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Factorials and Stirling numbers in the algebra of formal Laurent series II: $z^a - z^b = t$

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Abstract

In Part I, Stirling numbers of both kinds were used to define a binomial (Laurent) series of integer degree in the formal variable x. The binomial series in turn served as coefficient of t^n in a formal series that reasonably well reflects the properties of $(1+t)^x$. Analogously, generalized Stirling numbers (like central factorial numbers) are now used to define a kind of generalized Catalan series. By a different method, the Catalan series can be shown to generate a formal series that has the properties of $z(t)^x$, where $z(t)^a - z(t)^b = t$. As in the case of ordinary Stirling numbers, not all the necessary coefficients can be described by generalized Stirling numbers alone. But they can all be explicitly expressed as an ordinary double sum of powers and factorials.

Key words: Stirling numbers; Factorial numbers; Bernoulli polynomials; Umbral calculus

1. Introduction, and a word on notation

Special formal power series are usually an umbra of special (analytical) functions. To make them visible, discrete mathematicians are forced to use familiar but ambiguous notation. That practice can be confusing to the uninitiated. For example, if 1 denotes the multiplicative identity in some field K (of characteristic 0), the short power series $1+x$ and its translations $E^1(1+x) = 1+(x+1) = 2+x$, and $3+x, \ldots$ are well-defined members of $K[[x]]$, and so are their reciprocals $1/(1+x)$, $1/(2+x), 1/(3+x), \ldots$. But $1/(2+x)$ is no longer a translation of $1/(1+x)$, because $1/(1+(1+x))$ cannot be defined. There is no translation operator on $K[[x]]$, even though some power series (all polynomials) can be translated. Then again, $\frac{1}{2}(1/(1+(x/2)))$ is straightfoward and equals $1/(2+x)$. We exchange this notational inconsistency for another if we work with formal (lower) Laurent series, where a whole group of translations can be defined (Section 2, (i)). We pay for this by distinguishing between the polynomial $1+x$ and the series x translated by 1. I denote the latter by $\{x+1\}$, and it equals $x+1+\frac{1}{2}x^{-1} - \frac{1}{6}x^{-2} + \cdots$. It adds to the confusion that

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$\{x+1\}^{-1}$ is the reciprocal of $1+x$, and not of $\{x+1\}$. Instead, $\{x+1\}^{-1}\{x+1\}=1+\frac{1}{2}x^{-2}-\frac{2}{3}x^{-3}+\cdots$. Here is a list of possible notations, all 'pragmatic' except for the first and last one.

$$E^1\{x\}^{-1}=\{x+1\}^{-1}=\frac{1}{1+x}=\frac{1}{\{x+1\}}=\frac{1}{x}\frac{1}{1+1/x}=\sum_{i\geq 0}(-1)^i x^{-i-1}.$$

Of course we cannot expect $1+\{x+1\}$ to equal $\{x+2\}$, but on the other hand it is easy to check that

$$\frac{1}{1+\{x+1\}}=\sum_{i\geq 1}(-1)^{i+1}x^{-i}\left(1+\frac{1}{x}\right)^{-i}=\frac{1}{\{x+2\}}=\frac{1}{2}\frac{1}{1+x/2}.$$

In comparison, the more complicated construction $(1+t)^x \in K[x][[t]]$ is reasonably well-behaved. We discussed in a previous paper how $(1+t)^x$ can be extended using 'Factorials and Stirling numbers in the algebra of formal Laurent series' [9]. In that process the part played by the nth degree x-polynomials which occur in the expansion of $(1+t)^x$ as coefficient of t^n is taken by (lower) Laurent series of degree n. The resulting double series $(1+t)^{\{x\}}$ has $(1+t)^x$ as projection into $K[x][[t]]$, but also possesses the 'right' properties

$$(1+t)^{\{x+d\}}=(1+t)^{\{x\}}(1+t)^d, \qquad D_x(1+t)^{\{x\}}=(1+t)^{\{x\}}\ln(1+t),$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(1+t)^{\{x\}}=x(1+t)^{\{x-1\}}, \qquad (1+t)^{\{dx\}}=((1+t)^d)^{\{x\}},$$

where $(1+t)^d$ and $\ln(1+t)$ are in $K[[t]]$. In the expression $(1+t)^{\{x\}}$, I enclosed x in braces without any translating. I will use the braces for formal Laurent series whenever necessary to avoid confusion.

The formal variable t in $(1+t)^{\{x\}}$ can be replaced by any other formal power series of order 1. In a direct approach to our problem — the formal powers $z(t)^{\{x\}}$ when $z(t)^a - z(t)^b = t$ — we would solve for $z(t)\in K[[t]]$ first [10, Problem No. 212], and substitute $z(t)-1$ for t in $(1+t)^{\{x\}}$. But the technical difficulty of finding all integer powers of $z(t)-1$ can be avoided by the technique introduced in Section 2. That section is the framework for our 'generic' example, demonstrating how much of the structure is shared by 'almost all' sequences of lower Laurent series of binomial type. In particular, the orthogonality of the generalized Stirling numbers is such a structure. There is nothing new about the 'Umbral calculus' in Section 2 except perhaps the point of view. Other points of view may be preferred by the reader (transfer formula, dual operators, coalgebras, etc), so the general theory is not emphasized. At the end of Section 3 we introduce a larger class of applications from which the present problem has been selected as the 'natural' example.

What makes $z^a - z^b = t$ so natural are the generalized Stirling numbers (= Bernoulli numbers of higher order) that occur in this connection. Their importance in the case $a=-b$ has been demonstrated by Butzer et al. [2], and some aspects of the general case have been studied in [8]. The first kind, $f^{a,b}(n,k)$, already shows up when we expand $z(t)^x$, the projection of $z(t)^{\{x\}}$ into $K[x][[t]]$:

$$z(t)^x = \sum_{n\geq 0} t^n \frac{x}{x-nb}\binom{(x-nb)/(a-b)}{n} = \sum_{n\geq 0} \frac{t^n}{n!} \sum_{k=0}^{n} f^{a,b}(n,k)x^k.$$

By orthogonality,

$$x^n = \sum_{k\geq 0}^{n} \frac{x}{x-kb}\binom{(x-kb)/(a-b)}{k} F^{a,b}(n,k)k!$$

must hold for the second kind, $F^{a,b}(k,n)$. We have shown in [8] that

$$F^{a,b}(n,k) = \sum_{i=0}^{k} \frac{(-1)^{k-i}}{i!(k-i)!}(ia+(k-i)b)^n. \tag{1}$$

The two different parameters a and b can be replaced by a single parameter $c := b/(a-b)$, and we will do that frequently. However, the combinatorial interpretations [8] of $f^{a,b}(n,k)$ and $F^{a,b}(n,k)$ in terms of colored permutations and partitions show that there are arguments for keeping two separate parameters a and b. Of all those interpretations we only want to mention the partitions of an n-element set into k blocks, *all of odd cardinality*, which are counted by $F^{-1,1}(n,k)/2^k$.

