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Modulated bicategories

Aurelio Carboni^a, Scott Johnson^{†,b}, Ross Street^{c,*}, Dominic Verity^c

^a *Dipartimento di Matematica, via C. Saldini 50, 20133 Milano, Italy*

^b *Department of Mathematics, Australian National University, Canberra, ACT 2601, Australia*

^c *Department of Mathematics, Macquarie University, NSW 2109, Australia*

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Abstract

The concept of regular category [1] has several 2-dimensional analogues depending upon which special arrows are chosen to mimic monics. Here, the choice of the conservative arrows, leads to our notion of faithfully conservative bicategory \mathcal{K} in which two-sided discrete fibrations become the arrows of a bicategory $\mathcal{F} = \text{DFib}(\mathcal{K})$. While the homcategories $\mathcal{F}(B, A)$ have finite limits, it is important to have conditions under which these finite “local” limits are preserved by composition (on either side) with arrows of \mathcal{F} . In other words, when are all fibrations in \mathcal{K} flat? Novel axioms on \mathcal{K} are provided for this, and we call a bicategory \mathcal{K} modulated when \mathcal{K}^{op} is such a \mathcal{K} . Thus, we have constructed a proarrow equipment $(\)_* : \mathcal{K} \rightarrow \mathcal{M}$ (in the sense of [28]) with $\mathcal{M} = \mathcal{F}^{\text{coop}}$. Moreover, \mathcal{M} is locally finitely cocomplete and certain collages exist [23].

In the converse direction, if \mathcal{M} is any locally countably cocomplete bicategory which admits finite collages [23], then the bicategory \mathcal{M}^* of maps in \mathcal{M} is modulated. (Recall from [26, p 266], that a 1-cell in a bicategory is called a *map* when it has a right adjoint.)

Introduction

The characterisation of bicategories $\mathcal{W}\text{-Mod}$, whose objects are categories with homs enriched in the base bicategory \mathcal{W} and whose arrows are modules, was achieved in [23]. The main requirements on a bicategory \mathcal{M} that it should be biequivalent to $\mathcal{W}\text{-Mod}$ are the existence of coproducts, Kleisli constructions and local colimits preserved by composition on either side with an arrow. We call such an \mathcal{M} a *cosmos*. Any “Cauchy generating” set of objects of \mathcal{M} would then form a suitable \mathcal{W} . Also see [5].

* Corresponding author. Email: street@mpce.mq.edu.au.

† Deceased.

An arrow of \mathcal{M} is called [26] a *map* when it has a right adjoint. An enriched category is called *Cauchy complete* when it admits all absolute (weighted) limits (or, equally, colimits) [13, 25]. The maps $f:A \rightarrow B$ in $\mathcal{W}\text{-Mod}$ can be identified with \mathcal{W} -functors $f:\mathcal{Q}A \rightarrow \mathcal{Q}B$ between Cauchy completions. We write \mathcal{M}^* for the sub-bicategory of \mathcal{M} obtained by restricting the arrows to maps. A suitable notation for $(\mathcal{W}\text{-Mod})^*$ is thus $(\mathcal{W}\text{-Cat})_{\text{cc}}$.

The main motivation for the present paper was to attempt to characterise bicategories of the form $(\mathcal{W}\text{-Cat})_{\text{cc}}$. This means we looked for conditions on a bicategory \mathcal{H} (involving limits, colimits, inherent factorisation systems, and the like) which would ensure that \mathcal{H} was equivalent to \mathcal{M}^* for a cosmos \mathcal{M} . Apart from its intrinsic interest for enriched category theory, this question is relevant to the work of Pitts [17] and to the theory of quantales [4, 15, 16].

We were also guided by the challenge of keeping external set theory to a minimum. So our development requires only first-order (“elementary”) concepts. The replacement of “finite” by “small” is easily achieved.

Because of results of Street [22] and Wood [29], there was no doubt about how to construct \mathcal{M} from \mathcal{H} ; the arrows had to be two-sided codiscrete cofibrations in \mathcal{H} . The problem was to find elementary conditions on \mathcal{H} to ensure that \mathcal{M} became a bicategory and had the desirable properties, especially local finite cocompleteness. Because the reader may be familiar with the construction of the bicategory of relations from a regular category, and since our situation resembles the dual of this, we have decided to present the work in terms of $\mathcal{K} = \mathcal{H}^{\text{op}}$ rather than \mathcal{H} itself. This has the advantage that fibrations are generally more familiar than cofibrations, and preservation of finite limits by composition (“tensoring”) with a fibration is a familiar flatness condition.

In the Sections 2 and 3, we review the required limits, special arrows and factorisation systems in a bicategory. An arrow is *conservative* when a 2-cell into its source can be declared invertible if its composite with our arrow is invertible. The dual of conservative is *liberal*, while *strong liberals* can be defined from conservatives similarly to the way that strong epimorphisms are derived from monomorphisms [9]. *Faithfully conservational bicategories* are defined analogously to regular categories in [6] beginning with conservative arrows in place of monomorphisms. The main result (Theorem 2.19) of these sections is that stability of strong liberals under pseudopull-back implies each strong liberal is a coinverter of some 2-cell. In Section 3, we interpret the concepts of the earlier sections in the 2-categories **Cat**, **Cat**_{cc}, **Lex**, **Rex**, and their duals. The calculus of discrete fibrations in a faithfully conservative bicategory \mathcal{K} is developed in Section 4; in particular, we construct the bicategory **DFib**(\mathcal{K}) whose objects are those of \mathcal{K} and whose arrows are two-sided discrete fibrations.

The definition of *faithful* arrow given in Section 2 deserves some comment. This is known as the *representable* definition, since all it says is that an arrow j is faithful precisely when, for each object X , the functor obtained by applying the representable $\mathcal{K}(X, -)$ to j is faithful. The faithful arrows in $\mathcal{V}\text{-Cat}$ are simply those \mathcal{V} -functors

which have a faithful underlying ordinary functor, rather than the more appropriate \mathcal{V} -faithful ones. In fact, many of the definitions of Section 2 are inappropriate when interpreted in \mathcal{V} -Cat, and so we do not do that here, rather we are interested in considering them in the context of the strong dual $(\mathcal{V}\text{-Cat}_{cc})^{\text{op}}$. Such definitions do make good sense in $(\mathcal{V}\text{-Cat}_{cc})^{\text{op}}$; it is that bicategory which is of central importance here, just as it was for Street in [22].

Notice that in Section 6 we give a definition of *fully faithful* map which is not the representable one, and does coincide with \mathcal{V} -fully faithfulness when interpreted for maps in $\mathcal{V}\text{-Mod}$. There is a corresponding definition of faithful map, and therefore it might be said that we should have adopted the prefix “representable” for the notion considered in earlier sections. We have not done so, simply because we only apply the representable notion in the very different context of the bicategory $(\mathcal{V}\text{-Cat}_{cc})^{\text{op}}$. We hope that contextual variety will serve to distinguish the two possible uses of the word “faithful”.

In Section 5, a bicategory \mathcal{H} is defined to be modulated when \mathcal{H}^{op} is faithfully conservational and two extra axioms about strong conservatives in \mathcal{H} are satisfied. The reader should keep in mind the example of $\mathcal{H} = (\mathcal{V}\text{-Cat})_{cc}$ where the strong conservatives are fully faithful functors between the Cauchy complete \mathcal{V} -enriched categories. We show that the bicategory $\mathcal{M} = \text{CodCofib}(\mathcal{H}) = \text{DFib}(\mathcal{H}^{\text{op}})^{\text{coop}}$ of codiscrete cofibrations in a modulated bicategory \mathcal{H} is locally finitely cocomplete. This is deduced from the result that the “inclusion” $(\)_*: \mathcal{H} \rightarrow \mathcal{M}$ preserves tensors with certain finite categories. This inclusion also preserves, so that \mathcal{M} has, finite coproducts. A principal result here is the construction of the collage of any arrow in \mathcal{M} . Some questions still remain. We are unable to show that $(\)_*: \mathcal{H} \rightarrow \mathcal{M}$ preserves pseudopushouts; perhaps another axiom is required. A related problem is that we are unable to construct general finite collages in \mathcal{M} . Then there is the question of whether each map in \mathcal{M} is isomorphic to f_* for some arrow f of \mathcal{H} . These questions need to be resolved before our motivating problem can be fully answered.

Finally, in Section 6, we show that, if \mathcal{M} is any locally finitely cocomplete bicategory which admits finite collages and free monads on endo-arrows, then \mathcal{M}^* is modulated and the inclusion $\mathcal{M}^* \rightarrow \mathcal{M}$ preserves finite colimits.

1. Review of limits in bicategories

1.1. The appropriate limits for bicategories were introduced in [22]. These limits differ from those appropriate for 2-categories [21] in that they are only unique up to equivalence, not up to isomorphism. Our purpose in this section is to recall the general definition and to give examples needed in the present paper. Since the general definition is a representable one, it suffices to give the examples in the bicategory **Cat** of categories, functors and natural transformations. In order to avoid the confusing “bi” prefix in terminology such as “bilimit” and “biproduct” (since these could also mean that they, at the same time, provide the dual notion), we have adopted the policy

of using the 2-categorical name for the limit when it is understood that we are working bicategorically (even if we are working in a 2-category). However, one should be aware that some 2-categorical limits (pullbacks and equalisers for example) do not survive this passage to bicategories, and so have no rôle for them.

1.2. Suppose $J, S: \mathcal{A} \rightarrow \mathbf{Cat}$ are homomorphisms of bicategories. Write S^J for the category whose objects are strong transformations (= pseudo-natural transformations) $J \rightarrow S$, and whose arrows are modifications [2]. Observe that, for all X , there is a canonical equivalence of categories

$$\mathbf{Cat}(X, S^J) \xrightarrow{\sim} \mathbf{Cat}(X, S)^J$$

1.3. Now suppose $J: \mathcal{A} \rightarrow \mathbf{Cat}$, $S: \mathcal{A} \rightarrow \mathcal{K}$ are homomorphisms of bicategories. A J -weighted limit of S consists of an object S^J of \mathcal{K} and a strong transformation $\lambda: J(-) \rightarrow \mathcal{K}(S^J, S(-))$ such that, for all objects X of \mathcal{K} , the functor

$$\mathcal{K}(X, S^J) \xrightarrow{\sim} \mathcal{K}(X, S)^J, \quad u \mapsto \mathcal{K}(u, S) \circ \lambda$$

is an equivalence of categories. The representable nature of this general definition allows us to describe various examples by merely explaining them in \mathbf{Cat} . We do not need to exhibit the weight homomorphism J in each case. The reader who has had no experience in this may refer to [11], [21] and [22].

A J -weighted colimit of a homomorphism $T: \mathcal{A}^{\text{op}} \rightarrow \mathcal{K}$ is a J -weighted limit of T regarded as a homomorphism $\mathcal{A} \rightarrow \mathcal{K}^{\text{op}}$. For each of our examples below there is a dual version for which the prefix “co” is used. We have no need here for examples which give new constructions when \mathcal{K} is replaced by the weak dual \mathcal{K}^{co} .

1.4. The *product* of two categories is defined in the obvious way by taking, as the set of arrows, the cartesian product of the sets of arrows. The *product* of objects A, B in \mathcal{K} consists of an object $A \times B$, and arrows

$$A \xleftarrow{P_A} A \times B \xrightarrow{P_B} B$$

(called *projections*) in \mathcal{K} , such that the induced functor

$$\mathcal{K}(X, A \times B) \xrightarrow{\sim} \mathcal{K}(X, A) \times \mathcal{K}(X, B)$$

is an equivalence for all objects $X \in \mathcal{K}$. Of course, products of families of objects are now defined in the obvious way. In particular (the empty family case), a *terminal object* 1 for \mathcal{K} is defined by the condition that each $\mathcal{K}(X, 1)$ is equivalent to the terminal category $\mathbf{1}$.

1.5. The comma category $f \downarrow g$ of two functors forming a cospan

$$A \xrightarrow{f} C \xleftarrow{g} B$$

is a familiar construction [14]. The objects are triples $(a, \gamma: fa \rightarrow gb, b)$ where a, b are objects of A, B and γ is an arrow of C . An arrow $(\alpha, \beta): (a, \gamma, b) \rightarrow (a', \gamma', b')$ consists of arrows $\alpha: a \rightarrow a', \beta: b \rightarrow b'$ of A, B such that $\gamma' \circ f(\alpha) = g(\beta) \circ \gamma$. There is a diagram

$$\begin{array}{ccc}
 f \downarrow & g \xrightarrow{d_1} & B \\
 d_0 \downarrow & \cong & \downarrow g \\
 A & \xrightarrow{f} & C
 \end{array} \tag{1}$$

where λ is the natural transformation whose component at the object (a, γ, b) is the arrow γ . The *comma object* of a cospan f, g as above, but now in our bicategory \mathcal{K} , consists of a diagram in \mathcal{K} in the form of the last square, which induces an equivalence of categories

$$\mathcal{K}(X, f \downarrow g) \xrightarrow{\sim} \mathcal{K}(X, f) \downarrow \mathcal{K}(X, g).$$

1.6. The *pseudopullback* of functors $f: A \rightarrow C, g: B \rightarrow C$ is the full subcategory $A_f \times_g B$ (or, less precisely $A \times_C B$) of $f \downarrow g$ consisting of those objects $(a, \gamma: fa \rightarrow gb, b)$ for which the arrow γ is invertible. This sits in a square like the one for the comma category (1), but this time the 2-cell in our square is invertible. The left arrow in this square is sometimes denoted by pr_1 and the top arrow by pr_2 . It should be clear now how to define the *pseudopullback* of a cospan in \mathcal{K} . We also use the terminology that the arrow on the top (respectively, at the left) of a pseudopullback square is a *pseudopullback of the arrow* on the bottom (respectively, at the right).

1.7. **Proposition.** Suppose that the right-hand square in the diagram

$$\begin{array}{ccccc}
 K & \xrightarrow{s} & H & \xrightarrow{q} & B \\
 r \downarrow & \cong & p \downarrow & \Rightarrow & \downarrow g \\
 D & \xrightarrow{h} & A & \xrightarrow{f} & C
 \end{array}$$

exhibits H as a comma object of f, g . The whole pasted diagram exhibits K as a comma object of f, h, g if and only if the left-hand square is a pseudopullback of h, p . \square

1.8. For categories \mathbb{C}, \mathbb{D} we have the functor category $\mathbb{D}^{\mathbb{C}}$. This leads to the *cotensor product* $A^{\mathbb{C}}$ of the object A of \mathcal{K} with a category \mathbb{C} , which comes equipped with a functor $\mathbb{C} \rightarrow \mathcal{K}(A^{\mathbb{C}}, A)$, inducing an equivalence of categories

$$\mathcal{K}(X, A^{\mathbb{C}}) \xrightarrow{\sim} \mathcal{K}(X, A)^{\mathbb{C}}.$$

(Of course, the dual of cotensor is *tensor*, not cocotensor!)

1.9. The *pseudoequaliser* of two functors $f, g: A \rightarrow B$ is the category C whose objects are pairs $(a, \beta: fa \xrightarrow{\sim} ga)$ where a is an object of A and β is an invertible arrow of B , and whose arrows $\alpha: (a, \beta) \rightarrow (a', \beta')$ are arrows $\alpha: a \rightarrow a'$ of A such that $\beta' \circ f(\alpha) = g(\alpha) \circ \beta$. Also, we have the forgetful functor $u: C \rightarrow A$ and a canonical invertible 2-cell $fu \Rightarrow gu$. This leads to the corresponding limit in \mathcal{K} : a *pseudoequaliser of a pair of arrows* $f, g: A \rightarrow B$ is an arrow $u: C \rightarrow A$ together with an invertible 2-cell $fu \Rightarrow gu$ which induces an equivalence between $\mathcal{K}(X, C)$ and the pseudoequaliser of the functors $\mathcal{K}(X, f), \mathcal{K}(X, g)$ for all X .

1.10. The *inverter* of a natural transformation $\sigma: f \Rightarrow g: A \rightarrow B$ is the full subcategory of A consisting of the objects a for which the component $\sigma_a: fa \rightarrow ga$ is invertible. An *inverter* for a 2-cell $\sigma: f \Rightarrow g: A \rightarrow B$ in \mathcal{K} is thus an arrow $u: C \rightarrow A$ which induces an equivalence between the category $\mathcal{K}(X, C)$ and the inverter of the natural transformation $\mathcal{K}(X, \sigma)$.

1.11. The *invertee* of a functor $f: A \rightarrow B$ is the full subcategory C of the category A^2 of arrows of A consisting of those arrows inverted by f . There is a canonical 2-cell $\lambda: u \Rightarrow v: C \rightarrow A$ with $f\lambda$ invertible. The reader should now be able to define an *invertee* for an arrow in \mathcal{K} .

1.12. The *equifier* (resp. *equinverter*) of two natural transformations $\sigma, \tau: f \Rightarrow g: A \rightarrow B$ is the full subcategory of A consisting of the objects a for which $\sigma_a = \tau_a$ (resp. $\sigma_a = \tau_a$ and σ_a is invertible). The bicategorical limit should be clear.

1.13. A bicategory \mathcal{K} is said to be *finitely complete* when it admits all limits weighted by homomorphisms $J: \mathcal{A} \rightarrow \mathbf{Cat}$ such that \mathcal{A} is a finite bicategory and each $J(A)$ is a finitely presentable category.

1.14. Theorem (Street [22, 27]). *Every bicategory which admits a terminal object, pseudopullbacks, and cotensor products with the arrow category $\mathbb{2}$ is finitely complete.* \square

1.15. A *pseudopushout* of a pair of arrows $j: X \rightarrow Y, f: X \rightarrow A$ in \mathcal{K} is of course their pseudopullback in \mathcal{K}^{op} . As a special example, we shall describe the pseudopushout

P of j, f in \mathbf{Cat} in the special case where X is a full subcategory of Y , closed under isomorphisms (replete), and j is the inclusion. First the equality $\text{obj}(P) = \text{obj}(A) + (\text{obj}(Y) - \text{obj}(X))$ determines the objects of P . The category A is a full subcategory of P , let the inclusion be called $g: A \rightarrow P$. The other homsets of P are given by coends

$$P(a, y) = \int^x A(a, fx) \times Y(jx, y),$$

$$P(y, a) = \int^x Y(y, jx) \times A(fx, a),$$

$$P(y, y') = \int^{x, x'} Y(y, jx) \times A(fx, fx') \times Y(jx', y'),$$

where $a \in A, y, y' \in Y, y, y' \notin X$. Composition in P is given in the obvious way using the compositions of A and Y . There is a functor $h: Y \rightarrow P$ given by $hy = y$ for $y \notin X$, and $hx = fx$ for $x \in X$. So we have an actual equality $hj = gf$, not just an isomorphism. In fact (P, g, h) provide a pushout for j, f as well as a pseudopushout. This extends Proposition 6.17 of [22].

2. Conservative and liberal arrows

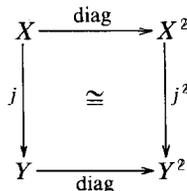
2.1. An arrow $j: X \rightarrow Y$ in a bicategory \mathcal{K} is called *conservative* when, for all 2-cells $\sigma: u \Rightarrow v: K \rightarrow X$, if $j\sigma$ is invertible then so is σ itself. That is, when, for all objects K , the functor

$$\mathcal{K}(K, X) \xrightarrow{\mathcal{K}(K, j)} \mathcal{K}(K, Y)$$

is conservative (= reflects isomorphisms). This condition holds precisely when the identity 2-cell of the identity arrow of X is an invertible (1.11) for j . Clearly, j is conservative in \mathcal{K} iff it is conservative in \mathcal{K}^{co} .

2.2. An arrow in \mathcal{K} is called *liberal* when it is conservative in \mathcal{K}^{op} .

2.3. Suppose \mathcal{K} admits cotensoring with the arrow category $\mathbf{2}$. Then $j: X \rightarrow Y$ is conservative iff the following square is a pseudopullback.

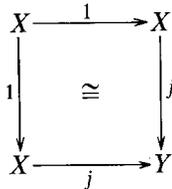


2.4. Any arrow isomorphic to a conservative arrow is conservative. Any composite of conservative arrows is conservative. If kj is conservative then so is j . Conservative arrows are stable under pseudopullback.

2.5. Suppose $J: \mathcal{A} \rightarrow \mathbf{Cat}$, $S, T: \mathcal{A} \rightarrow \mathcal{K}$ are homomorphisms of bicategories and suppose \mathcal{K} admits the J -weighted limits S^J, T^J of S, T . If each component of a strong transformation $q: S \rightarrow T$ is conservative then so is the induced arrow $q^J: S^J \rightarrow T^J$.

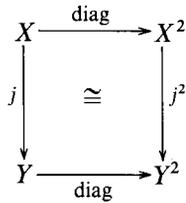
2.6. An arrow $j: X \rightarrow Y$ is *faithful* when, for all 2-cells $\sigma, \tau: u \Rightarrow v: K \rightarrow X$, if $j\sigma = j\tau$ then $\sigma = \tau$. That is, when, for all objects K , the functor $\mathcal{K}(K, j)$ is faithful.

2.7. An arrow $j: X \rightarrow Y$ is *pseudomononic* when it is faithful and, for all pairs of arrows $a, b: K \rightarrow X$ and all invertible 2-cells $\zeta: ja \Rightarrow jb$, there exists a 2-cell $\eta: a \Rightarrow b$ such that $j\eta = \zeta$. (It follows that η is uniquely determined and invertible.) Note that j is pseudomononic iff the following square is a pseudopullback.



Each pseudomononic is conservative. An arrow is *pseudoepic* in \mathcal{K} when it is pseudomononic in \mathcal{K}^{op} .

