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Global solutions for micro–macro models of polymeric fluids

Zhen Lei [a,*], Yun Wang [b]

[a] School of Mathematical Sciences; LMNS and Shanghai Key Laboratory for Contemporary Applied Mathematics, Fudan University, Shanghai 200433, China
[b] The Institute of Mathematical Sciences, The Chinese University of Hong Kong, Shatin, N.T., Hong Kong

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ABSTRACT

We provide a new proof for the global well-posedness of systems coupling fluids and polymers in two space dimensions. Compared to the well-known existing method based on the losing a priori estimates, our method is more direct and much simpler. The co-rotational FENE dumbbell model and the coupling Smoluchowski and Navier–Stokes equations are studied as examples to illustrate our main ideas.

1. Introduction

The dynamics of many polymeric fluids are described by two-scale micro–macro models. The systems usually consist of a macroscopic momentum equation and a microscopic Fokker–Planck type equation. The fluid is described by the macroscopic equation (sometimes the incompressible Navier–Stokes equations), with an induced elastic stress. The stress is the micro–macro interaction. The particles in the system are represented by a probability distribution $\psi(t,x,R)$ or $\psi(t,x,m)$ that depends on time t, macroscopic physical location x and particle configuration R or m. The Lagrangian transport of the particles is modeled using a Taylor expansion of the velocity field, which accounts for a drift term that depends on the spatial gradient of velocity. The system attempts to describe the behavior of this complex mixture of polymers and fluids. For more physical and mechanical backgrounds, see [3,11].

* Corresponding author.
E-mail addresses: leizhn@yahoo.com, zlei@fudan.edu.cn (Z. Lei), wangyun2616@hotmail.com (Y. Wang).

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The FENE (Finite Extensible Nonlinear Elastic) dumbbell model is one of the typical and extensively studied micro–macro models. In this model a polymer is idealized as an "elastic dumbbell" consisting of two "beads" joined by a spring which can be modeled by a vector R. Mathematically, this system reads

$$\begin{cases} \partial_t u + (u\cdot\nabla)u + \nabla p = \nu\Delta u + \operatorname{div}\tau, \quad x\in\mathbb{R}^2,\\ \operatorname{div} u = 0, \quad x\in\mathbb{R}^2,\\ \partial_t\psi + (u\cdot\nabla)\psi = \operatorname{div}_R\big[-W(u)\cdot R\psi + \beta\nabla_R\psi + \nabla_R\mathcal{U}\psi\big], \quad (x,R)\in\mathbb{R}^2\times B(0,R_0),\\ (\nabla_R\mathcal{U}\psi + \beta\nabla_R\psi)\cdot\mathbf{n} = 0, \quad \text{on } \partial B(0,R_0),\\ t=0: \quad u(t,x) = u_0(x), \qquad \psi(t,x,R) = \psi_0(x,R). \end{cases} \tag{1.1}$$

In the above system, $u = u(t,x)$ denotes the velocity field of the fluid, $p = p(t,x)$ denotes the pressure, $\psi(t,x,R)$ is the distribution function for the internal configuration, $\nu > 0$ is the viscosity of the fluid and β is related to the temperature of the system. Moreover, the spring potential $\mathcal{U}$ and the induced elastic stress τ is given by

$$\mathcal{U}(R) = -k\ln\big(1-|R|^2/|R_0|^2\big), \qquad \tau_{ij} = \int\limits_{B(0,R_0)} (R_i\otimes\nabla_{R_j}\mathcal{U})\psi(t,x,R)\,dR. \tag{1.2}$$

Here $k > 0$ is a constant. The boundary condition insures the conservation of the polymer density. Assume that $W(u) = \frac{\nabla u-(\nabla u)^t}{2}$, which corresponds to the co-rotational case. For simplicity of writing, assume that $\beta = 1$ and $R_0 = 1$ and denote $B(0,R_0)$ by B. In what follows, without special claim, ∇ represents ∇_x, and div represents div_x.

The Smoluchowski equation coupled with the incompressible Navier–Stokes equations is another extensively studied micro–macro model. The Smoluchowski equation describes the temporal evolution of the probability distribution function ψ for directions of rod-like particles in a suspension. Mathematically, the system reads

$$\begin{cases} \partial_t u + (u\cdot\nabla)u + \nabla p = \nu\Delta u + \operatorname{div}\tau, \quad x\in\mathbb{R}^2,\\ \operatorname{div} u = 0, \quad x\in\mathbb{R}^2,\\ \partial_t\psi + (u\cdot\nabla)\psi + \operatorname{div}_g\big(G(u,\psi)\psi\big) - \Delta_g\psi = 0, \quad (x,m)\in\mathbb{R}^2\times M,\\ t=0: \quad u(t,x) = u_0(x), \qquad \psi(t,x,m) = \psi_0(x,m). \end{cases} \tag{1.3}$$

Here M is a d-dimensional smooth compact Riemannian manifold without boundary of and dm is the Riemannian volume element of M, $u = u(t,x)$ and $p = p(t,x)$ denote the velocity field and the pressure of fluid respectively, $\psi = \psi(t,x,m)$ is the distribution function, $G(u,\psi) = \nabla_g\mathcal{U} + W$ stands for a meanfield potential resulting from the excluded volume effects due to steric forces between molecules with $W = c_\alpha^{ij}(m)\partial_j u_i$. Besides, the added stress tensor τ and the potential $\mathcal{U}$ are given by

$$\begin{cases} \mathcal{U}(t,x,m) = \displaystyle\int\limits_M K(m,q)\psi(t,x,q)\,dq,\\ \tau_{ij}(t,x) = \displaystyle\int\limits_M \gamma_{ij}^{(1)}(m)\psi(t,x,m)\,dm\\ \qquad\qquad + \displaystyle\int\limits_M\int\limits_M \gamma_{ij}^{(2)}(m_1,m_2)\psi(t,x,m_1)\psi(t,x,m_2)\,dm_1\,dm_2. \end{cases} \tag{1.4}$$

Here the kernel K is a smooth symmetric function defined on $M \times M$. $\gamma_{ij}^{(1)}$ and $\gamma_{ij}^{(2)}$ are smooth, time independent and x independent.

At present there have been extensive and systematical studies on the existence and regularity theories of those 2D micro–macro models of polymeric fluids [2,8,7,9,10,20,23]. For example, the first global well-posedness result for FENE (1.1) was derived by Lin, Zhang and Zhang [22] for $k > 6$. Masmoudi [23] extends it to the case of $k > 0$ by a crucial observation on the linear operator. Very recently the global existence of weak solutions to the FENE dumbbell model of polymeric flows for a very general class of potentials was also obtained by Masmoudi [24]. The global well-posedness of nonlinear Fokker–Planck system coupled with Navier–Stokes equations (1.3) in 2D has been proven by Constantin and Masmoudi in [10]. When the nonlinear Fokker–Planck equation is driven by a time averaged Navier–Stokes system in 2D, global well-posedness has been obtained by Constantin, Fefferman, Titi and Zarnescu [9].

