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Generalizations of a property of orthogonal projectors

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Abstract

Generalizing the result in Lemma of Baksalary and Baksalary [J.K. Baksalary, O.M. Baksalary, Commutativity of projectors, Linear Algebra Appl. 341 (2002) 129–142], Baksalary et al. [J.K. Baksalary, O.M. Baksalary, T. Szulc, A property of orthogonal projectors, Linear Algebra Appl. 354 (2002) 35–39] have shown that if $\mathbf{P}_1$ and $\mathbf{P}_2$ are orthogonal projectors, then, in all nontrivial situations, a product of any length having $\mathbf{P}_1$ and $\mathbf{P}_2$ as its factors occurring alternately is equal to another such product if and only if $\mathbf{P}_1$ and $\mathbf{P}_2$ commute, in which case all products involving $\mathbf{P}_1$ and $\mathbf{P}_2$ reduce to the orthogonal projector $\mathbf{P}_1\mathbf{P}_2$ $(=\mathbf{P}_2\mathbf{P}_1)$. In the present paper, further generalizations of this property are established. They consist in replacing a product of the type specified above, appearing on the left-hand side (say) of the equality under considerations, by an affine combination of two or three such products. Comments on the problem when the number of components in a combination exceeds three are also given.

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1. Introduction and statement of the results

Let $\mathbb{C}$ and $\mathbb{C}_{n,n}$ denote the set of complex numbers and the set of $n \times n$ complex matrices, respectively. The considerations of this paper are concerned with the subset of $\mathbb{C}_{n,n}$ denoted by $\mathbb{C}_n^{\mathsf{OP}}$, which consists of orthogonal projectors in $\mathbb{C}_{n,1}$, i.e.,

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$$\mathbb{C}_n^{\mathsf{OP}} = \{\mathbf{P} \in \mathbb{C}_{n,n}\colon\ \mathbf{P}^2 = \mathbf{P} = \mathbf{P}^*\}, \tag{1.1}$$

where $\mathbf{P}^*$ stands for the conjugate transpose of $\mathbf{P}$.

One of the essential properties of a pair of orthogonal projectors $\mathbf{P}_1$ and $\mathbf{P}_2$, say, is their commutativity

$$\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2\mathbf{P}_1. \tag{1.2}$$

Firstly, (1.2) is a necessary and sufficient condition for the product $\mathbf{P}_1\mathbf{P}_2$ of $\mathbf{P}_1, \mathbf{P}_2 \in \mathbb{C}_n^{\mathsf{OP}}$ to be a projector, i.e., to satisfy

$$\mathbf{P}_1\mathbf{P}_2 = (\mathbf{P}_1\mathbf{P}_2)^2. \tag{1.3}$$

If this is the case, then interpreting (1.2) as $\mathbf{P}_1\mathbf{P}_2 = (\mathbf{P}_1\mathbf{P}_2)^*$ and referring to (1.1) shows that $\mathbf{P}_1\mathbf{P}_2$ (and, consequently, $\mathbf{P}_2\mathbf{P}_1$ as well) is actually an orthogonal projector. It should be emphasized that for idempotent $\mathbf{P}_1$ and $\mathbf{P}_2$ not required to be Hermitian, condition (1.2), still being sufficient for (1.3), is no longer necessary; cf. [4, § 42]. Secondly, there are several interesting relations between the property (1.2) and various problems in mathematical statistics, for instance such as (i) characterizations of the situations where all nonzero canonical correlations are equal to one, (ii) comparisons between the ordinary least-squares estimator, generalized least-squares estimator, and best linear unbiased estimator of the vector of expectations in the general Gauss–Markov model, and (iii) comparisons between two linear models: one involving nuisance parameters and the other being free of them; see [1, Section 3]. Hence it follows that there is a substantial interest in the results enabling to recognize whether (1.2) is satisfied or not and the purpose of the present paper is to meet this need.

Baksalary et al. [3, Theorem], have shown that several characterizations of (1.2) known in the literature, which refer to two different products having orthogonal projectors $\mathbf{P}_1$ and $\mathbf{P}_2$ as their factors, are in fact particular cases of the theorem below.