Variations of these numbers have been studied earlier in the literature; for example, see [3, 6, 15]. In particular, the noncentral Stirling numbers of Koutras [6] take the part of ordinary Stirling numbers in the expansion of $(1+t)^{\{x+d\}}$, where $d \in K$ is the noncentrality parameter (Section 4.3). The following schematic diagram explains the function of both generalizations. An arrow stands for the word 'generate'

Stirling numbers	$\rightarrow$ bionomial series $\{ {x \atop n} \}$	$\rightarrow (1+t)^{\{x\}}$,
noncentral numbers	$\rightarrow \{ {x+d \atop n} \}$	$\rightarrow (1+t)^{\{x+d\}}$,
factorial numbers	$\rightarrow$ Catalan series	$\rightarrow z(t)^{\{x\}}$.

2. Background material

We recall some definitions and results from [9].

(a) $K((x))_{\leqslant} := \{\sum_{i\in\mathbb{Z}} k_i x^i \mid k^i \in K$; there is an $n \in \mathbb{Z}$ such that $k_i = 0$ for all $i > n\}$ (the K-algebra of *lower Laurent series*).

(b) If $\sum_{i\in\mathbb{Z}} k_i x^i \in K((x))_{\leqslant}$, then $\deg(\sum_{i\in\mathbb{Z}} k_i x_i) := \max\{i \mid k_i \neq 0\}$.

(c) $\mathscr{D}(x,t) := \{\sum_{n\in\mathbb{Z}} f_n(x)t^n \mid f_n(x) \in K((x))_{\leqslant}$ and $\deg(f_n(x)) \to -\infty$ as $n \to -\infty\}$ (the K-algebra of continuous double series).

(d) If $f(x) = \sum_{i\in\mathbb{Z}} k_i x^i \in K((x))_{\leqslant}$, then $f(x)^+ := \sum_{i\geqslant 0} k_i x^i$ is the *projection* of $f(x)$ into $K[x]$ (the polynomial part of $f(x)$).

(e) $x^{(n)} = \begin{cases} x(x-1)\cdots(x-n+1) & \text{if } n\geqslant 0 \\ 1/((x+1)\cdots(x-n)) & \text{if } n<0 \end{cases}$ (falling factorial powers). Each factor

$1/(x+i)$ has to be understood as the lower Laurent series $x^{-1}(1/(1+(i/x)))$ so we better write $\{x\}^{(n)}$ for negative n.

(f) $n! := (-1)^{n+1}/(-n-1)!$ for all negative integers n.

(g) $\lfloor {n \atop i} \rceil := n!/(i!(n-i)!)$ for all integers n and i (the *Roman coefficient*).

Note: In general, $\lfloor {n \atop i} \rceil \neq \lfloor {n-1 \atop i} \rceil + \lfloor {n-1 \atop i-1} \rceil$. For negative n and $i \geqslant 0$, $\lfloor {n \atop i} \rceil = \binom{n}{i}$.

(h) $D_x x^k/k! := x^{k-1}/(k-1)!$ for all $k \in \mathbb{Z}$ (the 'derivative' on $K((x))_\leqslant$).

(i) $\{x+d\}^n := E^d x^n := e^{dD_x} x^n = \sum_{i \leqslant n} \lfloor {n \atop i} \rceil x^i d^{n-i}$ for all $k \in \mathbb{Z}$, $d \in K$ (the translation operator on $K((x))_\leqslant$). For negative n, $\{x+d\}^n = (x+d)^n = x^n(1+d/x)^n$ in the usual sense (use (g) above).

(j) $e^{yt}(t/(e^t-1))^x =: \sum_{n \geqslant 0} B_n^{(x)}(y) t^n/n!$

(the Bernoulli polynomials $B_n^{(x)}(y)$ of order x in y).

It is convenient to use the notation $\lfloor n \rceil$ where $\lfloor n \rceil$ equals n for all $n \neq 0$, and $\lfloor 0 \rceil := 1$. Hence, $\lfloor n \rceil = n!/(n-1)!$ for all $n \in \mathbb{Z}$.

The following exercises on translations are easy to check:

$$\{x+d\}^{-1} = \sum_{i \geqslant 1} (-1)^{i-1} x^{-i} d^{i-1} = \frac{1}{d} \frac{1}{1+x/d} = \frac{1}{1+\{x+d-1\}}$$

for all $d \in K$. Only for negative n, $\{x+d\}^{-1}\{x+d\}^n = \{x+d\}^{n-1}$ (see (i)). Therefore,

$$\{x+d\}^{(n)} = (x+d)^{(n)} \quad \text{and} \quad \{x+d\}^{-1}\{x+d\}^{(n)} = \{x+d-1\}^{(n-1)} \tag{2}$$

for all negative integers n.

A *delta operator* Q on $K((x))_\leqslant$ is a continuous linear operator on $K((x))_\leqslant$ such that, for all $f(x) \in K((x))_\leqslant$,

$$\begin{array}{lll} \deg(Qf(x)) = \deg(f(x)) - 1 & & \text{(degree reducing), and} \\ E^d Q f(x) = Q E^d f(x) & \text{for all } d \in K & \text{(translation invariant).} \end{array}$$

D_x is an example of a delta operator. Every delta operator Q can be expanded in powers of D_x, $Q = \gamma(D_x) = \sum_{k \geqslant 0} \gamma_k D_x^k$. We are interested in sequences $\{q_n(x) \in K((x))_\leqslant \mid \deg(q_n(x)) = n,\ n \in \mathbb{Z}\}$ such that

$$Q q_n(x) = \lfloor n \rceil q_{n-1}(x) \quad \text{for all } n \in \mathbb{Z}. \tag{3}$$

Solutions to such a system are called *Sheffer* sequences (in $K((x))_\leqslant$). For every delta operator Q there is always exactly one Sheffer sequence that is of binomial type, i.e.

$$q_n(x+d) = \sum_{i \geqslant 0} \left\lfloor {n \atop i} \right\rceil q_{n-i}(x) q_i(d)^+ \quad \text{for all } n \in \mathbb{Z},\ d \in K.$$

The generating function (a recursive matrix [1]) of such a sequence of binomial type is in $\mathscr{D}(x,t)$, and has the form

$$\sum_{n \in \mathbb{Z}} q_n(x) t^n/n! = \sum_{m \in \mathbb{Z}} x^m \beta(t)^m/m! =: e^{\{x\}\beta(t)}, \tag{4}$$

where $\beta(t)$ is the compositional inverse of $\gamma(t)$ ($\gamma(D_x) = Q$). For proofs and more results see [11].