2.8. Suppose \mathcal{K} admits cotensoring with the discrete category $2 \stackrel{\text{def}}{=} \mathbb{1} + \mathbb{1}$. Then $j: X \rightarrow Y$ is pseudomononic iff the following square is a pseudopullback.



2.9. All inverters and equinverters are pseudomononic.

2.10. A functor $f: A \rightarrow B$ is an equivalence iff it is pseudomononic and the functor $f^2: A^2 \rightarrow B^2$ is surjective up to isomorphism on objects.

2.11. An arrow $e: A \rightarrow B$ in a bicategory \mathcal{K} is called *strong liberal* (abbreviated to *slib*) when, for all conservative arrows $j: X \rightarrow Y$, the following square is a pseudopullback of categories.

$$\begin{array}{ccc}
 \kappa(B, X) & \xrightarrow{\kappa(e, X)} & \kappa(A, X) \\
 \kappa(B, j) \downarrow & \cong & \downarrow \kappa(A, j) \\
 \kappa(B, Y) & \xrightarrow{\kappa(e, Y)} & \kappa(A, Y)
 \end{array}$$

An arrow in \mathcal{K} is *strong conservative* (abbreviated to *scon*) when it is slib in \mathcal{K}^{op} .

2.12. Strong liberals are closed under composition. If eh is strong liberal and h is either pseudoepic or strong liberal then e is strong liberal. An arrow which is both strong liberal and conservative is an equivalence.

2.13. Every slib $e: A \rightarrow B$ satisfies the following condition: *each square*

$$\begin{array}{ccc}
 A & \xrightarrow{e} & B \\
 u \downarrow & \cong & \downarrow v \\
 X & \xrightarrow{j} & Y
 \end{array}$$

in which j is conservative, factorises uniquely up to isomorphism as

$$\begin{array}{ccc}
 A & \xrightarrow{e} & D \\
 u \downarrow & \cong & \downarrow v \\
 X & \xrightarrow{j} & Y \\
 & \swarrow w & \nearrow \\
 & & D
 \end{array}$$

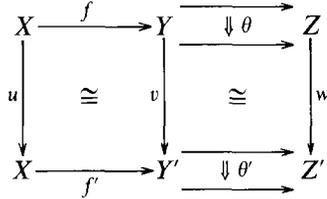
When \mathcal{K} admits cotensoring with the arrow category $\mathbf{2}$, this condition implies e is slib. (Apply the condition to both j and j^2 to get the pseudopullback of 2.11.)

2.14. If \mathcal{K} admits all inverters then all strong liberals are liberal. For, suppose $e: A \rightarrow B$ is slib, and suppose θe invertible. Let j be an inverter for θ ; so $e \cong ju$ for some u . By 2.12, since j is conservative, there exists w such that $u \cong we$ and $juw \cong 1$. The latter gives θ invertible (since θjw is). So e is liberal. \square

2.15. If \mathcal{K} admits cotensoring with the discrete category $\mathbf{2}$ then strong liberal arrows are pseudoepic. For all objects K , the diagonal $K \rightarrow K^2$ is conservative. Substituting this for j in the pseudopullback of 2.11, we see from 2.8 that $\mathcal{K}(e, K)$ is pseudomonic. So e is pseudoepic.

2.16. Each left adjoint with invertible counit is the coinverter of the unit. Dually, each right adjoint with invertible unit is the coinverter of the counit. Our next result shows that this gives two interesting classes of strong liberals.

2.17. Each coinverter is strong liberal. This follows representably from the fact that, in the following diagram of ordinary categories



if f, f' are inverters of θ, θ' and w is conservative then the square u, f', v, f is a pseudopullback. \square

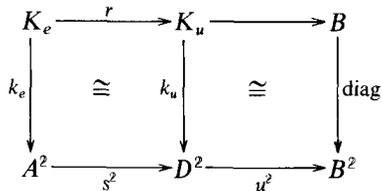
2.18. Definition. A bicategory \mathcal{K} will be called *conservational* when it satisfies the following conditions:

- (i) it is finitely complete;
- (ii) each pseudopullback of a strong liberal is strong liberal;
- (iii) cotensoring $()^2$ with the arrow category $\mathcal{2}$ preserves strong liberal arrows;
- (iv) for each arrow f , there exists a conservative arrow j , a strong liberal arrow e , and an invertible 2-cell $f \Rightarrow je$.

2.19. Theorem. In a conservational bicategory, every strong liberal arrow is a coinverter.

Proof. For any arrow $e: A \rightarrow B$, the invertee $\kappa_e: K_e \Rightarrow A$ of e corresponds to the pseudopullback $k_e: K_e \rightarrow A^2$ of the diagonal $B \rightarrow B^2$ along $e^2: A^2 \rightarrow B^2$. We shall use this notation below.

Assume e is slib. We shall show that e is the coinverter of κ_e . In view of 2.15, 2.10 and the existence of $()^2$, it suffices to show that each $g: A \rightarrow C$, with $g\kappa_e$ invertible, factors up to isomorphism through e . For this apply 2.18(iv) to the arrow $(e, g): A \rightarrow B \times C$ to obtain a conservative $(u, v): D \rightarrow B \times C$ and a slib $s: A \rightarrow D$ with $us \cong e, vs \cong g$. Since s, e are slib, so too is u (2.12). We have pseudopullbacks:



Now $(u, v)\kappa_u r \cong (u, v)s\kappa_e \cong (e, g)\kappa_e \cong (e\kappa_e, g\kappa_e)$ is invertible. Since (u, v) is conservative, $\kappa_u r$ is invertible. By 2.18(ii), (iii) and 2.14, r is liberal; so κ_u is invertible. So u is conservative. So u is an equivalence 2.10. So $g \cong vs \cong vwe$, where w is any inverse equivalence for u . \square

2.20. Remark. Perhaps the reader will have noticed that in the above proof we needed only the weaker form of 2.18(ii): a pseudopullback of a strong liberal is liberal.

2.21. In a conservational bicategory, if $e:A \rightarrow B$, $e':A' \rightarrow B'$ are strong liberal, so is $e \times e':A \times A' \rightarrow B \times B'$. This follows from the fact that $e \times A'$, $B \times e'$ are pullbacks of e, e' along projections, and $e \times e'$ is isomorphic to their composite (2.12).

2.22. An object A of a bicategory \mathcal{K} is called *groupoidal* when every 2-cell $\sigma:u \Rightarrow v:X \rightarrow A$ is invertible. This is the same as saying that (the unique up to isomorphism) $A \rightarrow 1$ is conservative (provided \mathcal{K} has a terminal object 1). Write $G\mathcal{K}$ for the full subcategory of \mathcal{K} consisting of the groupoidal objects.

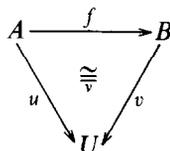
2.23. If \mathcal{K} satisfies 2.18 then the inclusion of $G\mathcal{K}$ in \mathcal{K} has a left biadjoint $\pi:\mathcal{K} \rightarrow G\mathcal{K}$ whose value at A is given by factoring $A \rightarrow 1$ as a strong liberal $A \rightarrow \pi A$ followed by a conservative $\pi A \rightarrow 1$.

2.24. Proposition. If \mathcal{K} is conservational then the homomorphism $\pi:\mathcal{K} \rightarrow G\mathcal{K}$ preserves products.

Proof. By 2.21, $A \times B \rightarrow \pi A \times \pi B$ is slib; and by 2.4 the composite of the pair $\pi A \times \pi B \rightarrow 1 \times 1 \xrightarrow{\sim} 1$ is conservative. So $\pi(A \times B) \xrightarrow{\sim} \pi A \times \pi B$. \square

2.25. Definition. A bicategory \mathcal{K} is called *faithfully conservational* when it is conservational and every conservative arrow is faithful (2.6). In this case, every groupoidal object A is *discrete* in the sense that each hom category $\mathcal{K}(X, A)$ is equivalent to a discrete category. We write $D\mathcal{K}$ instead of $G\mathcal{K}$.

2.26. There is a *slice bicategory* \mathcal{K}/U obtained from any bicategory \mathcal{K} and an object U thereof. The hom category $(\mathcal{K}/U)(u, v)$ is the pseudoequaliser (1.9) of the two functors $u!, v \circ -: \mathcal{K}(A, B) \rightarrow \mathcal{K}(A, U)$ which are constant at u and composition with v , respectively. The arrows $(f, v):u \rightarrow v$ of \mathcal{K}/U are pictured as triangles:



The composition of \mathcal{K}/U is obtained by pasting triangles in the obvious way.

2.27. Proposition. If \mathcal{K} is (faithfully) conservational then so is each slice \mathcal{K}/U . The forgetful homomorphism $\mathcal{K}/U \rightarrow \mathcal{K}$ preserves and reflects pseudopullbacks, conservatives and strong liberals.

Proof. The pseudopullback of $(f, v): u \rightarrow w$ and $(g, \mu): v \rightarrow w$ in \mathcal{K}/U is just obtained from the pseudopullback of f and g . A terminal object for \mathcal{K}/U is the identity $1_U: U \rightarrow U$. The cotensor product of the object $u \in \mathcal{K}/U$ with $\mathcal{2}$ is either of the isomorphic arrows $K_u \rightarrow U$ coming from the invertee K_u of u . So \mathcal{K}/U is finitely complete. An arrow $(f, v): u \rightarrow v$ with f conservative in \mathcal{K} is clearly conservative in \mathcal{K}/U . An arrow $(f, v): u \rightarrow v$ with f a coinverter in \mathcal{K} is clearly a coinverter in \mathcal{K}/U . This last result implies another similar one for strong liberals by 2.17 and Theorem 2.19.

Now given an arrow $(f, v): u \rightarrow v$, factorise f as $f \cong je$ with j conservative and e slib in \mathcal{K} , and then we obtain the desired factorisation $u \rightarrow vj \rightarrow v$ of (f, v) in \mathcal{K}/U . It follows that (f, v) is slib in \mathcal{K}/U iff f is in \mathcal{K} . It then follows that a pseudopullback of a slib in \mathcal{K}/U is slib. Since the arrow $K_u \rightarrow K_v$ induced on invertees by $e: A \rightarrow B$ can be obtained as the pseudopullback of e^2 along $K_v \rightarrow B^2$, we also have 2.18(iii) for \mathcal{K}/U . \square

2.28. Proposition. *Suppose M is a doctrine [22] (2.10) on the (faithfully) conservational bicategory \mathcal{K} such that the underlying endo-homomorphism of M preserves strong liberal arrows. Then the bicategory \mathcal{K}^M [22] (2.19) of algebras for M is (faithfully) conservational. The forgetful homomorphism $\mathcal{K}^M \rightarrow \mathcal{K}$ preserves and reflects finite limits and strong liberal arrows.*

Proof. Straightforward. \square

3. Examples and non-examples

3.1. Proposition. *An arrow $e: A \rightarrow B$ in \mathbf{Cat} is liberal iff each object $b \in B$ is a retract of an object of the form ea for some $a \in A$.*

Proof. Suppose $e: A \rightarrow B$ satisfies the condition. Let $\sigma: u \Rightarrow v: B \rightarrow C$ be a natural transformation for which σe is invertible. Then each component σ_{ea} is invertible. By naturality, each σ_b is a retract of a σ_{ea} , and so is invertible itself. Therefore σ is invertible and it follows that e is liberal.

Conversely, suppose $e: A \rightarrow B$ is liberal. Then its coinverter is invertible. Since the usual inclusion $\mathbf{Cat} \rightarrow \mathbf{Mod}$ preserves weighted colimits [3], the coinverter of e in \mathbf{Mod} is invertible. Hence e is “Cauchy dense” in \mathbf{Mod} in the sense of [23]. So the canonical function

$$\int^a B(b, ea) \times B(ea, b') \longrightarrow B(b, b')$$

is surjective [23, Proposition 1]. So there exists $b \rightarrow ea$, $ea \rightarrow b$ with composite the identity of b , as required. \square

3.2. Let \mathbf{Cat}_{cc} denote the full subcategory of \mathbf{Cat} consisting of those small categories in which idempotents split.

3.3. Remark. *Neither \mathbf{Cat} nor \mathbf{Cat}_{cc} is conservational.*

Proof. Let A be the free category generated by the graph

$$a \xrightarrow{\theta} b \xleftarrow{\alpha} c \xrightarrow{\phi} d.$$

Let X be the category generated by the graph

$$x \xrightarrow{\rho} y \xrightarrow{\sigma} x$$

subject to the relation $\sigma\rho = 1_x$. The functor $f: A \rightarrow X$, given by

$$f\theta = \rho, \quad f\alpha = 1_y, \quad f\phi = \sigma,$$

is in \mathbf{Cat}_{cc} . The functor $e: A \rightarrow \mathfrak{3}$, taking a, b, c, d to $0, 1, 1, 2$ is a coinverter for the inverttee of f . If \mathbf{Cat} or \mathbf{Cat}_{cc} were conservational, we would have, by Theorem 2.19, a factorisation $f \cong je$ with $j: \mathfrak{3} \rightarrow X$ conservative. But such a j must take $0, 1, 2$ to x, y, x and so invert the non-invertible $0 \rightarrow 2$ in $\mathfrak{3}$, contrary to j being conservative. \square

3.4. Proposition. *The bicategory \mathbf{Cat}_{cc}^{op} is (faithfully) conservational while \mathbf{Cat}^{op} fails only to satisfy 2.18(ii). The strong conservative arrows $j: X \rightarrow Y$ in \mathbf{Cat} are the fully faithful functors such that each retract of an object of the form jx with $x \in X$ is isomorphic to an object of the same form. The strong conservative arrows in \mathbf{Cat}_{cc} are the fully faithful functors.*

Proof. We begin by showing that each fully faithful $j: X \rightarrow Y$, “closed under retracts” (as in the Proposition), is scon in \mathbf{Cat} . Take a square in \mathbf{Cat} as in 2.13 with e liberal. In order to define w as in 2.13, take $b \in B$; this is a retract of some ea with $a \in A$, by Proposition 3.1. Then vb is a retract of $vea \cong jua$. By the condition on j , we have $vb \cong jwb$ for some $wb \in X$. This choice defines w on objects. The definition of w on arrows is forced since j is fully faithful. This clearly gives a factorisation of the square which is unique up to isomorphism. Since \mathbf{Cat} admits tensoring with $\mathfrak{2}$, we have shown (2.13) that each such j is scon.

If idempotents split in X , notice that any fully faithful functor $j: X \rightarrow Y$ is automatically retract closed (since there is an idempotent in X which maps to the idempotent generated by any retract of a jx). So $j: X \rightarrow Y$ in \mathbf{Cat}_{cc} is scon if it is fully faithful.

Both \mathbf{Cat} and \mathbf{Cat}_{cc} are finitely cocomplete bicategories. The former follows from the fact that the 2-category \mathbf{Cat} admits all weighted pseudocolimits [21], and these provide the bicategorical colimits required. The latter uses the idempotent (“Cauchy”) completion homomorphism $\mathcal{Q}: \mathbf{Cat} \rightarrow \mathbf{Cat}_{cc}$ to turn the constructions in \mathbf{Cat} into those for \mathbf{Cat}_{cc} .

Thus we have 2.18(i) for \mathbf{Cat}^{op} and $\mathbf{Cat}_{\text{cc}}^{\text{op}}$. To prove 2.18(iv), take any functor $f:A \rightarrow Y$. Let X be the full subcategory of Y consisting of all retracts of objects of the form $fa, a \in A$. Then we get $f = je$ where j is the inclusion of X in Y and e is liberal in \mathbf{Cat} . From the beginning of this proof we see that j is scon in \mathbf{Cat} and in \mathbf{Cat}_{cc} . So we have 2.18(iv) as required.

We can now see that scons in \mathbf{Cat} and \mathbf{Cat}_{cc} are of the type claimed in the Proposition. For, take any scon $f:A \rightarrow Y$ and factorise it $f = je$ as above. Since f, j are scon, so is e (by the dual of 2.12). Since e is also liberal, it is an equivalence. But j is a scon of the claimed type. This “type” clearly includes equivalences and is closed under composition. So $f = je$ is of that type.

Tensoring with \mathcal{Q} in \mathbf{Cat} is simply given by cartesian product $\mathcal{Q} \times -$. This is also true in \mathbf{Cat}_{cc} (since \mathbf{Cat}_{cc} is closed under exponentiation in \mathbf{Cat} which implies \mathcal{Q} preserves finite products). With our characterisation of scons, it is clear that, if $j:X \rightarrow Y$ is scon, so is $\mathcal{Q} \times j: \mathcal{Q} \times X \rightarrow \mathcal{Q} \times Y$. This proves 2.18(iii).

It remains to prove 2.18(ii). The “inclusion” $\mathbf{Cat} \rightarrow \mathbf{Mod}$ preserves pseudopushouts [3] and takes each $A \rightarrow \mathcal{Q}A$ to an equivalence. So any pseudopushout

$$\begin{array}{ccc}
 X & \xrightarrow{j} & Y \\
 f \downarrow & \cong & \downarrow h \\
 A & \xrightarrow{g} & P
 \end{array}$$

in \mathbf{Cat} or \mathbf{Cat}_{cc} remains so in \mathbf{Mod} . If j is fully faithful then the adjunction $j \dashv j^*$ has invertible unit $1_X \cong j^*j$. Hence there exists a module $m:P \rightarrow A$ with $1_A \cong mg$ and $mh \cong fj^*$. Using the universal property of the pseudopushout, we obtain a 2-cell $gm \Rightarrow 1_P$ and prove $m \cong g^*$. So $g \dashv g^*$ has invertible unit, in other words g is fully faithful. This completes the proof in the case of \mathbf{Cat}_{cc} .

To see that \mathbf{Cat} does not satisfy 2.18(ii), we use the construction of pseudopushout given in 1.15. Take Y to be the category generated by the graph $x \rightarrow y \rightarrow x'$, and let A be the category generated by the graph $a \rightarrow b \rightarrow a$ subject to the relation that the composite is the identity of a . Let X be the full subcategory of Y consisting of x, x' . Let $f:X \rightarrow A$ take $x \rightarrow x'$ to $a \rightarrow b$. Then the inclusion $j:X \rightarrow Y$ is scon in \mathbf{Cat} , while y is a retract of a in P not isomorphic to either object of A . So the inclusion $g:A \rightarrow P$ is not scon. \square

3.5. Notice that \mathcal{H} is conservational iff \mathcal{H}^{co} is.

3.6. **Proposition.** *The class of functors $e:A \rightarrow B$ in \mathbf{Cat} satisfying the three conditions:*

- (i) *each object of B is isomorphic to one of the form ea for some $a \in A$;*

- (ii) each $\beta \in B(ea, ea')$ has the form $\beta = (e\alpha)(e\sigma)^{-1}$ where α, σ are arrows of A and σ is inverted by e ;
- (iii) if $(e\alpha)(e\sigma)^{-1} = (e\alpha')(e\sigma')^{-1}$ then there exist arrows γ, γ' of A inverted by e such that $\sigma\gamma = \sigma'\gamma'$ and $\alpha\gamma = \alpha'\gamma'$;

is contained in the class of strong liberal arrows in **Cat** and is closed under pseudopull-back.

Proof. Consider a square in **Cat** as in 2.13 where e is in the class described in the proposition and j is conservative. We need to define a diagonal w for the square. For each $b \in B$, choose $a \in A$ and an isomorphism $b \cong ea$ (which is possible by (i)). Define w on objects by $wb = ua$. Each arrow $\beta: b \rightarrow b'$ in B is isomorphic by the chosen isomorphisms to an arrow $ea \rightarrow ea'$ which, by (ii), has the form $(e\alpha)(e\sigma)^{-1}$. Notice that $e\sigma$ invertible implies $ve\sigma$, and hence $ju\sigma$ invertible; so $u\sigma$ is invertible (using j conservative). So we can define $w\beta = (u\alpha)(u\sigma)^{-1}$ which is independent of choice of α, σ by (iii). This gives w and $u \cong we, jw \cong v$ as desired in 2.13. So e is slib.

The stability under pseudopullbacks is straightforward, and is a worthwhile exercise for the reader. \square

3.7. Let **Lex** (resp. **Rex**) denote the subcategory of **Cat** consisting of the categories which admit finite limits (resp. finite colimits), the functors which preserve these, and all the natural transformations between such functors.

3.8. Proposition. *The bicategories **Lex** and **Rex** are both conservational.*

Proof. Notice that taking duals gives an isomorphism $\mathbf{Rex}^{\text{co}} \cong \mathbf{Lex}$. So it suffices by 3.5 to prove that **Lex** is conservational. Limits in this bicategory are formed as in **Cat** (one reason is that the inclusion $\mathbf{Lex} \rightarrow \mathbf{Cat}$ has a left biadjoint [24]). Also, the conservative arrows are precisely the conservative functors.

If $f: A \rightarrow Y$ is an arrow of **Lex** then the arrows of A in the invertee of f form a calculus of right fractions [7]. By looking at the construction of the category of fractions for a calculus of fractions [7], we see that the coinverter of this invertee is a functor e in the class specified in Proposition 3.6, the category X has finite limits, the functor $j: X \rightarrow Y$ satisfying $f = je$ is conservative, and e, j preserve finite limits.

In the diagrams of 2.13 in the case of **Cat**, notice that, since j is conservative, the functor w will preserve whatever limits the categories have and $v(\cong jw)$ preserves. It follows from Proposition 3.6 that $e: A \rightarrow X$ in **Lex** satisfying the three conditions are slib in **Lex** and are closed under pseudopullback. Because of the factorisation of the last paragraph, the arrows in the class of Proposition 3.6 which are in **Lex** are precisely the slib arrows in **Lex**. Thus we have proved 2.18(i), (ii) and (iv) for **Lex**.