Theorem 1. *Let $P_{(m;i)}$ denote an m-factor product of $\mathbf{P}_1, \mathbf{P}_2 \in \mathbb{C}_n^{\mathsf{OP}}$, with $\mathbf{P}_i$ being the first factor and $\mathbf{P}_i, \mathbf{P}_j$ occurring alternately, $i, j = 1, 2;\ i \neq j$. Then the following statements are equivalent*:

(a) $P_{(p;i)} = P_{(q;i_0)}$ *for some $p, q \geqslant 2$ and some $i, i_0 \in \{1, 2\}$, except for the trivial case where simultaneously $p = q$ and $i = i_0$,*
(b) $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2\mathbf{P}_1$,
(c) $P_{(p;i)} = P_{(q;i_0)}$ *for every $p, q \geqslant 2$ and $i, i_0 \in \{1, 2\}$.*

In this paper, Theorem 1 is generalized by considering the replacement of the product occurring on the left-hand side (say) of the equality in (a) and (c) by affine combinations of products of such a type. Explicit presentations of these generalizations in the cases when the number of components in combinations under considerations is two or three are provided in Theorems 2 and 3. In addition, at the end of the paper, comments on the problem when the number of components exceeds three are given.

Theorem 2. *Let $c_1, c_2 \in \mathbb{C}$ be nonzero and such that $c_1 + c_2 = 1$ and let $P_{(m;i)}$ denote an m-factor product of $\mathbf{P}_1, \mathbf{P}_2 \in \mathbb{C}_n^{\mathsf{OP}}$, with $\mathbf{P}_i$ being the first factor and $\mathbf{P}_i, \mathbf{P}_j$ occurring alternately, $i, j = 1, 2;\ i \neq j$. Then the following statements are equivalent*:

(a) $c_1P_{(p_1;i_1)} + c_2P_{(p_2;i_2)} = P_{(q;i_0)}$ *for some $p_1, p_2, q \geqslant 2$ and some $i_1, i_2, i_0 \in \{1, 2\}$, excluding the situations where p_1, p_2, and q are all even or all odd and, simultaneously, $i_1 = i_2 = i_0$,*

(b) $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2\mathbf{P}_1$,
(c) $c_1P_{(p_1;i_1)} + c_2P_{(p_2;i_2)} = P_{(q;i_0)}$ *for every* $p_1, p_2, q \geqslant 2$ *and* $i_1, i_2, i_0 \in \{1, 2\}$.

Theorem 3. *Let* $c_1, c_2, c_3 \in \mathbb{C}$ *be nonzero and such that* $c_1 + c_2 + c_3 = 1$ *and let* $P_{(m;i)}$ *denote an* m*-factor product of* $\mathbf{P}_1, \mathbf{P}_2 \in \mathbb{C}_n^{\mathsf{OP}}$, *with* $\mathbf{P}_i$ *being the first factor and* $\mathbf{P}_i, \mathbf{P}_j$ *occurring alternately,* $i, j = 1, 2$; $i \neq j$. *Then the following statements are equivalent*:

(a) $c_1P_{(p_1;i_1)} + c_2P_{(p_2;i_2)} + c_3P_{(p_3;i_3)} = P_{(q;i_0)}$ *for some* $p_1, p_2, p_3, q \geqslant 2$ *and some* $i_1, i_2, i_3, i_0 \in \{1, 2\}$, *excluding the situations in which the set* $\{i_1, i_2, i_3\}$ *coincides with a set* $\{i_a, i_b, i_c\}$ *having the property that* $i_a = i_0, i_b = i_c$ *for some different* $a, b, c \in \{1, 2, 3\}$ *and, moreover,* p_1, p_2, p_3, q *are all even or all odd, or* p_a, q *are even while* p_b, p_c *are odd or* p_a, q *are odd while* p_b, p_c *are even*,
(b) $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2\mathbf{P}_1$,
(c) $c_1P_{(p_1;i_1)} + c_2P_{(p_2;i_2)} + c_3P_{(p_3;i_3)} = P_{(q;i_0)}$ *for every* $p_1, p_2, p_3, q \geqslant 2$ *and* $i_1, i_2, i_3, i_0 \in \{1, 2\}$.