If we write $\langle\phi\rangle_k$ for the coefficient of the kth power of the variable in a formal series ϕ, we can derive from (4) that

$$\frac{\langle q_n\rangle_m}{n!}=\frac{\langle\beta^m\rangle_n}{m!} \tag{5}$$

for all integers $m\leqslant n$. Lagrange inversion [5] tells us that $\langle\gamma^k\rangle_n=(k/n)\langle\beta^{-n}\rangle_{-k}$ for all $n\geqslant k\geqslant 1$. Thus for such values of n and k

$$\frac{\langle q_{-k}\rangle_{-n}}{n!}=(-1)^{n-k}\frac{\langle\gamma^k\rangle_n}{k!}. \tag{6}$$

The coefficients in $\{q_n(x)^+\in K[x]\,|\,n=0,1,\ldots\}$ and $\{q_{-n}(x)\in K((x))_\leqslant\,|\,n=0,1,\ldots\}$ are very closely related as we saw above. In fact, they always form an orthogonal pair.

Theorem 2.1. *If $\{q_n(x)\in K((x))_\leqslant\,|\,n\in\mathbb{Z}\}$ is a sequence of binomial type, then*

$$((-1)^{n-k}\langle q_{-k}\rangle_{-n})_{n,k=1,2,\ldots}\quad\text{and}\quad(\langle q_k\rangle_i)_{k,i=1,2,\ldots}$$

are an orthogonal pair, i.e. for all positive integers i

$$\sum_{k=i}^{n}(-1)^{n-k}\langle q_{-k}\rangle_{-n}\langle q_k\rangle_i=\delta_{i,n}.$$

Proof. Let $Q=\gamma(D_x)$ denote the delta operator associated with $\{q_n(x)\in K((x))_\leqslant\,|\,n\in\mathbb{Z}\}$, and let $\beta(t)$ be the inverse of $\gamma(t)$. Apply (5) and (6). For all integers $1\leqslant i\leqslant n$

$$\delta_{i,n}=\sum_{k=i}^{n}\langle\gamma^k\rangle_n\langle\beta^i\rangle_k=\sum_{k=i}^{n}(-1)^{n-k}\langle q_{-k}\rangle_{-n}\frac{k!}{n!}\langle q_k\rangle_i\frac{i!}{k!}.\qquad\square$$

Because of this orthogonality we can easily find the inverse basis transformation between $\{x^n\}$ and $\{q_n(x)\}$.

Corollary 2.1. *If $\{q_n(x)\in K((x))_\leqslant\,|\,n\in\mathbb{Z}\}$ is a sequence of binomial type, then for all integers $n\geqslant 0$*

$$x^n=\sum_{k=0}^{n}q_k(x)^+(-1)^{n-k}\langle q_{-k}\rangle_{-n},$$

and for all $n>0$

$$x^{-n}=\sum_{k\geqslant n}q_{-k}(x)(-1)^{n-k}\langle q_k\rangle_n.$$

The second expansion in the corollary above is a special case of Corollary 2 in [12, p. 159].

There is 'almost always' an easier way to determine the coefficients $\langle q_n\rangle_m$ than using (5). The following theorem carries some of the results for 'Sequences of polynomial type with polynomial coefficients' [7] from $K[x]$ to $K((x))_\leqslant$.

Theorem 2.2. *Suppose, $\{q_n(x) \in K((x))_{\leqslant} \mid n \in \mathbb{Z}\}$ is a sequence of binomial type with associated delta operator $Q = \gamma(D_x)$ and generating function $e^{\{x\}\beta(t)}$. The following statements are equivalent.*

(i) $\langle q_2 \rangle_1 \neq 0$,

(ii) $\langle q_{-1} \rangle_{-2} \neq 0$,

(iii) $\langle \gamma^1 \rangle_2 \neq 0$,

(iv) $\langle \beta^1 \rangle_2 \neq 0$,

(v) *there exists a polynomial sequence $\{r_n(x) \in K[x] \mid n = 0, 1, \ldots\}$ of binomial type such that for all integers $m \leqslant n$*

$$\langle q_n \rangle_m = \langle \beta \rangle_1^n \left\lfloor \begin{matrix} n \\ n-m \end{matrix} \right\rceil r_{n-m}(m).$$

Proof. (ii) and (iv) are equivalent because $\beta(t)$ and $\gamma(t)$ are an inverse pair. (i)$\Leftrightarrow$ (iv) because of (5), and (ii)$\Leftrightarrow$(iii) because of (6). In [7] we have demonstrated that (iv) is equivalent to the existence of a sequence $\{r_n(x) \in K[x] \mid n = 0, 1, \ldots\}$ of binomial type such that

$$\langle \beta^m \rangle_n = \langle \beta \rangle_1^n r_{n-m}(m)/(n-m)! \tag{7}$$

for all integers $n \geqslant m$. Substituting (5) gives (iv)$\Leftrightarrow$(v). $\square$

3. The construction of $z(t)^{\{x\}}$

If the series $z(t) \in K[[t]]$ solves for given $a, b \in K$, $a \neq b$, the equation $z^a - z^b = t$, then the powers $z(t)^n$ are explicitly known [10, Problem No. 212] for all positive integers n. But what should we expect from the symbol $z(t)^x$? There are at least two immediate answers: We hope that

$$z(t)^{x+a} - z(t)^{x+b} = z(t)^x z(t)^a - z(t)^x z(t)^b = z(t)^x t, \tag{8}$$

and we would also like $z(t)^x$ to equal $e^{x \ln z(t)}$. Knowing $z(t)^n$ we try

$$z(t)^x = \sum_{n \geqslant 0} \frac{x}{x - nb} \binom{(x-nb)/(a-b)}{n} t^n, \tag{9}$$

and check by straightforward verification that (8) holds. Does this $z(t)^x$ really agree with $e^{x \ln z(t)}$, which would allow us to say that $z(t)^{(dx)} = (z(t)^d)^x$? It is best to answer this question before one knows $\ln z(t)$, because the only way to go for now is substitution of its compositional inverse $e^{at} - e^{bt}$ for t. This reduces the problem to a simple question: is $z(e^{at} - e^{bt})^x = e^{xt}$?