In fact, any arrow $e: A \rightarrow B$ in **Lex** satisfies condition (iii) of Proposition 3.6 automatically. Just take the pullback τ, τ' of σ, σ' , then take the equaliser ρ of $\alpha\sigma, \alpha\sigma'$, and finally put $\gamma = \tau\rho, \gamma' = \tau'\rho$.

It remains to show that, if $e: A \rightarrow B$ in **Lex** satisfies conditions 3.6(i) and (ii), then so does $e^2: A^2 \rightarrow B^2$. For (i), take an object $\beta: b \rightarrow b'$ in B^2 . By (i) for e we have $ea \cong b \rightarrow b' \cong ea'$, which by (ii) for e , is of the form $(e\alpha)(e\sigma)^{-1}$; so β is isomorphic to $e\alpha$ as objects of B^2 . To prove (ii) for e^2 , take a commutative square in B of the form

$$\begin{array}{ccc} ea & \xrightarrow{\beta} & ex \\ e\xi \downarrow & & \downarrow e\eta \\ ea' & \xrightarrow{\beta'} & ex' \end{array}$$

Apply (ii) for e to obtain $\beta' = (e\alpha')(e\sigma')^{-1}$. Let σ_1, ξ_1 form a pullback for σ', ξ and let α_1, η_1 form a pullback for α', η . Then $e\alpha \cdot e\xi_1 = e\eta \cdot \beta \cdot e\sigma_1$, so there exists a unique β_1 with $e\eta_1 \cdot \beta_1 = e\xi_1 \cdot \beta$, $e\alpha_1 \cdot \beta_1 = \beta \cdot e\sigma_1$. Apply (ii) for e to obtain $\beta_1 = (e\alpha)(e\sigma)^{-1}$. Apply (iii) for e to obtain ρ, ρ' inverted by e and such that $\sigma\rho = 1 \cdot \rho', \eta_1\alpha\rho = \xi_1\rho'$. Then we have arrows $(\sigma_1\rho', \sigma'): \eta_1\alpha\rho \rightarrow \xi, (\alpha_1\alpha\rho, \alpha'): \eta_1\alpha\rho \rightarrow \eta$ in B^2 , the former inverted by e^2 , with

$$(\beta, \beta') = e^2(\alpha_1\alpha\rho, \alpha') \cdot [e^2(\sigma_1\rho', \sigma')]^{-1}$$

as desired. \square

4. Discrete fibrations

We begin by reviewing a few concepts from [22] and [24].

4.1. Consider a span $(p, E, q): B \rightarrow A$ (sometimes abbreviated to $E: B \rightarrow A$) from B to A in the bicategory \mathcal{X} ; that is, an ordered pair of arrows in \mathcal{X} displayed diagrammatically as follows:

$$\begin{array}{ccc} & E & \\ p \swarrow & & \searrow q \\ A & & B \end{array}$$

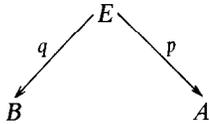
A 2-cell $\chi: e' \Rightarrow e: K \rightarrow E$ is said to be *left cartesian* (with respect to our span) when:

- (i) $q\chi$ is invertible;
- (ii) for each triple (g, ξ, α) consisting of an arrow $g: L \rightarrow K$ and 2-cells $\xi: e'' \Rightarrow eg: L \rightarrow E, \alpha: pe'' \Rightarrow pe'g$ such that $p\xi = p\chi g \cdot \alpha$, there exists a unique 2-cell $\xi': e'' \Rightarrow e'g$ with $\xi = \chi g \cdot \xi'$ and $p\xi' = \alpha$.

Notice that if χ is left cartesian for our span, then so is χg for any arrow $g: L \rightarrow K$.

4.2. Notice that spans in \mathcal{X} and \mathcal{X}^{co} coincide. By convention, on re-interpreting a span $(p, E, q): B \rightarrow A$ of \mathcal{X} in \mathcal{X}^{co} , we will always reverse its orientation. In other

words, in \mathcal{K}^{co} our canonical span becomes $(q, E, p): A \rightarrow B$ and we redraw it as



We say that a 2-cell in \mathcal{K} is *right cartesian* with respect to (p, E, q) precisely when it is left cartesian in \mathcal{K}^{co} with respect to the corresponding span (q, E, p) .

Later on we will see that this convention is related to a close link between weak duality in bicategories of functors and strong duality in bicategories of profunctors.

4.3. An arrow of spans $(\kappa, f, \nu): (p, E, q) \rightarrow (p', E', q')$ from B to A consists of an arrow $f: E \rightarrow E'$ and isomorphisms $\kappa: p'f \cong p, \nu: q \cong q'f$. A 2-cell $\theta: (\kappa, f, \nu) \Rightarrow (\lambda, g, \mu)$ between such arrows is a 2-cell $\theta: f \Rightarrow g$ such that $\kappa = \lambda \cdot p'\theta$ and $q'\theta \cdot \nu = \mu$. Let $\text{Spn}(K)(B, A)$ denote the bicategory of spans so obtained. The convention of 4.2 identifies $\text{Spn}((\mathcal{K}^{\text{co}}))(A, B)$ with $\text{Spn}(\mathcal{K})(B, A)$.

4.4. When \mathcal{K} has products, we can identify each span (p, E, q) with an object $(p, q): E \rightarrow A \times B$ of the slice bicategory $\mathcal{K}/A \times B$ (2.26), thus we may identify $\text{Spn}(\mathcal{K})(B, A)$ with $\mathcal{K}/A \times B$.

4.5. A span $(p, E, q): B \rightarrow A$ in \mathcal{K} is called a *left fibration* when the following condition holds:

(*left path lifting*) for all arrows $e: K \rightarrow E$ and 2-cells $\alpha: a \Rightarrow pe: K \rightarrow A$, there exists a left cartesian $\chi: e * \alpha \Rightarrow e$ for which there is an isomorphism $a \cong p(e * \alpha)$ whose composite with $p\chi$ is α .

Dually $(p, E, q) \in \mathcal{K}$ is a *right fibration* if it is a left fibration in \mathcal{K}^{co} , or more explicitly when it satisfies:

(*right path lifting*) for all arrows $e: K \rightarrow E$ and 2-cells $\beta: qe \Rightarrow b: K \rightarrow B$, there exists a right cartesian $\zeta: e \Rightarrow \beta * e$ for which there is an isomorphism $q(\beta * e) \cong b$ whose composite with $q\zeta$ is β .

4.6. A span $(p, E, q): B \rightarrow A$ is called a *discrete fibration* from B to A if it is both a left and right fibration and discrete (Definition 2.25) as an object of the bicategory $\text{Spn}(\mathcal{K})(B, A)$. The discreteness of (p, E, q) in $\text{Spn}(\mathcal{K})(B, A)$ may be more explicitly expressed as follows:

(*discreteness*) for all $\xi, \eta: e \Rightarrow e': K \rightarrow E$, if $p\xi = p\eta$ with $p\xi, q\xi$ invertible then $\xi = \eta$ with ξ invertible.

4.7. Proposition. *A span (p, E, q) is a discrete fibration iff the following three conditions hold:*

(unique left path lifting) *for all arrows $e: K \rightarrow E$ and 2-cells $\alpha: a \Rightarrow pe$, the category, whose objects are pairs $(\chi: e' \Rightarrow e, \nu: a \cong pe')$ with $p\chi \cdot \nu = \alpha$ and $q\chi$ invertible, and whose arrows are the obvious ones, is essentially discrete (i.e. an equivalence relation) and nonempty;*

(unique right path lifting) *for all arrows $e: K \rightarrow E$ and 2-cells $\beta: qe \Rightarrow b$, the category, whose objects are pairs $(\zeta: e \Rightarrow e', \mu: qe' \cong b)$ with $\mu \cdot q\zeta = \beta$ and $p\zeta$ invertible, and whose arrows are the obvious ones, is essentially discrete and nonempty;*

(factorisation) *each 2-cell $\xi: e' \Rightarrow e'': K \rightarrow E$ is a composite $\xi = \chi \cdot \zeta$ where $q\chi$ and $p\zeta$ are invertible.*

Proof. Straightforward. \square

4.8. If (p', E', q') satisfies the discreteness condition of 4.6 then there is at most one 2-cell $\theta: (\kappa, f, \nu) \Rightarrow (\lambda, g, \mu): (p, E, q) \rightarrow (p', E', q')$; and, if there is one, it is invertible. Any arrow of spans between discrete fibrations preserves left and right cartesian 2-cells.

4.9. Write $\text{DFib}(\mathcal{K})(B, A)$ for the category whose objects are the discrete fibrations (p, E, q) from B to A and whose arrows are the isomorphism classes $[\kappa, f, \nu]$ of arrows (κ, f, ν) of spans between such.

4.10. Proposition [20, 22]. *Any comma object of a pair $f: A \rightarrow C, g: B \rightarrow C$ of arrows in any bicategory gives a discrete fibration $(d_0, f \downarrow g, d_1): B \rightarrow A$. In **Cat**, every discrete fibration arises this way.* \square

4.11. Suppose that \mathcal{K} has whatever finite limits are needed. For any span (p, E, q) , the arrow $d_1: A \downarrow p \rightarrow E$ has a right adjoint $i: E \rightarrow A \downarrow p$ (or i_p when confusion is likely), with isomorphisms $\varepsilon: d_1 i \cong 1_E$, this being the counit, and $\tau: d_0 i \cong p$. All this follows from the universal property of the comma object $A \downarrow p$.

The objects $E, A \downarrow p$ have canonical interpretations as objects of \mathcal{K}/A , namely $p: E \rightarrow A$ and $d_0: A \downarrow p \rightarrow A$, which we shall assume unless otherwise stated. Then the pair (i, τ) becomes an arrow from E to $A \downarrow p$ in \mathcal{K}/A , and again if we talk about the arrow i in \mathcal{K}/A we will assume that it is accompanied by the isomorphism τ .

Notice that the canonical (strict) homomorphism $\mathcal{K}/A \rightarrow \mathcal{K}$ takes an adjunction $\varepsilon, \eta: (f, \mu) \dashv (u, \nu)$ in \mathcal{K}/A to an adjunction $\varepsilon, \eta: f \dashv u$ in \mathcal{K} . Conversely if we are given arrows (f, μ) and (u, ν) and a 2-cell $\varepsilon: (f, \mu)(u, \nu) \Rightarrow 1$ in \mathcal{K}/A then $(f, \mu) \dashv (u, \nu)$ with counit ε iff $f \dashv u$ with counit ε in \mathcal{K} .

4.12. Proposition. *A span (p, E, q) is a left fibration iff the arrow $(i, \tau): E \rightarrow A \downarrow p$ has a right adjoint $(r, \kappa): A \downarrow p \rightarrow E$ in \mathcal{K}/A with counit $\varepsilon: ir \Rightarrow 1_{A \downarrow p}$ such that $qd_1\varepsilon$ is invertible.*

Proof. (\Leftarrow) Suppose we have an adjoint $r: A \downarrow p \rightarrow E$ as in the proposition. By applying d_1 to the counit $\varepsilon: ir \Rightarrow 1_{A \downarrow p}$ and using $1_E \cong d_1i$, we obtain a 2-cell $\chi: r \Rightarrow d_1$. It can be seen, using the adjunction $i \dashv r$ and the fact that $qd_1\varepsilon$ is invertible, that χ is a left cartesian arrow. Furthermore, the rule for ε as a 2-cell of \mathcal{K}/A ensures that the isomorphism $\kappa: d_0 \cong pr$ composed with $p\chi$ is equal to $\lambda: d_0 \Rightarrow pd_1$, the canonical 2-cell of the comma object $A \downarrow p$. Now consider any 2-cell $\alpha: a \Rightarrow pe: K \rightarrow A$, which, by the comma object property, is isomorphic to $\lambda c: d_0c \Rightarrow pd_1c$ with $d_1c \cong e$ for some arrow $c: K \rightarrow A \downarrow p$. But we have already shown that χ is a left path lift of λ , therefore it follows that the composite of $\chi c: rc \Rightarrow d_1c$ and $d_1c \cong e$ is a left path lift of α .

(\Rightarrow) Conversely suppose the left path lifting property holds. We have the canonical 2-cell $\lambda: d_0 \Rightarrow pd_1: A \downarrow p \rightarrow A$ to which we apply left path lifting to get a left cartesian $\chi: r \Rightarrow d_1: A \downarrow p \rightarrow E$ and an isomorphism $\kappa: pr \cong d_0$ (which makes r into an arrow from E to $A \downarrow p$ in \mathcal{K}/A). The universal property of the comma object $A \downarrow p$ implies that for fixed arrows $e: K \rightarrow E$, $e': K \rightarrow A \downarrow p$, 2-cells $\gamma: ie \Rightarrow e'$ correspond to pairs $\hat{\gamma}_0: pe \Rightarrow d_0e'$, $\hat{\gamma}_1: pe \Rightarrow pd_1e'$ such that $\lambda e' \cdot \gamma_0 = \gamma_1$. These in turn, using the left cartesian-ness of χ and the isomorphism $\kappa: pr \cong d_0$, correspond to 2-cells $\hat{\gamma}: e \Rightarrow re'$.

Now define a 2-cell $\varepsilon: ir \Rightarrow 1_{A \downarrow p}$ in \mathcal{K}/A , using the universal property of $A \downarrow p$, to be the unique ε such that $d_0\varepsilon$, $d_1\varepsilon$ are equal to the composites $d_0ir \cong pr \cong d_0$ and $d_1ir \cong r \Rightarrow d_1$ respectively. In particular this definition ensures that $qd_1\varepsilon$ is invertible since χ is left cartesian. It is now straightforward to show that if the 2-cell $\hat{\gamma}: e \Rightarrow re'$ corresponds to $\gamma: ie \Rightarrow e'$ under the bijection of the last paragraph then it is the unique such 2-cell satisfying $\gamma = \varepsilon \cdot i\hat{\gamma}$. In other words $i \dashv r$ with counit ε in \mathcal{K} , and ε is a suitable 2-cell in \mathcal{K}/A , therefore, by 4.11, $(i, \tau) \dashv (r, \kappa)$ in \mathcal{K}/A with counit ε such that $qd_1\varepsilon$ is invertible as required. \square

4.13. If (p, E, q) is a left fibration then the unit η of the adjunction $i_p \dashv r$ is an isomorphism. We show this by first noting that one of the triangle identities for $i_p \dashv r$ implies that $\chi i_p \cdot \eta$ is an isomorphism. Now $1_p = \lambda i_p = p\chi i_p: p \Rightarrow p$ so χi_p is a cartesian lift of 1_p , as is any isomorphic 2-cell with domain 1_E , like for instance $\chi i_p \cdot \eta$. But η is the unique map factoring one of these lifts through the other, and so, by the usual essential uniqueness argument, it is an isomorphism.

It follows, by 2.16, that for any left fibration $(p, E, q): B \rightarrow A$ the right adjoint $r: A \downarrow p \rightarrow E$ to i_p is a strong liberal.

4.14. Notice that, if we know that (p, E, q) is discrete, as in 4.6, then it satisfies the property given in Proposition 4.12 iff $i \dashv r$ in \mathcal{K} (with counit ε such that $qd_1\varepsilon$ is invertible) and there exists some isomorphism $pr \cong d_0$.

4.15. Clearly, if the span $(p, E, q): B \rightarrow A$ is a left fibration then so is the derived span $(p, E, qq): C \rightarrow A$ for all arrows $g: B \rightarrow C$. However, discreteness does not carry over under this process.

4.16. For all arrows $b: K \rightarrow B$, if the span $(p, E, q): B \rightarrow A$ is a left fibration then so is $(p \text{ pr}_1, E_q \times_b K, \text{pr}_2)$.

Proof. Using the isomorphism $qr \cong qd_1$, obtained from $d_1i \cong 1_E$ and $qd_1\varepsilon: qd_1ir \Rightarrow qd_1$, we may make the adjunction $(i, \tau) \dashv (r, \kappa)$ of Proposition 4.12 into an adjunction in $\text{Spn}(\mathcal{X})(B, A)$, between spans (p, E, q) and $(d_0, A \downarrow p, qd_1)$. Pseudopull-back along $b: K \rightarrow B$ determines a homomorphism of bicategories

$$\text{Spn}(\mathcal{X})(B, A) \xrightarrow{b^*} \text{Spn}(\mathcal{X})(K, A),$$

and $b^*(p, E, q) \simeq (p \text{ pr}_1, E_q \times_b K, \text{pr}_2)$. All that remains is to note that b^* takes our adjunction in $\text{Spn}(\mathcal{X})(B, A)$ to the one in $\text{Spn}(\mathcal{X})(K, A)$ (and therefore in \mathcal{X}/A) which is needed to demonstrate that $(p \text{ pr}_1, E_q \times_b K, \text{pr}_2)$ is a left fibration (cf. Proposition 4.12). \square

4.17. Write $\pi_{A,B}$ for the replacement of the homomorphism π when the bicategory \mathcal{X} is replaced by $\text{Spn}(\mathcal{X})(B, A)$ (see 2.23, Definition 2.25 and 4.4).

4.18. Proposition. In a conservational bicategory, if $(p, E, q): B \rightarrow A$ is a left fibration then so is $\pi_{A,B}(p, E, q): B \rightarrow A$.

Proof. $\pi_{A,B}(p, E, q) = (u, D, v)$ is obtained from (p, E, q) by factoring the arrow $(p, q): E \rightarrow A \times B$ up to isomorphism as a slib $e: E \rightarrow D$ composed with a conservative $(u, v): D \rightarrow A \times B$. The squares in the following diagram are pseudopullbacks.

$$\begin{array}{ccccc} A \downarrow p & \xrightarrow{e'} & A \downarrow u & \xrightarrow{\quad} & A^2 \times B \\ \downarrow d_1 & \cong & \downarrow d_1 & \cong & \downarrow d_1 \times B \\ E & \xrightarrow{e} & D & \xrightarrow{(u, v)} & A \times B \end{array}$$

Since \mathcal{X} is conservational, e' is slib. Replacing the left, right-hand downward pointing arrows in the above diagram by r (which exists since (p, E, q) is a left fibration), $d_0 \times B$ respectively, and applying 2.13, we obtain a new vertical middle arrow $s: A \downarrow u \rightarrow D$ making the two new squares commute up to isomorphism. This arrow s is our candidate for the required right adjoint to $i_u: D \rightarrow A \downarrow u$.

To produce a counit $\zeta: i_us \Rightarrow 1_{A \downarrow u}$ we shall use the fact that composing on the right with e' is fully faithful (this is because e' is a coinverter by Theorem 2.19, so the functor

$\mathcal{K}(e', -)$ is an inverter, and hence fully faithful 1.10). So we define ζ by the requirement that $\zeta e' : i_u s e' \Rightarrow e'$ is the composite of $i_u s e' \cong i_u e r \cong e' i_p r$ and $e' e : e' i_p r \Rightarrow e'$. Checking that this is indeed a counit for an adjunction $i_u \dashv s$ is straightforward, then the remainder of the conditions on ζ and s , as required by 4.14, follow easily from those for $i_p \dashv r$. \square

4.19. Proposition. *In a conservational bicategory, for all spans $(p, E, q) : B \rightarrow A$ and $(u, F, v) : C \rightarrow B$, the canonical strong liberal $F \rightarrow \pi_{B,C} F$ induces an equivalence $\pi_{A,C}(E \times_B F) \xrightarrow{\sim} \pi_{A,C}(E \times_B \pi_{B,C}(F))$.*

Proof. The two squares in the following diagram are pseudopullbacks.

$$\begin{array}{ccccc}
 E \times_B F & \longrightarrow & E \times_B \pi_{B,C}(F) & \longrightarrow & E \times C \\
 \downarrow & & \downarrow & & \downarrow (p, q) \times 1_C \\
 A \times F & \longrightarrow & A \times \pi_{B,C}(F) & \longrightarrow & A \times B \times C
 \end{array}$$

$\cong \qquad \qquad \qquad \cong$

The bottom composite is the factorisation of $1_A \times (u, v)$ into a slib and cons. Since \mathcal{K} is conservational, the top left arrow is slib. In order to obtain $\pi_{A,C}(E \times_B \pi_{B,C}(F))$, we factor $E \times_B \pi_{B,C}(F) \rightarrow E \times C \rightarrow A \times C$ into a slib and a cons. Composing this slib with the slib $E \times_B F \rightarrow E \times_B \pi_{B,C}(F)$, we obtain a factorisation of $E \times_B F \rightarrow A \times C$ into a slib and a cons. The result follows. \square

4.20. Theorem. *Suppose \mathcal{K} is a faithfully conservational bicategory. There is a bicategory*

$$\mathcal{F} = \text{DFib}(\mathcal{K})$$

whose objects are those of \mathcal{K} , and whose homcategories are the categories $\mathcal{F}(B, A) = \text{DFib}(\mathcal{K})(B, A)$ of 4.9. The identity arrow $A \rightarrow A$ of this bicategory is the discrete fibration (d_0, A^2, d_1) . The composition functor

$$\mathcal{F}(B, A) \times \mathcal{F}(C, B) \longrightarrow \mathcal{F}(C, A)$$

takes discrete fibrations $E : B \rightarrow A, F : C \rightarrow B$ to $E \circ F = \pi_{A,C}(E \times_B F) : C \rightarrow A$.