It is noteworthy that Theorem 2 is a generalization of Theorem 1. In fact, if in the former the products $P_{(p_1;i_1)}$ and $P_{(p_2;i_2)}$ are both taken to be $P_{(p;i)}$, then on account of $c_1 + c_2 = 1$ the left-hand side of the equality occurring in parts (a) and (c) of Theorem 2 reduces to $P_{(p;i)}$, as in the corresponding parts of Theorem 1. Similar arguments show that Theorem 3 is a generalization of Theorem 2. Namely, if $p_3 = p_2$, $i_3 = i_2$, then the equality occurring in parts (a) and (c) of Theorem 3 takes the form $c_1P_{(p_1;i_1)} + \tilde{c}_2P_{(p_2;i_2)} = P_{(q;i_0)}$ with $\tilde{c}_2 = c_2 + c_3$, which clearly satisfies $c_1 + \tilde{c}_2 = 1$. Analogous conclusions are obtained when considering the versions $p_3 = p_1$, $i_3 = i_1$ and $p_2 = p_1$, $i_2 = i_1$.

2. Proofs of theorems and comments

It is clear that (1.2) implies the equalities $P_{(m;i)} = \mathbf{P}_1\mathbf{P}_2$ and $P_{(m;j)} = \mathbf{P}_1\mathbf{P}_2$ for every $m \geqslant 2$ and any $i, j \in \{1, 2\}$. Hence it follows immediately that part (b) of Theorem 2 entails its part (c). Since the implication (c) $\Rightarrow$ (a) holds trivially, it remains to establish that (a) $\Rightarrow$ (b). In order to do it, we consider 24 cases depending on which of the integers p_1, p_2, q (indicating the lengths of product chains in (a) and (c)) are even and which are odd, and depending also on the mutual relations between i_1, i_2, and i_0, i.e., whether $i_1 = i_2$ is combined with $i_0 = i_1$ or $i_0 \neq i_1$ or whether $i_1 \neq i_2$ is combined with $i_0 = i_1$ or $i_0 = i_2$. Adopting the notation $p_1 = 2s$ or $p_1 = 2s + 1$, $p_2 = 2t$ or $p_2 = 2t + 1$, and $q = 2v$ or $q = 2v + 1$, with positive integers s, t, and v, the complete list of cases to be analyzed can be expressed as the set of the equations

$$A_\alpha^{(2)} = B_\beta, \quad \alpha = 1, \ldots, 6; \quad \beta = 1, \ldots, 4, \tag{2.1}$$

where

$$A_1^{(2)} = c_1P_{(2s;i)} + c_2P_{(2t;i)},$$
$$A_2^{(2)} = c_1P_{(2s;i)} + c_2P_{(2t;j)},$$
$$A_3^{(2)} = c_1P_{(2s;i)} + c_2P_{(2t+1;i)},$$
$$A_4^{(2)} = c_1P_{(2s+1;i)} + c_2P_{(2t;j)},$$

$$A_5^{(2)} = c_1 P_{(2s+1;i)} + c_2 P_{(2t+1;i)},$$
$$A_6^{(2)} = c_1 P_{(2s+1;i)} + c_2 P_{(2t+1;j)}$$

and

$$B_1 = P_{(2v;i)},$$
$$B_2 = P_{(2v;j)},$$
$$B_3 = P_{(2v+1;i)},$$
$$B_4 = P_{(2v+1;j)}.$$

The first step in our proof of Theorem 2 is to show that, as pointed out in part (a), the equalities $A_1^{(2)} = B_1$ and $A_5^{(2)} = B_3$ do not in general ensure the commutativity of $\mathbf{P}_1$ and $\mathbf{P}_2$. For example, if

$$\mathbf{P}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{P}_2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \tag{2.2}$$

then, for any $c_1, c_2 \in \mathbb{C}$ such that $4c_1 + c_2 = 2$,

$$c_1 P_{(2;1)} + c_2 P_{(6;1)} = P_{(4;1)} \quad \text{and} \quad c_1 P_{(3;1)} + c_2 P_{(7;1)} = P_{(5;1)},$$

although $\mathbf{P}_1\mathbf{P}_2 \neq \mathbf{P}_2\mathbf{P}_1$. Clearly, the choice of $c_1 = \frac{1}{3}$, $c_2 = \frac{2}{3}$ provides desired counterexamples, as $4c_1 + c_2 = 2$ is then satisfied along with $c_1 + c_2 = 1$.