Lemma 3.1. *If*

$$z(t)^x = \sum_{n \geqslant 0} t^n \frac{x}{x - nb} \binom{(x-nb)/(a-b)}{n}$$

then $z(t)^x = e^{x \ln z(t)}$.

Proof.

$$z(e^{at}-e^{bt})^x=\sum_{n\geqslant 0}(e^{at}-e^{bt})^n\frac{x}{x-nb}\binom{(x-nb)/(a-b)}{n}$$
$$=\sum_{k\geqslant 0}\frac{t^k}{k!}\sum_{n\geqslant 0}^{k}\frac{x}{x-nb}\binom{(x-nb)/(a-b)}{n}n!F^{a,b}(k,n)\quad\text{(see (1))}$$
$$=\sum_{k\geqslant 0}\frac{x^k t^k}{k!}.\qquad\square$$

Now that we have straightened out any possible doubts about the nature of $z(t)^x$ we will use the symbols $f(t)^x$ and $f(t)^{\{x\}}$ only in the sense $e^{x\ln f(t)}\in K[x][[t]]$ and $e^{\{x\}\ln f(t)}\in\mathscr{D}(x,t)$, respectively.[1]

Let $\partial^{a,b}$ denote the general difference operator

$$\partial^{a,b}:\{x\}^n\mapsto\{x+a\}^n-\{x+b\}^n.$$

$\partial^{a,b}$ has the representation $\gamma(D_x)=e^{aD_x}-e^{bD_x}$. That $\ln z(t)$ is the compositional inverse to $\gamma(t)$ gives us the important information that

$$e^{\{x\}\ln z(t)}=z(t)^{\{x\}}=\sum_{n\in\mathbb{Z}}\frac{q_n(x)}{n!}t^n, \tag{10}$$

where $\{q_n(x)\,|\,n\in\mathbb{Z}\}$ is the sequence of binomial type associated with $\partial^{a,b}$. Theorem 2.2 can be applied for determining this sequence, because $\gamma(t)=(a-b)t+(a^2-b^2)t^2/2+\cdots$ is of the right form as long as $a^2\neq b^2$. In this case, there exists a polynomial sequence $\{r_n(x)\}$ of binomial type such that

$$\langle q_n\rangle_m=(a-b)^n\left\lfloor\begin{matrix}n\\ n-m\end{matrix}\right\rceil r_{n-m}(m)$$

for all integers $m\leqslant n$ (Theorem 2.2(v)). The polynomials $p_n^{a,b}(x):=(x/(x-n))r_n(x-n)$ are again of binomial type (see [13]), and solve the umbral equation $r_n(x)=q_n(p^{a,b}(x))$, as we have shown in [7, Theorem 1(*k*)]. The same theorem also shows that

$$\sum_{n\geqslant 0}p_n^{a,b}(x)t^n/n!=\left(\frac{(a-b)t}{\gamma(t)}\right)^x=\left(\frac{(a-b)t}{e^{at}-e^{bt}}\right)^x. \tag{11}$$

We can now express $p_n^{a,b}(x)$ in terms of Bernoulli polynomials,

$$(a-b)^{-n}p_n^{a,b}(x):=B_n^{(x)}\left(\frac{-bx}{a-b}\right)=B_n^{(x)}(-cx)=p_n^{1+c,c}(x),\quad n=0,1,\ldots,$$

where $c=b/(a-b)$. We call them *generalized Stirling polynomials* because $p_n^{1,0}(x)=p_n(x)$, the basic stirling polynomial as defined in [14,9]. The generalized

[1] In general, the logarithm of a formal power series $z(t)\in K[[t]]$ is defined as long as $z(t)$ has the constant term 1 and contains only powers that are multiples of the same positive integer.

Stirling polynomials can be easily expanded in terms of basic Stirling polynomials by comparing coefficients of t^i in

$$\left(\frac{(a-b)t}{e^{at}-e^{bt}}\right)^x = e^{-xbt}\left(\frac{(a-b)t}{e^{t(a-b)}-1}\right)^x = e^{-xbt}\sum_{i\geq 0} p_i(x)(t(a-b))^i/i!\,.$$

That gives the expansion

$$B_i^{\{x\}}(-cx) = \frac{p_i^{a,b}(x)}{(a-b)^i} = \sum_{k=0}^{i}\binom{i}{k}p_k(x)(-xc)^{i-k}. \tag{12}$$

If $a=-b$ then only even powers of t occur in the generating function $(2at/(e^{at}-e^{-at}))^x$, and we can obtain a polynomial sequence $\{\hat{p}_n^a(x)\,|\,n=0,1,\ldots\}$ of binomial type by defining

$$\sum_{n\geq 0}\hat{p}_n^a(x)t^{2n}/n! = (2at/(e^{at}-e^{-at}))^x = (at\sinh(at))^x, \tag{13}$$

i.e., $\hat{p}_n^a(x)/n! = p_{2n}^{a,-a}(x)/(2n)!$.

To sum up: Theorem 2.2 tells us that the lower Laurent series $q_n(x)$ in (10) can be written as

$$q_n(x) = (a-b)^{-n}\sum_{i\geq 0} x^{n-i}\left[\begin{matrix} n \\ n-i \end{matrix}\right]\frac{n-i}{n}p_i^{a,b}(n)$$

for *all* integers n.[2] It follows from Eq. (9) that for $n\geq 0$

$$\frac{q_n(x)^+}{n!} = \frac{x}{x-nb}\binom{\frac{x-nb}{a-b}}{n}.$$

These polynomials are sometimes called Catalan polynomials. Therefore I want to call

$$C_n(x) := \sum_{i\geq 0}\frac{(n-i)x^{n-i}}{i!(n-i)!n}B_i^{(n)}(-cn) = q_n(x(a-b))/n! \tag{14}$$

the *Catalan series* (of degree $n\in\mathbb{Z}$, and translation $c\in K$). Hence

$$C_n(x)^+ = \frac{x}{x-nc}\binom{x-nc}{n}, \tag{15}$$

and

$$z(t)^{\{x\}} = \sum_{n\in\mathbb{Z}} C_n\left(\frac{x}{a-b}\right)t^n.$$

[2] *For* $i>0$ the polynomial $p_i^{a,b}(x)$ has a root at 0 thus the expression $((n-i)/n)p_i^{a,b}(n)$ $(=(a-b)^n((n-i)/n)B_i^{(n)}(-cn))$ makes sense even if $n=0$. For $i=0$ the expression is identical 1.

