR] D. Bao, J. Lafontaine, and T. Ratiu, On a nonlinear equation related to the geometry of the diffeomorphism group, *Pacific J. Math.* **158** (1993), 223–242.

[E] D. Ebin, The motion of slightly compressible fluids viewed as a motion with strong constraining force, *Ann. of Math.* **105** (1977), 141–200.

[EM] D. Ebin and J. Marsden, Groups of diffeomorphisms and the motion of an incompressible fluid, *Ann. of Math.* **92** (1970), 102–163.
[EMis] D. Ebin and G. Misiolek, The exponential map on $\mathscr{D}^s_\mu$, preprint.
[FHT] C. Foias, D. D. Holm, and E. S. Titi, The three dimensional viscous Cammassa–Holm Equations and their relation to the Navier–Stokes equations and turbulence theory, preprint.
[HKMRS] D. D. Holm, S. Kouranbaeva, J. E. Marsden, T. S. Ratiu, and S. Shkoller, A nonlinear analysis of the averaged Euler equations, *Fields Inst. Comm.*, Arnoldsfest Vol. 2, AMS, 1998, to appear.
[HMR1] D. D. Holm, J. E. Marsden, and T. S. Ratiu, Euler–Poincaré equations and semidirect products with applications to continuum theories, *Adv. in Math.* **137** (1998), 1–81.
[HMR2] D. D. Holm, J. E. Marsden, and T. S. Ratiu, Euler–Poincaré models of ideal fluids with nonlinear dispersian, *Phys. Rev. Lett.* **80** (1998), 4273–4277.
[La] S. Lang, "Differentiable and Riemannian Manifolds," Springer-Verlag, New York, 1995.
[L1] A. M. Lukatskii, On the curvature of the group of measure-preserving diffeomorphisms of an n-dimensional torus, *Comm. Moscow Math. Soc.* (1980), 179–180.
[L2] A. M. Lukatskii, Curvature of the group of measure-preserving diffeomorphisms of an n-dimensional torus, *Sibirskii Math. J.* **25** (6) (1984), 76–88.
[L3] A. M. Lukatskii, Structure of the curvature tensor of the group of measure-preserving diffeomorphisms of a compact two-dimensional manifold, *Sibirskii Math. J.* **29**(6) (1988), 95–99.
[MEF] J. E. Marsen, D. G. Ebin, and A. E. Fischer, Diffeomorphism groups, hydrodynamics and relativity, *in* "Proceedings of the Thirteenth Biennial Seminar of the Canadian Mathematical Congress" (J. R. Vanstone, Ed.), Montreal, 1972.
[MR] J. E. Marsden and T. S. Ratiu, "Introduction to Mechanics and Symmetry," Springer-Verlag, Berlin/New York, 1995.
[M1] G. Misiolek, Stability of flows of ideal fluids and the geometry of the group of diffeomorphisms, *Indiana Univ. Math. J.* **2** (1993), 215–235.
[M2] G. Misiolek, Conjugate points in $\mathscr{D}_\mu(\mathbb{T}^2)$, *Proc. Amer. Math. Soc.* **124** (1996), 977–982.
[M3] G. Misiolek, A shallow water equation as a geodesic flow on the Bott–Viasoro group, *J. Geom. Phys.* **24** (1998), 203–208.
[M4] G. Misiolek, The exponential map on the free loop space is Fredholm, *Geom. Funct. Anal.* **7** (1997), 954–969.
[Mor] C. B. Morrey, "Multiple Integrals in the Calculus of Variations," Springer-Verlag, Berlin/New York, 1966.
[R] S. Rosenberg, The Laplacian on a Riemannian manifold, *London Math. Soc.* (1997).
[S1] A. I. Shnirelman, The geometry of the group of diffeomorphisms and the dynamics of an ideal incompressible fluid, *Mat. Sb.* (*N.S.*) **128** (1985), 82–109. [in Russian]
[S2] A. I. Shnirelman, Generalized fluid flows, their approximation and applications, *Geom. Funct. Anal.* **4** (1994), 586–620.
[T] F. Treves, "Introduction to Pseudodifferential and Fourier Integral Operators," Plenum, New York, 1982.

Printed in Belgium