Proof. If $(p, E, q) : B \rightarrow A, (u, F, v) : C \rightarrow B$ are discrete fibrations then the span $(p \text{ pr}_1, E_q \times_u F, v \text{ pr}_2)$ is both a left and right fibration (4.15, 4.16). So $\pi_{A,C}(E \times_B F)$ is also a left and right fibration (Proposition 4.18); but it is also discrete (Definition 2.25 and 2.26). Therefore the proposed composite $E \circ F$ is a discrete fibration. By Proposition 4.19, we have canonical equivalences between the ternary composite

$E \circ F \circ G = \pi_{A,D}(E \times_B F \times_C G)$ and the bracketed iterated binary composites $(E \circ F) \circ G$ and $E \circ (F \circ G)$; these provide the canonical associativity isomorphisms $(E \circ F) \circ G \xrightarrow{\sim} E \circ (F \circ G)$ in $\mathcal{F}(B, A)$. Also, we have the equivalence $A^2 \times_A E \xrightarrow{\sim} A \downarrow p$ and the slib $r: A \downarrow p \rightarrow E$ (2.16 and Proposition 4.12); so we have a canonical isomorphism $A^2 \circ E \xrightarrow{\sim} E$ (and dually, $E \circ A^2 \xrightarrow{\sim} E$) in $\mathcal{F}(B, A)$. The coherence properties of these isomorphisms, as required for a bicategory, are easily verified because of the universal nature of the construction. \square

4.21. Remark. Recall the convention introduced in 4.2 concerning spans in \mathcal{K} and \mathcal{K}^{co} . We defined right fibrations in terms of left fibrations using that convention, under which we now see that discrete fibrations in \mathcal{K} correspond to discrete fibrations in \mathcal{K}^{co} , by a bijection which *reverses their orientation*. In fact there is a canonical strict isomorphism of bicategories $\text{DFib}(\mathcal{K})^{op} \cong \text{DFib}(\mathcal{K}^{co})$.

4.22. There is an *embedding homomorphism* $(\)_*: \mathcal{K} \rightarrow \mathcal{F}$ which we shall now describe. It is the identity on objects. The arrow $f: B \rightarrow A$ of \mathcal{K} is taken to the discrete fibration $f_* = (d_0, A \downarrow f, d_1): B \rightarrow A$. The 2-cell $\sigma: f \Rightarrow g: B \rightarrow A$ is taken to the (isomorphism class of the) arrow $A \downarrow \sigma: (d_0, A \downarrow f, d_1) \rightarrow (d_0, A \downarrow g, d_1)$ of spans induced, using the comma object property of $A \downarrow g$, by the 2-cell obtained by pasting σ to the square which exhibits $A \downarrow f$ as a comma object. It is straightforward (compare [22]) to see that we do have a homomorphism of bicategories and that it is locally fully faithful.

4.23. Proposition. For each arrow $f: B \rightarrow A$ in \mathcal{K} , the discrete fibration $f^* = (d_0, f \downarrow A, d_1): A \rightarrow B$ is right adjoint to $f_*: B \rightarrow A$ in $\mathcal{F} = \text{DFib}(\mathcal{K})$.

Proof. The “composition” arrow $d_1: A^3 \rightarrow A^2$ is slib (2.16, 2.17), and it provides an arrow from (d_0, d_2) to (d_0, d_1) in $\mathcal{K}/A \times A$. Applying pseudopullback along $f \times f: B \times B \rightarrow A \times A$, we obtain a slib $(f \downarrow A) \times_A (A \downarrow f) \rightarrow f \downarrow f$ in $\mathcal{K}/B \times B$. Hence $f^* \circ f_*$ is just $f \downarrow f$. The unit $n: B^2 \rightarrow f \downarrow f$ for the adjunction is the arrow of spans induced by $f \lambda: f d_0 \Rightarrow f d_1: B^2 \rightarrow A$.

The composite $f_* \circ f^*$ in \mathcal{F} can be obtained by factoring the composite of the obvious arrow $(A \downarrow f) \times_B (f \downarrow A) \rightarrow A^3$ and $d_1: A^3 \rightarrow A^2$ into a slib $(A \downarrow f) \times_B (f \downarrow A) \rightarrow f_* \circ f^*$ and a conservative $m: f_* \circ f^* \rightarrow A^2$; this last arrow m is the counit. One of the adjunction triangle conditions follows from the diagram below, which is commutative up to isomorphism, while the other follows by weak duality in \mathcal{K} . \square

$$\begin{array}{ccccccc}
 (A \downarrow f) \times_B B^2 & \longrightarrow & (A \downarrow f) \times_B (f \downarrow A) \times_A (A \downarrow f) & \longrightarrow & A^3 \times_A (A \downarrow f) & \longrightarrow & A^2 \times_A (A \downarrow f) \\
 \downarrow & \searrow & \downarrow & \searrow & \downarrow & \searrow & \downarrow \\
 & & (A \downarrow f) \times_B (f \downarrow f) & & f_* \circ f^* \times_A (A \downarrow f) & & \\
 & & \downarrow & \searrow & \downarrow & \searrow & \\
 A \downarrow f & \longrightarrow & f_* \circ f^* \circ f_* & \longrightarrow & A \downarrow f & & \\
 & \searrow & & \searrow & & \searrow & \\
 & & A \times B & & & &
 \end{array}$$

4.24. Remark. Notice that, by taking mates of the structural 2-cells of the homomorphism $(\)_*: \mathcal{K} \rightarrow \mathcal{F}$ under the adjunctions $f_* \dashv f^*$, we get a homomorphism $(\)^*: \mathcal{K}^{\text{co}} \rightarrow \mathcal{F}^{\text{op}}$. In fact, the strict isomorphism $\mathcal{F}^{\text{op}} = \text{DFib}(\mathcal{K})^{\text{op}} \cong \text{DFib}(\mathcal{K}^{\text{co}})$ (cf. Remark 4.21) identifies $(\)^*$ with $(\)_*: \mathcal{K}^{\text{co}} \rightarrow \text{DFib}(\mathcal{K}^{\text{co}})$.

4.25. Proposition. For each arrow $(p, E, q): B \rightarrow A$ in \mathcal{F} , there is a canonical isomorphism $E \cong p_* \circ q^*$.

Proof. Let $P = (A \downarrow p) \times_E (q \downarrow B)$. Since (p, E, q) is a left fibration, there is a right adjoint $r: A \downarrow p \rightarrow E$ to i_p with invertible unit. Following the argument of 4.16 we consider $i_p \dashv r$ as an adjunction in $\text{Spn}(\mathcal{K})(A, B)$, then apply the homomorphism

$$\text{Spn}(\mathcal{K})(A, B) \xrightarrow{- \times_B (d_0, B^2, d_1)} \text{Spn}(\mathcal{K})(A, B).$$

Notice that we have canonical equivalences $P \simeq (A \downarrow p)_{q d_1 \times_{d_0} B^2}$ and $E_{q \times_{d_0} B^2}$, which we compose with $r \times_E B^2$ to get an arrow $s: P \rightarrow q \downarrow B$ such that $(p d_0, d_1) s \cong (d_0 p r_1, d_1 p r_2)$.

Now r is a right adjoint with invertible unit, as is $r \times_E B^2$, since homomorphisms preserve such things, and it follows that s is as well, so s is slib by 2.16. We also have the slib $r': q \downarrow B \rightarrow E$, by 4.13, such that $(p, q) r' \cong (p d_0, d_1)$. Finally we construct $p_* \circ q^*$ by factoring $(d_0 p r_1, d_1 p r_2): P \rightarrow A \times B$ into a slib followed by a cons, but $(d_0 p r_1, d_1 p r_2) \cong (p d_0, d_1) s \cong (p, q) r' s$ where $r' s$ is slib (being a composite of such) and (p, q) is cons, therefore $E \cong p_* \circ q^*$ as required. \square

4.26. Proposition. Suppose that $(p, E, q): B \rightarrow A$ is an arrow in \mathcal{F} .

- (a) For any arrow $f: C \rightarrow A$ in \mathcal{K} , one has $f^* \circ E \cong (\text{pr}_1, C_f \times_p E, q \text{pr}_2)$.
- (b) For any left fibration $g: A \rightarrow C$ in \mathcal{K} , one has $g_* \circ E \cong \pi_{C, B}(g p, E, q)$.

Proof. (a) By 4.15 and 4.16 $(\text{pr}_1, C_f \times_p E, q \text{pr}_2)$ satisfies left and right lifting; discreteness is obtained using the discreteness of (p, E, q) and the pseudopullback property of $C_f \times_p E$. So $(\text{pr}_1, C_f \times_p E, q \text{pr}_2)$ is a discrete fibration. We also have a factorisation of $(f \downarrow A) \times_A E \simeq f \downarrow p \rightarrow C \times B$ as a composite $f \downarrow p \rightarrow C_f \times_p E \rightarrow C \times B$ where the first

arrow is a right adjoint for the canonical $C_f \times_p E \rightarrow f \downarrow p$ with invertible unit. So, by 2.16 the result follows.

(b) Using the left fibration properties of (p, E, q) and g , we obtain a right adjoint r to the canonical $i: E \rightarrow C \downarrow gp$ with invertible unit, and this adjunction naturally sits in $\text{Spn}(\mathcal{K})(C, B)$, cf. 4.16. This gives a factorisation of $(C \downarrow g) \times_A E \simeq C \downarrow gp$ into the sliab r followed by $(gp, q): E \rightarrow C \times B$. Hence the result. \square

5. Modulated bicategories

5.1. In this section we introduce some elementary conditions on \mathcal{K} which imply some desirable properties of the bicategory $\mathcal{F} = \text{DFib}(\mathcal{K})$ which we constructed in the last section. In particular we show that \mathcal{F} has lax limits of arrows (these are dual to the *collages* of [23]). As corollaries we will see that $(\)_*: \mathcal{K} \rightarrow \mathcal{M}$ preserves the terminal object, binary products and some cotensors. This, in turn, is enough to show that \mathcal{F} has all *finite local limits* (cf. [5, 23]).

5.2. A bicategory \mathcal{K} is called *comodulated* when it is faithfully conservational and satisfies two extra axioms:

- (v) for all pairs of arrows $f: A \rightarrow C, g: B \rightarrow C$, both of the canonical projections $d_0: f \downarrow g \rightarrow A, d_1: f \downarrow g \rightarrow B$ are strong liberal;
- (vi) for all conservative arrows $(a_1, a_2, a_3): X \rightarrow A_1 \times A_2 \times A_3$ with factorisations

$$(a_2, a_3) \cong j_1 e_1: X \rightarrow A_2 \times A_3,$$

$$(a_1, a_3) \cong j_2 e_2: X \rightarrow A_1 \times A_3,$$

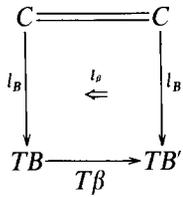
$$(a_1, a_2) \cong j_3 e_3: X \rightarrow A_1 \times A_2$$

into conservative j_i and strong liberal e_i , if $f: K \rightarrow X$ is such that $e_1 f, e_2 f, e_3 f$ are all strong liberal then f is strong liberal.

A bicategory \mathcal{K} is called *modulated* when \mathcal{K}^{op} is comodulated.

5.3. Remark. The weak dual \mathcal{K}^{co} of a (co)modulated bicategory \mathcal{K} is again (co)modulated.

5.4. We refer to [22] and [21] for some background to the following definitions. Given a small bicategory \mathcal{B} , let $\text{Bicat}(\mathcal{B}^{\text{co}}, \mathcal{F}^{\text{co}})^{\text{co}}$ denote the usual bicategory of *comorphisms*, *optransformations* and *modifications*. The *lax limit* of a comorphism $T: \mathcal{B} \rightarrow \mathcal{F}$ consists of an object $\text{laxlim}(T) \in \mathcal{F}$ and an optransformation $l: \Delta C \rightarrow T$,



composition with which induces an equivalence of categories

$$\mathcal{F}(X, \text{llim}(T)) \simeq \text{Bicat}(\mathcal{B}^{\text{co}}, \mathcal{F}^{\text{co}})^{\text{co}}(\Delta X, T)$$

for each object $X \in \mathcal{F}$. For instance, a comonad in \mathcal{F} is no more nor less than a comorphism $\mathbb{1} \rightarrow \mathcal{F}$; the lax limit of a comonad (if it exists) is its associated object of Eilenberg–Moore coalgebras.

Notice that [23] and [5] make heavy use of lax colimits, which they call *collages*. In fact we are interested in precisely the same *glueing* constructions, but in the context of \mathcal{F} they present themselves as lax limits. This difference stems from the fact that the bicategories of [23] and [5] correspond to the dual $\mathcal{F}^{\text{coop}}$.

5.5. The principal result of this section will show that \mathcal{F} has lax limits of all arrows; that is lax limits of *normal* comorphisms with domain $\mathcal{2}$. A lax cone over a discrete fibration $(p, E, q): B \rightarrow A$ with vertex C comprises a pair of arrows $(u_0, F_0, v_0): C \rightarrow A$ and $(u_1, F_1, v_1): C \rightarrow B$ accompanied by a 2-cell $[s, \phi]: F_0 \Rightarrow E \circ F_1: C \rightarrow A$. In particular notice that the pair $p_*: E \rightarrow A, q_*: E \rightarrow B$ may be made into a lax cone over (p, E, q) :



Here the 2-cell γ is constructed by composing $p_* \circ \eta_q: p_* \Rightarrow p_* \circ q^* \circ q_*$, where η_q is the unit of the adjunction $q_* \dashv q^*$ of Proposition 4.23, with the isomorphism obtained by applying $- \circ q_*$ to the 2-cell $p_* \circ q^* \cong E$ of Proposition 4.25.

5.6. Recall from [5, 23] that we say that \mathcal{F} has *local finite limits* if each homset $\mathcal{F}(C, B)$ has all finite limits and furthermore these are preserved by left and right compositions

$$\mathcal{F}(C, B) \xrightarrow{- \circ F} \mathcal{F}(D, B), \quad \mathcal{F}(C, B) \xrightarrow{E \circ -} \mathcal{F}(C, A),$$

for all discrete fibrations $(p, E, q): B \rightarrow A$ and $(u, F, v): D \rightarrow C$.

5.7. Remark. As an example of the relationship between local limits and lax limits, which will be useful later on, we consider the following. Let \mathbb{C} be a small category and $\mathbf{2}(\mathbb{C})$ the bicategory with two objects 0 and 1, with $\mathbf{2}(\mathbb{C})(1, 0) = \mathbb{C}$, and with no other non-trivial 1- or 2-cells. Now suppose that \mathcal{F} has local limits parameterised by \mathbb{C} . Then we may calculate the lax limit of any normal comorphism $T: \mathbf{2}(\mathbb{C}) \rightarrow \mathcal{F}$ as that of the arrow $(\varprojlim_{\mathbb{C}} T(-)): T(0) \rightarrow T(1)$ (where one of these lax limits exists iff the other one does).

To establish this result it is enough to provide (natural) equivalences between categories of lax cones over these diagrams. An lax cone γ over T with vertex X consists of a pair of arrows $\gamma_0: X \rightarrow T(0)$, $\gamma_1: X \rightarrow T(1)$ and a family of 2-cells $\gamma_c: \gamma_0 \Rightarrow T(c) \circ \gamma_1$ for $c \in \text{obj}(\mathbb{C})$. The coherence conditions that these 2-cells must satisfy, with respect to the 2-cellular structure of $\mathbf{2}(\mathbb{C})$, amount to saying that they form a cone over the functor $T(-) \circ \gamma_1: \mathbb{C} \rightarrow \mathcal{F}(X, T(0))$ with vertex γ_0 .

The appropriate observation at this stage is that the preservation of local limits of type \mathbb{C} by composition ensures that $\varprojlim_{\mathbb{C}} (T(-) \circ \gamma_1) \cong (\varprojlim_{\mathbb{C}} T(-)) \circ \gamma_1$. So the family of 2-cells associated with the lax cone γ , which we know forms a cone over $T(-) \circ \gamma_1$, factors to yield a unique 2-cell $\bar{\gamma}: \gamma_0 \Rightarrow (\varprojlim_{\mathbb{C}} T(-)) \circ \gamma_1$. In other words, we get the 2-cell in a lax cone $(\gamma_0, \gamma_1, \bar{\gamma})$ over the arrow $\varprojlim_{\mathbb{C}} T(-)$. This construction is clearly reversible, giving us the required equivalences of lax cone categories. \square

5.8. Remark. It is straightforward to show that the hom-categories of \mathcal{F} possess finite limits. First notice that there is a reflection from the slice $\mathcal{K}/A \times B$ into its full sub-bicategory of discrete fibrations from B to A . But slices and reflective full sub-bicategories of the finitely complete bicategory \mathcal{K} are also finitely complete. Now the category $\mathcal{F}(B, A)$ was constructed by factoring out isomorphism classes of arrows in this (locally discrete) bicategory of discrete fibrations. All that remains for us is to remark that under this process the finite bicategorical limits, which we know exist, factor to corresponding categorical limits in $\mathcal{F}(B, A)$.

We must work harder to prove that these finite limits in the hom-categories of \mathcal{F} are preserved by left and right composition. To do so we will ultimately use the following result:

5.9. Lemma. *If \mathcal{K} is a faithfully conservational bicategory and the homomorphism $(\)_*: \mathcal{K} \rightarrow \mathcal{F}$ preserves cotensors with the finite category \mathbb{C} , then each hom category $\mathcal{F}(B, A)$ has finite limits parameterised by \mathbb{C} and, for each discrete fibration $F: \mathbb{C} \rightarrow B$, these are preserved by $- \circ F: \mathcal{F}(B, A) \rightarrow \mathcal{F}(C, A)$.*

Proof. For any object $A \in \mathcal{K}$ there is a cotensor $A^{\mathbb{C}} \in \mathcal{K}$, which comes equipped with a functor $\pi: \mathbb{C} \rightarrow \mathcal{K}(A^{\mathbb{C}}, A)$ inducing an equivalence

$$\mathcal{K}(B, A^{\mathbb{C}}) \xrightarrow{\sim} \mathcal{K}(B, A)^{\mathbb{C}} \quad (3)$$

for each object $B \in \mathcal{K}$ (cf. 1.8). The assumption that this is preserved by $()_*$ simply means that the composite

$$\mathbb{C} \xrightarrow{\pi} \mathcal{K}(A^{\mathbb{C}}, A) \xrightarrow{(\)_*} \mathcal{F}(A^{\mathbb{C}}, A) \tag{4}$$

induces an equivalence

$$\mathcal{F}(B, A^{\mathbb{C}}) \xrightarrow{\sim_{\hat{\pi}'}} \mathcal{F}(B, A)^{\mathbb{C}} \tag{5}$$

for each object B .

For any category \mathbb{B} let $\Delta_{\mathbb{B}}: \mathbb{B} \rightarrow \mathbb{B}^{\mathbb{C}}$ denote the usual functor, which takes an object $b \in \mathbb{B}$ to the functor $\Delta_b: \mathbb{C} \rightarrow \mathbb{B}$ “constant at b ”. By definition the category \mathbb{B} admits limits of type \mathbb{C} iff $\Delta_{\mathbb{B}}$ has a right adjoint. The equivalence (3) ensures that there is an (essentially unique) arrow $\Delta: A \rightarrow A^{\mathbb{C}}$ such that $\hat{\pi}(\Delta) \cong \Delta_{\mathcal{K}(A, A)}(1_A)$. Notice that, since the map $\hat{\pi}'$ in (5) is induced by the composite in (4), we have $\hat{\pi}'(\Delta_*) \cong \Delta_{\mathcal{F}(A, A)}(1_A)$ and a diagram

$$\begin{array}{ccc} & \mathcal{F}(B, A) & \\ \Delta_* \circ - \swarrow & \cong & \searrow \Delta_{\mathcal{F}(B, A)} \\ \mathcal{F}(B, A^{\mathbb{C}}) & \xrightarrow{\sim_{\hat{\pi}'}} & \mathcal{F}(B, A)^{\mathbb{C}} \end{array} \tag{6}$$

It follows that $\Delta_{\mathcal{F}(B, A)}$ has a right adjoint iff the functor $\Delta_* \circ -$ does, but Δ_* has a right adjoint Δ^* in \mathcal{F} (see Proposition 4.23), which gives rise to an adjunction $\Delta_* \circ - \dashv \Delta^* \circ -: \mathcal{F}(B, A^{\mathbb{C}}) \rightarrow \mathcal{F}(B, A)$.

It remains to prove the preservation property mentioned in the lemma. Firstly we may recast this to postulate that the mate [12] of the invertible 2-cell

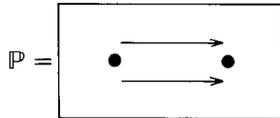
$$\begin{array}{ccc} \mathcal{F}(B, A) & \xrightarrow{\Delta_{\mathcal{F}(B, A)}} & \mathcal{F}(B, A)^{\mathbb{C}} \\ \downarrow - \circ F & \cong & \downarrow (- \circ F)^{\mathbb{C}} \\ \mathcal{F}(C, A) & \xrightarrow{\Delta_{\mathcal{F}(C, A)}} & \mathcal{F}(B, A)^{\mathbb{C}} \end{array} \tag{7}$$

under the adjunctions $\Delta_{\mathcal{F}(B, A)} \dashv \lim_{\mathbb{C}}$ and $\Delta_{\mathcal{F}(C, A)} \dashv \lim_{\mathbb{C}}$, is an isomorphism as well. But the canonical isomorphisms in the squares

$$\begin{array}{ccc}
 \mathcal{F}(B, A) & \xrightarrow{\Delta_{*} \circ -} & \mathcal{F}(B, A^{\mathbb{C}}) \\
 \downarrow - \circ F & \cong & \downarrow - \circ F \\
 \mathcal{F}(C, A) & \xrightarrow{\Delta_{*} \circ -} & \mathcal{F}(C, A^{\mathbb{C}})
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathcal{F}(B, A^{\mathbb{C}}) & \xrightarrow{\Delta^{*} \circ -} & \mathcal{F}(B, A) \\
 \downarrow - \circ F & \cong & \downarrow - \circ F \\
 \mathcal{F}(C, A^{\mathbb{C}}) & \xrightarrow{\Delta^{*} \circ -} & \mathcal{F}(C, A)
 \end{array}$$

are mates under the adjunctions $\Delta_{*} \circ - \dashv \Delta^{*} \circ -$, and the left-hand square may be obtained by pasting copies of the triangle in (6) onto the square in (7). It follows, from the right-hand square, that the mate of (7) is also an isomorphism. \square

5.10. Remark. To obtain the conclusion of the last lemma for all finite limits, it is enough to check that $(\)_{*} : \mathcal{K} \rightarrow \mathcal{F}$ preserves the terminal object, binary products and cotensors with \mathbb{P} , where \mathbb{P} is the category consisting of a parallel pair of arrows between two different objects:



This simply reflects the fact that the finite limits in a hom category $\mathcal{F}(B, A)$ may be constructed using its terminal object, binary products and equalisers, when these exist.