Most proofs of the above global well-posedness theorems are based on an important analytic technique called losing a priori estimate in the spirit of Bahouri and Chemin [2] and Chemin and Masmoudi [7]. In [17], we studied the blow-up criteria of a macroscopic viscoelastic Oldroyd-B system avoiding using the losing a priori estimates. The main purpose of this paper is to extend the method in [17] to micro–macro models and provide a new proof for global well-posedness of the co-rotational FENE dumbbell model (1.1) and the coupling Smoluchowski and incompressible Navier–Stokes equations (1.3). Compared to the proofs of the theorems in [23,10], which are based on the technique of losing a priori estimates, ours are direct and much simpler.

For the co-rotational FENE model (1.1), we have

Theorem 1.1. *Assume that $u_0 \in H^s(\mathbb{R}^2)$ $(s > 2)$, $\psi_0 \in H^s(\mathbb{R}^2; \mathcal{L}^r \cap \mathcal{L}^2)$ for some $r \geqslant 2$ such that $(r-1)k > 1$ with $\int_B \psi_0\, dR = 1$, a.e. in x. Then there exists a unique global solution (u, ψ) of the FENE model (1.1) in $C([0,\infty); H^s) \times C([0,\infty); H^s(\mathbb{R}^2; \mathcal{L}^2))$. Moreover, $u \in L^2_{loc}(0,\infty; H^{s+1})$ and $\psi \in L^2_{loc}(0,\infty; H^s(\mathbb{R}^2; \mathcal{L}^{1,r}))$.*

For the definition of $\mathcal{L}^r$ and $\mathcal{L}^{1,r}$, please refer to Section 2.

Similarly, for the coupling Smoluchowski and Navier–Stokes system (1.3), we have

Theorem 1.2. *Take $u_0 \in W^{1+\epsilon,r}(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ and $\psi_0 \in W^{1,r}(\mathbb{R}^2, H^{-s}(M))$, for some $r > 2$, $\epsilon > 0$, $s > \frac{d}{2}+1$ and $\psi_0 \geqslant 0$. $\int_M \psi_0\, dm \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. Then (1.3) has a global solution in $u \in L^\infty_{loc}(0,\infty; W^{1,r}) \cap L^2_{loc}(0,\infty; W^{2,r})$ and $\psi \in L^\infty_{loc}(0,\infty; W^{1,r}(\mathbb{R}^2; H^{-s}(M)))$. Moreover, for $T > T_0 > 0$, we have $u \in L^\infty((T_0,T); W^{2-\epsilon,r})$.*

We end this introduction by mentioning some other results on micro–macro models. Global existence of weak solutions can be found in [4,21] and local existence of strong solutions are studied in [12,13,25]. For macroscopic models, we refer the readers to [14,15,18,19,16,6] as references.

The paper is organized as follows. In Section 2, we give some preliminaries. Then we give some a priori estimates for the FENE model (1.1) in Section 3 and the coupling Smoluchowski and Navier–Stokes equations (1.3) in Section 4. The a priori estimates obtained in Sections 3 and 4 are enough to get the global existence of systems (1.1) and (1.3) and prove Theorems 1.1 and 1.2 [23,10].

2. Definitions and useful lemmas

We will use the Littlewood–Paley decomposition in the following sections. Define $\mathcal{C}$ to be the ring

$$\mathcal{C} = \left\{\xi \in \mathbb{R}^2\colon\ \frac{3}{4} \leqslant |\xi| \leqslant \frac{8}{3}\right\},$$

and define $\mathcal{D}$ to be the ball

$$\mathcal{D} = \left\{\xi \in \mathbb{R}^2\colon\ |\xi| \leqslant \frac{4}{3}\right\}.$$

Let χ and φ be two smooth nonnegative radial functions supported respectively in $\mathcal{D}$ and $\mathcal{C}$, such that

$$\chi(\xi)+\sum_{q\geqslant 0}\varphi\big(2^{-q}\xi\big)=1 \quad \text{for } \xi\in\mathbb{R}^2, \quad \text{and} \quad \sum_{q\in\mathbb{Z}}\varphi\big(2^{-q}\xi\big)=1 \quad \text{for } \xi\in\mathbb{R}^2\setminus\{0\}.$$

Let us denote by $\mathcal{F}$ the Fourier transform on $\mathbb{R}^2$ and denote

$$h=\mathcal{F}^{-1}\varphi, \qquad \tilde{h}=\mathcal{F}^{-1}\chi.$$

The frequency localization operator is defined by

$$\Delta_q u=\mathcal{F}^{-1}\big[\varphi\big(2^{-q}\xi\big)\mathcal{F}(u)\big]=2^{2q}\int_{\mathbb{R}^2} h\big(2^q y\big)u(x-y)\,dy,$$

and

$$S_q u=\mathcal{F}^{-1}\big[\chi\big(2^{-q}\xi\big)\mathcal{F}(u)\big]=2^{2q}\int_{\mathbb{R}^2} \tilde{h}\big(2^q y\big)u(x-y)\,dy.$$

Hence, for $s<2/p$ or $s=2/p$ and $r=1$, the homogeneous Besov space $\dot{B}^s_{p,r}$ is defined as the closure of compactly supported smooth functions under the norm $\|\cdot\|_{\dot{B}^s_{p,r}}$,

$$\|u\|_{\dot{B}^s_{p,r}}=\big\|\big(2^{qs}\|\Delta_q u\|_{L^p}\big)_{q\in\mathbb{Z}}\big\|_{l^r(\mathbb{Z})},$$

when $p=r=\infty$, $s=k+\alpha$ where $k\in\mathbb{N}$ and $\alpha\in(0,1)$, then $\dot{B}^s_{p,r}$ turns to be the homogeneous Hölder space $\dot{C}^{k+\alpha}$.

Another kind of space to be used is $\tilde{L}^p(t_1,t_2;\dot{C}^r)$, which is the space of distributions u such that

$$\|u\|_{\tilde{L}^p(t_1,t_2;\dot{C}^r)}\triangleq\sup_{q\in\mathbb{Z}}2^{qr}\|\Delta_q u\|_{L^p(t_1,t_2;L^\infty)}<\infty.$$

For the FENE model, let

$$\psi_\infty(R)=\frac{e^{-\mathcal{U}(R)}}{\int_B e^{-\mathcal{U}(R)}\,dR}=c\big(1-|R|^2\big)^k,$$

where the constant c is given such that $\int_B \psi_\infty\,dR=1$. In fact, $(u,\psi)=(0,\psi_\infty)$ defines a stationary solution of (1.1).

For $r\geqslant 1$, denote $\mathcal{L}^r$ and $\mathcal{L}^{1,r}$ the weighted spaces

$$\mathcal{L}^r=\left\{\psi\colon\ \|\psi\|^r_{\mathcal{L}^r}=\int_B \psi_\infty\left|\frac{\psi}{\psi_\infty}\right|^r dR<\infty\right\},$$

$$\mathcal{L}^{1,r}=\left\{\psi\in\mathcal{L}^r\colon\ |\psi|^r_{\dot{\mathcal{L}}^{1,r}}=\int_B \psi_\infty\left|\nabla_R\left(\frac{\psi}{\psi_\infty}\right)\right|^r dR<\infty\right\}.$$

We will need to use the following well-known inequalities.

