JOURNAL OF FUNCTIONAL ANALYSIS 111, 76–96 (1993)

On Toeplitz-Invariant Subspaces of the Bergman Space

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Communicated by D. Sarason

Received December 2, 1991

A subspace $M \subset L_a^2(\Delta) = A_2$ is called an e-subspace if (i) $\dim M < \infty$; (ii) $1 \in M$; (iii) $M \subset H^\infty$; (iv) for every $f \in A_2$ such that $(|f|^2 - 1)$ is orthogonal to M, and every $g \in M$, $\|fg\| \geqslant \|g\|$. Define the operator T by $(Tf)(z) = \int_\Delta |f(w)|^2 \overline{K(z, w)}\, dA(w)$, where $K(z, w) = (1/\pi)(1 - \bar{z}w)^{-2}$ is the Bergman kernel in Δ. A subspace $M \subset A_2$ satisfying (i), (ii), (iii) is called a T-subspace if $TM \subset M$. It is proved that M is an e-subspace if and only if M is a T-subspace. In particular, a finite dimensional linear space M of polynomials is an e-subspace if and only if $M = \operatorname{span}\{z^{kj}\}_{j=0}^N$ where $k > 0$ and $N \geqslant 0$ are integers. For $k = 1$ this implies a sharper form of a theorem of H. Hedenmalm. © 1993 Academic Press, Inc.

INTRODUCTION

Let A_p be the Bergman space of holomorphic functions $f(z) = \sum_0^\infty c_k z^k$ in the unit disk $\Delta = \{z \in \mathbf{C} : |z| < 1\}$ with the norm

$$\|f\|_{A_p} = \left(\frac{1}{\pi} \int_\Delta |f(z)|^p \, dA(z)\right)^{1/p}, \tag{1}$$

where $dA(z)$ is the Lebesgue area measure on Δ. The Hardy space H_p consists of holomorphic functions in Δ such that

$$\|f\|_{H_p} = \sup_{0 < r < 1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta\right)^{1/p} < \infty. \tag{2}$$

* Partially supported by a grant from the National Science Foundation.

0022-1236/93 $5.00

If $p=2$ both A_2 and H_2 are Hilbert spaces, and

$$\|f\|_{A_2}^2=\frac{1}{\pi}\int_{\Delta}|f(z)|^2\,dA(z)=\sum_0^{\infty}|c_k|^2/(k+1), \tag{3}$$

$$\|f\|_{H_2}^2=\frac{1}{2\pi}\int_0^{2\pi}|f(e^{i\theta})|^2\,d\theta=\sum_0^{\infty}|c_k|^2, \tag{3'}$$

where $f(e^{i\theta})$ are the radial limit values of f.

The factorization theorem for H_p is classical: every $f\in H_p$ can be represented as a product $f=gh$ where g is an inner function, that is, $g\in H_\infty$ and $|g(e^{i\theta})|=1$ a.e., and h is an outer function (see [1, p. 24]). Thus f/g is holomorphic in Δ, and $\|f/g\|_{H_p}=\|f\|_{H_p}$.

For a long time a similar result for A_p was unknown and, moreover, seemed to be implausible. However, in two recent papers [3, 2] a factorization theorem was established for the Bergman spaces. In the first of them [3] it was proved that for every divisor D in Δ (i.e., a finite or countable set $\{z_n\}\subset\Delta$ with corresponding "multiplicities" $m_n\in\mathbf{N}$) which is a zero set of a function in A_2, there is a unique (up to a constant unimodular factor) function $G_D\in A_2$, with $\|G_D\|_{A_2}=1$, such that G_D vanishes on D, and for every $f\in A_2$ that vanishes on D $\|f/G_D\|_{A_2}\leqslant\|f\|_{A_2}$. In the second paper [2] this result was extended to an arbitrary $p\geqslant 1$. The function G_D is the solution of the extremal problem

$$\sup\{|f^{(k)}(0)| : f\in A_p,\ \|f\|_{A_p}\leqslant 1,\ f|_D=0\}, \tag{4}$$

where k is the multiplicity of D at 0 (if $0\notin\{z_n\}$, $k=0$). A standard variational argument shows that the function $G=G_D$ that realizes the supremum in (4) satisfies

$$\int_{\Delta}(|G(z)|^p-1)\,g(z)\,dA(z)=0 \tag{5}$$

for every $g\in H_\infty$. The general pattern of proof in both papers was the same and included two step: first it was proved that (5) implies $\|Gg\|_{A_p}\geqslant\|g\|_{A_p}$ for all $g\in H_\infty$, and then it had to be shown that G_D has no extra zeros; the proofs, of course, were quite different, as the Hilbert space technique of [3] does not work for $p\neq 2$. The first step suggests that functions satisfying (5) should be viewed as "inner functions" in A_p. Since a function $G\in H_p$ is inner if and only if

$$\frac{1}{2\pi}\int_0^{2\pi}(|G(e^{i\theta})|^p-1)\,g(e^{i\theta})\,d\theta=0 \tag{5'}$$

for all $g\in H_\infty$, we thus obtain a common definition of an inner function in both the Hardy and Bergman cases:

DEFINITION 1. A function $G \in A_p$ ($G \in H_p$) is called *inner* in A_p (respectively, H_p) if $|G|^2 - 1$ is orthogonal to H_∞, i.e., (5) (respectively, (5′)) holds for all $g \in H_\infty$.

DEFINITION 2. A function $G \in A_p$ is called a *(norm-) expansive multiplier* if $\|G\|_{A_p} = 1$ and $\|Gg\|_{A_p} \geq \|g\|_{A_p}$ for all $g \in H_\infty$.

If, for some fixed $G \in A_p$, we consider the functional l_G on H_∞

$$l_G(g) = \|Gg\|^p_{A_p}, \tag{6}$$

then Definition 2 simply means that G is norm-expansive if $\|G\|_{A_p} = 1$, and $g = 1$ is the point of minimum for l_G restricted to the H_∞ part of the unit sphere of A_p. On the other hand, a standard variational argument shows that (5) expresses the fact that 1 is a critical point for l_G, at least for $p > 1$. Thus, if G is an expansive multiplier then G is inner. However, the converse is also true:

THEOREM (Hedenmalm [3] for $p = 2$; Duren, Khavinson, Shapiro, and Sundberg [2] for an arbitrary $p \geq 1$). *If $G \in A_p$ is inner then G is a norm-expansive multiplier*:

$$\|Gg\|_{A_p} \geq \|g\|_{A_p}, \qquad \forall g \in H_\infty.$$

In the case of H_p, a norm-expansive multiplier G is an inner function, i.e., $|G(e^{it})| = 1$ a.e., and therefore G is a norm-preserving multiplier.

Now a natural question arises. Let L be a subspace of A_p (or H_p), $1 \in L$, and G be a norm-expansive L-multiplier, which means that $\|G\| = 1$ and $\|Gg\|_{A_p} \geq \|g\|_{A_p}$ for every $g \in L$. The necessary condition for a minimum implies that (5) holds for every $g \in L$, that is, G is an "L-inner function." The problem is to characterize those subspaces $L \subset A_p$ (or $L \subset H_p$) so that every L-inner function is a norm-expansive L-multiplier. Such a characterization is obtained in this paper for A_2 and H_2, provided that L is a finite dimensional subspace consisting of H_∞ functions. Our main focus is on A_2, which is a more interesting (and more difficult) case. One application of the theory is a description of subspaces of polynomials having the above property.