5.11. Remark. Suppose we knew the conclusion of Lemma 5.9 held for every bicategory of discrete fibrations $\mathcal{F} = \text{DFib}(\mathcal{K})$ constructed from a comodulated bicategory \mathcal{K} . Of course \mathcal{K}^{co} is comodulated iff \mathcal{K} is (Remark 5.3) and $\mathcal{F}^{\text{op}} = \text{DFib}(\mathcal{K}^{\text{co}})$ (Remark 4.21), so it follows that both \mathcal{F} and \mathcal{F}^{op} satisfy the conclusion of the lemma. This would serve to demonstrate that \mathcal{F} has local finite limits, as defined in 5.6, since the preservation property of the lemma, for the dual \mathcal{F}^{op} , is no more than the second one given in 5.6 for \mathcal{F} itself.

5.12. For the remainder of this section we will assume that we are working within a comodulated bicategory \mathcal{K} .

5.13. Proposition. If (p, E, q) is a left (resp. right) fibration then p (resp. q) is strong liberal.

Proof. The arrow $i: A \rightarrow A \downarrow p$ has a right adjoint $r: A \downarrow p \rightarrow A$ in \mathcal{K}/A (Proposition 4.12) with invertible counit, therefore r is a strong liberal, by 4.13 and $pr \cong d_0: A \downarrow p \rightarrow A$. Now, by 5.2(v), d_0 is strong liberal, therefore it follows from 2.12 that p is strongly liberal as well. \square

5.14. Remark. In what follows it will sometimes aid intuition and make arguments easier to adopt the terminology and informal notation of “generalised elements”. When we write $a \in_X A$, and say a is an *object of A defined at X* , we are really referring to an arrow $a: X \rightarrow A$. Similarly $(\alpha: a \Rightarrow a') \in_X A$, or in words α is a *morphism or arrow of A from a to a' defined at X* , denotes a 2-cell $\alpha: a \Rightarrow a': X \rightarrow A$.

Of course the foundation on which such a notation rests is simply the ubiquitous bicategorical Yoneda lemma [22] and so its advantages extend beyond saving space and time. This is especially apparent when working with all kinds of limit construction. For instance it is natural to write the generalised objects of a comma object $f \downarrow g$ as triples $(a \in_X A, b \in_X B, \gamma: fa \Rightarrow ga \in_X C)$ (or simply $(a, b, \gamma: fa \Rightarrow gb)$), with generalised morphisms from (a, b, γ) to (a', b', γ') as pairs $(\alpha \in_X A, \beta \in_X B)$ with $\gamma' \cdot \beta\alpha = g\beta \cdot \gamma$.

We do not propose to spend any time developing a formal theory supporting this notation. Rather we encourage the reader to gain an intuition for it by translating some of the proofs given in earlier sections.

5.15. Proposition. *If $(p, E, q): B \rightarrow A$ is a discrete fibration and $g: A \rightarrow C$ any right fibration then one has $g_* \circ E \cong \pi_{C, B}(gp, E, q)$ (compare with Proposition 4.26(b)).*

Proof. First consider the arrow $\bar{g}: A \downarrow p \rightarrow C \downarrow gp$ which is induced, via the universal property of $C \downarrow gp$, by the 2-cell $g\lambda: gd_1 \Rightarrow gpd_0: A \downarrow p \rightarrow C$. Here $\lambda: d_1 \Rightarrow pd_0$ is the canonical 2-cell associated with the comma object $A \downarrow p$. In other words the arrow \bar{g} carries an object $(a, e, \alpha: a \Rightarrow pe) \in_X A \downarrow p$ to $(ga, e, g\alpha: ga \Rightarrow gpe) \in_X C \downarrow gp$, and a morphism $(\tau, \kappa): (a, e, \alpha) \Rightarrow (a', e', \alpha') \in_X A \downarrow p$ to $(g\tau, \kappa) \in_X C \downarrow gp$.

Notice that a 2-cell $(\rho, \chi): (a, e, \alpha) \Rightarrow (a', e', \alpha') \in_X A \downarrow p$ is right cartesian for \bar{g} if $\rho: a \Rightarrow a' \in_X A$ is right cartesian for g , so how would we obtain a cartesian lift of a morphism $(\mu, \chi): \bar{g}(a, e, \alpha) \Rightarrow (c, e', \gamma) \in_X C \downarrow gp$. First lift $\mu: ga \Rightarrow c \in_X C$ along g to a right cartesian $\rho: a \Rightarrow a' \in_X A$ (accompanied by an isomorphism $\tau: ga' \cong c$ such that $\tau \cdot gp = \mu$), then notice that the pair (μ, χ) is a morphism of $C \downarrow gp$ so we know that $g(p\chi \cdot \alpha) = \gamma \cdot \mu$. Therefore, since ρ is right cartesian for g , $p\chi \cdot \alpha$ factors as $\alpha' \cdot \rho$ for a unique $\alpha': a' \Rightarrow pe'$ with $g\alpha = \gamma \cdot \tau$. Now we have constructed a object (a', e', α') , a right cartesian $(\rho, \chi): (a, e, \alpha) \Rightarrow (a', e', \alpha')$ for \bar{g} and an isomorphism $(\tau, 1_e): \bar{g}(a', e', \alpha') \Rightarrow (c, e', \gamma)$ with $(\tau, 1_e) \cdot \bar{g}(\rho, \chi) = (\mu, \chi)$, or in other words, this is the required cartesian lift. It follows that \bar{g} is a right fibration and as such Proposition 5.13 implies that it is a strong liberal.

We construct $g_* \circ E$ by factoring the arrow $(d_0, qd_1): C \downarrow gp \rightarrow C \times B$ into a strong liberal $e: C \downarrow gp \rightarrow g_* \circ E$ followed by a conservative $(u, v): g_* \circ E \rightarrow C \times B$. The composite $(d_0, qd_1)\bar{g}i_p: E \rightarrow C \times B$ is isomorphic to (gp, q) so we have established the proposition if we can prove that $e\bar{g}i_p$ is a strong liberal.

Consider the counit $\varepsilon: i_p r \Rightarrow 1_{A \downarrow p}$ of the adjunction $i_p \dashv r$ associated with the left fibration (p, E, q) . We know, from the statement of Proposition 4.12, that $qd_1\varepsilon$ is an isomorphism, and the third paragraph of its proof demonstrates that $d_0\varepsilon$ is an isomorphism as well. Now the arrow $(gd_0, qd_1): A \downarrow p \rightarrow C \times B$ may be factored as $e\bar{g}: A \downarrow p \rightarrow g_* \circ E$ followed by $(u, v): g_* \circ E \rightarrow C \times B$ which is conservative so, since we

know that $(gd_0, qd_1)\varepsilon$ is an isomorphism, it follows that $e\bar{g}\varepsilon$ provides us with an isomorphism between $e\bar{g}i_p r$ and $e\bar{g}$. Of course we know that r (as a right adjoint with invertible unit), e and \bar{g} are all strong liberals therefore we may use the composition and cancellation rules of 2.12 to show that $e\bar{g}i_p$ is a strong liberal as required. \square

5.16. Construction. The aim of the next few propositions will be to establish that the lax cone in (2) presents E as the lax limit of the arrow (p, E, q) in \mathcal{F} . The following construction will be crucial to that end.

Suppose (F_0, F_1, s) is a lax cone over E as above, then the composite $E \circ F_1$ is formed by factoring $E \times_B F_1 \rightarrow C \times A$ into a strong liberal $l: E \times_B F_1 \rightarrow E \circ F_1$ followed by a conservative $(k, w): E \circ F_1 \rightarrow C \times A$. Now form the pullback

$$\begin{array}{ccc}
 H & \xrightarrow{t} & E \times_B F_1 \\
 \downarrow m & \lrcorner & \downarrow l \\
 & \cong & \\
 F_0 & \xrightarrow{s} & E \circ F_1
 \end{array} \tag{8}$$

It is useful to think of H as the collage of the *gamut* (E, F_0, F_1, s) (cf. [22]), although we will not need to prove that result. All we need is the following proposition:

5.17. Proposition. *The span $(\text{pr}_E t, H, v_0 m): C \rightarrow E$ is a discrete fibration, as is $(u_0 m, H, \text{pr}_{F_1} t): F_1 \rightarrow A$.*

Proof. First notice that the two results we wish to prove are dual, we obtain the second by reinterpreting the first in the bicategory \mathcal{X}^{co} . This works simply because reversing the 2-cells of \mathcal{X} corresponds to reversing the arrows of \mathcal{F} (cf. 4.2 and Remark 4.21).

Discreteness: It is important to notice that we have a chain of isomorphisms $v_0 m \cong wsm \cong wlt \cong v_1 \text{pr}_{F_1} t$ so $(\text{pr}_E t, v_0 m) \cong (\text{pr}_E, v_1 \text{pr}_{F_1})t$, and dually an isomorphism $p \text{pr}_E t \cong u_0 m$. Now we know that t is conservative, since it is a pullback of the conservative s , but how about $(\text{pr}_E, v_1 \text{pr}_{F_1})$? Suppose that the morphism $\alpha: h \Rightarrow h' \in_X H$ has both $\text{pr}_E \alpha$ and $v_1 \text{pr}_{F_1} \alpha$ invertible then, since $q \text{pr}_E \cong u_1 \text{pr}_{F_1}$, we also know that $u_1 \text{pr}_{F_1} \alpha$ is an isomorphism. But (u_1, v_1) is conservative which implies that $\text{pr}_{F_1} \alpha$ is an isomorphism, which in turn allows us to infer that α is invertible from the fact that $(\text{pr}_E, \text{pr}_{F_1})$ is conservative as well. Dually $(u_0 m, \text{pr}_{F_1} t)$ is conservative.

Left fibration: Firstly it is easily seen that a morphism $\sigma: h \Rightarrow h' \in_X H$ is left cartesian for $(\text{pr}_E t, H, v_0 m)$ if both of $m\sigma$ and $\text{pr}_{F_1} t\sigma$ are left cartesian for $(u_0 F_0, v_0)$ and (u_1, F_1, v_1) respectively. So how might we lift a morphism $\alpha: e \Rightarrow \text{pr}_E t h \in_X E$ to such a left cartesian arrow?

(1) Apply $q: E \rightarrow A$ to α and compose the result with the isomorphism $q \text{pr}_E t h \cong u_1 \text{pr}_{F_1} t h$ and then lift the result to obtain a left cartesian arrow $\chi: f_1 \Rightarrow \text{pr}_F t h$

for the span (u_1, F_1, v_1) along with an isomorphism $\tau: u_1 f_1 \cong qe$ such that the diagram

$$\begin{array}{ccc} u_1 f_1 & \xrightarrow[\tau]{\sim} & qe \\ \downarrow u_1 \chi & & \downarrow q\alpha \\ u_1 \text{pr}_F t & \xrightarrow[\sim]{} & q\text{pr}_E t \end{array}$$

commutes. This of course means that we have formed an object $(e, f_1, \tau) \in_X E \times_B F_1$ and a morphism $(\alpha, \chi): (e, f_1, \tau) \Rightarrow th$.

(2) Apply $p: E \rightarrow A$ to α and compose with the isomorphism obtained by right application of h to $p \text{pr}_E t \cong u_0 m$ (cf. proof of discreteness above), then lift the result to obtain a left cartesian arrow $\rho: f_0 \Rightarrow mh$ for the span (u_0, F_0, v_0) along with an isomorphism $\kappa: u_0 f_0 \cong pe$.

(3) Now s is a map of discrete fibrations so $s\rho: sf_0 \Rightarrow smh$ is left cartesian for $(k, E \circ F_1, w)$. From the proof of 4.16 it is clear that $l(\alpha, \chi)$ is also left cartesian for $(k, E \circ F_1, w)$ since χ is left cartesian for (u_1, F_1, v_1) . Composing the isomorphisms $ksf_0 \cong u_0 f_0$, $\kappa: u_0 f_0 \cong pe$ and $pe \cong kl(e, f_1, \tau)$ we get an isomorphism κ' which is easily shown to make the square

$$\begin{array}{ccc} ksf_0 & \xrightarrow[\kappa']{\sim} & kl(e, f_1, \tau) \\ \downarrow ksp & & \downarrow kl(\alpha, \chi) \\ ksmh & \xrightarrow[\kappa v h]{\sim} & klth \end{array}$$

commute. So by the universal properties of $s\rho$ and $l(\alpha, \chi)$ there exists a unique isomorphism $\lambda: sf_0 \cong l(e, f_1, \tau)$ such that the diagram

$$\begin{array}{ccc} sf_0 & \xrightarrow[\lambda]{\sim} & l(e, f_1, \tau) \\ \downarrow s\rho & & \downarrow l(\alpha, \chi) \\ smh & \xrightarrow[\kappa v h]{\sim} & klth \end{array} \tag{9}$$

commutes and $k\lambda = \kappa'$.

The isomorphism λ completes the definition of an object $((e, f_0, \tau), f_1, \lambda) \in_X H$ and the commutative square (9) ensures that the pair $((\alpha, \chi), \rho)$ becomes an arrow $((e, f_1, \tau), f_0, \lambda) \Rightarrow h$. As we mentioned at the beginning of this proof, the fact that χ and ρ are left cartesian (for F_0 and F_1 resp.) implies that $((\alpha, \chi), \rho)$ itself is left cartesian for $(\text{pr}_E t, H, v_0 m)$ and it is clearly a lift of α as required. So $(\text{pr}_E t, H, v_0 m)$ is a left fibration, and dually $(u_0 m, H, \text{pr}_F t)$ is a right fibration.

Right fibration: We know that $(u_0m, H, \text{pr}_{F_1}t)$ and (u_1, F_1, v_1) are both right fibrations, therefore the composite $v_1 \text{pr}_{F_1}t: H \rightarrow C$ is a right fibration. We also have a sequence of isomorphisms $v_1 \text{pr}_{F_1}t \cong \text{wlt} \cong \text{wsm} \cong v_0m$ so $v_0m: H \rightarrow C$ is a right fibration as well. To lift an arrow $\beta: v_1 \text{pr}_{F_1}th \Rightarrow c \in_X C$ we first lift it to a right cartesian for (u_1, F_1, v_1) and then lift again (as above) to H thereby obtaining an arrow $\sigma: h \Rightarrow h'$ with $\text{pr}_{F_1}t\sigma, \text{pr}_E t\sigma$ and $m\sigma$ all right cartesian for F_1, E and F_0 respectively. Therefore $p \text{pr}_E t\sigma$ and $u_1 \text{pr}_{F_1}t\sigma$ are both isomorphisms, but $u_1 \text{pr}_{F_1}t \cong q \text{pr}_E t$ so $(p, q) \text{pr}_E t\sigma$ is invertible. The pair (p, q) is conservative which implies that $\text{pr}_E t\sigma$ is invertible and so σ is right cartesian for the span $(\text{pr}_E t, H, v_0m)$. This establishes that $(\text{pr}_E t, H, v_0m)$ is a right fibration and dually $(u_0m, H, \text{pr}_{F_1}t)$ is a left fibration. \square

5.18. Theorem. *The lax cone (p_*, q_*, γ) in diagram (2) displays E as the lax limit of the arrow $(p, E, q): B \rightarrow A$ in \mathcal{F} .*

Proof. For each object $C \in \mathcal{F}$ composition with the cone (p_*, q_*, γ) gives a functor

$$\mathcal{F}(C, E) \xrightarrow{\phi_C} \text{Bicat}(\mathbb{1}, \mathcal{F}^{co})^{co}(\Delta C, (p, E, q)),$$

$$(f, G, g) \longmapsto (p_* \circ G, q_* \circ G, \gamma \circ G),$$

which we must prove to be an equivalence, but we have:

(a) $(u_0, F_0, v_0) \stackrel{\text{def}}{=} p_* \circ (f, G, g) \cong \pi_{A,C}(pf, G, g)$ by Proposition 4.26(b), since p is a left fibration. Of course this means that there is a strong liberal $e_0: G \rightarrow F_0$ such that $(pf, g) \cong (u_0, v_0)e_0$.

(b) $(u_1, F_1, v_1) \stackrel{\text{def}}{=} q_* \circ (f, G, g) \cong \pi_{B,C}(qf, G, g)$ by Proposition 5.15, since q is a right fibration. This means that there is a strong liberal $e_1: G \rightarrow F_1$ along with an isomorphism $\theta: (qf, g) \cong (u_1, v_1)e_1$.

Now consider the diagram

$$\begin{array}{ccc}
 G & \xrightarrow{(f, e_1, \theta_B)} & E \times_B F_1 \\
 e_0 \downarrow & \cong \xi & \downarrow l \\
 F_0 & & E \circ F_1 \\
 (u_0, v_0) \searrow & & \swarrow (k, w) \\
 & A \times C &
 \end{array} \tag{10}$$

where the isomorphism $\theta_B: qf \cong u_1 e_1$, comprising part of the data for the top arrow in this diagram, is in fact the second component of the isomorphism $\theta: (qf, g) \cong (u_1, v_1)e_1$ in (b) above and ξ is constructed in the obvious way from the various isomorphisms $(k, w)l(f, e_1, \theta_B) \cong (p \text{pr}_E, v_1 \text{pr}_{F_1})(f, e_1, \theta_B) \cong (pf, v_1 e_1), (u_0, v_0)e_0 \cong (pf, g)$ and $v_1 e_1 \cong g$.

But e_0 is strong liberal and (k, w) is conservative so our diagram factorises uniquely (up to isomorphism) as

$$\begin{array}{ccc}
 G & \xrightarrow{(f, e_1, \theta_B)} & E \times_B F_1 \\
 \downarrow e_0 & \cong & \downarrow l \\
 F_0 & \xrightarrow{s} & E \circ F_1 \\
 \searrow (u_0, v_0) & \cong & \swarrow (k, w) \\
 & A \times C &
 \end{array}
 \tag{11}$$

(cf. 2.13). It is straightforward to check that the isomorphism class $[s, \phi]$ is precisely the composite $\gamma \circ H$. We leave the action of Φ_C on 2-cells $\alpha: H \Rightarrow H': C \rightarrow E$ up to the reader to determine.

Our description of Φ_C , and Proposition 5.17, clearly indicate that Construction 5.16 may be used to provide a potential equivalence inverse:

$$\text{Bicat}(\mathbb{1}, \mathcal{F}^{\text{co}})^{\text{co}}(\Delta C, (p, E, q)) \xrightarrow{\Phi_C} \mathcal{F}(C, E),$$

$$(F_0, F_1, s) \longmapsto (\text{pr}_{E t}, H, v_0 m).$$

Examining the definition of $(\text{pr}_{E t}, H, v_0 m)$ reveals that there is a natural way to define the action of Φ'_C on modifications of lax cones, but again we leave the details up to the reader. It remains to provide natural isomorphisms $\Phi_C \Phi'_C \cong 1$ and $\Phi'_C \Phi_C \cong 1$.

$\Phi_C \Phi'_C \cong 1$: Recall the isomorphisms $u_0 m \cong p \text{pr}_{E t}, v_0 m \cong v_1 \text{pr}_{F_1 t}$, from the “discreteness” portion of the proof of Proposition 5.17, and $q \text{pr}_E \cong u_1 \text{pr}_{F_1}$ which give us $(p \text{pr}_{E t}, v_0 m) \cong (u_0, v_0) m$ and $(q \text{pr}_{E t}, v_0 m) \cong (u_1, v_1) \text{pr}_{F_1 t}$. But $m: H \rightarrow F_0$, as a pull-back of the strong liberal $l: E \times_B F_1 \rightarrow E \circ F_1$, is strong liberal, so $p_* \circ H \cong \pi_{A,C}(p \text{pr}_{E t}, H, v_0 m) \cong (u_0, F_0, v_1)$. Similarly Propositions 5.17 and 5.13 imply that $\text{pr}_{F_1 t}$ is a right fibration and thus a strong liberal, therefore $q_* \circ H \cong \pi_{B,C}(q \text{pr}_{E t}, H, v_0 m) \cong (u_1, F_1, v_1)$.

It remains to show that composing these isomorphisms with $\gamma \circ H: p_* \circ H \Rightarrow E \circ (q_* \circ H)$ gives us the 2-cell $[s, \phi]$ that we started with. This though is quite straightforward, because in this case diagram (10) reduces to one of the form

$$\begin{array}{ccc}
 H & \xrightarrow{t} & E \times_B F_1 \\
 \downarrow m & \cong & \downarrow l \\
 F_0 & & E \circ F_1 \\
 \searrow (u_0, v_0) & & \swarrow (k, w) \\
 & A \times C &
 \end{array}$$

which can be shown to factor, as in (11), into a composite of the pullback square in (8) and the triangle associated with the original map of discrete fibrations $[s, \phi]$.

In summary, for each lax cone (F_0, F_1, s) over E we have constructed the required isomorphism $\Phi_C \Phi'_C(F_0, F_1, s) \cong (F_0, F_1, s)$. We leave the naturality of these up to the reader to check.