Lemma 2.1 *(Bernstein inequalities). (See [5].) For $s \in \mathbb{R}$, $1 \leqslant p \leqslant r \leqslant \infty$ and $q \in \mathbb{Z}$, one has*

$$\|\Delta_q u\|_{L^r(\mathbb{R}^d)} \leqslant C \cdot 2^{d(\frac{1}{p}-\frac{1}{r})q} \|\Delta_q u\|_{L^p(\mathbb{R}^d)},$$
$$c2^{qs}\|\Delta_q u\|_{L^p} \leqslant \|\nabla^s \Delta_q u\|_{L^p} \leqslant C2^{qs}\|\Delta_q u\|_{L^p},$$
$$\||\nabla|^s S_q u\|_{L^p} \leqslant C2^{qs}\|u\|_{L^p},$$
$$ce^{-C2^{2q}t}\|\Delta_q u\|_{L^\infty} \leqslant \|e^{t\Delta}\Delta_q u\|_{L^\infty} \leqslant Ce^{-c2^{2q}t}\|\Delta_q u\|_{L^\infty}.$$

Here C and c are positive constants independent of s, p and q.

We will also need the following lemma, whose proof can be found in [17].

Lemma 2.2. *Assume that $\beta > 0$. Then there exists a positive constant $C > 0$ such that*

$$\int_t^T \|\nabla g(s,\cdot)\|_{L^\infty}\,ds$$
$$\leqslant C\left(1 + \int_t^T \|g(s,\cdot)\|_{L^2}\,ds + \sup_q \int_t^T \|\Delta_q \nabla g(s,\cdot)\|_{L^\infty}\,ds \ln\left(e + \int_t^T \|\nabla g(s,\cdot)\|_{\dot{C}^\beta}\,ds\right)\right).$$

3. Proof of Theorem 1.1

Since local existence of smooth solutions has been derived by N. Masmoudi [23], here we only focus on the a priori estimate which is sufficient for proving Theorem 1.1. As explained in [22] or [23], to get the global existence we just need to control the L^∞-norm of ∇u, i.e., $\|\nabla u\|_{L^\infty}$. Define the flow associated with u by $\Phi(t,x)$, which means Φ satisfies the ODEs,

$$\begin{cases} \partial_t \Phi(t,x) = u\big(t, \Phi(t,x)\big), \\ \Phi(t=0,x) = x. \end{cases} \tag{3.1}$$

Step I: Uniform estimates for ψ and τ with respect to t.

Due to the special structure of the equation about $\psi(t,x,R)$, we have the following bounded estimates of ψ. For $r > 1$, multiplying the third equation of (1.1) by $r|\frac{\psi}{\psi_\infty}|^{r-2}\frac{\psi}{\psi_\infty}$, and integrating over B, then we get

$$\partial_t \int_B \psi_\infty \left|\frac{\psi}{\psi_\infty}\right|^r dR + u\cdot\nabla \int_B \psi_\infty \left|\frac{\psi}{\psi_\infty}\right|^r dR = -\frac{4(r-1)}{r}\int_B \psi_\infty \left|\nabla_R\left(\frac{\psi}{\psi_\infty}\right)^{\frac{r}{2}}\right|^2 dR.$$

Therefore,

$$\int_B \psi_\infty \left|\frac{\psi}{\psi_\infty}\right|^r \big(t, \Phi(t,x), R\big)\,dR \leqslant \int_B \psi_\infty \left|\frac{\psi_0}{\psi_\infty}\right|^r (x,R)\,dR.$$

Since the flow is incompressible, then

$$\|\psi\|_{L^\infty_{t,x}(\mathcal{L}^r)} \leqslant \|\psi_0\|_{L^\infty_x(\mathcal{L}^r)}, \qquad \|\psi\|_{L^\infty_t(L^2_x(\mathcal{L}^r))} \leqslant \|\psi_0\|_{L^2_x(\mathcal{L}^r)}. \tag{3.2}$$

To estimate τ, we need a lemma.

Lemma 3.1. *For any p such that $pk > 1$, it holds that*

$$\int_B \frac{|\psi|}{1-|R|}\,dR \leqslant C\left(\int_B \frac{|\psi|^{p+1}}{\psi_\infty^p}\right)^{\frac{1}{p+1}} = \|\psi\|_{\mathcal{L}^{p+1}}.$$

Proof. By the Hölder's inequality,

$$\int_B \frac{|\psi|}{1-|R|}\,dR \leqslant C\int_B \frac{1}{(1-|R|)^{1-kp/(p+1)}}\frac{|\psi|}{\psi_\infty^{p/(p+1)}}\,dR$$
$$\leqslant C\left(\int_B \frac{1}{(1-|R|)^{1+1/p-k}}\right)^{\frac{p}{p+1}}\left(\int_B \frac{|\psi|^{p+1}}{\psi_\infty^p}\,dR\right)^{\frac{1}{p+1}}.$$

Since $pk > 1$, the result follows. □

Noting that

$$\tau_{ij}(t,x) = \int_B (R_i \otimes \nabla_{R_j}\mathcal{U})\psi(t,x,R)\,dR,$$

one has

$$\begin{aligned}\big|\tau(t,x)\big| &\leqslant C\int_B |\nabla_R \mathcal{U}|\cdot\big|\psi(t,x,R)\big|\,dR\\ &\leqslant Ck\int_B \frac{|R|}{1-|R|^2}\cdot\big|\psi(t,x,R)\big|\,dR\\ &\leqslant C\int_B \frac{|\psi(t,x,R)|}{1-|R|}\,dR\\ &\leqslant C\big\|\psi(t,x,\cdot)\big\|_{\mathcal{L}^r},\end{aligned}$$

where in the last step we used Lemma 3.1.

Hence, we have

$$\|\tau\|_{L^\infty(0,T;L^2)} \leqslant C\|\psi_0\|_{L^2_x(\mathcal{L}^r)}, \qquad \|\tau\|_{L^\infty(0,T;L^\infty)} \leqslant C\|\psi_0\|_{L^\infty_x(\mathcal{L}^r)}. \tag{3.3}$$

Step II: A priori estimates for u.

We need a useful lemma whose proof was established by Chemin and Masmoudi [7] (see also [17]).

Lemma 3.2 *(Chemin–Masmoudi). Let v be a solution of the Navier–Stokes equations with initial data $v_0 \in L^2(\mathbb{R}^2)$, and an external force $f \in \tilde{L}^1(0,T;\dot{C}^{-1}) \cap L^2(0,T;\dot{H}^{-1})$:*

$$\begin{cases} \partial_t v - \Delta v + (v \cdot \nabla) v + \nabla p = f, & \textit{in } \mathbb{R}^2 \times (0, T), \\ \operatorname{div} v = 0, & \textit{in } \mathbb{R}^2 \times (0, T), \\ v(t = 0, x) = v_0(x), & \textit{in } \mathbb{R}^2. \end{cases} \tag{3.4}$$

Then we have the following a priori estimates,

$$\|v\|^2_{L^\infty(0,T;L^2)} + 2\|\nabla v\|^2_{L^2(0,T;L^2)} \leqslant \|v_0\|^2_{L^2} + \|f\|^2_{L^2(0,T;\dot{H}^{-1})},$$

and

$$\|v\|_{\tilde{L}^1(0,T;\dot{C}^1)} \leqslant C\Big(\sup_q \|\Delta_q v_0\|_{L^2}\big(1 - \exp\{-c2^{2q}T\}\big) + \big(\|v_0\|_{L^2} + \|f\|_{L^2(0,T;\dot{H}^{-1})}\big)\|\nabla v\|^2_{L^2(0,T;L^2)} + \sup_q \int_0^T \big\|2^{-q}\Delta_q f(s)\big\|_{L^\infty}\, ds\Big).$$