1. e- AND T-SUBSPACES: STATEMENT OF RESULTS

Let $L \subset A_2$ be a subspace. We call an element $G \in A_2$ a *(norm-) expansive L-multiplier* if

(i) $\|G\|_{A_2} = 1$;

(ii) $gG \in A_2$ for every $g \in L$;

(iii) $\|gG\|_{A_2} \geq \|g\|_{A_2}$ for every $g \in L$. (7)

Remark. In the terminology of [5], (7) means that $G \succ 1$ on L.

We call $G \in A_2$ an *L-inner function* if conditions (i) and (ii) hold, and

$$\frac{1}{\pi} \int_{\Delta} (|G|^2 - 1) g \, dA = 0 \tag{8}$$

for every $g \in L$. Note that condition (ii) implies the convergence of the integral in (8).

DEFINITION 1. A subspace $L \subset A_2$ is called an e-subspace if

(i) $1 \in L$;

(ii) every L-inner function is a norm-expansive L-multiplier.

L is called an e^*-subspace if L satisfies (i) and

(ii′) every bounded L-inner function is a norm- expansive L-multiplier.

Our first result is the following

APPROXIMATION LEMMA. *Let $L \subset H_\infty$, $\dim L < \infty$, $1 \in L$, and let $G \in A_2$ be an L-inner function. Then there is a sequence $\{G_n\}_1^\infty$ of polynomials such that $G_n \to G$ in A_2, and all G_n are L-inner functions.*

COROLLARY. *If $L \subset H_\infty$, $\dim L < \infty$, and L is an e^*-subspace of A_2, then L is an e-subspace of A_2.*

Let $K(z, w) = (1 - z\bar{w})^{-2}$ be the Bergman kernel in Δ. Convolution with $K(z, w) = K(w, z)$ is a bounded projector from $L_p(\Delta, (1/\pi)\, dA)$ onto A_p, for $1 < p < \infty$ (see [6, p. 122]), and is an orthogonal projector for $p = 2$. The following operator plays an important role in what follows:

$$(Tg)(z) := \frac{1}{\pi} \int_{\Delta} |g(w)|^2 K(z, w) \, dA(w). \tag{9}$$

We call T the *quadratic Toeplitz operator* because $Tg = B(g, g)$, where $B(g, h) = T_g(h)$; here $T_g(h)$ is the Toeplitz operator with the symbol $\bar{g}$.

DEFINITION 2. A subspace $L \subset A_2$ is called *Toeplitz-invariant*, or a T-subspace, if $TL \subset L$.

The main result of this paper is the following:

THEOREM 1. *Let* $L \subset H_\infty$, $1 \in L$, $\dim L < \infty$. *Then* L *is an e-subspace of* A_2 *if and only if* L *is a T-subspace.*

Using this theorem it is easy to check that the subspace $\mathscr{P}_n$ of all polynomials of degree $\leqslant n$ is an e-subspace of A_2. However, we can say more. The following theorem describes all subspaces of polynomials that are e-subspaces.

THEOREM 2. *Let* L *be a finite-dimensional linear space of polynomials, and* $1 \in L$. *Then* L *is an e-subspace of* A_2 *if and only if there are integers* $k \geqslant 1$ *and* $n \geqslant 0$ *such that* $L = \operatorname{span}\{z^{kj};\ j = 0, 1, ..., n\}$.

COROLLARY. *Let* $M_k = \{f(z^k) : f \in H_\infty\}$. *Then* M_k *is an e-subspace of* A_2.

Note that for $k = 1$ we obtain Hedenmalm's theorem (see above).

Although Theorem 1 gives a necessary and sufficient condition for a finite-dimensional subspace $L \subset H^\infty$ to be an e-subspace of A_2, it is not clear a priori how large the class of e-subspaces is. The next theorem shows that this class is rather large. Let D be a finite divisor in Δ, that is, $D = m_1 a_1 + \cdots + m_n a_n$, $m_1, ..., m_n \in \mathbf{N}$, $a_1, ..., a_n \in \Delta$. Set

$$L_D = \operatorname{span}\{1, K(z, a_1), ..., K_{m_1 - 1}(z, a_1), ..., K(z, a_n), ..., K_{m_n - 1}(z, a_n)\}, \qquad (10)$$

where

$$K_s(z, w) = \frac{\partial^s}{\partial \bar{w}^s} K(z, w).$$

THEOREM 3. L_D *is an e-subspace of* A_2 *for every finite divisor* D.

All the above theorems have their analogues for the Hardy space H_2.

Given a subspace $L \subset H_2$, we call a function $G \in H_2$ an *L-inner function* if

(i) $\|G\|_{H_2} = 1$;

(ii) $gG \in H_2$ for every $g \in L$;

(iii) $(1/2\pi) \int_0^{2\pi} (|G(e^{i\theta})|^2 - 1)\, g(e^{i\theta})\, d\theta = 0$, $\forall g \in L$.

An element $G \in H_2$ is called an *L-multiplier* if $\|gG\|_{H_2} = \|g\|_{H_2}$ for all $g \in L$. A subspace $L \subset H_2$ is an e-subpsace of H_2 if $1 \in L$ and every L-inner function is an L-multiplier.

Finally, we define the operator $\tilde{T}$ by

$$(\tilde{T}g)(z) = \frac{1}{2\pi} \int_0^{2\pi} |g(e^{i\theta})|^2\, \tilde{K}(z, e^{i\theta})\, d\theta,$$

where $\tilde{K}(z, w) = (1 - z\bar{w})^{-1}$ is the Cauchy kernel. We call a subspace $L \subset H_2$ a $\tilde{T}$-subspace if L is $\tilde{T}$-invariant, i.e., $\tilde{T}L \subset L$.

THEOREM 1'. *Let L be a finite-dimensional subspace of H_∞ such that $1 \in L$. Then L is an e-subspace of H_2 if and only if L is a $\tilde{T}$-subspace.*

THEOREM 2'. *Let L be a finite-dimensional linear space of polynomials such that $1 \in L$. Then L is an e-subspace of H_2 if and only if there are integers $k \geqslant 1$ and $n \geqslant 0$ such that $L = \operatorname{span}\{z^{kj} : j = 0, ..., n\}$.*

THEOREM 3'. *Let D be a finite divisor, and let $\tilde{L}_D$ be defined by (10), with $\tilde{K}$ instead of K. Then $\tilde{L}_D$ is an e-subspace of H_2.*

A comparison of Theorem 2 and Theorem 2' shows that $L = \operatorname{span}\{z^{kj} : j = 0, ..., n\}$ is an e-subpsace of both A_2 and H_2. We are tempted to conjecture that there are no other finite-dimensional $L \subset H_\infty$ that are e-subspaces of both A_2 and H_2.