The second step is to establish the validity of (1.2) in the remaining 22 cases. An essential role in this part of the proof is played by the observations that, for any positive integer k and $i, j = 1, 2$, $i \neq j$,

$$\mathbf{P}_i P_{(2k;i)} = P_{(2k;i)} = P_{(2k;i)}\mathbf{P}_j, \tag{2.3}$$
$$\mathbf{P}_i P_{(2k+1;i)} = P_{(2k+1;i)} = P_{(2k+1;i)}\mathbf{P}_i, \tag{2.4}$$
$$\mathbf{P}_i P_{(2k;j)} = P_{(2k+1;i)} = P_{(2k;i)}\mathbf{P}_i, \tag{2.5}$$
$$\mathbf{P}_i P_{(2k+1;j)} = P_{(2k+2;i)} = P_{(2k+1;i)}\mathbf{P}_j. \tag{2.6}$$

On account of the first parts of (2.3) and/or (2.4) combined with the first part of (2.5) it follows that comparing $A_1^{(2)} = B_2$, $A_3^{(2)} = B_2$, and $A_5^{(2)} = B_2$ with the corresponding versions of these equalities obtained by premultiplying by $\mathbf{P}_i$ leads to

$$P_{(2v;j)} = P_{(2v+1;i)}. \tag{2.7}$$

Analogous arguments, with (2.5) replaced by (2.6), show that each of the equalities $A_1^{(2)} = B_4$, $A_3^{(2)} = B_4$, and $A_5^{(2)} = B_4$ implies

$$P_{(2v+1;j)} = P_{(2v+2;i)}. \tag{2.8}$$

Moreover, in view of the second parts of (2.3) and/or (2.4) combined with the second part of either (2.5) or (2.6), comparing $A_4^{(2)} = B_1$, $A_5^{(2)} = B_1$, and $A_4^{(2)} = B_4$ with their counterparts obtained by postmultiplying by $\mathbf{P}_i$ leads to

$$P_{(2v;i)} = P_{(2v+1;i)} \tag{2.9}$$

in the first two cases and to

$$P_{(2v+1;j)} = P_{(2v+2;j)} \tag{2.10}$$

in the third one. Similar arguments applied to $A_1^{(2)} = B_3$, but with the postmultiplier $\mathbf{P}_j$ instead of $\mathbf{P}_i$, yield

$$P_{(2v+1;i)} = P_{(2v+2;i)}. \tag{2.11}$$

On account of equalities (2.7)–(2.11), in all 10 cases discussed above the commutativity condition (1.2) follows immediately from Theorem 1.

In the analysis of the remaining 12 cases, the nonzero scalars c_1, c_2 come into play. This, however, makes the considerations only seemingly more involved. In view of combinations of the first parts of (2.3) and/or (2.4) with the first part of (2.5), it follows that comparing $A_2^{(2)} = B_1$, $A_2^{(2)} = B_3$, and $A_4^{(2)} = B_3$ with the corresponding versions of these equalities obtained by premultiplying by $\mathbf{P}_i$ and referring to $c_2 \neq 0$ leads in each of these cases to

$$P_{(2t;j)} = P_{(2t+1;i)}. \tag{2.12}$$

Referring again to $c_2 \neq 0$, other conditions involving t, namely

$$P_{(2t;j)} = P_{(2t+1;j)}, \quad P_{(2t+1;i)} = P_{(2t+2;i)}, \quad \text{and} \quad P_{(2t+1;j)} = P_{(2t+2;i)}, \tag{2.13}$$

are obtained when $A_2^{(2)} = B_4$ is compared with $A_2^{(2)}\mathbf{P}_j = B_4\mathbf{P}_j$, when $A_3^{(2)} = B_1$ is compared with $A_3^{(2)}\mathbf{P}_j = B_1\mathbf{P}_j$, and when $A_6^{(2)} = B_3$ is compared with $\mathbf{P}_i A_6^{(2)} = \mathbf{P}_i B_3$, respectively.