Furthermore, if $f \in L^1(0, T; \dot{C}^{-1})$, then $\forall \epsilon > 0$, there exists $t_0(\epsilon) \in (0, T)$ such that

$$\|v\|_{\tilde{L}^1(t_0,T;\dot{C}^1)} \leqslant \epsilon.$$

Particularly for our problem, since we have shown that $\tau \in L^\infty(0, T; L^2) \cap L^\infty(0, T; L^\infty)$, applying Lemma 3.2, we know that

$$u \in L^\infty\big(0, T; L^2\big) \cap L^2\big(0, T; \dot{H}^1\big) \cap \tilde{L}^1\big(0, T; \dot{C}^1\big)$$

and

$$\forall \epsilon > 0,\ \exists t_0(\epsilon) \in (0, T),\ \text{such that} \quad \|u\|_{\tilde{L}^1(t_0,T;\dot{C}^1)} \leqslant \epsilon. \tag{3.5}$$

Step III: Hölder estimates for u.

For $0 \leqslant t < T$, choose some α satisfying $0 < \alpha < \min\{s - 2, 1\}$, define

$$N_q^r(t, x) = \int_B \psi_\infty \left|\frac{\Delta_q \psi(t, x, R)}{\psi_\infty}\right|^r dR = \|\Delta_q \psi\|^r_{\mathcal{L}^r}(t, x),$$

$$A(t) = \sup_{0 \leqslant s < t} \big\|u(s, \cdot)\big\|_{\dot{C}^{1+\alpha}}, \qquad B(t) = \sup_{0 \leqslant s < t} \big\|\tau(s, \cdot)\big\|_{\dot{C}^\alpha},$$

$$D(t) = \sup_{0 \leqslant s < t} \sup_{q \in \mathbb{Z}} 2^{\alpha q}\big\|N_q(s, \cdot)\big\|_{L^\infty}.$$

Here Δ_q is the frequency operator with respect to x.

Before the detailed estimates, we will introduce an inequality for later use, which can be considered as an extension of Hölder inequality.

Lemma 3.3. *For any $u \in L^4(\mathbb{R}^2) \cap \dot{C}^{1+\alpha}(\mathbb{R}^2)$, there holds that*

$$\|u \otimes u\|_{\dot{C}^{\frac{1}{2}+\alpha}} \leqslant C\|u\|_{L^4} \cdot \|u\|_{\dot{C}^{1+\alpha}},$$

with some constant C independent of u.

Proof. For any $q \in \mathbb{Z}$, using Bony's para-product decomposition [1], we have

$$\begin{aligned}\big\|\Delta_q(u\otimes u)\big\|_{L^\infty}\cdot 2^{(\frac{1}{2}+\alpha)q} &\leqslant 2\sum_{|p-q|\leqslant 5}\big\|\Delta_q(S_{p-1}u\otimes\Delta_p u)\big\|_{L^\infty}\cdot 2^{(\frac{1}{2}+\alpha)q}\\ &\quad+\sum_{p\geqslant q-3}\sum_{|p-r|\leqslant 1}\big\|\Delta_q(\Delta_p u\otimes\Delta_r u)\big\|_{L^\infty}\cdot 2^{(\frac{1}{2}+\alpha)q}\\ &\triangleq 2I_1+I_2.\end{aligned}$$

I_1 can be estimated as

$$\begin{aligned}I_1 &\leqslant C\sum_{|p-q|\leqslant 5}\|S_{p-1}u\otimes\Delta_p u\|_{L^4}\cdot 2^{(1+\alpha)q}\\ &\leqslant C\sum_{|p-q|\leqslant 5}\|S_{p-1}u\|_{L^4}\cdot\|\Delta_p u\|_{L^\infty}\cdot 2^{(1+\alpha)q}\\ &\leqslant C\sum_{|p-q|\leqslant 5}\|u\|_{L^4}\cdot 2^{(1+\alpha)p}\|\Delta_p u\|_{L^\infty}\cdot 2^{(1+\alpha)(q-p)}\\ &\leqslant C\|u\|_{L^4}\cdot\|u\|_{\dot{C}^{1+\alpha}},\end{aligned}$$

here the first inequality is due to Lemma 2.1.

While I_2 can be estimated as

$$\begin{aligned}I_2 &= \sum_{p\geqslant q-3}\sum_{|p-r|\leqslant 1}\big\|\Delta_q(\Delta_p u\otimes\Delta_r u)\big\|_{L^\infty}\cdot 2^{(\frac{1}{2}+\alpha)q}\\ &\leqslant C\sum_{p\geqslant q-3}\sum_{|p-r|\leqslant 1}\|\Delta_p u\|_{L^\infty}\cdot\|\Delta_r u\|_{L^4}\cdot 2^{(\frac{1}{2}+\alpha)q}\cdot 2^{\frac{1}{2}r}\\ &\leqslant C\sum_{p\geqslant q-3}2^{(1+\alpha)p}\|\Delta_p u\|_{L^\infty}\cdot\|u\|_{L^4}\cdot 2^{(\frac{1}{2}+\alpha)(q-p)}\\ &\leqslant C\|u\|_{\dot{C}^{1+\alpha}}\cdot\|u\|_{L^4},\end{aligned}$$

here in the second inequality we also used Lemma 2.1. These above estimates complete the proof of Lemma 3.3. □

First, applying Δ_q to the first equation of the FENE system, then we obtain

$$\partial_t\Delta_q u-\nu\Delta\Delta_q u+\nabla\Delta_q p=\nabla\cdot\Delta_q(\tau-u\otimes u),$$

hence

$$\Delta_q u=e^{\nu t\Delta}\Delta_q u_0+\int_0^t e^{\nu(t-s)\Delta}\mathbb{P}\big(\Delta_q\nabla\cdot(\tau-u\otimes u)\big)\,ds,$$

where $\mathbb{P}$ is the Helmoltz–Weyl projection operator.

$$
\begin{aligned}
&2^{q(1+\alpha)}\|\Delta_q u\|_{L^\infty}(t)\\
&\quad\leqslant Ce^{-c\nu 2^{2q}t}\|\Delta_q u_0\|_{L^\infty}2^{q(1+\alpha)}+C\int_0^t e^{-c\nu 2^{2q}(t-s)}2^{q(1+\alpha)}\big\|\Delta_q\nabla\cdot(\tau-u\otimes u)\big\|_{L^\infty}\,ds\\
&\quad\leqslant C\|u_0\|_{\dot C^{1+\alpha}}+C\int_0^t e^{-c2^{2q}(t-s)}\cdot 2^{q(1+\alpha)}\big(\|\nabla\cdot\Delta_q\tau\|_{L^\infty}+\big\|\nabla\cdot\Delta_q(u\otimes u)\big\|_{L^\infty}\big)\,ds. \qquad (3.6)
\end{aligned}
$$