2. SOME PROPERTIES OF THE DIRICHLET SPACE

To prove our theorems we need several lemmas, some of them may be of independent interest. Let f be an analytic function in Δ, $f = \sum_0^\infty a_k z^k$. Consider the operators:

$$(Sf)(z) := zf(z) = \sum_0^\infty a_{k-1} z^k, \qquad a_{-1} = 0; \tag{11}$$

$$(Rf)(z) := \frac{f(z) - f(0)}{z} = \sum_0^\infty a_{k+1} z^k; \tag{12}$$

$$(Df)(z) := f'(z) = \sum_0^\infty (k+1)\, a_{k+1} z^k. \tag{13}$$

There are obvious relations between these operators:

$$RS = I, \qquad DSR = D.$$

Let $f \in H_2$, and let U_f be the least harmonic majorant of $|f|^2$:

$$U_f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(1 - |z|^2)\, |f(e^{i\theta})|^2}{|e^{i\theta} - z|^2}\, d\theta, \qquad z \in \Delta.$$

Put $\Phi_f(z) = U_f(z) - |f(z)|^2$. The following result is due to H. Hedenmalm [3]. To make our presentation self-contained, we prove it here. Our proof differs from that given in [3].

Let $\mathscr{D}_p$ be the space of analytic functions modulo constants in Δ with the norm

$$\|f\|_{\mathscr{D}_p}=\left(\frac{1}{\pi}\int_\Delta |f'(z)|^p\,dA(z)\right)^{1/p}.$$

For $p=2$, we write $\mathscr{D}_2=\mathscr{D}$; $\mathscr{D}$ is called the Dirichlet space.

LEMMA 1. *If* $f\in\mathscr{D}$ *then*

$$\Phi_f(z)=(1-|z|^2)\sum_1^\infty |(R^jf)(z)|^2. \tag{14}$$

Proof. First check that both sides of (14) have equal Laplacians $\partial\bar\partial=(1/4)(\partial^2/\partial x^2+\partial^2/\partial y^2)$. We have

$$\partial\bar\partial\Phi_f(z)=-\partial\bar\partial\,|f(z)|^2=-|f'(z)|^2.$$

On the other hand,

$$\begin{aligned}\partial\bar\partial\{(1-|z|^2)\sum_1^\infty |(R^jf)(z)|^2\}&=\partial\bar\partial\left\{\sum_1^\infty |(R^jf)(z)|^2-\sum_1^\infty |(SR^jf)(z)|^2\right\}\\&=\sum_1^\infty |(DR^jf)(z)|^2-\sum_1^\infty |(DSR^jf)(z)|^2\\&=\sum_1^\infty (DR^jf)(z)|^2\\&\quad-\sum_1^\infty |(DR^{j-1}f)(z)|^2=-|f'(z)|^2.\end{aligned}$$

This proves (14) if f is sufficiently smooth on $\bar\Delta$. Replace f by $f_r=f(rz)$ $(0<r<1)$. Since for $f\in H_2$

$$|f(z)|\leqslant\|f\|_{H_2}(1-|z|^2)^{-1/2}\qquad(z\in\Delta),$$

we obtain

$$\begin{aligned}\sum_1^\infty (1-|z|^2)\,|(R^jf)(z)|^2\leqslant\sum_1^\infty \|R^jf\|^2_{H_2}&=\sum_{j=1}^\infty\sum_{k=0}^\infty |a^{k+j}|^2\\&=\sum_1^\infty n\,|a_n|^2=\|f\|^2_{\mathscr{D}}\end{aligned}$$

and

$$
\begin{aligned}
&(1-|z|^2)\left|\sum_1^\infty |(R^jf)(z)|^2-\sum_1^\infty |(R^jf_r)(z)|^2\right| \\
&\leqslant \sum_1^\infty (1-|z|^2)\,|R^j(f-f_r)(z)|\,(|(R^jf)(z)|+|(R^jf_r)(z)|) \\
&\leqslant \left(\sum_1^\infty (1-|z|^2)\,|R^j(f-f_r)(z)|^2\right)^{1/2} \\
&\quad\times\left(\sum_1^\infty 2(1-|z|^2)[|(R^jf)(z)|^2+|(R^jf_r)(z)|^2]\right)^{1/2} \\
&\leqslant \|f-f_r\|_{\mathscr{D}}\,(2\,\|f\|_{\mathscr{D}}^2+2\,\|f_r\|_{\mathscr{D}}^2)^{1/2}\leqslant 2\,\|f\|_{\mathscr{D}}\,\|f-f_r\|_{\mathscr{D}},
\end{aligned}
$$

and this tends to 0 as $r\to 1$. Thus $\Phi_{f_r}(z)$ converges uniformly on Δ to the right hand side of (14) as $r\to 1$, while the pointwise limit of $\Phi_{f_r}(z)$ is $\Phi_f(z)$, which proves the lemma.

Let $\mathscr{M}$ be the set of points $\xi\in\partial\Delta$ where $f(z)$ and all $(R^jf)(z)$ have finite radial limit values

$$\lim_{r\to 1}(R^jf)(r\xi)=(R^jf)(\xi),\qquad j=0,1,2,\ldots.$$

For each $\xi\in\mathscr{M}$ we can define

$$q_f(\xi)=\lim_{r\to 1}\frac{\Phi_f(r\xi)}{1-r}=\frac{\partial}{\partial v}\Phi_f(\xi)=2\sum_1^\infty |(R^jf)(\xi)|^2, \tag{15}$$

where $\partial/\partial v$ is the normal derivative into Δ, and the right hand side may be equal to $+\infty$. We have

$$\|f\|_{\mathscr{D}}^2=\sum_1^\infty n\,|a_n|^2=\sum_1^\infty \|R^jf\|_{H_2}^2=\frac{1}{4\pi}\int_0^{2\pi} q_f(e^{i\theta})\,d\theta. \tag{16}$$

If we replace f by f_r, we can see that $\|q_{f_r}\|_{L_1}\nearrow\|q_f\|_{L_1}$ as $r\to 1$. This implies $\|q_{f_r}-q_f\|_{L_1}\to 0$ $(r\to 1)$.

The purpose of the next lemma is to justify application of the Green formula in a special situation where the functions involved are not smooth enough, as required by the classical Green formula.

Let $g\in H_1$, $g(0)=0$. Define $g_1=2\,\mathrm{Re}(\bar zg)=\bar zg+z\bar g$, and let $V_g(z)$ be the Poisson integral of $g_1(e^{i\theta})$. If $\Psi_g=V_g-g_1$, then

$$\Psi_g(z)=(1-|z|^2)[(Rg)(z)+\overline{(Rg)(z)}], \tag{17}$$

where R is the "backward shift" operator (12). In fact, let $g(z)=\sum_1^\infty b_k z^k$. Then it is easily seen that

$$g_1(z)=2\,|z|^2\,\mathrm{Re}(Rg)(z), \qquad V_g(z)=2\,\mathrm{Re}(Rg)(z),$$

and (17) follows. Since H_1-functions are $o(1/(1-|z|))$, formula (17) shows that Ψ_g extends to a continuous function on $\bar{\Delta}$ with $\Psi_g=0$ on $\partial\Delta$. Moreover,

$$s_g(\xi) := \lim_{r\to 1} \frac{\Psi_g(r\xi)}{1-r} = \frac{\partial}{\partial v}\,\Psi_g(\xi) = 2[(Rg)(\xi)+\overline{(Rg)(\xi)}], \tag{18}$$

and $\|s_{g_r}-s_g\|_{L_1}\to 0$ $(r\to 1)$.