The last six cases involve s and are dependent on the assumption $c_1 \neq 0$. In view of (2.3)–(2.5), combining $A_2^{(2)} = B_2$ with $A_2^{(2)}\mathbf{P}_i = B_2\mathbf{P}_i$ and $A_3^{(2)} = B_3$ with $A_3^{(2)}\mathbf{P}_i = B_3\mathbf{P}_i$ yields

$$P_{(2s;i)} = P_{(2s+1;i)}. \tag{2.14}$$

The next two situations are described by the equalities $A_6^{(2)} = B_1$ and $A_6^{(2)} = B_4$. In view of (2.3), (2.4), and (2.6), combining them with $A_6^{(2)}\mathbf{P}_j = B_1\mathbf{P}_j$ and $A_6^{(2)}\mathbf{P}_j = B_4\mathbf{P}_j$, respectively, results in

$$P_{(2s+1;i)} = P_{(2s+2;i)}. \tag{2.15}$$

When considering $A_4^{(2)} = B_2$ and $A_6^{(2)} = B_2$, the postmultiplication by $\mathbf{P}_j$ is replaced by the premultiplication by this matrix in which case, in view of (2.3), (2.4), and (2.6), analogous arguments as above show that

$$P_{(2s+1;i)} = P_{(2s+2;j)}. \tag{2.16}$$

On account of equalities (2.12)–(2.16), the commutativity condition (1.2) again follows immediately from Theorem 1. The proof of Theorem 2 is thus complete.

A structure of the proof of Theorem 3 is similar to the previous one, but now 40 cases are to be considered when establishing that (a) $\Rightarrow$ (b). This is a quite substantial increase compared to the number of 24 cases, which have arisen in the proof of Theorem 2. Their complete list can be expressed as the set of the equalities

$$A_\alpha^{(3)} = B_\beta, \quad \alpha = 1, \ldots, 10; \quad \beta = 1, \ldots, 4, \tag{2.17}$$

where $B_1, \ldots, B_4$ are the same as in the earlier proof, while $A_1^{(3)}, \ldots, A_{10}^{(3)}$ are specified as follows:

$$A_1^{(3)} = c_1 P_{(2s;i)} + c_2 P_{(2t;i)} + c_3 P_{(2u;i)},$$
$$A_2^{(3)} = c_1 P_{(2s;i)} + c_2 P_{(2t;i)} + c_3 P_{(2u;j)},$$
$$A_3^{(3)} = c_1 P_{(2s;i)} + c_2 P_{(2t;i)} + c_3 P_{(2u+1;i)},$$

$$A_4^{(3)} = c_1 P_{(2s;i)} + c_2 P_{(2t;i)} + c_3 P_{(2u+1;j)},$$
$$A_5^{(3)} = c_1 P_{(2s;i)} + c_2 P_{(2t;j)} + c_3 P_{(2u+1;i)},$$
$$A_6^{(3)} = c_1 P_{(2s;i)} + c_2 P_{(2t+1;i)} + c_3 P_{(2u+1;i)},$$
$$A_7^{(3)} = c_1 P_{(2s;i)} + c_2 P_{(2t+1;i)} + c_3 P_{(2u+1;j)},$$
$$A_8^{(3)} = c_1 P_{(2s;i)} + c_2 P_{(2t+1;j)} + c_3 P_{(2u+1;j)},$$
$$A_9^{(3)} = c_1 P_{(2s+1;i)} + c_2 P_{(2t+1;i)} + c_3 P_{(2u+1;i)},$$
$$A_{10}^{(3)} = c_1 P_{(2s+1;i)} + c_2 P_{(2t+1;i)} + c_3 P_{(2u+1;j)}.$$