Applying Lemma 2.1, we obtain

$$
\begin{aligned}
\int_0^t e^{-c2^{2q}(t-s)}2^{q(1+\alpha)}\|\nabla\cdot\Delta_q\tau\|_{L^\infty}\,ds &\leqslant C\int_0^t e^{-c2^{2q}(t-s)}2^{q(1+\alpha)}2^q\|\Delta_q\tau\|_{L^\infty}\,ds\\
&\leqslant C\int_0^t e^{-c2^{2q}(t-s)}2^{2q}\cdot 2^{\alpha q}\|\Delta_q\tau\|_{L^\infty}\,ds\\
&\leqslant CB(t).
\end{aligned}
$$

On the other hand, applying Lemma 2.1 again, we obtain

$$
\begin{aligned}
2^{q(1+\alpha)}\int_0^t e^{-c2^{2q}(t-s)}\big\|\nabla\cdot\Delta_q(u\otimes u)\big\|_{L^\infty}\,ds &\leqslant C\int_0^t e^{-c2^{2q}(t-s)}2^{q(2+\alpha)}\big\|\Delta_q(u\otimes u)\big\|_{L^\infty}\,ds\\
&\leqslant C\int_0^t e^{-c2^{2q}(t-s)}2^{\frac{3}{2}q}\|u\otimes u\|_{\dot C^{\frac{1}{2}+\alpha}}\,ds\\
&\leqslant C\int_0^t e^{-c2^{2q}(t-s)}2^{\frac{3}{2}q}\|u\|_{L^4}\cdot\|u\|_{\dot C^{1+\alpha}}\,ds\\
&\leqslant C\bigg(\int_0^t \|u\|_{L^4}^4\|u\|_{\dot C^{1+\alpha}}^4\,ds\bigg)^{\frac{1}{4}}\cdot\bigg(\int_0^t e^{-c2^{2q}(t-s)}2^{2q}\,ds\bigg)^{\frac{3}{4}}\\
&\leqslant C\bigg(\int_0^t \|u\|_{L^2}^2\|\nabla u\|_{L^2}^2\|u\|_{\dot C^{1+\alpha}}^4\,ds\bigg)^{\frac{1}{4}},
\end{aligned}
$$

where we used Lemma 3.3 and interpolation inequality for the third and last inequality respectively.

Taking the supreme of both sides of (3.6) with respect to q, one gets

$$
\big\|u(t)\big\|_{\dot C^{1+\alpha}}^4\leqslant C\big(\|u_0\|_{\dot C^{1+\alpha}}+B(t)\big)^4+C\bigg(\int_0^t \big\|u(s)\big\|_{L^2}^2\big\|\nabla u(s)\big\|_{L^2}^2\big\|u(s)\big\|_{\dot C^{1+\alpha}}^4\,ds\bigg).
$$

By the Gronwall's inequality and the estimates in Step II, we have

$$A(t) \leqslant C\big(\|u_0\|_{\dot{C}^{1+\alpha}} + B(t)\big) \leqslant C\big(1 + B(t)\big). \tag{3.7}$$

As a result of the relationship between τ and ψ and Lemma 3.1,

$$\begin{aligned} B(t) &= \sup_{0\leqslant s<t} \sup_q 2^{\alpha q} \big\| \Delta_q \tau(s,\cdot)\big\|_{L^\infty} \\ &\leqslant \sup_{0\leqslant s<t} \sup_q 2^{\alpha q} \bigg\| \int_B |\Delta_q \psi| \cdot |\nabla_R \mathcal{U}|\, dR \bigg\|_{L^\infty} \\ &\leqslant C \sup_{0\leqslant s<t} \sup_q 2^{\alpha q} \|\Delta_q \psi\|_{L^\infty_x(\mathcal{L}^r)}(s) \\ &= CD(t). \end{aligned}$$

Combining the above inequality and (3.7),

$$A(t) \leqslant C\big(1 + D(t)\big). \tag{3.8}$$

Step IV: Hölder estimates of ψ.

The remaining part is to estimate the $\dot{C}^\alpha$-norm of ψ. Take the operator Δ_q to the third equation, then

$$\begin{aligned} \partial_t \Delta_q \psi + u \cdot \nabla \Delta_q \psi = {} & \mathrm{div}_R\big(-W(u) \cdot R\Delta_q \psi\big) + \mathrm{div}_R\bigg(\psi_\infty \nabla_R\bigg(\frac{\Delta_q \psi}{\psi_\infty}\bigg)\bigg) \\ & + \mathrm{div}_R\big(\big[\Delta_q, -W(u)\big] \cdot R\psi\big) + [u\cdot\nabla, \Delta_q]\psi. \end{aligned} \tag{3.9}$$

Multiply (3.9) by $r|\frac{\Delta_q\psi}{\psi_\infty}|^{r-2}\frac{\Delta_q\psi}{\psi_\infty}$ and integrate over B,

$$\begin{aligned} & \partial_t N_q^r + u \cdot \nabla N_q^r + \frac{4(r-1)}{r} \int_B \psi_\infty \bigg|\nabla_R\bigg(\frac{\Delta_q\psi}{\psi_\infty}\bigg)^{r/2}\bigg|^2 dR \\ & \quad \leqslant \int_B \mathrm{div}_R\big(\big[\Delta_q, -W(u)\big]\cdot R\psi\big) \cdot \bigg|\frac{\Delta_q\psi}{\psi_\infty}\bigg|^{r-2} \frac{\Delta_q\psi}{\psi_\infty} r\, dR + \int_B \big|[u\cdot\nabla, \Delta_q]\psi\big| \cdot \bigg|\frac{\Delta_q\psi}{\psi_\infty}\bigg|^{r-1} r\, dR \\ & \quad \triangleq J_1 + J_2. \end{aligned}$$

By the Young's inequality and the Hölder inequality,

$$|J_1| \leqslant C\big\|\big[\Delta_q, W(u)\big]\cdot R\psi\big\|_{\mathcal{L}^r}^2 \cdot N_q^{r-2} + \frac{r-1}{r}\int_B \psi_\infty \bigg|\nabla_R\bigg(\frac{\Delta_q\psi}{\psi_\infty}\bigg)^{r/2}\bigg|^2 dR,$$

$$J_2 \leqslant C\big\|[u\cdot\nabla, \Delta_q]\psi\big\|_{\mathcal{L}^r} \cdot N_q^{r-1}.$$

Hence

$$\partial_t N_q^r + u\cdot\nabla N_q^r \leqslant C\big\|\big[\Delta_q, W(u)\big]\cdot R\psi\big\|_{\mathcal{L}^r}^2 \cdot N_q^{r-2} + C\big\|[u\cdot\nabla, \Delta_q]\psi\big\|_{\mathcal{L}^r} \cdot N_q^{r-1},$$

which implies that

$$2^{\alpha qr}\|N_q\|^r_{L^\infty}(t) \leqslant C2^{\alpha qr}\int_0^t \big\|\big[\Delta_q, W(u)\big]\cdot R\psi\big\|^2_{L^\infty_x(\mathcal{L}^r)}\cdot\|N_q\|^{r-2}_{L^\infty}(s)\,ds$$
$$+ C2^{\alpha qr}\int_0^t \big\|[u\cdot\nabla, \Delta_q]\psi\big\|_{L^\infty_x(\mathcal{L}^r)}\cdot\|N_q\|^{r-1}_{L^\infty}(s)\,ds. \tag{3.10}$$