LEMMA 2. *Let $f\in\mathscr{D}$, $g\in\mathscr{D}_1$, and $h\in H_\infty$. Then*

(a)

$$\int_\Delta |h(z)|^2\,|f'(z)|^2\,dA(z)+\lim_{r\to 1}\int_\Delta \left|\frac{d}{dz}\,h_r(z)\right|^2 \Phi_f(z)\,dA(z)$$
$$=\frac{1}{4}\int_0^{2\pi} |h(e^{i\theta})|^2\,q_f(e^{i\theta})\,d\theta; \tag{19}$$

(b)

$$2\int_\Delta |h(z)|^2\,\mathrm{Re}\,g'(z)\,dA(z)+\lim_{r\to 1}\int_\Delta \left|\frac{d}{dz}\,h_r(z)\right|^2 \Psi_g(z)\,dA(z)$$
$$=\frac{1}{4}\int_0^{2\pi} |h(e^{i\theta})|^2\,s_g(e^{i\theta})\,d\theta; \tag{20}$$

(c)

$$\int_\Delta h(z)\,|f'(z)|^2\,dA(z)=\frac{1}{4}\int_0^{2\pi} h(e^{i\theta})\,q_f(e^{i\theta})\,d\theta; \tag{21}$$

(d)

$$2\int_\Delta h(z)\,\mathrm{Re}\,g'(z)\,dA(z)=\frac{1}{4}\int_0^{2\pi} h(e^{i\theta})\,s_g(e^{i\theta})\,d\theta; \tag{22}$$

(e) *The limits in* (a) *and* (b) *exist.*

Proof. To simplify notation, let us agree that h_r' means $(h_r)'$, not $(h')_r$:

$$h_r'(z)=\frac{d}{dz}\,h_r(z)=rh'(rz).$$

We have, for $0<r<1$, $0<\rho<1$, by the classical Green formula,

$$\int_{\Delta} |h_r(z)|^2 \, |f'_\rho(z)|^2 \, dA(z) = -\int_{\Delta} |h_r(z)|^2 \, \partial\bar{\partial}\Phi_{f_\rho}(z)$$

$$= -\int_{\Delta} |h'_r(z)|^2 \, \Phi_{f_\rho}(z) \, dA(z)$$

$$+\frac{1}{4}\int_{\partial\Delta} |h_r(e^{i\theta})|^2 \, q_{f_\rho}(e^{i\theta}) \, d\theta.$$

Fixing $r<1$, we allow ρ to tend to 1. Since $f'_\rho \to f'$ in A_2, $q_{f_\rho} \to q_f$ in $L_1(\partial\Delta)$, $h_r \in H_\infty$, and $\Phi_{f_\rho} \to \Phi_f$ uniformly in $\bar{\Delta}$, we obtain

$$\int_{\Delta} |h_r(z)|^2 \, |f'(z)|^2 \, dA(z) + \int_{\Delta} |h'_r(z)|^2 \, \Phi_f(z) \, dA(z)$$

$$=\frac{1}{4}\int_{\partial\Delta} |h_r(e^{i\theta})|^2 \, q_f(e^{i\theta}) \, d\theta.$$

Using the dominated convergence theorem we find that of the three integrals above the first and the third tend to a limit as $r \to 1$. Therefore the second integral also has a limit, which proves the required relation.

(b) Since $g \subset \mathcal{D}_1 \subset H_1$, we can construct $g_1 = 2\,\mathrm{Re}(\bar{z}g)$ and $\Psi_g = V_g - g_1$. Clearly, Ψ_g does not depend on $g(0)$, so we can assume that $g(0)=0$. We have

$$\partial\bar{\partial}\Psi_g = -\partial\bar{\partial}(\bar{z}g + z\bar{g}) = -2\,\mathrm{Re}\, g'(z).$$

Repeating the argument used above for proving (19) we easily obtain (20). Here again the proof depends on two facts: $s_{g_r} \to s_g$ in $L_1(\partial\Delta)$, and $\Psi_{g_r} \to \Psi_g$ uniformly on $\bar{\Delta}$, as $r \to 1$.

Parts (c) and (d) are proved similarly.

COROLLARY. *If $f \in \mathcal{D}$, $g \in \mathcal{D}_1$, and*

$$\int_{\Delta} (|f'(z)|^2 - 2\,\mathrm{Re}\, g'(z)) \, h(z) \, dA = 0$$

for all $h \in H_\infty$, then $q_f(e^{i\theta}) - s_g(e^{i\theta}) = 0$ a.e.

LEMMA 3. *Let $f \in \mathcal{D}$, $g \in \mathcal{D}_1$, and let $q(z) = q_f(z)$ and $s(z) = s_g(z)$ be the harmonic extensions to Δ of $q_f(\xi)$ and $s_g(\xi)$, respectively. Then*

$$\Phi_f(z) - \Psi_g(z) \leqslant \frac{1}{2}(1-|z|^2)(q(z)-s(z)). \tag{23}$$

Proof. If f and g are smooth enough on $\bar{\Delta}$ then (23) follows from $(\Phi_f(z) - \Psi(z))/(1-|z|^2)$ being subharmonic (Lemma 1 and (17)) with boundary values $(1/2)(q(\xi) - s(\xi))$. Therefore

$$\Phi_{f_r}(z) - \Psi_{g_r}(z) \leqslant \frac{1}{2}\,(1-|z|^2)(q_{f_r}(z) - s_{f_r}(z)),$$

where $q_{f_r}(z)$ and $s_{g_r}(z)$ are the Poisson integrals of $q_{f_r}(\xi)$ and $s_{g_r}(\xi)$, respectively. Since $q_{f_r} \to q_f$ and $s_{g_r} \to s_g(r \to 1)$ in $L_1(\partial\Delta)$, we have $q_{f_r}(z) \to q_f(z)$ and $s_{g_r}(z) \to s_g(z)$ $(r \to 1)$ for $z \in \Delta$. On the other hand, $\Phi_{f_r}(z) \to \Phi_f(z)$ and $\Psi_{g_r}(z) \to \Psi_g(z)$ as $r \to 1$, which proves the lemma.

3. Auxiliary Lemmas

Let $M \subset A_2$ be a finite dimensional subspace of A_2 such that $1 \in M \subset A_4$. Denote $M_0 = \{f \in M : f(0) = 0\}$ and define $T_0 : M_0 \to A_2$ by

$$\begin{aligned}(T_0 g)(z) &= T(1+g)(z) - \|1+g\|^2_{A_2}\\ &= \frac{1}{\pi}\int_\Delta |1+g(w)|^2\,(K(z,w)-1)\,dA(w),\end{aligned} \tag{24}$$

where T is the quadratic Toeplitz operator (9).