$\Phi'_C \Phi_C \cong 1$: Starting with a discrete fibration $(f, G, g): C \rightarrow E$, the construction of $(F_0, F_1, s) \stackrel{\text{def}}{=} \Phi_C(f, G, g)$ adumbrated above produces, as a by-product, the factored diagram (11). But H is formed by pulling l back along s , as in (8), so its universal property ensures that we get an (essentially) unique arrow $d: G \rightarrow H$ and a factorisation of the square in (11), viz.

$$\begin{array}{ccc}
 G & \xrightarrow{(f, e_1, \theta_B)} & E \times_B F_1 \\
 \searrow d & \cong \downarrow & \downarrow l \\
 & H & \xrightarrow{t} E \times_B F_1 \\
 \swarrow e_0 & \lrcorner & \downarrow l \\
 & m & \cong \downarrow \\
 & F_0 & \xrightarrow{s} E \circ F_1
 \end{array} \tag{12}$$

Of course we have $v_0 e_0 \cong g$ (by definition of e_0 , see (a) above), and $\text{pr}_E(f, e, \theta_B) \cong f$ which we compose with the isomorphisms in the triangles of our diagram, then combine to get $\psi: (f, g) \cong (\text{pr}_E t, v_0 m)d$. In this way we have constructed a 2-cell $[d, \psi]: (f, G, g) \Rightarrow (\text{pr}_E t, H, v_0 m)$ in \mathcal{F} , what is more, the collection of these form a natural transformation $1 \Rightarrow \Phi'_C \Phi_C$. Now d , as a map of discrete fibrations, is conservative; so in order to prove that $[d, \psi]$ is an isomorphism in \mathcal{F} all that remains is to show that it is strong liberal as well.

It is here that we finally get to apply axiom 5.2(vi) from the definition of a comodulated bicategory. The maps $(p, q): E \rightarrow A \times B$ and $(\text{pr}_E t, v_0 m): H \rightarrow E \times C$ are conservative so it follows that $(p \text{pr}_E t, q \text{pr}_E t, v_0 m): H \rightarrow A \times B \times C$ is conservative as well. We have already come across the first two of the following factorisations into conservatives following strong liberals (cf. the proof of $\Phi_C \Phi'_C \cong 1$):

$$\begin{aligned}
 (q \text{pr}_E t, v_0 m) &\cong (u_0, v_0)m: H \rightarrow B \times C, \\
 (p \text{pr}_E t, v_0 m) &\cong (u_1, v_1) \text{pr}_{F_1} t: H \rightarrow A \times C, \\
 (p \text{pr}_E t, q \text{pr}_E t) &\cong (p, q) \text{pr}_E t: X \rightarrow A \times B.
 \end{aligned}$$

For the last one, (p, q) is conservative (by the discreteness of (p, E, q)) and Propositions 5.17 and 5.13 demonstrate that $\text{pr}_E t$ is a left fibration and thus a strong liberal. Composing d with each of these strong liberals in turn, and using the isomorphisms in (12), we get $md \cong e_0$, $\text{pr}_{F_1} td \cong \text{pr}_{F_1}(f, e_1, \theta_B) \cong e_1$ and $\text{pr}_E td \cong \text{pr}_E(f, e_1, \theta_B) \cong f$. Notice though that e_0 and e_1 are strong liberals (by definition) as is f (since it is a left

fibration), so we may apply axiom 5.2(vi) and infer that d is a strong liberal as well. \square

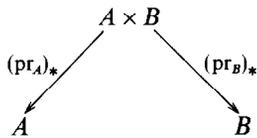
5.19. Proposition. *The inclusion $(\)_*: \mathcal{K} \rightarrow \mathcal{F}$ preserves the terminal object of \mathcal{K} .*

Proof. The category $\mathcal{F}(A, 1)$ is non-empty, since it contains the discrete fibration $(\square, A, \text{id}_A)$, where $\square: A \rightarrow 1$ is the (essentially) unique arrow required by the terminality of 1. Any other discrete fibration $(p, E, q): A \rightarrow 1$ has $(p, q): E \rightarrow 1 \times A$ conservative and equivalent to $q: E \rightarrow A$, which is strongly liberal by Proposition 5.13. So q is an equivalence. It follows that $\mathcal{F}(A, 1)$ is discrete, and so equivalent to $\mathbb{1}$. \square

5.20. Proposition. *The inclusion $(\)_*: \mathcal{K} \rightarrow \mathcal{F}$ preserves the binary products of \mathcal{K} .*

Proof. By applying Proposition 5.9 to the result of Proposition 5.19 we establish that \mathcal{F} possesses local terminal objects. Remark 5.8 demonstrates that the terminal object in $\mathcal{F}(B, A)$ is the discrete fibration $(\text{pr}_A, A \times B, \text{pr}_B)$.

Clearly the lax limit of the normal comorphism $\mathbb{1} \amalg \mathbb{1} \rightarrow \mathcal{F}$, which picks out a pair of objects $A, B \in \mathcal{F}$, is no more or less than their product in \mathcal{F} . Remark 5.7 implies that we can express this lax limit as that of the terminal arrow $(\text{pr}_A, A \times B, \text{pr}_B) \in \mathcal{F}(B, A)$, so by Theorem 5.18 the diagram



displays a product in \mathcal{F} as required. \square

5.21. Proposition. *The inclusion $(\)_*: \mathcal{K} \rightarrow \mathcal{F}$ preserves cotensors with \mathbb{P} .*

Proof. By applying Proposition 5.9 to the result of Proposition 5.20 we establish that \mathcal{F} possesses all local binary (and therefore finite) products. Remark 5.8 demonstrates that the product of discrete fibrations (p, E, q) and (p', E', q') in $\mathcal{F}(B, A)$ is simply the discrete fibration $(p \text{pr}_E, E \times_{A \times B} E', q \text{pr}_E)$.

The lax limit of the comorphism $T: \mathbb{P} \rightarrow \mathcal{F}$, which sends both non-identity arrows of \mathbb{P} to the identity arrow $(d_0, A^2, d_1): A \rightarrow A$, is precisely the cotensor of A with \mathbb{P} in \mathcal{F} . We have seen that \mathcal{F} has local finite products, so Remark 5.7 can be applied to show that the lax limit of T may be obtained as that of the arrow obtained by taking the product of two copies of (d_0, A^2, d_1) in $\mathcal{F}(A, A)$. But the pullback $A^2 \times_{A \times A} A^2$, which occurs in the construction of this product of arrows, is equivalent to $A^{\mathbb{P}}$, so the lax limit of T is also equivalent to that of the discrete fibration $(d_0, A^{\mathbb{P}}, d_1)$. Theorem 5.18 demonstrates that the lax limit of this final arrow (and so of T) in \mathcal{F} is $A^{\mathbb{P}}$ as required. \square

5.22. Theorem. \mathcal{F} has all local finite limits.

Proof. Finally, applying Proposition 5.9 to the result of Proposition 5.21 we establish that \mathcal{F} also has local equalisers. This completes the list of local limits required for the construction of all finite local limits (cf. Remark 5.10). \square

6. Constructing modulated bicategories

6.1. In this section we will examine the principal construction giving rise to modulated bicategories. We start with a bicategory \mathcal{M} (which in practice will be some bicategory of enriched or internal profunctors) possessing:

- (i) finite bicategorical coproducts;
- (ii) the Kleisli object for each monad;
- (iii) local finite colimits (which we should recall are, by definition, preserved by composition on both sides).

We call such a bicategory \mathcal{M} a *finitary cosmos* [23].

An arrow f in \mathcal{M} is called a *map* when it has a right adjoint f^* . We will adopt the convention of using $\eta_f: 1 \Rightarrow f^*f$ and $\varepsilon_f: ff^* \Rightarrow 1$ to denote the unit and counit of such a map. Let \mathcal{M}^* denote the (locally full) sub-bicategory of \mathcal{M} with the same objects and only those arrows which are maps.

6.2. For a monad m on an object A in a bicategory \mathcal{M} , we always denote the unit and multiplication by $\eta: 1_A \Rightarrow m$ and $\mu: mm \Rightarrow m$. An *m-algebra into X* is a pair (a, α) where $a: A \rightarrow X$ is an arrow and $\alpha: am \Rightarrow a$ is a 2-cell satisfying $\alpha \cdot a\eta = 1_a$ and $\alpha \cdot a\mu = \alpha \cdot \alpha m$. The category $\mathcal{M}(A, X)^{\mathcal{M}(m, X)}$ of *m-algebras into X* is the Eilenberg–Moore category for the monad $\mathcal{M}(m, X)$ on the category $\mathcal{M}(A, X)$. An *m-algebra (a, α) into K* presents that object as the *Kleisli object* of m if composition with (a, α) provides us with an equivalence

$$\mathcal{M}(K, X) \xrightarrow{\sim} \mathcal{M}(A, X)^{\mathcal{M}(m, X)}$$

for each object X . In fact a monad m is precisely a morphism of bicategories $\mathbb{1} \rightarrow \mathcal{M}$, *m-algebras into X* are no more than lax cones under that diagram (with vertex X) and the Kleisli object of m (if it exists) is its lax (bi)colimit.

6.3. Recall that an arrow in a category is called an *extremal epic* when it is epic, and any monic into its target, through which it factors, is invertible.

A map $e: A \rightarrow B$ in \mathcal{M} is called *Cauchy dense* when its counit $\varepsilon_e: ee^* \Rightarrow 1_B$ is an extremal epic in the category $\mathcal{M}(B, B)$. It was essentially proved in Proposition 1 of [23] that the following conditions are equivalent:

- (a) $e: A \rightarrow B$ is Cauchy dense;
- (b) the diagram

$$ee^*ee^* \begin{array}{c} \xrightarrow{ee^*\varepsilon_e} \\ \xrightarrow{\varepsilon_e ee^*} \end{array} ee^* \xrightarrow{\varepsilon_e} 1_B$$

- is a coequaliser in $\mathcal{M}(B, B)$;
- (c) for all objects X , the functor $\mathcal{M}(e, X): \mathcal{M}(B, X) \rightarrow \mathcal{M}(A, X)$ is conservative, in other words e is liberal as an arrow of \mathcal{M} (cf. 2.1 and 2.2);
- (d) the object B , arrow e and 2-cell ε_e provide a Kleisli construction for the monad on A generated by $e \dashv e^*$;
- (e) the object B , arrow e^* and 2-cell $e^* \varepsilon_e$ provide an Eilenberg–Moore construction for the monad on A generated by $e \dashv e^*$;
- (f) e^* is conservative as an arrow of \mathcal{M} .

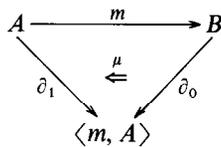
Proof. The equivalence of (b), (c) and (d) was given in Proposition 1 of [23]. The equivalence of (b), (e) and (f) follows from the same proposition, but this time interpreted in the dual bicategory \mathcal{M}^{op} . That (b) implies (a) is trivial, since regular epics are extremal. It remains to see that (a) implies (b).

Since composition preserves local epics, each $m\varepsilon_e: mee^* \Rightarrow m$ is epic; so $\mathcal{M}(e, X)$ is faithful for all X when ε_e is epic. Take the coequaliser $\gamma: ee^* \Rightarrow n$ of $ee^*\varepsilon_e$ and $\varepsilon_e e^*$ in $\mathcal{M}(B, B)$. Then there exists $\mu: n \Rightarrow 1_B$ with $\varepsilon_e = \mu\gamma$. Since $\mathcal{M}(e, B)$ takes the diagram of (b) to a split coequaliser and preserves the coequaliser γ , we have that $\mu\varepsilon$ is invertible. But $\mathcal{M}(e, B)$ is faithful, so μ is monic. Finally if ε_e is extremal epic, it factors through the monic μ , which is therefore invertible, proving that (a) implies (b). \square

6.4. A map $f: A \rightarrow B$ is called *fully faithful* when its unit $\eta_f: 1_A \Rightarrow f^*f$ is an isomorphism in $\mathcal{M}(A, A)$.

It was shown in Proposition 1 of [23] that each map $f: A \rightarrow B$ factors up to isomorphism as je where j is a fully faithful map and e is Cauchy dense. The map e is obtained by taking the Kleisli construction of the monad associated with the adjunction $f \dashv f^*$.

6.5. For each arrow $m: A \rightarrow B$, there is a universal diagram



in \mathcal{M} ; that is, the cocomma object $\langle m, A \rangle$ of $m, 1_A$ exists. The cospan $(\partial_1, \langle m, A \rangle, \partial_0)$ from A to B is called the *cone on m* . To construct $\langle m, B \rangle$ one takes the Kleisli construction for the monad $(\begin{smallmatrix} 1_A & 0 \\ m & 1_B \end{smallmatrix})$ on the coproduct $A \oplus B$ (cf. [29] for details of this matrix notation). It follows that ∂_1 and ∂_0 are fully faithful maps such that $(\partial_1, \partial_0): A \oplus B \rightarrow \langle m, A \rangle$ is Cauchy dense, the 2-cell $m \Rightarrow \partial_1^* \partial_0$ induced by μ is invertible, and $\partial_0^* \partial_1$ is initial in $\mathcal{M}(B, A)$ (cf. [23, Proposition 1]). When $m = 1_A$, note that $\langle m, A \rangle$ is the *tensor product* $2 * A$ of the category 2 with the object $A \in \mathcal{M}$.

could arise by doing so. We say that a monad morphism (u, ϕ) is *strong* if ϕ is an isomorphism. Monad morphisms generalise algebras, for instance an m -algebra into X is precisely a monad morphism from m to the identity monad on X .

Let (a, α) present K as the Kleisli object of m , and (b, β) present L as the Kleisli object of n . Using the compatibility conditions on ϕ , it is easily checked that the pair $(bu, \beta u \cdot b\phi)$ is an m -algebra into L ; so there is an essentially unique arrow $\overline{(u, \phi)}: K \rightarrow L$ with $\overline{(u, \phi)}(a, \alpha) \cong (bu, \beta u \cdot b\phi)$, and this is a map if u is.

If u is a map then the transpose of $\beta u \cdot b\phi: bum \Rightarrow bu$, under the adjunction $bu \dashv u^*b^*$, can be obtained via the following calculation:

$$\frac{\frac{bum \xrightarrow{b\phi} bnu \xrightarrow{\beta u} bu}{um \xrightarrow{\phi} nu \xrightarrow{\hat{\beta}u} b^*bu} \quad b \dashv b^*}{m \xrightarrow{\eta_u m} u^*um \xrightarrow{u^*\phi} u^*nu \xrightarrow{u^*\hat{\beta}u} u^*b^*bu} \quad u \dashv u^*$$

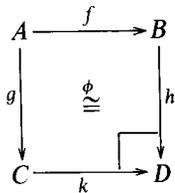
Now (b, β) displays a Kleisli object so $\hat{\beta}$ is an isomorphism; therefore when u is fully faithful and (u, ϕ) is strong, the bottom line of this calculation is an isomorphism. But this is precisely the case considered in 6.6, and it follows that, under these conditions, $\overline{(u, \phi)}$ is itself fully faithful.

As an example consider the tensors 2^*A in 6.5 above, which were constructed by taking the Kleisli construction for a monad $(\begin{smallmatrix} 1_A & 0 \\ 1_A & 1_A \end{smallmatrix})$ on $A \oplus A$. For any arrow $j: A \rightarrow B$, the canonical arrow $2^*j: 2^*A \rightarrow 2^*B$ that it induces may be described in terms of a monad morphism. This consists of an arrow $(\begin{smallmatrix} j & 0 \\ 0 & j \end{smallmatrix}): A \oplus A \rightarrow B \oplus B$ and the isomorphism

$$\left(\begin{smallmatrix} j & 0 \\ 0 & j \end{smallmatrix} \right) \left(\begin{smallmatrix} 1_A & 0 \\ 1_A & 1_A \end{smallmatrix} \right) \cong \left(\begin{smallmatrix} j & 0 \\ j & j \end{smallmatrix} \right) \cong \left(\begin{smallmatrix} 1_B & 0 \\ 1_B & 1_B \end{smallmatrix} \right) \left(\begin{smallmatrix} j & 0 \\ 0 & j \end{smallmatrix} \right).$$

Now suppose that j is a fully faithful map, then so is $(\begin{smallmatrix} j & 0 \\ 0 & j \end{smallmatrix})$ (since its right adjoint is $(\begin{smallmatrix} j^* & 0 \\ 0 & j^* \end{smallmatrix})$ with unit $(\begin{smallmatrix} \eta_j & 0 \\ 0 & \eta_j \end{smallmatrix})$), so the conditions of the last paragraph apply and therefore 2^*j is fully faithful.

6.8. Proposition. *In the bicategory \mathcal{M}^* of maps in a finitary cosmos \mathcal{M} , the pseudopushout*



exists iff f is fully faithful and is preserved by the inclusion $\mathcal{M}^ \rightarrow \mathcal{M}$. Moreover, k is fully faithful.*

Proof. First form $B \oplus C$ and consider the matrix

$$m = \begin{pmatrix} 1_B + fg^*gf^* & fg^* \\ gf^* & 1_C \end{pmatrix}$$

representing an endo-arrow on $B \oplus C$. We define a candidate identity 2-cell, for a monad with underlying arrow m , as a matrix

$$\eta = \begin{pmatrix} c_0 & !_{fg^*} \\ !_{gf^*} & 1_{1_C} \end{pmatrix} : \begin{pmatrix} 1_B & 0 \\ 0 & 1_C \end{pmatrix} \Rightarrow \begin{pmatrix} 1_B + fg^*gf^* & fg^* \\ gf^* & 1_C \end{pmatrix}$$

where $c_0 : 1_B \Rightarrow 1_B + fg^*gf^*$ is the canonical coprojection, and $!_h : 0 \Rightarrow h : X \rightarrow Y$ denotes the unique 2-cell into h from the initial object of $\mathcal{M}(X, Y)$. To construct a multiplication 2-cell for this monad, first notice that (traditional) matrix composition gives

$$\begin{pmatrix} 1_B + fg^*gf^* & fg^* \\ gf^* & 1_C \end{pmatrix}^2 = \begin{pmatrix} 1_B + 3 \bullet fg^*gf^* + fg^*gf^*fg^*gf^* & 2 \bullet fg^* + fg^*gf^*fg^* \\ 2 \bullet gf^* + gf^*fg^*gf^* & 1_C + gf^*fg^* \end{pmatrix}$$

where

$$n \bullet h \stackrel{\text{def}}{=} \underbrace{h + h + \dots + h}_{n \text{ terms}}$$

for $n \in \mathbb{N}$ and any arrow $h \in \mathcal{M}$. We provide a multiplication $\mu : m^2 \Rightarrow m$ “componentwise”, by supplying a 2-cell between each pair of corresponding matrix entries. These can be constructed by combining various *fold* maps $\nabla : n \bullet h \Rightarrow h$ and the composite

$$\gamma = gf^*fg^* \xrightarrow{g\eta_f^{-1}g^*} gg^* \xrightarrow{c_g} 1_C$$

(for which we need the assumption that f is fully faithful). For instance the first component is induced, via the universal property of $1_B + 3 \bullet fg^*gf^* + fg^*gf^*fg^*gf^*$, by the 2-cells

$$1_B \xrightarrow{c_0} 1_B + fg^*gf^*,$$

$$3 \bullet fg^*gf^* \xrightarrow{\nabla} fg^*gf^* \xrightarrow{c_1} 1_B + fg^*gf^*,$$

$$fg^*gf^*fg^*gf^* \xrightarrow{fg^*\gamma gf^*} fg^*gf^* \xrightarrow{c_1} 1_B + fg^*gf^*,$$

where c_c and c_1 are the canonical coprojections associated with $1_B + fg^*gf^*$. The remaining components are defined as variations on this theme so we leave these

details, along with the verification that (m, μ, v) is a monad, up to the reader.

Consider the m -algebras into X . An arrow $a: B \oplus C \rightarrow X$ is determined by a row vector (b, c) of arrows $b: B \rightarrow X$ and $c: C \rightarrow X$. Matrix multiplication gives

$$am = (b \ c) \begin{pmatrix} 1_B + fg^*gf^* & fg^* \\ gf^* & 1_C \end{pmatrix} = (b + bfg^*gf^* + cgf^* \ bfg^* + c)$$

so a 2-cell $\alpha: am \Rightarrow a$ is determined by giving 2-cells:

$$\alpha_0: b \Rightarrow b, \quad \alpha_1: bfg^*gf^* \Rightarrow b, \quad \alpha_2: cgf^* \Rightarrow b, \\ \alpha_3: bfg^* \Rightarrow c, \quad \alpha_4: c \Rightarrow c.$$

Unravelling the definition of η , we see that the unit condition on (a, α) reduces to the stipulation that $\alpha_0 = 1_b$ and $\alpha_4 = 1_c$. Compatibility with μ reduces to the commutativity of the following three diagrams:

(i)

$$\begin{array}{ccc} bfg^*gf^* & \xrightarrow{\alpha_3gf^*} & cgf^* \\ & \searrow \alpha_1 & \downarrow \alpha_2 \\ & & b \end{array}$$

(ii)

$$\begin{array}{ccc} cgf^*fg^* & \xrightarrow{\alpha_2fg^*} & bfg^* \\ c\eta_f g^* \uparrow & & \downarrow \alpha_3 \\ cgg^* & \xrightarrow{c\epsilon_g} & c \end{array}$$

(13)

(iii)

$$\begin{array}{ccc} bfg^*gf^*fg^*gf^* & \xrightarrow{\alpha_1fg^*gf^*} & bfg^*gf^* \\ \uparrow bfg^*g\eta_f g^*gf^* & & \downarrow \alpha_1 \\ bfg^*gg^*gf^* & \xrightarrow{bfg^*\epsilon_g gf^*} & bfg^*gf^* \xrightarrow{\alpha_1} b \end{array}$$

The first of these eliminates α_1 , so substitute for α_1 in (13) (iii) and simplify (by applying triangle identities and (13) (ii)). A little effort reveals that the commutativity of our pentagon, in the presence of the first two conditions, corresponds to the commutativity of

$$\begin{array}{ccc} bfg^*gf^* & \xrightarrow{\alpha_3gf^*} & cgf^* \\ b\eta_g f^* \downarrow & & \downarrow \alpha_2 \\ bff^* & \xrightarrow{b\epsilon_f} & b \end{array} \tag{14}$$

Now α_2 and α_3 correspond, under the adjunctions $f \dashv f^*$ and $g \dashv g^*$, to 2-cells $\hat{\alpha}_2: cg \Rightarrow bf$ and $\hat{\alpha}_3: bf \Rightarrow cg$ respectively. Finally, by applying triangle identities, we may demonstrate that our remaining conditions (13) (ii) and (14) correspond to the

identities $\hat{\alpha}_2 \hat{\alpha}_3 = 1_{bf}$, $\hat{\alpha}_3 \hat{\alpha}_2 = 1_{cg}$. Following the action of this construction on 2-cells, a process we leave up to the reader, it should be clear that we have constructed an equivalence

$$\mathcal{M}(B \oplus C, X)^{\mathcal{M}(m, X)} \xrightarrow{\sim} \mathcal{M}(B, X) \times_{\mathcal{M}(A, X)} \mathcal{M}(C, X)$$

By assumption there is a Kleisli object D , presented by an m -algebra (r, ψ) , and let $(h, k, \phi: hf \cong kg)$ be the corresponding cocone under (f, g) , then we have a diagram

$$\begin{array}{ccc} & \mathcal{M}(D, X) & \\ \cong \swarrow & & \searrow \\ \mathcal{M}(B \oplus C, X)^{\mathcal{M}(m, X)} & \xrightarrow{\sim} & \mathcal{M}(B, X) \times_{\mathcal{M}(A, X)} \mathcal{M}(C, X) \end{array}$$

where the diagonals are induced by (r, ψ) and (h, k, ϕ) respectively. So the right-hand diagonal is an equivalence as well, which means that (h, k, ϕ) is the pseudopushout of f along g in \mathcal{M} , and consequently in \mathcal{M}^* (since factoring through the Kleisli object D respects maps), as required.