Note that

$$\big|\big[\Delta_q, W(u)\big]\cdot R\psi\big| = \left|\int_{\mathbb{R}^2} h(y)\big[W(u)(x) - W(u)\big(x-2^{-q}y\big)\big]\cdot\big(R\psi\big(x-2^{-q}y\big)\big)\,dy\right|$$
$$\leqslant C2^{-\alpha q}\big\|W(u)\big\|_{\dot{C}^\alpha}\int_{\mathbb{R}^2}\big|h(y)\big|\cdot\big|R\psi\big(x-2^{-q}y\big)\big|\,dy.$$

Then we get that

$$\big\|\big[\Delta_q, W(u)\big]\cdot R\psi\big\|_{L^\infty_x(\mathcal{L}^r)} \leqslant C\|\nabla u\|_{\dot{C}^\alpha}\|\psi\|_{L^\infty_x(\mathcal{L}^r)}\cdot 2^{-\alpha q} \leqslant C\|\nabla u\|_{\dot{C}^\alpha}\cdot 2^{-\alpha q}.$$

And by the Bony's para-product formula,

$$\big|[u\cdot\nabla, \Delta_q]\psi\big| \leqslant \sum_{p\geqslant q-3}\sum_{|p-q'|\leqslant 1}\big|[\Delta_p u\cdot\nabla, \Delta_q]\Delta_{q'}\psi\big| + \sum_{|q-q'|\leqslant 5}\big|[S_{q'-1}u\cdot\nabla, \Delta_q]\Delta_{q'}\psi\big|$$
$$+ \sum_{|q-q'|\leqslant 5}\big|[\Delta_{q'}u\cdot\nabla, \Delta_q]S_{q'-1}\psi\big|$$

with each term having the following estimate:

Since

$$[\Delta_p u\cdot\nabla, \Delta_q]\Delta_{q'}\psi(s,x,R)$$
$$= \int_{\mathbb{R}^2} h(y)\big[\Delta_p u(s,x) - \Delta_p u\big(s, x-2^{-q}y\big)\big]\cdot\nabla\Delta_{q'}\psi\big(s, x-2^{-q}y, R\big)\,dy,$$

and

$$\nabla\Delta_{q'}\psi(x) = \int_{\mathbb{R}^2} 2^{2q'}h\big(2^{q'}y\big)\nabla\psi(x-y)\,dy = -\int_{\mathbb{R}^2} 2^{3q'}\nabla h\big(2^{q'}y\big)\psi(x-y)\,dy$$

then

$$\begin{aligned}
&\sum_{p\geqslant q-3}\sum_{|p-q'|\leqslant 1} 2^{\alpha q}\big\|[\Delta_p u\cdot\nabla,\Delta_q]\Delta_{q'}\psi\big\|_{L^\infty_x(\mathcal{L}^r)}(s)\\
&\quad\leqslant C\sum_{p\geqslant q-3}\sum_{|p-q'|\leqslant 1} 2^{\alpha q}2^{-q}\|\nabla\Delta_p u\|_{L^\infty}(s)\cdot\|\nabla\Delta_{q'}\psi\|_{L^\infty_x(\mathcal{L}^r)}(s)\\
&\quad\leqslant C\sum_{p\geqslant q-3}\sum_{|p-q'|\leqslant 1} 2^{\alpha q}2^{-q}\|\nabla\Delta_p u\|_{L^\infty}(s)\cdot 2^{q'}\|\psi\|_{L^\infty_x(\mathcal{L}^r)}(s)\\
&\quad\leqslant C\sum_{p\geqslant q-3} 2^{\alpha p}\|\nabla\Delta_p u\|_{L^\infty}(s)\cdot 2^{\alpha(q-p)}\|\psi\|_{L^\infty_x(\mathcal{L}^r)}(s)\\
&\quad\leqslant CA(s)\cdot\|\psi\|_{L^\infty_x(\mathcal{L}^r)}(s)\leqslant CA(s).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\sum_{|q-q'|\leqslant 5} 2^{\alpha q}\big\|[S_{q'-1}u\cdot\nabla,\Delta_q]\Delta_{q'}\psi\big\|_{L^\infty_x(\mathcal{L}^r)}(s)\\
&\quad\leqslant C\sum_{|q-q'|\leqslant 5} 2^{\alpha q-q}\|\nabla S_{q'-1}u\|_{L^\infty}(s)\|\nabla\Delta_{q'}\psi\|_{L^\infty_x(\mathcal{L}^r)}(s)\\
&\quad\leqslant C\sum_{|q-q'|\leqslant 5} \|\nabla u\|_{L^\infty}(s)\cdot 2^{\alpha q'}\|\Delta_{q'}\psi\|_{L^\infty_x(\mathcal{L}^r)}(s)\\
&\quad\leqslant C\|\nabla u\|_{L^\infty}(s)\cdot D(s),
\end{aligned}$$

$$\begin{aligned}
&\sum_{|q'-q|\leqslant 5} 2^{\alpha q}\big\|[\Delta_{q'}u\cdot\nabla,\Delta_q]S_{q'-1}\psi\big\|_{L^\infty_x(\mathcal{L}^r)}(s)\\
&\quad\leqslant C\sum_{|q'-q|\leqslant 5} 2^{\alpha q-q}\|\nabla\Delta_{q'}u\|_{L^\infty}(s)\cdot\|S_{q'-1}\nabla\psi\|_{L^\infty_x(\mathcal{L}^r)}(s)\\
&\quad\leqslant CA(s)\|\psi\|_{L^\infty_x(\mathcal{L}^r)}(s)\leqslant CA(s).
\end{aligned}$$

Therefore, taking the supreme of (3.10) with respect to q, by the Young's inequality and (3.8),

$$\begin{aligned}
D(t)^r&\leqslant\int_0^t C\big[A(s)^r+D(s)^r\big]\,ds+C\int_0^t\|\nabla u\|_{L^\infty}(s)D(s)^r\,ds\\
&\leqslant\int_0^t C\big[1+D(s)^r\big]+C\|\nabla u\|_{L^\infty}D(s)^r\,ds.
\end{aligned}$$

Then the Gronwall's inequality implies that

$$D(t)\leqslant C\big(D(0)+1\big)e^{C\int_0^t(\|\nabla u\|_{L^\infty}+1)\,ds}.$$

Hence according to Lemma 2.2,

$$\begin{aligned} e + D(t) &\leqslant C\big(D(0)+1\big)C_* \exp\left\{C\int_{t_*}^{t}\big(\|\nabla u\|_{L^\infty}+1\big)\,ds\right\} \\ &\leqslant CC_*\big(D(0)+1\big)\exp\left\{C\left(1+\int_{t_*}^{t}\|u\|_{L^2}\,ds+\epsilon\ln\big(e+(t-t_*)A(t)\big)\right)\right\} \\ &\leqslant CC_*\big(D(0)+1\big)\exp\big\{C\big[1+\epsilon\ln\big(e+D(t)\big)\big]\big\} \\ &\leqslant CC_*\big(D(0)+1\big)\big(e+D(t)\big)^{C\epsilon}, \end{aligned}$$

where C_* is some positive constant depending on the solution u on $[0, t_*]$. Choosing $\epsilon = \frac{1}{2C}$,

$$D(T) \leqslant \big[CC_*\big(D(0)+1\big)\big]^2.$$

Then by (3.8), $A(T)$ is bounded, which implies that $\|\nabla u\|_{L^\infty}$ is bounded on $[0, T]$ since

$$\|\nabla u\|_{L^\infty} \leqslant C\big(\|u\|_{L^2} + \|u\|_{\dot{C}^{1+\alpha}}\big).$$

4. Proof of Theorem 1.2

The proof of Theorem 1.2 is similar to that of Theorem 1.1, we just give the sketch. As above, to get the global existence we only need to control $\|\nabla u\|_{L^\infty}$, see [10].