Lemma 4. *T_0 is a continuously* **R**-*differentiable operator from M_0 to A_2, and $T_0'(0)$ is the imbedding $M_0 \to A_2$.*

Proof. For a fixed $g \in M_0$ and variable $h \in M_0$ we have

$$\begin{aligned}T_0(g+h) &= \frac{1}{\pi}\int_\Delta [1+g(w)+h(w)][1+\overline{g(w)}+\overline{h(w)}]\\ &\quad\times [K(z,\omega)-1]\,dA(w)\\ &= T_0(g) + \frac{1}{\pi}\int_\Delta [h(w)(1+\overline{g(w)}) + \overline{h(w)}(1+g(w))]\\ &\quad\times (K(z,w)-1)\,dA(w)\\ &\quad+\frac{1}{\pi}\int_\Delta |h(w)|^2\,(K(z,w)-1)\,dA(w).\end{aligned} \tag{25}$$

If we consider A_2 (and M_0) as normed linear spaces over the field **R**, then the second term of the right hand side of (25) is an **R**-linear operator $T_0'(g)$: $M_0 \to A_2$. In fact, $M \subset A_4$ implies that $T_0 g$ and $T_0(g+h)$ are in A_2, so $T(g)$ is well-defined on M_0. Since $\dim M_0 < \infty$ and $T_0'(g)$ depends linearly on $h \in M_0$, $T_0'(g)$ is a continuous linear operator on M_0. The convolution with the Bergman kernel K is the orthogonal projection from $L_2(\Delta, dA)$ onto A_2 and therefore

$$\left\| \frac{1}{\pi} \int_\Delta |h(w)|^2 (K(\cdot, w) - 1)\, dA(w) \right\|_2 \leqslant \|h^2\|_{L_2} + \|h\|_{A_2}^2$$
$$\leqslant \|h\|_{A_4}^2 + \|h\|_{A_2}^2. \tag{26}$$

Since M_0 is finite dimensional, the A_4 and A_2 norms are equivalent on M_0. Therefore there is a constant C such that the right hand side of (26) is $\leqslant C\, \|h\|_{A_2}^2$, which proves that $T_0'(g)$ is in fact the derivative of T_0 at g. Clearly, $T_0'(g)$ is continuous in g. Finally, since $h(0) = 0$ if $h \in M_0$, we obtain

$$(T_0'(0)h)(z) = \frac{1}{\pi} \int_\Delta [h(w) + \overline{h(w)}](K(z, w) - 1)\, dA(w) = h(z),$$

i.e., $T_0'(0)$ is the identity operator. The lemma is proved.

Let $L \subset H_\infty$, $\dim L < \infty$, $1 \in L$, $L_0 = \{f \in L : f(0) = 0\}$, $G \in A_2$. Pick a basis $\{\varphi_1, \varphi_2, ..., \varphi_n\}$ in L_0, and define $\Phi: A_2 \to \mathbf{R}^{2n}$ by

$$(\Phi h)_j = \operatorname{Re} \int_\Delta |G + h|^2\, \varphi_j\, dA,$$

$$(\Phi h)_{n+j} = \operatorname{Im} \int_\Delta |G + h|^2\, \varphi_j\, dA, \qquad j = 1, 2, ..., n \tag{27}$$

(again we consider A_2 as a normed linear space over the field of real numbers).

Lemma 5. *Φ is a continuously differentiable mapping, and if $G \neq 0$ then Φ is regular at 0, which means that $\Phi'(0)$ is a bounded linear operator from A_2 onto $\mathbf{R}^{2n}$.*

Proof. We have

$$\int_\Delta |G + h|^2\, \varphi_j\, dA = \int_\Delta |G|^2\, \varphi_j\, dA + \int_\Delta (\bar{G}h + G\bar{h})\, \varphi_j\, dA$$
$$+ \int_\Delta |h|^2\, \varphi_j\, dA. \tag{28}$$

The second term of this sum is a continuous $\mathbf{R}$-linear complex-valued functional on A_2. Hence

$$(\Phi'(0)h)_j = \operatorname{Re} \int_\Delta (\bar{G}h + G\bar{h})\, \varphi_j \, dA = \frac{1}{2} \int_\Delta (\bar{G}h + G\bar{h})(\varphi_j + \bar{\varphi}_j)\, dA,$$

$$(\Phi'(0)h)_{n+j} = \operatorname{Im} \int_\Delta (\bar{G}h + G\bar{h})\, \varphi_j \, dA = -\frac{i}{2} \int_\Delta (\bar{G}h + G\bar{h})(\varphi_j - \overline{\varphi_j})\, dA.$$

Suppose that $\Phi'(0)$ is not "onto." Then there are constants $\alpha_1, \alpha_2, ..., \alpha_n$, $\beta_1, \beta_2, ..., \beta_n \in \mathbf{R}$, not all $= 0$, such that

$$\frac{1}{2} \sum_{j=1}^{n} \left[\alpha_j \int_\Delta (\bar{G}h + Gh)(\varphi_j + \bar{\varphi}_j)\, dA - i\beta_j \int_\Delta (\bar{G}h + G\bar{h})(\varphi_j - \bar{\varphi}_j)\, dA \right] = 0$$

for all $h \in A_2$. Put $\varphi = \sum (\alpha_j - i\beta_j)\, \varphi_j$; then we get

$$\int_\Delta (\bar{G}h + G\bar{h})(\varphi + \bar{\varphi})\, dA = 0 \tag{29}$$

for all $h \in A_2$. Put ih instead of h in (29);

$$\int_\Delta (i\bar{G}h - iG\bar{h})(\varphi + \bar{\varphi})\, dA = 0. \tag{30}$$

Therefore

$$\int_\Delta \bar{G}h(\varphi + \bar{\varphi})\, dA = 0, \qquad \forall h \in A_2. \tag{31}$$

Put $h = G\varphi$ in (31). We obtain

$$\int_\Delta |G|^2\, |\varphi|^2\, dA = -\int_\Delta |G|^2\, \varphi^2\, dA.$$

This yields, if $\varphi = u + iv$,

$$\int_\Delta |G|^2\, u^2\, dA = 0.$$

Therefore $u \equiv 0$, which implies $\varphi = \text{const}$. Since $\varphi(0) = 0$, $\varphi \equiv 0$ and $\alpha_1 = \cdots = \alpha_n = \beta_1 = \cdots = \beta_n = 0$. The lemma is proved.