It is now easy to see that k is fully faithful, the monad m is isomorphic to that associated with $r \dashv r^*$, but $r^*r = \begin{pmatrix} h^* & h \\ k^* & k \end{pmatrix} = \begin{pmatrix} h^* & h & h^* & k \\ k^* & h & k^* & k \end{pmatrix}$. So the compatibility of the isomorphism

$$\begin{pmatrix} \theta_0 & \theta_1 \\ \theta_2 & \theta_3 \end{pmatrix} : \begin{pmatrix} 1_B + fg^*gf^* & fg^* \\ gf^* & 1_C \end{pmatrix} \cong \begin{pmatrix} h^* & h & h^* & k \\ k^* & h & k^* & k \end{pmatrix}$$

with the units η and $\begin{pmatrix} \eta_0 & 0 \\ 0 & \eta_k \end{pmatrix}$ implies that $\eta_k = \theta_3$, which is invertible. \square

6.9. Theorem. *A map $e: A \rightarrow B$ is Cauchy dense iff, for all fully faithful maps $j: X \rightarrow Y$, all maps $u: A \rightarrow X$, $v: B \rightarrow Y$, and all invertible 2-cells $\tau: ju \cong ve$, there exists a map $w: B \rightarrow X$ and an invertible 2-cell $v: jw \cong v$.*

Proof. (\Rightarrow) The map j is fully faithful, so we can form a composite isomorphism

$$\lambda = u \xrightarrow[\cong]{\eta_j u} j^* j u \xrightarrow[\cong]{j^* \tau} j^* v e$$

which we use to make u into an m -algebra, where m is the monad associated with e , by supplementing it with a 2-cell:

$$\alpha = u e^* e \xrightarrow[\cong]{\lambda e^* e} j^* v e e^* e \xrightarrow{j^* v \varepsilon_e e} j^* v e \xrightarrow[\cong]{\lambda^{-1}} u$$

But e is Cauchy dense, which means that $(e, \varepsilon_e e)$ presents B as the Kleisli object of m , therefore there is an essentially unique arrow $w: B \rightarrow X$ such that $w(e, \varepsilon_e e) \cong (u, \alpha)$. Furthermore, Kleisli objects in \mathcal{M} respect maps so w is a map since u is.

Notice that α is defined to ensure that the 2-cell $\tau:ju \cong ve$ gives an isomorphism of m -algebras (into Y) between $j(u, \alpha) = (ju, j\alpha)$ and $v(e, \varepsilon_e e) = (ve, v\varepsilon_e e)$. It follows that $jw(e, \varepsilon_e e) \cong j(u, \alpha) \cong v(e, \varepsilon_e e)$, or in other words the maps jw and v are obtained by factoring the same m -algebra through the limiting algebra $(e, \varepsilon_e e)$, so the essential uniqueness of such factorings implies that $jw \cong v$ as required.

(\Leftarrow) Consider the condition on e in the statement of the theorem. From the fact that j is fully faithful we can infer more about the induced map $w: B \rightarrow Y$. For instance, if $w': B \rightarrow Y$ is another map along with an isomorphism $v': jw' \cong v$, then the composite

$$w \xrightarrow[\cong]{\eta_j w} j*jw \xrightarrow[\cong]{j^*v} j*v \xrightarrow[\cong]{j^*(v')^{-1}} j*jw' \xrightarrow[\cong]{\eta_j^{-1}w'} w'$$

demonstrates that $w \cong w'$. Reasoning similarly, the composite

$$we \xrightarrow[\cong]{\eta_j we} j*jwe \xrightarrow[\cong]{j^*ve} j*ve \xrightarrow[\cong]{j^*\tau} j*ju \xrightarrow[\cong]{\eta_j^{-1}u} u$$

provides an isomorphism $\xi: we \cong u$, which satisfies the pasting identity

$$\begin{array}{ccc} A & \xrightarrow{e} & B \\ u \downarrow & \cong \tau & \downarrow v \\ X & \xrightarrow{j} & Y \end{array} = \begin{array}{ccc} A & \xrightarrow{e} & B \\ u \downarrow & \cong \xi & \downarrow v \\ X & \xrightarrow{j} & Y \\ & \swarrow w & \downarrow v \\ & & Y \end{array}$$

So suppose that e is a map satisfying the condition in the statement of this theorem. We know that there is a factorisation $e \cong ju$, with $j: C \rightarrow B$ fully faithful and $u: A \rightarrow C$ Cauchy dense, so setting $v = 1_B$ we may “fill the diagonal” to get a map $w: B \rightarrow C$ and isomorphisms $v: jw \cong 1_B$, $\xi: we \cong u$. But the composite

$$wj \xrightarrow[\cong]{\eta_j kj} j*jwj \xrightarrow[\cong]{j^*vj} j*j \xrightarrow[\cong]{\eta_j^{-1}} 1_C$$

demonstrates that $wj \cong 1_C$, so j is an equivalence with inverse w . Therefore e , as a composite of this equivalence and the Cauchy dense map u , is itself Cauchy dense. \square

6.10. Proposition 6.8 gives examples of pseudopushouts of maps which exist in any finitary cosmos, but in order to get all such pseudopushouts we need to adopt extra assumptions. An example of a finitary cosmos without pseudopushouts of maps is the bicategory of categories and profunctors internal to any elementary topos which lacks a natural numbers object.

6.11. If $r: A \rightarrow A$ is an endo-arrow on A then an r -algebra into X , (a, θ) , consists of an arrow $a: A \rightarrow X$ and a 2-cell $\theta: ar \Rightarrow a$. A morphism $\xi: (a, \theta) \Rightarrow (a', \theta')$ of r -algebras into

X consists of a 2-cell $\xi: a \Rightarrow a'$ such that $\xi \cdot \theta = \theta' \cdot \zeta r$. For each X we get a category $\text{Alg}(\mathcal{M})(r, X)$ of r -algebras into X , and their morphisms.

An algebraically free monad on an endo-arrow $r: A \rightarrow A$ is a monad m on A and a 2-cell $\lambda: r \Rightarrow m$ such that, for each object X , the functor

$$\begin{array}{ccc}
 \mathcal{M}(A, X)^{\mathcal{M}(m, X)} & \xrightarrow{\tilde{\lambda}} & \text{Alg}(\mathcal{M})(r, X) \\
 (a, \alpha) & \xrightarrow{\quad} & (a, \alpha \cdot a\lambda) \\
 \downarrow \xi & \xrightarrow{\quad} & \downarrow \xi \\
 (a', \alpha') & \xrightarrow{\quad} & (a', \alpha' \cdot a'\lambda)
 \end{array} \tag{15}$$

is an isomorphism. This condition is not really as strong as it might seem, if $\tilde{\lambda}$ is an equivalence it is easily demonstrated that it must necessarily be an isomorphism.

We define an iterative cosmos to be a finitary cosmos in which each endo-arrow has an algebraically free monad upon it.

6.12. In a finitary cosmos, the algebraically free monad m upon an endo-arrow $r: A \rightarrow A$ is also free, in the sense that if n is another monad on A , and $\xi: r \Rightarrow n$ is any 2-cell, then there is a unique monad morphism $\phi: m \Rightarrow n$ with $\phi \cdot \lambda = \xi$.

To prove this, first fix algebras (a, α) , presenting K as a Kleisli object for m , and (b, β) , presenting L as a Kleisli object for n . Then $(b, \beta \cdot b\xi)$ is an r -algebra, so there is an m -algebra (b, γ) such that $\gamma \cdot b\lambda = \beta \cdot b\xi$ and, since (a, α) is Kleisli, there is an (essentially unique) map $c: K \rightarrow L$ such that $c(a, \alpha) \cong (b, \gamma)$. But the adjoint transposes of $\alpha: am \Rightarrow a, \beta: bn \Rightarrow b$ provide us with isomorphisms $m \cong a^*a, n \cong b^*b$, so define ϕ to be the composite:

$$m \xrightarrow{\cong} a^*a \xrightarrow{a^*\eta_c a} a^*c^*ca \xrightarrow{\cong} b^*b \xrightarrow{\cong} n.$$

We leave it up to the reader to check that this is indeed a monad morphism, and the unique such with $\xi = \phi \cdot \lambda$.

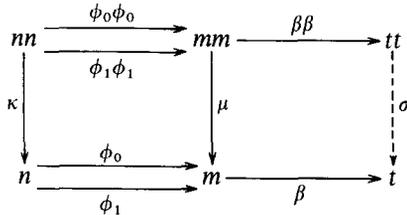
6.13. An endo-arrow $r: A \rightarrow A$ together with a 2-cell $\rho: 1_A \Rightarrow r$ is called a pointed endo-arrow on A . An (r, ρ) -algebra into X is an r -algebra (a, θ) into X satisfying $\theta \cdot a\rho = 1_a$, and there is a category $\text{Alg}(\mathcal{M})(r, \rho, X)$ of (r, ρ) -algebras into X . We say that a monad m on A is the algebraically free monad on (r, ρ) if there is a 2-cell $\lambda: r \Rightarrow m$, with $\lambda \cdot \rho = \eta$, such that the obvious functor

$$\mathcal{M}(A, X)^{\mathcal{M}(m, X)} \xrightarrow{\tilde{\lambda}} \text{Alg}(\mathcal{M})(r, \rho, X)$$

is an isomorphism for each object X . In a finitary cosmos \mathcal{M} it is again true that an algebraically free monad on a pointed endo-arrow is free in the usual sense.

If a pointed endo-arrow (r, ρ) satisfies the extra condition $r\rho = \rho r$ then it is called *well pointed* (cf. [10]).

6.14. Remark. Suppose that (m, μ, η) and (n, κ, τ) are monads on $A \in \mathcal{M}$, and the 2-cells $\phi_0, \phi_1: n \Rightarrow m$ are a *reflexive* pair of monad morphisms (so there is some 2-cell $\zeta: m \Rightarrow n$ with $\phi_0 \cdot \zeta = \phi_1 \cdot \zeta = 1_m$). Form the coequaliser $\beta: m \Rightarrow t$ of this pair in $\mathcal{M}(A, A)$, then the compatibility of ϕ_0, ϕ_1 with the multiplications of n and m ensures that we get a serially commutative diagram:



The preservation of such local coequalisers by composition in \mathcal{M} , along with the reflexivity condition on our pair, allow us to show that the top line of this diagram is a coequaliser (cf. [8, Lemma 0.17]), which explains the existence of the induced (dashed) 2-cell σ . This, along with a unit $\beta \cdot \eta: 1_A \Rightarrow t$, makes t into a monad, and β becomes a morphism of monads $(m, \mu, \eta) \Rightarrow (t, \sigma, \beta \cdot \eta)$. In fact t is the coequaliser of ϕ_0, ϕ_1 in the category of monads on A .

Given any morphism of monads $\phi: n \Rightarrow m$, we get a functor

$$\begin{array}{ccc}
 \mathcal{M}(A, X)^{\mathcal{M}(m, X)} & \xrightarrow{\tilde{\phi}} & \mathcal{M}(A, X)^{\mathcal{M}(n, X)} \\
 (a, \alpha) & \longmapsto & (a, \alpha \cdot a\phi)
 \end{array}$$

for each object X . An important property of the coequaliser constructed in the last paragraph is that the diagram

$$\mathcal{M}(A, X)^{\mathcal{M}(t, X)} \xrightarrow{\tilde{\beta}} \mathcal{M}(A, X)^{\mathcal{M}(m, X)} \begin{array}{c} \xrightarrow{\tilde{\phi}_0} \\ \xrightarrow{\tilde{\phi}_1} \end{array} \mathcal{M}(A, X)^{\mathcal{M}(n, X)} \quad (16)$$

is a strict (2-categorical) equaliser of categories, for each X . This follows directly from the assumption that local coequalisers are preserved by composition in \mathcal{M} .

In what follows we will not want to apply this result to reflexive pairs, but pairs where (n, κ, τ) is the algebraically free monad on some endo-arrow $r: A \rightarrow A$. To do so we take the coproduct of arrows $r + m$ in $\mathcal{M}(A, A)$, and let (n', κ', τ') be the algebraically free monad on $r + m$, as presented by a 2-cell $\lambda': r + m \Rightarrow n'$. Using the universal properties of algebraically free monads and coproducts, we can demonstrate a natural bijection between monad morphisms $\phi: n \Rightarrow m$ and $\phi': n' \Rightarrow m$ with $\phi' \cdot \lambda' \cdot c_m = 1_m$ (where $c_m: m \Rightarrow r + m$ is the coprojection of m into $r + m$). So from ϕ_0, ϕ_1 we get a reflexive pair $\phi'_0, \phi'_1: n' \Rightarrow m$, to which we may apply the above; the resulting coequaliser is also that of the pair we started with. Finally, it is easily checked that the strict equaliser of (16) arises from the corresponding one for the pair (ϕ'_0, ϕ'_1) .

6.15. Proposition. For a finitary cosmos \mathcal{M} the following are equivalent:

- (a) any endo-arrow r admits an algebraically free monad thereon;
- (b) any pointed endo-arrow (r, ρ) admits an algebraically free monad thereon;
- (c) any well-pointed endo-arrow (r, ρ) admits an algebraically free monad thereon.

Proof. (a) \Rightarrow (c) Suppose that (r, ρ) is a well-pointed endo-arrow on A (as in (c)), then (a) ensures that we may form an algebraically free monad m upon the endo-arrow r , presented by a 2-cell $\lambda: r \Rightarrow m$. Also let $\bar{1}_A$ denote the algebraically free monad on the identity 1_A , as presented by $v: 1_A \Rightarrow \bar{1}_A$. Now m has two points, its unit $\eta: 1_A \Rightarrow m$ and the composite $\lambda \cdot \rho: 1_A \Rightarrow m$, so the freeness property of $\bar{1}_A$, as given in 6.12, implies that we get unique monad morphisms $\phi_0, \phi_1: \bar{1}_A \Rightarrow m$ with $\phi_0 \cdot v = \eta$, $\phi_1 \cdot v = \lambda \cdot \rho$. Taking the coequaliser of ϕ_0 and ϕ_1 we get a monad (t, σ, ζ) and a map $\beta \cdot \lambda: r \Rightarrow t$. cf. Remark 6.14.

Now we form a serially commutative diagram

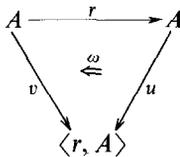
$$\begin{array}{ccccc}
 \mathcal{M}(A, X)^{\mathcal{A}(t, X)} & \xrightarrow{\tilde{\beta}} & \mathcal{M}(A, X)^{\mathcal{A}(m, X)} & \xrightarrow[\tilde{\phi}_1]{\tilde{\phi}_0} & \mathcal{M}(A, X)^{\mathcal{A}(\bar{1}_A, X)} \\
 & & \downarrow \tilde{\lambda} \cong & & \downarrow \cong \tilde{v} \\
 & & \text{Alg}(\mathcal{M})(r, \rho, X) & \xrightarrow[F_1]{F_0} & \text{Alg}(\mathcal{M})(1_A, X)
 \end{array}$$

in which the upper line is the strict equaliser from diagram (16). To calculate the actions of F_0, F_1 we use the defining equations $F_i \tilde{\lambda} = \tilde{v} \tilde{\phi}_i$, $= \tilde{\phi}_i v$, along with the fact that $\tilde{\lambda}$ is an isomorphism. The equalities that we used to define ϕ_0 and ϕ_1 imply that $\tilde{\phi}_0 v = \tilde{\eta}$ and $\tilde{\phi}_1 v = \tilde{\lambda} \cdot \rho$, but if (a, α) is an m -algebra then $\alpha \cdot a\eta = 1_a$ so $\tilde{\eta}(a, \alpha) = (a, 1_a)$, also $\tilde{\lambda} \cdot \rho(a, \alpha) = (a, (\alpha \cdot a\lambda) \cdot a\rho)$. Now if (a, θ) is an r -algebra, there is a unique m -algebra (a, α) with $\theta = \alpha \cdot a\lambda$, and we get

$$\begin{aligned}
 F_0(a, \theta) &= F_0 \tilde{\lambda}(a, \alpha) = (a, 1_a), \\
 F_1(a, \theta) &= F_1 \tilde{\lambda}(a, \alpha) = (a, (\alpha \cdot a\lambda) \cdot a\rho) = (a, \theta \cdot a\rho).
 \end{aligned}$$

The functor $\tilde{\beta} \cdot \tilde{\lambda}$ provides us with an isomorphism between $\mathcal{M}(A, X)^{\mathcal{A}(t, X)}$ and the strict equaliser of F_0 and F_1 , but our descriptions of these functors reveal that this is the full subcategory of r -algebras (a, α) with $1_a = \alpha \cdot a\lambda$, or in other words the category of (r, ρ) -algebras. This is precisely what is needed to prove that $\beta \cdot \lambda$ presents t as the algebraically free monad on the (well-)pointed endo-arrow (r, ρ) , cf. 6.13.

(c) \Rightarrow (b) Let (r, ρ) be a pointed endo-arrow on A . Form the cone on r



as in 6.5, and let $\gamma:vr \Rightarrow w$ be the coequaliser of the two 2-cells $ur \Rightarrow vr$ given by

$$ur \begin{array}{c} \xrightarrow{u\rho r} \\ \xrightarrow{u\rho} \\ \xrightarrow{ur\rho} \end{array} urr \xrightarrow{\omega r} vs$$

Now we define $t:\langle r, A \rangle \rightarrow \langle r, A \rangle$, via the universal property of $\langle r, A \rangle$, to be the essentially unique arrow with isomorphisms $\theta:tv \cong w$, $\phi:tu \cong v$ such that $\gamma \cdot r\phi = \theta \cdot tw$. This has a point $\tau:1_A \Rightarrow t$ defined, (again) via the universal property of $\langle r, A \rangle$, by $\theta \cdot v\tau = \gamma \cdot v\rho$ and $\phi \cdot u\tau = w \cdot u\rho$. One easily sees that $t\tau = \tau t$, and the category of (t, τ) -algebras into X is equivalent to that of (r, ρ) -algebras into X . So an algebraically free monad on (t, τ) gives one on (r, ρ) . Compare with [10, Section 17.1 p. 50].

(b) \Rightarrow (a) If $r:A \rightarrow A$ then $t = 1_A + r:A \rightarrow A$ becomes pointed by the coprojection $\tau:1_A \Rightarrow t$; moreover, the category of r -algebras into X is equivalent to that of (t, τ) -algebras into X . So an algebraically free monad on (t, τ) gives one on r . \square

6.16. The axioms for an iterative cosmos are chosen to be finitary, but if we are willing to allow ourselves some countable local colimits we may construct algebraically free monads in a straightforward manner. For instance, if \mathcal{M} has countable local coproducts (which, as usual, are assumed to be preserved by composition) then the algebraically free monad on an endo-arrow $r:A \rightarrow A$ can be constructed as $\sum_{n=0}^{\infty} r^n$.

An application of the last proposition is to showing that weaker infinitary assumptions suffice to ensure that a finitary cosmos is iterative. Suppose that \mathcal{M} has local colimits indexed by the ordered set of natural numbers, then the algebraically free monad on any well-pointed endo-arrow (r, ρ) can be obtained by taking the colimit of the chain:

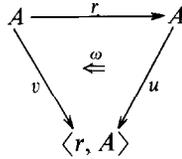
$$1_A \xrightarrow{\rho} r \xrightarrow{r\rho} r^2 \xrightarrow{r^2\rho} r^3 \xrightarrow{r^3\rho} r^4 \xrightarrow{r^4\rho} \dots$$

But Proposition 6.15 ensures that, in establishing that \mathcal{M} is iterative, we need only check that well-pointed endo-arrows have free monads.