We defined in [9, p. 57] the binomial series

$$\begin{Bmatrix} x \\ n \end{Bmatrix} := \frac{\{x\}^{(n)}}{n!} = \sum_{i \geqslant 0} \frac{(n-i)x^{n-i}}{i!(n-i)!n} B_i^{(n)}(0)$$

for all integers n. What is the relationship between the Catalan and the binomial series? Is there an analogy to the polynomial situation? To answer these questions we introduce a family of linear operators on $K((x))_{\leqslant}$ which reduce by 1 the degree of translated powers:

$$(X+c)^{-1}\{x+c\}^n := \{x+c\}^{n-1} \quad \text{for all } n \in \mathbb{Z}.$$

Lemma 3.2. *For all integers n,*

$$C_n(x) = x(X-nc)^{-1} \begin{Bmatrix} x-nc \\ n \end{Bmatrix} + \frac{B_n^{(n)}(0)x\{x-nc\}^{-1} - B_n^{(n)}(-nc)}{n!n}.$$

Note that $B_n^{(x)}(-cx)$ is zero for negative n. Only for such n is $(X+c)^{-1}\{x+c\}^n = \{x+c\}^{-1}\{x+c\}^n$ (see (2)). So we get for all $n<0$ the familiar form

$$n!\,C_n(x) = x(X-nc)^{-1}\{x-nc\}^{(n)} = x\{x-nc-1\}^{(n-1)}, \tag{16}$$

in complete analogy to the polynomial case. With careful Laurent series interpretation we could even write

$$\begin{aligned} C_n(x) &= \frac{x}{\{x-nc\}} \frac{\{x-nc\}^{(n)}}{n!} = \frac{1}{n} \frac{x}{\{x-nc\}} \frac{(-1)^n(-n)!}{(x-nc-n)^{(-n)}} \\ &= \frac{1}{n} \frac{x}{\{x-nc\}} \binom{nc-x-1}{-n}^{-1} \end{aligned} \tag{17}$$

for negative n.

Proof of Lemma 3.2. The lemma is true for $n=0$ because of $C_0(x) = \{\begin{smallmatrix} x \\ 0 \end{smallmatrix}\}$. Let $n \neq 0$. First, we write $\lfloor n-i \rceil$ instead of $n-i$ in (14) and get

$$C_n(x) := \sum_{i \geqslant 0} \frac{\lfloor n-i \rceil x^{n-i}}{i!(n-i)!n} B_i^{(n)}(-cn) - \frac{B_n^{(n)}(-cn)}{n!n}.$$

Substitute (12):

$$\begin{aligned} C_n(x) + \frac{B_n^{(n)}(-cn)}{n!n} &= \sum_{i \geqslant 0} \frac{\lfloor n-i \rceil x^{n-i}}{i!(n-i)!n} \sum_{k=0}^{i} \binom{i}{k} p_k(n)(-nc)^{i-k} \\ &= \sum_{k \geqslant 0} \frac{\lfloor n-k \rceil p_k(n)}{k!(n-1-k)!n} x\{x-nc\}^{n-k-1} \\ &= x(X-nc)^{-1} \sum_{k=0} \frac{(n-k)p_k(n)}{k!(n-1-k)!n} \{x-nc\}^{n-k} + \frac{p_n(n)x\{x-nc\}^{-1}}{n!n} \\ &= x(X-nc)^{-1} \begin{Bmatrix} x-nc \\ n \end{Bmatrix} + \frac{p_n(n)x\{x-nc\}^{-1}}{n!n}. \qquad \square \end{aligned}$$

The 'disturbance' $(B_n^{(n)}(0)x\{x-nc\}^{-1}-B_n^{(n)}(-nc))/(n!n)$ should remind us of the notational inconsistencies mentioned in the introduction. Yet $z(t)^{\{x\}}$ has all the right properties, as we show next.

Theorem 3.1. (1) $D_x z(t)^{\{x\}}=z(t)^{\{x\}}\ln z(t)$,

(2) $z(t)^{\{x+d\}}=z(t)^{\{x\}}z(t)^d$ *for all* $d\in K$,

(3) $\frac{\mathrm{d}}{\mathrm{d}t}z(t)^{\{x\}}=xz(t)^{\{x-1\}}\frac{\mathrm{d}}{\mathrm{d}t}z(t)=\frac{xz(t)^{\{x\}}}{az(t)^a-bz(t)^b}$

(4) $z(t)^{\{\mathrm{d}x\}}=(z(t)^d)^{\{x\}}$ *for* $d\neq 0$.

Proof. (1) $D_x z(t)^{\{x\}}=D_x \mathrm{e}^{\{x\}\ln z(t)}=\sum_{n\in\mathbb{Z}}\frac{x^{n-1}}{(n-1)!}(\ln z(t))^n=z(t)^{\{x\}}\ln z(t)$.

(2) By the binomial theorem,

$$z(t)^{\{x+d\}}=\sum_{n\in\mathbb{Z}}\frac{q_n(\{x+d\})}{n!}t^n=\sum_{n\in\mathbb{Z}}\frac{t^n}{n!}\sum_{i\leqslant n}\begin{bmatrix}n\\ i\end{bmatrix}q_i(x)q_{n-i}(d)^+ .$$

(3) $\frac{\mathrm{d}}{\mathrm{d}t}z(t)^{\{x\}}=\sum_{n\in\mathbb{Z}}\frac{x^n}{n!}n(\ln z(t))^{n-1}\frac{\mathrm{d}}{\mathrm{d}t}\ln z(t)=xz(t)^{\{x-1\}}\frac{\mathrm{d}}{\mathrm{d}t}z(t)$.
For the second form use that $t=z^a-z^b$, and therefore

$$1=(az(t)^{a-1}-bz(t)^{b-1})\frac{\mathrm{d}}{\mathrm{d}t}z(t).$$

(4) $z(t)^{\{\mathrm{d}x\}}=\sum_{n\in\mathbb{Z}}\frac{x^n}{n!}(d\ln z(t))^n=\sum_{n\in\mathbb{Z}}\frac{x^n}{n!}(\ln z(t)^d)^n$ (Lemma 3.1). □

The first three statements of the theorem should be seen as duality results in the sense of [4, 15]. First, we stated that the t-operator 'multiplication by $\ln z(t)$' is the dual of the x-operator D_x. Next, x-translation E^d is dual to multiplication by $z(t)^d$. Finally, multiplication by x is dual to the t-derivative followed by multiplication by $az(t)^a-bz(t)^b$.