Step I: Uniform estimates for ψ and τ.

Integrate the third equation of (1.3) on M, then

$$\partial_t \int_M \psi(t,x,m)\,dm + u\cdot\nabla\int_M \psi(t,x,m)\,dm = 0.$$

By the maximum principle for evolutionary equation, ψ is always nonnegative. Hence

$$\|\psi\|_{L^1_x\cap L^\infty_x(L^1(M))}(t) = \|\psi_0\|_{L^1_x\cap L^\infty_x(L^1(M))}, \tag{4.1}$$

$$\|\tau\|_{L^\infty(0,T;L^2)} \leqslant C\big(\|\psi_0\|^2_{L^4_x(L^1(M))} + \|\psi_0\|_{L^2_x(L^1(M))}\big), \tag{4.2}$$

$$\|\tau\|_{L^\infty(0,T;L^\infty)} \leqslant C\big(\|\psi_0\|^2_{L^\infty_x(L^1(M))} + 1\big). \tag{4.3}$$

Since $s > \frac{d}{2}+1$ and M is a smooth compact manifold without boundary,

$$\|\psi\|_{L^\infty_{t,x}(H^{-s}(M))} \leqslant C\|\psi_0\|_{L^\infty_x(L^1(M))}. \tag{4.4}$$

Step II: A priori estimates for u.

Applying Lemma 3.2 and the estimates (4.2), (4.3), we get that

$$u \in L^\infty\big(0,T;L^2\big)\cap L^2\big(0,T;\dot{H}^1\big)\cap \tilde{L}^1\big(0,T;\dot{C}^1\big)$$

and $\forall\epsilon > 0$, there exists $t_0(\epsilon) \in (0, T)$, such that $\|u\|_{\tilde{L}^1(t_0,T;\dot{C}^1)} \leqslant \epsilon$.

Step III: Hölder estimates for u.
Denote

$$H=(-\Delta_g+I)^{-\frac{s}{2}},\qquad N_q^2(t,x)=\int_M \left|H\Delta_q\psi(t,x,m)\right|^2 dm,$$
$$A(t)=\sup_{0\leqslant s<t}\left\|u(s,\cdot)\right\|_{\dot C^{1+\alpha}},\qquad B(t)=\sup_{0\leqslant s<t}\left\|\tau(s,\cdot)\right\|_{\dot C^{\alpha}},$$
$$D(t)=\sup_{0\leqslant s<t}\sup_{q\in\mathbb{Z}}2^{\alpha q}\left\|N_q(s,\cdot)\right\|_{L^\infty}.$$

As in Section 3, we have the estimates

$$A(t)\leqslant C\big(\|u_0\|_{\dot C^{1+\alpha}}+B(t)\big)\leqslant C\big(1+B(t)\big),$$

and by (4.4), $\forall q\in\mathbb{Z}$,

$$\left\|N_q(t,\cdot)\right\|_{L^\infty}\leqslant C\left\|H\psi(t,\cdot,\cdot)\right\|_{L_x^\infty(L^2(M))}\leqslant C\|\psi_0\|_{L_x^\infty(L^1(M))},$$

$$\begin{aligned}
&2^{\alpha q}\|\Delta_q\tau_{ij}(s,x)\|_{L^\infty}\\
&\quad\leqslant 2^{\alpha q}\left\|\int_M\int_M H_{m_1}^{-1}H_{m_2}^{-1}\gamma_{ij}^{(2)}\Delta_q\big(H_{m_1}\psi(s,x,m_1)H_{m_2}\psi(s,x,m_2)\big)\,dm_1\,dm_2\right\|_{L^\infty}\\
&\qquad+2^{\alpha q}\left\|\int_M H^{-1}\gamma_{ij}^{(1)}(m)H\Delta_q\psi(s,x,m)\,dm\right\|_{L^\infty}\\
&\quad\leqslant 2^{\alpha q}\sum_{|p-q|\leqslant 5}\left\|\int_M\int_M H_{m_1}^{-1}H_{m_2}^{-1}\gamma_{ij}^{(2)}(m_1,m_2)S_{p-1}H\psi(m_1)\Delta_pH\psi(m_2)\,dm_1\,dm_2\right\|_{L^\infty}\\
&\qquad+2^{\alpha q}\sum_{|p-q|\leqslant 5}\left\|\int_M\int_M H_{m_1}^{-1}H_{m_2}^{-1}\gamma_{ij}^{(2)}(m_1,m_2)\Delta_pH\psi(m_1)S_{p-1}H\psi(m_2)\,dm_1\,dm_2\right\|_{L^\infty}\\
&\qquad+2^{\alpha q}\sum_{p\geqslant q-3}\sum_{|p-r|\leqslant 1}\left\|\int_M\int_M H_{m_1}^{-1}H_{m_2}^{-1}\gamma_{ij}^{(2)}\Delta_pH\psi(m_1)\Delta_rH\psi(m_2)\,dm_1\,dm_2\right\|_{L^\infty}\\
&\qquad+C2^{\alpha q}\left\|N_q(s,\cdot)\right\|_{L^\infty}\\
&\quad\leqslant CD(s)\|H\psi\|_{L_x^\infty(L^2(M))}(s)+CD(s)\leqslant CD(s)
\end{aligned}$$

which implies that

$$B(t)\leqslant CD(t).$$

Therefore,

$$A(t)\leqslant C\big(1+D(t)\big).\tag{4.5}$$

Step IV: Hölder estimates for ψ.
Take the operator H and Δ_q to the third equation, multiply by $\Delta_q H\psi$ and integrate over M, then

$$\begin{aligned}
&\frac{1}{2}\partial_t\int_M |\Delta_q H\psi|^2\,dm + \frac{1}{2}u\cdot\nabla\int_M |\Delta_q H\psi|^2\,dm + \int_M |\nabla_g\Delta_q H\psi|^2\,dm \\
&= \int_M [u\cdot\nabla,\Delta_q]H\psi\cdot\Delta_q H\psi\,dm - \int_M \Delta_q H\,\mathrm{div}_g\big(G(u,\psi)\psi\big)\cdot\Delta_q H\psi\,dm \\
&= \int_M [u\cdot\nabla,\Delta_q]H\psi\cdot\Delta_q H\psi\,dm - \partial_j u_i\int_M H\,\mathrm{div}_g\big(c_\alpha^{ij}\Delta_q\psi\big)\cdot H\Delta_q\psi\,dm \\
&\quad + \int_M [\partial_j u_i,\Delta_q]H\big(c_\alpha^{ij}\psi\big)\cdot\nabla_g\Delta_q H\psi\,dm + \int_M \Delta_q(\nabla_g\mathcal{U}H\psi)\cdot\nabla_g\Delta_q H\psi\,dm \\
&\quad + \int_M \Delta_q\big[H\nabla_g\mathcal{U},H^{-1}\big]H\psi\cdot\nabla_g\Delta_q H\psi\,dm.
\end{aligned}$$