4. Proof of the Theorems

Proof of the Approximation Lemma. The assumption is that $L \subset H_\infty$ is a finite dimensional linear space, $1 \in L$, and $G \in A_2$ is such that (8) holds for every $g \in L$, which implies, in particular, that $\|G\|_{A_2} = A_1$. Let $L_0 = \{f \in L : f(0) = 0\}$; we have

$$\frac{1}{\pi} \int_A |G|^2 f \, dA = 0, \qquad \forall f \in L_0. \tag{32}$$

Our goal is to show that G can be approximated by polynomials P so that $\|P - G\|_{A_2} < \varepsilon$ ($\varepsilon > 0$ arbitrary) and

$$\frac{1}{\pi} \int_A |P|^2 f \, dA = 0, \qquad \forall f \in L_0; \tag{33}$$

in fact, $P/\|P\|_{A_2}$ will then be an L-inner polynomial approximating G. Pick a basis $\{\varphi_1, ..., \varphi_n\}$ in L_0 and consider the mapping Φ defined by (27) of A_2 into $\mathbf{R}^{2n}$; it follows from (32) that $\Phi(0) = 0$. By Lemma 5, $\Phi'(0)$ is an "onto" mapping, i.e., there is a $2n$-(real) dimensional subspace M of A_2, say $M = \text{span}\{h_1, ..., h_{2n}\}$, such that $\Phi'(0) M = \mathbf{R}^{2n}$. The restriction of Φ to M can be represented as $\Phi = \Phi'(0) - \Phi_1$, where $\Phi'(0)$ is an invertible linear mapping from M onto $\mathbf{R}^{2n}$ and Φ_1 is a nonlinear operator such that

$$\sup\{ \|\Phi_1 h\|_{\mathbf{R}^{2m}} : h \in M, \ \|h\|_{A_2} \leqslant \varepsilon\} \leqslant C\varepsilon^2 \tag{34}$$

(see (28)). Using von Neumann's series we deduce that Φ is invertible in a ball $D_\varepsilon = \{h \in M : \|h\|_{A_2} \leqslant \varepsilon\}$ if ε is sufficiently small. In fact, since $\Phi(0) = 0$, that is equivalent to the existence of Φ^{-1} in a ball $B_\delta \subset \mathbf{R}^{2n}$ of radius δ, centered at 0, and this follows from the convergence of von Neumann's series

$$\Phi^{-1} = \sum_{j=0}^{\infty} ([\Phi'(0)]^{-1} \Phi_1)^j \, [\Phi'(0)]^{-1}$$

for $\delta < C^{-1/2} \|(\Phi'(0))^{-1}\|^{-1}$, where C is the constant in (34). Fix such a δ, and find $\varepsilon > 0$ so that $\Phi(D_\varepsilon) \subset B_\delta$. Then consider $\Phi_N \colon M_N \to \mathbf{R}^{2n}$ defined by (27) on $M_N = \{h_N : h \in M\}$ but with $(G+h)_N$ instead of $G+h$, where h_N and $(G+h)_N$ is the Nth partial sum of the Taylor series of h and $G+h$, respectively. Because of the finite dimension of M, $(G+h)_N$ uniformly approximates $G+h$ in the A_2 metric if $h \in D_\varepsilon$ and $N \to \infty$. We can therefore choose N large enough to guarantee the existence of Φ_N^{-1} in $B_{\delta/2}$, by the convergence of the corresponding von Neumann's series. We thus obtain a polynomial of degree N, $P = P_N = \Phi_N^{-1}(0) = (G + \tilde{h})_N$, satisfying (33). Also

$$\|P_N - G\|_{A_2} \leqslant \|\tilde{h}_N\|_{A_2} + \|G - G_N\|_{A_2} \leqslant \varepsilon + \|G - G_N\| < 2\varepsilon,$$

provided that N is large enough. Since ε is arbitrary, we get the required result.

Proof of Theorem 1. Let $L \subset H_\infty$ be an e-subspace of A_2, with $\dim L < \infty$. Suppose that $Tg \notin L$ for some $g \in L$. Let $Tg = g_1 + g_2$ where $g_1 \in L$ and $g_2 \in L^\perp$, $g_2 \neq 0$. Note that $g_2(0) = 0$ since $1 \in L$. As $g \in H_\infty$, $Tg \in A_4$ and hence $g_2 \in A_4$. Set $M = \text{span}(L, \{g_2\})$, $M_0 = \{f \in M : f(0) = 0\}$; M_0 is a closed subspace of A_2. Let $P: A_2 \to M_0$ be the orthogonal projection. Set

$$T_1 : M_0 \to M_0, \qquad T_1 = PT_0,$$

where T_0 is given by (24). It follows from Lemma 4 that T_1 is a continuously differentiable operator, and $T_1'(0) = I$ (identity operator). Also, $T_1(0) = 0$. Now, the implicit function theorem implies that T_1 is invertible in some neighborhood of 0. It means that there is an $f \in M_0$ such that $T_1 f = -\varepsilon g_2$, if $\varepsilon > 0$ is sufficiently small, and $f = -\varepsilon g_2 + o(\varepsilon)$, where the last term denotes an element in M_0 with the norm of order ε^2. Hence $T_0 f = -\varepsilon g_2 + b(\varepsilon) h_\varepsilon$ and $T(1+f) = 1 + a(\varepsilon) - \varepsilon g_2 + b(\varepsilon) h_\varepsilon$ where $a(\varepsilon)$ and $b(\varepsilon)$ are scalar functions of order ε^2 (since $g_2(0) = 0$), and h_ε is orthogonal to M_0, with $\|h_\varepsilon\|_{A_2}$ bounded as $\varepsilon \to 0$. Moreover, $h_\varepsilon(0) = 0$ and, therefore, h_ε is orthogonal to L.

Put $G = (1+f)/\|1+f\|_{A_2}$. Since $f = -\varepsilon g_2 + o(\varepsilon)$ and $g_2(0) = 0$, we have $\|1+f\| = 1 + o(\varepsilon^2)$ and $G = 1 - \varepsilon g_2 + o(\varepsilon)$ where the last term denotes an element with the norm of order at least ε^2. Also, $\|G\|_{A_2} = 1$.

Now, for each $h \in L$, $h(0) = 0$, we have

$$\frac{1}{\pi}\int_\Delta |G(z)|^2 h(z)\, dA(z)$$

$$= \frac{1}{\pi^2}\int_\Delta |G(z)|^2\, dA(z) \int_\Delta K(z, w)\, h(w)\, dA(w)$$

$$= \frac{1}{\pi^2}\int_\Delta h(w)\, dA(w) \int_\Delta |G(z)|^2 K(z, w)\, dA(z)$$

$$= \frac{1}{\pi}\int_\Delta h(w)\overline{(TG)(w)}\, dA(w)$$

$$= \|1+f\|_{A_2}^{-2} \frac{1}{\pi}\int_\Delta h(w)\, \overline{T(1+f)(w)}\, dA(w)$$

$$= \|1+f\|_{A_2}^{-2} \frac{1}{\pi}\int_\Delta h(w)(1 + \overline{a(\varepsilon)} - \varepsilon\overline{g_2(w)} + \overline{b(\varepsilon)}\,\overline{h_\varepsilon(w)})\, dA(w)$$

$$= \|1+f\|_{A_2}^{-2} \left[(1 + \overline{a(\varepsilon)})\, h(0) - \varepsilon\langle h, g_2\rangle + \overline{b(\varepsilon)}\langle h, h_\varepsilon\rangle\right] = 0.$$