The same method that we used above to construct $z(t)^{\{x\}}$ can be applied to find a $Z(t)^{\{x\}}$ such that $u_1Z(t)^{a_1}+\cdots+u_NZ(t)^{a_N}=t$, if the constants (in K) satisfy the three conditions

$$\sum_{i=1}^N u_i=0,\qquad \sum_{i=1}^N u_ia_i\neq 0,\qquad \sum_{i=1}^N u_ia_i^2\neq 0.$$

These conditions ensure that $\gamma(t)=\sum_{i=1}^N u_i\mathrm{e}^{a_it}$ is of the right form to apply Theorem 2.2. If $P_n(x)$ denotes the polynomial that plays the part of $p_n^{a,b}(x)$, then

$$\sum_{n\geqslant 0}P_n(x)\frac{t^n}{n!}=\left(\frac{t\sum_{i=1}^N u_ia_i}{\sum_{i=1}^N u_i\mathrm{e}^{a_it}}\right)^x .$$

4. Some generalizations of Stirling numbers

4.1. The a, b-factorial numbers of the first kind

Combining the Eq. (14) and (15) allows us to define the connection coefficients $f^{1+c,c}(n,i)$ for any nonnegative n as follows:

$$\sum_{i=0}^{n} f^{1+c,c}(n,i)x^i := \sum_{i=0}^{n} x^i \binom{n}{i}\frac{i}{n} B_{n-i}^{(n)}(-cn) = n!\,C_n(x)^+ = x(x-nc-1)^{(n-1)}.$$

Obviously, $f^{1,0}(n,i)=s(n,i)$, the Stirling number of the first kind. For general $c=b/(a-b)$ we call the coefficients

$$f^{a,b}(n,i) := (a-b)^{-i} f^{1+c,c}(n,i) = (a-b)^{-n}\binom{n}{i}\frac{i}{n} p_{n-i}^{a,b}(n) \tag{18}$$

the *a, b-factorial numbers of the first kind* to distinguish them from other generalizations of the Stirling numbers. The matrix $(f^{a,b}(n,k))_{n,k=0,1,\ldots}$ transforms the basis $\{x^n \mid n=0,1,\ldots\}$ into the basis

$$\left\{ \frac{x}{a-b}\left(\frac{x-kb}{a-b}-1\right)^{(k-1)} \,\middle|\, k=0,1,\ldots \right\}$$

in $K[[x]]$. It follows from Corollary 2.1 that the transposed of this matrix is also a basis transformation, but in $K[1/x]$. For all $n>0$,

$$x^{-n} = \sum_{k\geqslant n} \frac{x}{a-b}\left\{\frac{x+kb}{a-b}-1\right\}^{(-k-1)} (-1)^{n-k} f^{a,b}(k,n).$$

The a,b-factorial numbers are not well-suited for recurrence relations with a constant number of terms, and fixed a and b. If there are positive integers u and v such that $ua+vb=0$, then $f^{a,b}(n,k)$ can be written as a sum of at most $u+v+1$ terms. Such recurrence formulas are known for the Stirling case ($a=1$, $b=0$) and the central case ($a=-b$, see [2]). At the next level of complexity we get (proof omitted):

$$\begin{aligned} 27f^{1,-2}(n,k) &= f^{1,-2}(n-3,k-3) + 3(n-2)f^{1,-2}(n-3,k-2) \\ &\quad + 3(n-3)f^{1,-2}(n-3,k-1) - 2(n-3)^2(2n-3)f^{1,-2}(n-3,k). \end{aligned}$$

4.2. The a, b-factorial numbers of the second kind

If k is a positive integer, the connection coefficients $F^{1+c,c}(n,k)$ in the identity

$$\begin{aligned} \sum_{n\geqslant k} F^{1+c,c}(n,k)x^{-n} &:= x\{x+kc-1\}^{(-k-1)} = (-k)!\,C_{-k}(x) \quad \text{(see (16))} \\ &= \sum_{n\geqslant k} x^{-n}\binom{-k}{n-k}\frac{n}{k} B_{n-k}^{(-k)}(ck) \end{aligned}$$

are the Stirling number of the second kind for $c=0$ (up to a factor $(-1)^{n-k}$). In general,

$$F^{a,b}(n,k)=(a-b)^n F^{1+c,c}(n,k)=(a-b)^k\binom{n}{k}p_{n-k}^{a,b}(-k)$$

are the *a, b-factorial numbers of the second kind*, and $F^{1,0}(n,k)=S(n,k)$. Theorem 2.1 is the origin of the orthogonality of the two kinds of Stirling numbers,

$$\sum_{k=i}^{N} F^{a,b}(n,k)f^{a,b}(k,i)=\delta_{n,i}.$$

So we could define $F^{a,b}(n,k)$ via the inverse relation as we did in the introduction.

The generating function for the second kind is easy to calculate from (11):

$$\sum_{n\geqslant 0}\sum_{k\geqslant 0} F^{a,b}(n,k)x^k t^n/n! = \exp\{x(e^{at}-e^{bt})\}. \tag{19}$$

4.3. Other generalizations of Stirling numbers

The 1988 review of Charalambides and Singh [3] on Stirling numbers contains more than 400 references. Among those are many generalizations of Stirling numbers. For example, some 'generalized Stirling numbers' are defined as the connection coefficients $s(n,k|\{a_i\})$ and $S(n,k|\{a_i\})$ between $\{x^k|k=0,1,\dots\}$ and $\{(x+a_0)(x+a_1\cdot\dots\cdot(x+a_{k-1})|k=0,1,\dots\}$, where $\{a_i\}$ is a given sequence of constants $a_0, a_1, \dots$ [3, Eqs. (4.15) and (4.16)]. But they are related. For instance,

$$f^{b-a,b}(n,k)=a^{-n}\sum_{j=k}^{n}\binom{j-1}{k-1}(-1)^{n-k-j}(nb)^{j-k}s(n,j|\{ia\}).$$