By the Young's inequality, we have

$$\begin{aligned}
&\frac{1}{2}\|\Delta_q H\psi\|^2_{L_x^\infty(L^2(M))}(t) \\
&\leqslant 2\int_0^t\bigg\|\int_M [u\cdot\nabla,\Delta_q]H\psi\cdot\Delta_q H\psi\,dm\bigg\|_{L^\infty}ds + \int_0^t\bigg\|\partial_j u_i\int_M H\,\mathrm{div}_g\big(c_\alpha^{ij}\Delta_q\psi\big)\Delta_q H\psi\,dm\bigg\|_{L^\infty}ds \\
&\quad + \frac{3}{4}\int_0^t\big\|[\partial_j u_i,\Delta_q]H\big(c_\alpha^{ij}\psi\big)\big\|^2_{L_x^\infty(L^2(M))}ds + \frac{3}{4}\int_0^t\|\Delta_q(\nabla_g\mathcal{U}H\psi)\|^2_{L_x^\infty(L^2(M))}ds \\
&\quad + \frac{3}{4}\int_0^t\big\|\Delta_q\big(\big[H\nabla_g\mathcal{U},H^{-1}\big]H\psi\big)\big\|^2_{L_x^\infty(L^2(M))}ds \\
&\triangleq \int_0^t (J_1+J_2+J_3+J_4+J_5)\,ds.
\end{aligned}\tag{4.6}$$

As in Section 3, applying (4.5), for every $q\in\mathbb{Z}$,

$$\begin{aligned}
2^{2\alpha q}J_1(s) &\leqslant CA(s)\|H\psi\|_{L_x^\infty(L^2(M))}\cdot D(s) + C\|\nabla u\|_{L^\infty}\cdot D(s)^2 \\
&\leqslant C\big(D(s)^2+1\big)\big(\|\nabla u\|_{L^\infty}(s)+1\big).
\end{aligned}$$

J_2, J_4 and J_5 are estimated as in [9],

$$\begin{aligned}
&\left|\int_M H\,\mathrm{div}_g\big(c_\alpha^{ij}\Delta_q\psi\big)\cdot\Delta_q H\psi\,dm\right| \\
&\quad\leqslant\left|\int_M \mathrm{div}_g\big(c_\alpha^{ij}H\Delta_q\psi\big)\cdot\Delta_q H\psi\,dm\right|+\left|\int_M\big(\big[H\,\mathrm{div}_g\,c_\alpha^{ij},H^{-1}\big]H\Delta_q\psi\big)\cdot\Delta_q H\psi\,dm\right| \\
&\quad\leqslant\left|\int_M \frac{1}{2}\big(\mathrm{div}_g\,c_\alpha^{ij}\big)|\Delta_q H\psi|^2\,dm\right|+\left|\int_M\big(\big[H\,\mathrm{div}_g\,c_\alpha^{ij},H^{-1}\big]H\Delta_q\psi\big)\cdot\Delta_q H\psi\,dm\right| \\
&\quad\leqslant C\|\Delta_q H\psi\|_{L^2(M)}^2,
\end{aligned}$$

which deduces that

$$2^{2\alpha q}J_2(s)\leqslant C2^{2\alpha q}\|\nabla u\|_{L^\infty}(s)\cdot N_q^2(s)\leqslant C\|\nabla u\|_{L^\infty}(s)\cdot D(s)^2.$$

Using Bony's decomposition,

$$\begin{aligned}
&2^{\alpha q}\big\|\Delta_q(\nabla_g\mathcal{U}H\psi)\big\|_{L_x^\infty(L^2(M))} \\
&\quad\leqslant 2^{\alpha q}\sum_{|p-q|\leqslant 5}\|S_{p-1}\nabla_g\mathcal{U}\cdot\Delta_p H\psi\|_{L_x^\infty(L^2(M))}+\sum_{|p-q|\leqslant 5}\|\Delta_p\nabla_g\mathcal{U}\cdot S_{p-1}H\psi\|_{L_x^\infty(L^2(M))} \\
&\qquad+\sum_{p\geqslant q-3}\sum_{|p-r|\leqslant 1}\|\Delta_p\nabla_g\mathcal{U}\cdot\Delta_r H\psi\|_{L_x^\infty(L^2(M))} \\
&\quad\leqslant C\|\nabla_g\mathcal{U}\|_{L_x^\infty(L^2(M))}\cdot\sup_p 2^{\alpha p}\|\Delta_p H\psi\|_{L_x^\infty(L^2(M))} \\
&\qquad+C\sup_p 2^{\alpha p}\|\Delta_p\nabla_g\mathcal{U}\|_{L_x^\infty(L^2(M))}\cdot\|H\psi\|_{L_x^\infty(L^2(M))}.
\end{aligned}$$

Combining the relationship between $\mathcal{U}$ and ψ,

$$2^{2\alpha q}J_4(s)\leqslant C2^{2\alpha q}\|H\psi\|_{L_x^\infty(L^2(M))}^2(s)\cdot N_q^2(s)\leqslant CD(s)^2.$$

Similarly,

$$2^{2\alpha q}J_5(s)\leqslant C\|H\psi\|_{L_x^\infty(L^2(M))}^2(s)\cdot\sup_p 2^{2\alpha p}N_p^2(s)\leqslant CD(s)^2.$$

Since

$$[\partial_j u_i,\Delta_q]H\big(c_\alpha^{ij}\psi\big)=\int_{\mathbb{R}^2}h(y)\big[\partial_j u_i(x)-\partial_j u_i\big(x-2^{-q}y\big)\big]H\big(c_\alpha^{ij}\psi\big)\big(x-2^{-q}y,m\big)\,dy,$$

$$\begin{aligned}
2^{2\alpha q}J_3(s)&\leqslant C2^{2\alpha q}\cdot 2^{-2\alpha q}\|\nabla u\|_{\dot C^\alpha}^2(s)\big\|H\big(c_\alpha^{ij}\psi\big)\big\|_{L_x^\infty(L^2(M))}^2(s) \\
&\leqslant CA(s)^2\big[\big\|c_\alpha^{ij}H\psi\big\|_{L_x^\infty(L^2(M))}^2+\big\|H\big[c_\alpha^{ij},H^{-1}\big]H\psi\big\|_{L_x^\infty(L^2(M))}^2\big] \\
&\leqslant CA(s)^2\|H\psi\|_{L_x^\infty(L^2(M))}^2 \\
&\leqslant CA(s)^2.
\end{aligned}$$

Therefore, taking the supreme of (4.6) $\times 2^{2\alpha q}$ with respect to t and q,

$$D(t)^2 \leqslant C \int_0^t \left(\|\nabla u\|_{L^\infty} + 1\right)\left(1 + D(s)^2\right) ds.$$

The remaining proof is just the same as that in Section 3. We omit the details.

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