Thus $|G|^2-1$ is orthogonal to all elements of L. At the same time

$$\|Gg\|_{A_2}^2=\frac{1}{\pi}\int_\Delta |G|^2\,|g|^2\,dA$$

$$=\frac{1}{\pi}\int_\Delta (1-\varepsilon g_2(z)+o(\varepsilon))(1-\varepsilon\overline{g_2(z)}+\overline{o(\varepsilon)})\,|g|^2\,dA(z)$$

$$=\|g\|_{A_2}^2-\frac{\varepsilon}{\pi}\int_\Delta (g_2(z)+\overline{g_2(z)})\,|g|^2\,dA(z)+o(\varepsilon)$$

$$=\|g\|_A^2-2\varepsilon\,\mathrm{Re}\,\frac{1}{\pi}\int_\Delta g_2(z)\,|g(z)|^2\,dA(z)+o(\varepsilon)$$

$$=\|g\|_{A_2}^2-2\varepsilon\,\mathrm{Re}\left(\frac{1}{\pi^2}\int_\Delta |g(z)|^2 \times dA(z)\int_\Delta K(z,w)\,g_2(w)\,dA(w)\right)+o(\varepsilon)$$

$$=\|g\|_{A_2}^2-2\varepsilon\,\mathrm{Re}\,\frac{1}{\pi}\int_\Delta g_2(w)\overline{(Tg)(w)}\,dA(w)+o(\varepsilon)$$

$$=\|g\|_{A_2}^2-2\varepsilon\,\mathrm{Re}\langle g_2, g_1+g_2\rangle+o(\varepsilon)$$

$$=\|g\|_{A_2}^2-2\varepsilon\,\|g_2\|_{A_2}^2+o(\varepsilon)<\|g\|_{A_2}^2$$

for sufficiently small $\varepsilon>0$. Therefore L is not an e-subspace. This contradiction shows that $TL\subset L$ if L is an e-subspace.

Conversely, let L be a T-subspace. By the Corollary to the Approximation Lemma, it is sufficient to prove that L is an e^*-subspace of A_2. Let G be a bounded L-inner function, and let

$$\tilde{G}(z)=\int_0^z G(w)\,dw,\qquad \tilde{g}(z)=\int_0^z (TG)(w)\,dw-\frac{z}{2}.$$

Let $h\in H_\infty$. We have

$$\frac{1}{\pi}\int_\Delta \overline{(TG)(z)}\,h(z)\,dA(z)=\int_\Delta h(z)\,dA(z)\int_\Delta |G(w)|^2\,K(w,z)\,dA(w)$$

$$=\frac{1}{\pi}\int_\Delta |G(w)|^2\,h(w)\,dA(w)$$

and

$$\frac{1}{\pi}\int_\Delta (TG)(z)\,h(z)\,dA(z)=(TG)(0)\cdot h(0)=\|G\|_{A_2}^2\,h(0)=h(0).$$

Hence for each $h \in H_\infty$ we have

$$\int_\Delta (|G(z)|^2 - (TG)(z) - \overline{(TG)(z)} + 1)\, h(z)\, dA(z) = 0. \tag{35}$$

Consider now $\Phi_{\tilde{G}} - \Psi_{\tilde{g}}$, with Φ and Ψ as defined in Section 2. Since $\tilde{G}$ and $\tilde{g}$ belong to $\mathscr{D}$, the Corollary to Lemma 2 can be applied to (35), which yields

$$q_{\tilde{G}} - s_{\tilde{g}} = \frac{\partial}{\partial v} (\Phi_{\tilde{G}} - \Psi_{\tilde{g}}) = 0 \tag{36}$$

a.e. on $\partial\Delta$. Combining now assertions (a) and (b) of Lemma 2 we obtain

$$\int_\Delta (|G(z)|^2 - (TG)(z) - \overline{(TG)(z)} + 1)\, |h(z)|^2\, dA(z)$$

$$= -\lim_{r \to 1} \int_\Delta |h_r'(z)|^2 (\Phi_{\tilde{G}}(z) - \Psi_{\tilde{g}}(z))\, dA(z). \tag{37}$$

By Lemma 3, $\Phi_{\tilde{G}}(z) - \Psi_{\tilde{g}}(z) \leqslant 0$, because of (36). Thus, the limit in (37) (which exists) is $\leqslant 0$, and

$$\int_\Delta (|G(z)|^2 - (TG)(z) - \overline{(TG)(z)} + 1)\, |h(z)|^2\, dA(z) \geqslant 0 \tag{38}$$

for all $h \in H_\infty$. Now, let $h \in L$. By the assumption, $Th \in L$ and $|G|^2 - 1 \perp Th$, since G is an L-inner function. We get

$$\frac{1}{\pi} \int_\Delta (TG)(z)\, |h(z)|^2\, dA(z)$$

$$= \frac{1}{\pi^2} \int_\Delta |h(z)|^2\, dA(z) \int_\Delta |G(w)|^2 K(z, w)\, dA(w)$$

$$= \frac{1}{\pi} \int_\Delta |G(w)|^2\, \overline{(Th)(w)}\, dA(w)$$

$$= \frac{1}{\pi} \int_\Delta (|G(w)|^2 - 1) \overline{(Th)(w)}\, dA(w) + \frac{1}{\pi} \int_\Delta \overline{(Th)(w)}\, dA(w)$$

$$= \overline{(Th)(0)} = \|h\|_{A_2}^2,$$

and similarly

$$\frac{1}{\pi} \int_\Delta \overline{(TG)(z)}\, |h(z)|^2\, dA(z) = \|h\|_{A_2}^2.$$

Thus (38) yields

$$\frac{1}{\pi}\int\int |G(z)\,h(z)|^2\,dA(z) - \|h\|^2_{A_2} = \|Gh\|^2_{A_2} - \|h\|^2_{A_2} \geqslant 0,$$

which proves that G is an expansive L-multiplier. The theorem is proved.

Proof of Theorem 2. Let $L = \operatorname{span}\{z^{kj} : j = 0, 1, ..., n\}$. A simple computation yields

$$\frac{1}{\pi}\int_{\Delta} w^l\,|w|^{2s}\,K(z, w)\,dA(w) = \frac{l+1}{l+s+1}\,z^l, \qquad l, s \geqslant 0 \tag{39}$$

and

$$\frac{1}{\pi}\int_{\Delta} \bar{w}^l\,|w|^{2s}\,K(z, w)\,dA(w) = 0, \qquad l > 0,\ s \geqslant 0.$$

Therefore, if $f = \sum_{j=0}^{n} a_j z^{jk} \in L$ then

$$(Tf)(z) = \sum_{j=0}^{n} \frac{|a_j|^2}{kj+1} + \sum_{j=1}^{n} z^{kj} \sum_{i=j}^{n} \frac{a_i \bar{a}_{i-j}[k(i-j)+1]}{kj+1} \in L.$$

So L is a T-subspace, and by Theorem 1, L is an e-subspace.

Conversely, suppose that L is an $(n+1)$-dimensional e-subspace of A_2 whose elements are polynomials. L has a basis, $\{P_k\}_{k=0}^{n}$, such that

(i) $P_0 = 1$,

(ii) $P_k(0) = 0$, $k = 1, ..., n$,

(iii) $\deg P_k > \deg P_{k-1}$, $k = 1, ..., n$.