Koutras [6] defined the noncentral Stirling numbers $s_d(n,k)$ and $S_d(n,k)$ by expanding

$$(x)^{(n)}=\sum_{k=0}^{n} s_d(n,k)(x-d)^k \quad\text{and}\quad (x-d)^n=\sum_{k=0}^{n} S_d(n,k)(x)^{(k)}$$

(see also [3, Eqs. (4.1) and (4.2)]). Here we have the simple relationship

$$f^{1+c,c}(n,k)=s_{-1-nc}(n-1,k-1) \quad\text{and}\quad F^{k-d,-d}(n,k)=k^n S_d(n,k).$$

In view of recurrence relations, the noncentral Stirling numbers are the natural extension of ordinary Stirling numbers. If we generate a matrix $(a(n,i))_{n\geqslant 0,\ i\in\mathbb{Z}}$ through the recurrence $a(n,i+1)=a(n,i)+(i-d)a(n-1,i)$ and the initial values $a(0,i)=1$ for all $i\in\mathbb{Z}$, and $a(n,0)=0$ for all $n\geqslant 1$, then

$$a(n,i)=\begin{cases} S_d(n-i,-i) & \text{for all } i<0,\\ |s_d(i,i-n)| & \text{for } 0\leqslant n\leqslant i,\end{cases}$$

analogously to the Stirling case in [9, p. 55]. The closeness of ordinary to noncentral Stirling numbers is also reflected in their respective double series, $(1+t)^{\{x\}}$ and $(1+t)^{\{x+d\}}$.

In [15, (6.21)] the 'generalized Stirling numbers' $S_{\alpha,\beta}(n,k)$ are defined, such that

$$F^{a,b}(n,k)=(a-b)^k S_{a-b,kb}(n,k), \quad \text{i.e., } S_{\alpha,\beta}(n,k)=\alpha^{-k}k^{-n}F^{\alpha k+\beta,\beta}(n,k).$$

5. Identities

5.1. The coefficient matrix of $z(t)^{\{x\}}$

In Theorem 3.1 we confirmed that for $d\neq 0$ the expected identity $z(t)^{\{dx\}}=(z(t)^d)^{\{x\}}$ holds. Here is another observation about $z(t)^{\{dx\}}$: the generating function (11) yields $d^{-n}p_n^{a,b}(x)=p_n^{a/d,b/d}(x)$, and therefore

$$z_{a,b}(t)^{\{dx\}}=z_{a/d,b/d}(t)^{\{x\}},$$

where we indicated the dependence of $z(t)$ on a,b in an obvious way.

If we define the coefficient matrix $(\zeta_{m,n})\in K^{\mathbb{Z}\times\mathbb{Z}}$ through the expansion

$$z(t)^{\{x\}}=:\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\zeta_{m,n}\frac{x^m}{m!}t^n,$$

then $(\zeta_{m,n})$ is a *recursive* matrix in the sense of [1]. Such matrices can be multiplied, and the coefficient matrix of $1/(1-\{x\}(e^{at}-e^{bt})):=\sum_{m\in\mathbb{Z}}x^m(e^{at}-e^{bt})^m$ is inverse to $(\zeta_{m,n})$. Hence,

$$z(t)^{\{x\}}\times\frac{1}{1-\{x\}(e^{at}-e^{bt})}=e^{\{x\}t}$$

in matrix interpretation. We know the upper triangular matrix $(\zeta_{m,n})$ from (14),

$$\zeta_{m,n}=(a-b)^{-n}\frac{m}{n}p_{n-m}^{a,b}(n)/(n-m)!=(a-b)^{-m}\frac{m}{n}B_{n-m}^{(n)}(-bn)/(n-m)!,$$

and for completeness we give an explicit expression for $(\zeta_{m,n})$. In the case of negative indices we get the simple sum

$$\begin{aligned}\zeta_{-m,-n}&=(a-b)^n\frac{m}{n}p_{m-n}^{a,b}(-n)/(m-n)!=\frac{(n-1)!}{(m-1)!}F^{a,b}(m,n)\\&=\frac{1}{n(m-1)!}\sum_{k=0}^{n}(-1)^{n-k}(ak+(n-k)b)^m\end{aligned}$$

for all $1\leqslant n\leqslant m$. Thus we know the values of the generalized Stirling polynomials for each degree at any negative integer. By extrapolation we find almost all other values:

$$\begin{aligned}\zeta_{m,n}=&\sum_{j=0}^{n-m}\frac{m(a-b)^{-n-j}}{(n-m-j)!(n-m+j)!}\frac{(2n-m)^{(n-m)}}{n+j}\\&\times\sum_{k=0}^{j}\frac{(-1)^k}{k!(j-k)!}(ka+(j-k)b)^{n-m+j}\end{aligned}$$

for all integers $n \geqslant m$, except $n=0$ ($\zeta_{m,n}=0$ if $n<m$). The restriction $n \neq 0$ is necessary, because we cannot obtain the values $(m/0)p_m^{a,b}(0)$ by extrapolation! Of course, $\zeta_{0,0}=1$. For the basic Stirling polynomials we found $(m/0)p_m(0)$ in [9], and for $m>0$ we get via (12)

$$\zeta_{-m,0}=\frac{-m}{0}p_m^{a,b}(0)/m!=\frac{(a-b)^m}{m!}(B_m+\delta_{m,1})$$
$$=\frac{(a-b)^m}{m!}\sum_{j=1}^{m+1}\sum_{k=1}^{j}\binom{j-1}{k-1}\frac{(-1)^{k+1}k^m}{j},$$

where $B_1, B_2, \ldots$ are the ordinary Bernoulli numbers, $B_m = B_m^{(1)}(0)$.

5.2. Factorial numbers

For $0 \leqslant k \leqslant n$, the explicit representation $\zeta_{k,n}=(k!/n!)f^{a,b}(n,k)$ results in

$$f^{a,b}(n,k)=\frac{k}{(a-b)^k}\binom{2n-k}{k}\sum_{j=0}^{n-k}\binom{2n-2k}{n-k-j}\frac{1}{j!(n+j)}$$
$$\times\sum_{v=0}^{j}\binom{j}{v}(-1)^v\left(v+\frac{bj}{a-b}\right)^{j+n-k}.$$

That formula is well-known for Stirling numbers ($a=1$, $b=0$) and central factorial numbers ($a=1/2$, $b=-1/2$) [2].