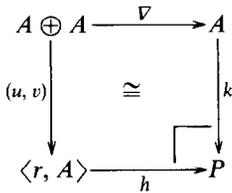
There are, of course, many finitary cosmoi which are iterative without satisfying any infinitary conditions. For instance, by arguing along the lines of [8, Section 6.4], we see that the finitary cosmos $\text{Prof}(\mathcal{E})$, of categories and profunctors internal to any elementary topos \mathcal{E} with natural numbers object, is iterative.

6.17. Proposition. *A finitary cosmos \mathcal{M} is iterative iff the associated bicategory of maps \mathcal{M}^* has pseudopushouts, and they are preserved by the inclusion $\mathcal{M}^* \rightarrow \mathcal{M}$.*

Proof. (\Leftarrow) Let r be an endo-arrow on A . First form the cone on r



then take the pseudopushout



in \mathcal{M}^* , which is also a pseudopushout in \mathcal{M} (since $\mathcal{M}^* \rightarrow \mathcal{M}$ preserves pseudopushouts). So the category $\mathcal{M}(P, X)$ is equivalent to a category whose objects are pairs (p, τ) where $p: \langle r, A \rangle \rightarrow X$ is an arrow and $\tau: pu \cong pv$ an invertible 2-cell. In turn, by using the universal property of the cone $\langle r, A \rangle$, we see that this latter category is equivalent to the category of r -algebras into X .

In 6.5 it was noted that $(u, v): A \oplus A \rightarrow \langle r, A \rangle$ is Cauchy dense, that is to say liberal in \mathcal{M} ; but any pseudopushout of a liberal is again liberal, so k is Cauchy dense as well. Now 6.3 shows that if m is the monad associated with $k \dashv k^*$, then $(k, \varepsilon_k k)$ presents P as the Kleisli object of m . So, composing the consequent equivalence $\mathcal{M}(A, X) \cong \mathcal{M}^{(m, X)}$ with $\mathcal{M}(P, X) \cong \text{Alg}(\mathcal{M})(r, X)$ from the last paragraph, we get an equivalence which is easily shown to be induced by the obvious 2-cell $\lambda: r \Rightarrow m$. This is precisely what is required to prove that m is the algebraically free monad on r .

(\Rightarrow) To construct the pseudopushout of the pair of maps $f: A \rightarrow B, g: A \rightarrow C$, we first form the coproduct $B \oplus C$, and consider the endo-arrow r upon it, determined by the matrix

$$r = \begin{pmatrix} 1_B & fg^* \\ gf^* & 1_C \end{pmatrix},$$

which has an obvious point:

$$\rho = \begin{pmatrix} 1_{1B} & !fg^* \\ !gf^* & 1_{1C} \end{pmatrix} : \begin{pmatrix} 1_B & 0 \\ 0 & 1_C \end{pmatrix} \longrightarrow \begin{pmatrix} 1_B & fg^* \\ gf^* & 1_C \end{pmatrix}.$$

By assumption, and Proposition 6.15, there is a monad (m, μ, η) and a 2-cell $\lambda: r \Rightarrow m$ (with $\lambda \cdot \rho = \eta$) displaying m as the algebraically free monad on (r, ρ) . Using the universal property of $B \oplus C$, it is easily shown that the category of (r, ρ) -algebras into X is equivalent to a category with objects consisting of a pair of arrows $b: B \rightarrow X, c: C \rightarrow X$, and an action $\theta: (bc)r \Rightarrow (bc)$ compatible with the point ρ . But

$(bc)r = (b + cgf^* bfg^* + c)$, so θ corresponds to a family of 2-cells

$$\theta_0 : b \Rightarrow b, \quad \theta_1 : c \Rightarrow c, \quad \theta_2 : cgf^* \Rightarrow b, \quad \theta_3 : bfg^* \Rightarrow c$$

and compatibility with the point ρ corresponds to saying that $\theta_0 = 1_b$ and $\theta_1 = 1_c$. So the category of m -algebras, which is isomorphic to the category of (r, ρ) -algebras, is equivalent to a category of pairs (bc) equipped with 2-cells $\hat{\theta}_2 : cg \Rightarrow bf, \hat{\theta}_3 : bf \Rightarrow cg$ (these correspond to θ_2, θ_3 under the adjunctions $f \dashv f^*, g \dashv g^*$). Somehow we must take a quotient of m , thereby imposing conditions ensuring that $\hat{\theta}_2$ and $\hat{\theta}_3$ are mutual inverses.

To this end, consider the endo-arrow

$$r^2 = \begin{pmatrix} 1_B + fg^*gf^* & 2 \bullet fg^* \\ 2 \bullet gf^* & 1_C + gf^*fg^* \end{pmatrix}$$

and the new matrix

$$s = \begin{pmatrix} ff^* & 0 \\ 0 & gg^* \end{pmatrix}.$$

There are two canonical 2-cells $\psi_0, \psi_1 : s \Rightarrow r^2$, given by matrices $(\begin{smallmatrix} \psi_{i0} & \psi_{i1} \end{smallmatrix}) (i = 0, 1)$, the first with non-trivial components

$$\psi_{00} = ff^* \xrightarrow{\varepsilon_f} 1_B \xrightarrow{c_0} 1_B + fg^*gf^*,$$

$$\psi_{01} = gg^* \xrightarrow{\varepsilon_g} 1_C \xrightarrow{c_0} 1_C + gf^*fg^*.$$

and the second with

$$\psi_{10} = ff^* \xrightarrow{f\eta_g f^*} fg^*gf^* \xrightarrow{c_1} 1_B + fg^*gf^*,$$

$$\psi_{11} = gg^* \xrightarrow{g\eta_f g^*} gf^*fg^* \xrightarrow{c_1} 1_C + gf^*fg^*.$$

Let (n, κ, τ) be the algebraically free monad on s , displayed by a 2-cell $v : s \Rightarrow n$. By composing ψ_0, ψ_1 with the 2-cell $\mu \cdot \lambda^2 : r^2 \Rightarrow m$, then applying the freeness property of n described in 6.12, we get monad morphisms $\phi_0, \phi_1 : n \Rightarrow m$ such that $\phi_0 \cdot v = \mu \cdot \lambda^2 \cdot \psi_0$ and $\phi_1 \cdot v = \mu \cdot \lambda^2 \cdot \psi_1$.

Now, following Remark 6.14, we take the coequaliser of ϕ_0, ϕ_1 thereby obtaining a new monad t and a serially commutative diagram

$$\begin{array}{ccccc} \mathcal{M}(A, X)^{\#(t, X)} & \xrightarrow{\tilde{\beta}} & \mathcal{M}(A, X)^{\#(m, X)} & \begin{array}{c} \xrightarrow{\tilde{\phi}_0} \\ \xrightarrow{\tilde{\phi}_1} \end{array} & \mathcal{M}(A, X)^{\#(n, X)} \\ & & \downarrow \tilde{\lambda} \cong & & \downarrow \cong \tilde{v} \\ & & \text{Alg}(\mathcal{M})((r, \rho), X) & \begin{array}{c} \xrightarrow{F_0} \\ \xrightarrow{F_1} \end{array} & \text{Alg}(\mathcal{M})(s, X) \end{array}$$

in which the upper line is the strict equaliser from diagram (16). To work out the action of F_0 (resp. F_1) on (r, ρ) -algebras, consider the action of $\tilde{v}\tilde{\phi}_i = F_i\tilde{\lambda}$ ($i = 0, 1$) on an m -algebra (a, α) :

$$\begin{aligned} \tilde{v}\tilde{\phi}_i(a, \alpha) &= (a, \alpha \cdot a(\phi_i \cdot v)) \\ &= (a, \alpha \cdot a(\mu \cdot \lambda^2 \cdot \psi_1)) && \text{(definition of } \phi_i) \\ &= (a, \alpha \cdot \alpha m \cdot a\lambda^2 \cdot a\psi_i) && \text{((} a, \alpha \text{) is an } m\text{-algebra)} \\ &= (a, \alpha \cdot (\alpha \cdot a\lambda)m \cdot a\lambda \cdot a\psi_i) \\ &= (a, (\alpha \cdot a\lambda) \cdot (\alpha \cdot a\lambda)r \cdot a\psi_i) && \text{(middle four interchange).} \end{aligned}$$

But $\tilde{\lambda}(a, \alpha) = (a, \alpha \cdot a\lambda)$, and $\tilde{\lambda}$ is an isomorphism, so for each (r, ρ) -algebra (a, θ) there exists an m -algebra (a, α) with $\theta = \alpha \cdot a\lambda$, and therefore $F_i(a, \theta) = \tilde{v}\tilde{\phi}_i(a, \alpha) = (a, (\alpha \cdot a\lambda) \cdot (\alpha \cdot a\lambda)r \cdot a\psi_i) = (a, \theta \cdot \theta r \cdot a\psi_i)$. It follows that the functor $\tilde{\lambda}\beta$ identifies $\mathcal{M}(A, X)^{\mathcal{M}(A, X)}$ with the full subcategory of $\text{Alg}(\mathcal{M})((r, \rho), X)$, on those (r, ρ) -algebras (a, θ) with $\theta \cdot \theta r \cdot a\psi_0 = \theta \cdot \theta r \cdot a\psi_1$.

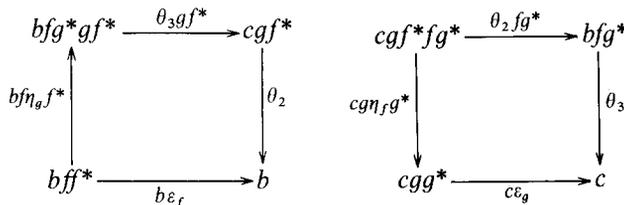
Recall that we have already re-expressed the category $\text{Alg}(\mathcal{M})((r, \rho), X)$ (up to equivalence) by using the universal property of $B \oplus C$. For a pair c^f arrows $b: B \rightarrow X$, $c: C \rightarrow X$ we have

$$\begin{aligned} (b \ c)r^2 &\cong (b \ c) \begin{pmatrix} 1_B + fg^*gf^* & 2\bullet fg^* \\ 2\bullet gf^* & 1_c + gf^*fg^* \end{pmatrix} \\ &\cong (b + bfg^*gf^* + 2\bullet cgf^* \quad 2\bullet bfg^* + c + cgf^*fg^*) \end{aligned}$$

and the 2-cell $\theta \cdot \theta r = (b \ c)r^2 \Rightarrow (b \ c)$ may be given in terms of six components:

$$\begin{array}{ccc} b \xrightarrow{1_b} b, & bfg^*gf^* \xrightarrow{\theta_3 gf^*} cgf^* \xrightarrow{\theta_2} b, \\ 2\bullet cgf^* \xrightarrow{\nabla} cgf^* \xrightarrow{\theta_2} b, & 2\bullet bfg^* \xrightarrow{\nabla} bfg^* \xrightarrow{\theta_3} c, \\ cgf^*fg^* \xrightarrow{\theta_2 fg^*} bfg^* \xrightarrow{\theta_3} c, & c \xrightarrow{1_c} c. \end{array}$$

Similarly $(b \ c)s = (bff^* \ cgg^*)$, so an s -algebra structure on $(b \ c)$ consists of a pair of 2-cells $bff^* \Rightarrow b$, $cgg^* \Rightarrow c$. Finally, putting all this together with the definitions of ψ_0 and ψ_1 , we see that the equation $\theta \cdot \theta r \cdot a\psi_0 = \theta \cdot \theta r \cdot a\psi_1$ reduces to the commutativity of diagrams:



Compare these with conditions (13) (ii) and (14) in (the proof of) Proposition 6.8. The rest is identical to that proof, our conditions correspond to the identities $\hat{\theta}_2\hat{\theta}_3 = 1$ and $\hat{\theta}_3\hat{\theta}_2 = 1$, and we get an equivalence

$$\mathcal{M}(B \oplus C, X)^{\mathcal{M}(t, X)} \xrightarrow{\sim} \mathcal{M}(B, X) \times_{\mathcal{M}(A, X)} \mathcal{M}(C, X)$$

from which we infer that the Kleisli object of the monad t provides us with a pseudopushout of f along g . \square

Finally we come to the principal result of this section:

6.18. Theorem. *If \mathcal{M} is an iterative cosmos then \mathcal{M}^* is a modulated bicategory.*

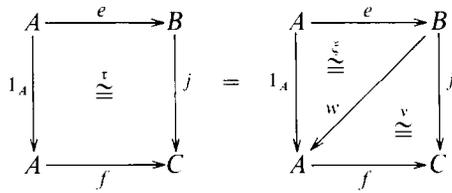
Proof. In showing that \mathcal{M}^* is modulated we must first show that $(\mathcal{M}^*)^{\text{op}}$ is faithfully conservational (cf. Definitions 2.18 and 2.25). Recall the comments in 6.5; tensors with $\mathbb{2}$ and terminal objects are both examples of collages, so \mathcal{M} possesses these. But collages respect maps so these are also the corresponding colimits in \mathcal{M}^* . Proposition 6.17 provides \mathcal{M}^* with pseudopushouts, which are also preserved by $\mathcal{M}^* \rightarrow \mathcal{M}$, so Theorem 1.14 implies that \mathcal{M}^* has all finite colimits and that they are preserved by the inclusion $\mathcal{M}^* \rightarrow \mathcal{M}$. This serves to establish 2.18(i).

The truth of the remaining conditions hinge on identifying the classes of liberals and strong conservatives in \mathcal{M}^* . Following 2.3 we know that a map $e: A \rightarrow B$ is liberal iff the following square is a pseudopushout in \mathcal{M}^* :

$$\begin{array}{ccc} \mathbb{2} * A & \xrightarrow{\nabla} & A \\ \mathbb{2} * e \downarrow & \cong & \downarrow e \\ \mathbb{2} * B & \xrightarrow{\nabla} & B \end{array}$$

But all finite colimits are preserved by $\mathcal{M}^* \rightarrow \mathcal{M}$, so this diagram is mapped to the corresponding one in \mathcal{M} , and this is a pseudopushout in \mathcal{M} iff it is so in \mathcal{M}^* . In other words the liberal arrows in \mathcal{M}^* are precisely those maps which are liberal as arrows in \mathcal{M} ; these are in turn exactly the Cauchy dense maps (by 6.3).

We know that \mathcal{M}^* has tensors with $\mathbb{2}$, so for a map to be strong conservative it need only satisfy (the appropriate dual of) of the “fill in” property of 2.13, with respect to the Cauchy dense maps. Theorem 6.9, and comments contained in its proof, demonstrate that fully faithful maps are certainly strong conservatives, but suppose conversely that $f: A \rightarrow B$ is a strong conservative map. We know that it factors as $\tau: f \cong je$ with $e: A \rightarrow B$ Cauchy dense and $j: B \rightarrow C$ fully faithful, so the diagonal “fill in” property, for f relative to e , gives w , ξ and v in:



But j is fully faithful, so we have the composite isomorphism

$$ew \xrightarrow{\cong} j^*ew \xrightarrow{\cong} j^*fw \xrightarrow{\cong} j^*j \xrightarrow{\cong} 1_B$$

which, along with ξ , establishes that w is an equivalence inverse to e . Since f is a composite of a fully faithful map and an equivalence, it is itself fully faithful. Our factorisation of maps, into a fully faithful following a Cauchy dense, now becomes that satisfying axiom 2.18(iv).

Proposition 6.8, which constructed pseudopushouts of fully faithful maps, shows that axiom 2.18(ii) holds; meanwhile 6.7 demonstrates that if j is fully faithful so is $2*j$, as required by 2.18(iii). Cauchy dense maps are clearly faithful arrows in $(\mathcal{M}^*)^{op}$, which completes the verification that this bicategory is faithfully conservational.

All that remains for us is to check the modulation axioms of 5.2. The first is easy, since the cocomma object of a pair of maps $f: A \rightarrow B$ and $g: A \rightarrow C$ may be constructed as the collage of the arrow $gf^*: B \rightarrow C$, and we observed in 6.5 that the coprojections into such a collage are fully faithful maps.

Finally we verify 5.2(vi), let $a = (a_1 \ a_2 \ a_3): A_1 \oplus A_2 \oplus A_3 \rightarrow X$ be Cauchy dense and $j_1(e_{12} \ e_{13}) \cong (a_2 \ a_3)$, $j_2(e_{21} \ e_{23}) \cong (a_1 \ a_3)$ and $j_3(e_{31} \ e_{32}) \cong (a_1 \ a_2)$ be the required factorisations into fully faithful maps following Cauchy dense ones. Suppose now that $f: X \rightarrow K$ is a map such that fj_m is fully faithful for $m = 1,2,3$. Since a is Cauchy dense, the map f is determined by the a^*a -algebra structure on $fa \cong (fa_1fa_2fa_3): A_1 \oplus A_2 \oplus A_3 \rightarrow K$ which has adjoint transpose (under $fa \dashv a^*f^*$) given by the 3×3 matrix

$$(a_k^* \eta_f a_l)_{k,l} : (a_k^* a_l)_{k,l} \longrightarrow (a_k^* f^* f a_l)_{k,l}$$

For each pair $1 \leq k, l \leq 3$ pick $1 \leq m \leq 3$ with $m \neq k, l$; then $a_l \cong j_m e_{ml}$ and $a_k^* \cong e_{mk}^* j_m^*$ so it follows that $a_k^* \eta_f a_l$ is invertible iff $e_{mk}^* j_m^* \eta_f j_m e_{ml}$ is. Of course both j_m and fj_m (which has unit $j_m^* \eta_f j_m \cdot \eta_{j_m}$) are fully faithful, by assumption, so $j_m^* \eta_f j_m$ is an isomorphism. In summary each component of our matrix is invertible, implying that the 2-cell it represents is also invertible, and therefore we can apply 6.6 to infer that f itself is fully faithful. \square

6.19. Qf particular interest is the special case wherein we restrict our bicategories to be *locally ordered*, meaning that each homcategory is a partially ordered set (which we

consider to be a category in the usual way). A bicategory \mathcal{H} is called *idyllic* when it is modulated, locally ordered and each liberal arrow is pseudoepic.

Of course we should provide examples of idyllic bicategories; to fill in the details of the following construction we refer the reader to [6]. Let \mathcal{E} be a regular category, there is a bicategory $\text{Idl}(\mathcal{E})$ whose objects are the (partially) ordered objects in \mathcal{E} , whose arrows are ideals, and whose local ordering is by “inclusion” of ideals. The composite of two ideals is that customary for relations. We also have a locally ordered bicategory $\text{Ord}(\mathcal{E})$ with the same objects, but with arrows which are order preserving functions, which are themselves ordered “pointwise”. Finally, there is a canonical homomorphism $()_* : \text{Ord}(\mathcal{E}) \rightarrow \text{Idl}(\mathcal{E})$, which acts as the identity on objects; the ideal f_* has a right adjoint f^* for each arrow $f \in \text{Ord}(\mathcal{E})$.

If \mathcal{E} is an elementary topos with natural numbers object then it is easily demonstrated that $\text{Idl}(\mathcal{E})$ is an iterative cosmos. The finite colimits of \mathcal{E} provide local finite colimits and global finite bicategorical coproducts. A monad R on the ordered object (A, \leq) is precisely a transitive relation on A containing \leq . So (A, R) is an object of $\text{Idl}(\mathcal{E})$ and the identity on A is an order preserving function $(A, \leq) \rightarrow (A, R)$; the associated adjunction of ideals presents (A, R) as the Kleisli object of I . Finally the algebraically free monad upon an endo-ideal I on (A, \leq) is simply the smallest transitive relation on A containing both \leq and I , which may be constructed by recursion as in [8, Section 6.3].

We now apply Theorem 6.18 and thereby infer that the category of maps $\text{Idl}(\mathcal{E})^*$ is a modulated bicategory. Since $\text{Idl}(\mathcal{E})$ is locally ordered the Cauchy dense maps in $\text{Idl}(\mathcal{E})$, which are the liberal arrows of $\text{Idl}(\mathcal{E})^*$, are precisely those with counit an equality. So these are split epics in $\text{Idl}(\mathcal{E})$, which implies that they are pseudoepic, both in there and in $\text{Idl}(\mathcal{E})^*$. This completes the proof that $\text{Idl}(\mathcal{E})^*$ is idyllic.

In fact, $\text{Ord}(\mathcal{E})$ is also idyllic with $\text{DFib}(\text{Ord}(\mathcal{E})^{\text{op}})^{\text{coop}} = \text{Idl}(\mathcal{E})$. However (see [6, Corollary 4]), we have $\text{Ord}(\mathcal{E}) = \text{Idl}(\mathcal{E})^*$ iff \mathcal{E} satisfies the axiom of choice.

6.20. Theorem 6.18 allows us to apply the constructions of previous sections to \mathcal{M}^* for any iterative cosmos \mathcal{M} . In fact [22], which deals specifically with enriched categories, and [19], which concerns a generalisation to a cosmos-like setting, show that the bicategory $\text{DFib}(\mathcal{M}^*)$ is canonically equivalent to the dual $\mathcal{M}^{\text{coop}}$. The novelty in our approach is the introduction of a factorisation system, with which to make sense of the composition of discrete fibrations. Earlier work used equinverters for this task, not altogether successfully.

Following the work of the last section we might like to prove a converse to the last theorem. In other words, starting with a comodulated bicategory \mathcal{H} we ask ourselves the question “is $\text{DFib}(\mathcal{H})^{\text{coop}}$ an iterative cosmos?” Our work has already made a step in this direction, by establishing that $\text{DFib}(\mathcal{H})^{\text{coop}}$ has local finite colimits and collages of arrows. It seems unlikely however that the answer to our question will be in the affirmative, with the principal task becoming that of adding axioms on \mathcal{H} to force the existence of free monads and more general collages in $\text{DFib}(\mathcal{H})^{\text{coop}}$.

A related question, which we have not discussed here, is that of determining the bicategory of maps in $\text{DFib}(\mathcal{K})$. Ideally we would like this to be \mathcal{K} itself, which it certainly contains, but again we may need further axioms to force equality.

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