Since L is an e-subspace, it is a T-subspace. Hence

$$TP_1 = \sum_{k=0}^{n} \alpha_k P_k.$$

It follows from (39) and (ii) that $\deg TP_1 < \deg P_1$. Therefore $\alpha_1 = \cdots = \alpha_n = 0$ and $TP_1 = \text{const}$, which implies that P_1 is a monomial, say, z^k. Prove that $P_m \in \operatorname{span}\{z^{kj} : j = 1, ..., m\}$ for $m = 1, ..., n$ or, equivalently, that $\operatorname{span}\{P_0, ..., P_m\} = \operatorname{span}\{1, z^k, ..., z^{mk}\}$. Suppose that it has already been proved that $\operatorname{span}\{P_0, ..., P_r\} = \operatorname{span}\{1, z^k, ..., z^{rk}\}$. Again (ii) and (iii) imply that $\deg TP_{r+1} < \deg P_{r+1}$ and therefore

$$TP_{r+1} = \operatorname{span}\{P_0, ..., P_r\} = \operatorname{span}\{1, z^k, ..., z^{rk}\}.$$

The same argument shows that

$$T(P_{r+1}+Q)\in \operatorname{span}\{1, z^k, \ldots, z^{rk}\}$$

for every $Q\in \operatorname{span}\{P_1, \ldots, P_r\}=\operatorname{span}\{z^k, \ldots, z^{rk}\}$. Write

$$P_{r+1}(z)=a_1 z+\cdots+a_s z^s, \qquad a_s\neq 0,$$

and set $Q=\alpha z^k$. If $s>k(r+1)$ then $T(P_{r+1}+Q)$ contains z^{s-k} with the coefficient

$$b_{s-k}=\sum_{j=0}^{k}\frac{s-k+1}{s-k+j+1}\,\bar{a}_j a_{s-k+j}+\bar{\alpha} a_s\,\frac{s-k+1}{s+1}.$$

Since $s-k>rk$, $b_{s-k}=0$ for every α, which implies $a_s=0$—a contradiction. Thus $\deg P_{r+1}=k(r+1)$. Set $Q=\sum_{i=1}^{r}\alpha_i z^{ki}$. If $1\leqslant j<k$ then the coefficient b_j (at z^j) of the polynomial $T(P_{r+1}+Q)$ is equal to

$$b_j=\sum_{i=j}^{k(r+1)-j}\frac{j+1}{i+j+1}\,\bar{a}_i a_{i+j}+\sum_{l=1}^{r}\alpha_l\bar{a}_{lk-j}\,\frac{j+l}{kl+1}$$
$$+\sum_{l=1}^{r}\frac{j+1}{lk+j+1}\,\bar{\alpha}_l a_{lk+j}.$$

Since $b_j=0$ for every $\alpha=(\alpha_1, \ldots, \alpha_r)\in \mathbf{C}^r$ we have

$$a_{lk+j}=a_{lk-j}=0, \qquad l=1, \ldots, r,\ j=1, \ldots, k-1,$$

and therefore $P_{r+1}\in \operatorname{span}\{z^k, \ldots, z^{k(r+1)}\}$. Since $P_{r+1}\notin \operatorname{span}\{P_0, \ldots, P_r\}=\operatorname{span}\{1, z^k, \ldots, z^{rk}\}$, we obtain

$$\operatorname{span}\{P_0, \ldots, P_{r+1}\}=\operatorname{span}\{1, z^k, \ldots, z^{k(r+1)}\}.$$

The theorem is proved.

Proof of the Corollary to Theorem 2. Let $f\in M_k$, i.e., $f=f(z^k)$, $f\in H^\infty$. Let $\{P_n\}$ be a sequence of polynomials converging weak* to f. If G is an M_k-inner function, G is inner on any space of polynomials $L=\operatorname{span}\{1, z^k, \ldots, z^{mk}\}$, and, by Theorem 2, G is an expansive multiplier for all $P_n(z^k)$. Hence

$$\|Gf(z^k)\|_{A_2}=\lim_{n\to\infty}\|GP_n(z^k)\|_{A_2}\geqslant \overline{\lim_{n\to\infty}}\,\|P_n(z^k)\|_{A^2}\geqslant \|f(z^k)\|_{A^2}.$$

(The last inequality follows from Fatou's lemma since clearly $P_n(z^k)\to f(z^k)$ uniformly on compact subset of Δ.)

Remark. It can be shown that the above corollary remains valid if M_k is replaced by its closure $\bar{M}_k$ in A_2: $\bar{M}_k = \{f(z^k) : f \in A_2\}$.

Proof of Theorem 3. Let $D = m_1 a_1 + \cdots + m_n a_n$. Obviously $L_D \subset H_\infty$, and dim $L_D = m_1 + \cdots + m_n$. Let $f \in L_D$,

$$f = \sum_{k=1}^{n} \sum_{j=0}^{m_k} b_{kj} K_j(z, a_k).$$

We have

$$|f(z)|^2 = \sum_{\substack{1 \leq s, l \leq n \\ 0 \leq j \leq m_s \\ 0 \leq i \leq m_l}} b_{sj} \overline{b_{li}} K_j(z, a_s) \overline{K_i(z, a_l)}$$

and

$$\frac{1}{\pi} \int K_j(w, a_s) \overline{K_i(w, a_l)} K(z, w)\, dA(w) = \overline{\frac{\partial^j}{\partial w^j} (K_i(w, a_l) K(w, z)) \big|_{w = a_s}}$$

$$= \sum_{r=0}^{j} \lambda_r K_r(z, a_s) \in L_D,$$

where

$$\lambda_r = \overline{\frac{\partial^{j-r+i}}{\partial z^{j-r+i}} K(z, a_l) \big|_{z = a_s}} = \frac{(j-r+i+1)!\, a_l^{j-r+i}}{(1 - a_l \bar{a}_s)^{j-r+i+2}}.$$

Thus $Tf \in L_D$, and L_D is a T- and, therefore, an e-subspace.

Proof of Theorem 1′. The implication e-subspace $\Rightarrow \tilde{T}$-subspace is proved by essentially the same argument as in Theorem 1. Now, let L be a $\tilde{T}$-subspace. Note that for every $g \in H_\infty$ we have $|g(\xi)|^2 = (\tilde{T}g)(\xi) + \overline{(\tilde{T}g)(\xi)} - \|g\|_{H_2}^2$, a.e. on $\partial \Delta$. Therefore

$$\|Gg\|_{H_2}^2 = \frac{1}{2\pi} \int_0^{2\pi} |G(e^{i\theta})|^2\, |g(e^{i\theta})|^2\, d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} |G(e^{i\theta})|^2 \left((\tilde{T}g)(e^{i\theta}) + \overline{(\tilde{T}g)(e^{i\theta})} - \|g\|_{H_2}^2\right) d\theta$$

$$= \|g\|_{H_2}^2,$$

which proves Theorem 1′.

The proofs of Theorem 2′ and Theorem 3′ also are quite similar to the proofs of Theorem 2 and Theorem 3, respectively. We leave the details to the reader.

ACKNOWLEDGMENTS

We thank D. Khavinson for useful discussions while this work was in progress. We also express our gratitude to the referee who detected numerous imperfections in the original manuscript.

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