Closer to combinatorics is the following sum for $f^{1+c,c}(n,k)$. We saw that for $m \geqslant 0$

$$(\ln z(t))^m=\sum_{n\geqslant m}\zeta_{m,n}t^n=\sum_{n\geqslant m}t^n\frac{m!}{n!}f^{a,b}(n,m)=\left(\sum_{i\geqslant 1}\frac{t^i}{i!}f^{a,b}(i,1)\right)^m.$$

The coefficient $(-1)^{n-1}(n(1+c)-1)^{(n-1)}$ of the linear term in (15) equals $f^{1+c,c}(n,1)/n!$. Hence,

$$f^{1+c,c}(n,m)=\frac{(-1)^{n-m}n!}{(1+c)^m}\sum_{\substack{k_1+\cdots k_n=m,\\ 1k_1+\cdots+nk_n=n}}\prod_{i=1}^{n}\frac{1}{i^{k_i}k_i!}\binom{i(1+c)}{i}^{k_i}$$

(all $k_i \geqslant 0$ in the above formula). For $c=0$ this is a well-known statistic on permutations, where k_i is the number of cycles of length i. Starting the same way, we can alternatively express $f^{1+c,c}(n,m)$ as

$$f^{1+c,c}(n,m)=(-1)^{n-m}\frac{n!}{m!}\sum_{l_1+\cdots+l_m=n}\prod_{i=1}^{m}\frac{(l_i(1+c)-1)^{(l_i-1)}}{l_i!},$$

where all $l_i \geqslant 1$. Finally, there is a third sum of products for the same number:

$$f^{1+c,c}(n,m)=(-1)^{n-m}\frac{(n-1)!}{(m-1)!}\sum_{l=m-1}^{n-1}\binom{n(c+1)-l-1}{n-1-l}\sum_{l_1+\cdots+l_{m-1}=l}\prod_{i=1}^{m-1}\frac{1}{l_i}.$$

This identity follows from $f^{1+c,c}(n,k)=s_{-1-nc}(n-1,k-1)$. Koutras derived the above sum for noncentral Stirling numbers of the first kind in [6].

The a,b-factorial numbers do not directly generate the coefficients $\zeta_{-m,n}$ when m and n are nonnegative integers. But these coefficients still can be expressed in terms of a,b-factorial numbers. The access to this road is given by the 'upgraded binomial theorem' which is property (iii) in [9]: if $a\neq -b$, then

$$\begin{aligned}&\frac{x+d+vn}{x+d+un}p_n^{a,b}(x+d+un)\\&=\sum_{i=0}^{n}\binom{n}{i}\frac{x+vi}{x+ui}p_i^{a,b}(x+ui)\frac{d}{d+u(n-i)}p_{n-i}^{a,b}(d+u(n-i)),\end{aligned}$$

for all $d,u,v\in K$, $n=0,1,\ldots$. Special choices of the parameters yield identities for a,b-factorial numbers of the first kind, the second kind (in [8] without proof), and mixed identities. We apply the latter to find a representation for $\zeta_{-m,n}$ in the case $a\neq -b$, choosing $v=0$, $u=1$, $x=-m-n-v$, and $d=n+v$, where v is any nonnegative integer.

$$\begin{aligned}&\zeta_{-m,n}\\&=(a-b)^{-n}\sum_{i=0}^{n+m}\frac{(n+v)!(i+v-1)!}{(n+m+v-1)!(n+v+i)!}F^{a,b}(n+m+v,i+v)f^{a,b}(n+i+v,n+v).\end{aligned}$$

A similar identity can be proven if $a=-b$, using the polynomials $\hat{p}_n^a(x)$. The above formula can be used for $n=0$ to expand the ordinary Bernoulli numbers $B_m, m=2,3,\ldots$.

Identities that express $f^{a,b}(n,k)$ in terms of Stirling numbers of the first kind are given in [8]. Ibidem, there are several identities derived from the generating function (19) for $F^{a,b}(n,k)$, including the explicit formula (1). Two additional examples are: (i) for all $a,b,d\in K$,

$$F^{a,b}(n,k)=\sum_{i,j}\binom{n}{i}F^{a,d}(i,j)F^{d,b}(n-i,k-j).$$

The factorial numbers on the right-hand side can be made into Stirling numbers by choosing $d=0$. (ii) The second result is equivalent to an identity of Verde–Star [15, (6.26)]:

$$F^{a,b}(n,k)=\sum_{i=k}^{n}\binom{n}{i}(dk)^{n-i}F^{a-d,b-d}(i,k).$$

5.3. Catalan identities

The Catalan series are of binomial type, and therefore

$$C_n(x+d)=\sum_{i\leqslant n} C_i(x)\frac{d}{d-c(n-i)}\binom{d-(n-i)c}{n-i}$$

for all integers n. In particular for negative n there are interesting ways of writing this identity, using (17). We leave this to the reader.

$\partial^{c+1,c}=E^c\Delta$ is the delta operator associated to the sequence $\{C_n(x)|n\in\mathbb{Z}\}$,

$$C_n(x+c+1)-C_n(x+c)=\lfloor n\rceil C_{n-1}(x),$$

and $\{\{x-cn\}^{(n)}|n\in\mathbb{Z}\}$ is a Sheffer sequence for the same operator (see [9, p. 60]). If $\{q_n(x)\}$ and $\{s_n(x)\}$ are Sheffer sequences for the same delta operator, $\{q_n(x)\}$ of binomial type, then

$$s_n(x+d)=\sum_{i\leqslant n}\begin{bmatrix} n\\ i\end{bmatrix} q_i(x)s_{n-i}(d)^+$$

by the binomial theorem for Sheffer sequences [11, Theorem 12]. The theorem also holds true in the form

$$s_n(x+d)=\sum_{i\leqslant n}\begin{bmatrix} n\\ i\end{bmatrix} s_i(x)q_{n-i}(d)^+ .$$

Now we can generalize in two ways an identity which is usually written for Catalan numbers:

$$\begin{Bmatrix} x+d\\ n\end{Bmatrix}=\sum_{i\leqslant n} C_i(x)\binom{d+ic}{n-i}$$
$$=\sum_{i\leqslant n}\begin{Bmatrix} x-ic\\ i\end{Bmatrix}\frac{d+nc}{d+ic}\binom{d+ic}{n-i}.$$

Again, the first equation could be further investigated for negative n.

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