Again the first step is to identify cases, in which an equality of the form (2.17) does not ensure the commutativity property (1.2). It appears that there are eight such situations. In all counterexamples, which are needed to justify this statement, it is assumed that $i = 1$, $j = 2$, and the orthogonal projectors $\mathbf{P}_1$ and $\mathbf{P}_2$ are as given in (2.2). The precise description of desired counterexamples is as follows:

$$A_1^{(3)} = B_1, \quad \text{with } s = 2, t = 3, u = 4, v = 1 \text{ and } c_1 = \tfrac{5}{2}, c_2 = -\tfrac{1}{2}, c_3 = -1,$$
$$A_2^{(3)} = B_2, \quad \text{with } s = 1, t = 2, u = 2, v = 1 \text{ and } c_1 = 1, c_2 = -2, c_3 = 2,$$
$$A_3^{(3)} = B_3, \quad \text{with } s = 1, t = 2, u = 2, v = 1 \text{ and } c_1 = 1, c_2 = -2, c_3 = 2,$$
$$A_4^{(3)} = B_4, \quad \text{with } s = 1, t = 2, u = 2, v = 1 \text{ and } c_1 = 1, c_2 = -2, c_3 = 2,$$
$$A_6^{(3)} = B_1, \quad \text{with } s = 2, t = 1, u = 2, v = 1 \text{ and } c_1 = 2, c_2 = 1, c_3 = -2,$$
$$A_8^{(3)} = B_1, \quad \text{with } s = 2, t = 1, u = 2, v = 1 \text{ and } c_1 = 2, c_2 = 1, c_3 = -2,$$
$$A_9^{(3)} = B_3, \quad \text{with } s = 2, t = 3, u = 4, v = 1 \text{ and } c_1 = \tfrac{5}{2}, c_2 = -\tfrac{1}{2}, c_3 = -1,$$
$$A_{10}^{(3)} = B_4, \quad \text{with } s = 1, t = 2, u = 2, v = 1 \text{ and } c_1 = 1, c_2 = -2, c_3 = 2.$$

The second step is to establish the validity of (1.2) in the remaining 32 cases. As in the proof of Theorem 2, we will frequently refer to equalities (2.3)–(2.6), but this time without precisely indicating which of them are utilized in a particular part of considerations.

Comparing $A_1^{(3)} = B_2, A_3^{(3)} = B_2, A_6^{(3)} = B_2, A_9^{(3)} = B_2$ and $A_1^{(3)} = B_4, A_3^{(3)} = B_4, A_6^{(3)} = B_4, A_9^{(3)} = B_4$ with the corresponding versions of these equalities obtained by premultiplying by $\mathbf{P}_i$ leads to (2.7) in the first four cases and to (2.8) in the further four. Modifications of (2.7) and (2.8) to the forms

$$P_{(2v;j)} = P_{(2v+1;j)} \quad \text{and} \quad P_{(2v+1;i)} = P_{(2v+2;i)} \tag{2.18}$$

are obtained when the conditions $A_4^{(3)} = B_2$, $A_8^{(3)} = B_2$ and $A_1^{(3)} = B_3$, $A_4^{(3)} = B_3$, $A_8^{(3)} = B_3$, respectively, are combined with their counterparts being the results of postmultiplying by $\mathbf{P}_j$. In addition, considering $A_9^{(3)} = B_1$ along with $A_9^{(3)}\mathbf{P}_i = B_1\mathbf{P}_i$ yields (2.9). Consequently, on account of conditions (2.7)–(2.9) and (2.18), in all 14 cases discussed above the commutativity condition (1.2) follows immediately from Theorem 1.

The next 16 cases will be analyzed by applying the same procedure as above, which consists in comparing an original equality with its modified version. Now, however, the conclusions depend on the assumption that c_1, c_2, c_3 are nonzero. To be precise, if $c_3 \neq 0$, then after postmultiplying $A_2^{(3)} = B_1$, $A_2^{(3)} = B_4$, and $A_3^{(3)} = B_1$ by $\mathbf{P}_j$ this procedure leads to

$$P_{(2u;j)} = P_{(2u+1;j)} \quad \text{and} \quad P_{(2u+1;i)} = P_{(2u+2;i)} \tag{2.19}$$

in the first two situations and in the third one, respectively. Similarly, when $A_2^{(3)} = B_3$, $A_4^{(3)} = B_1$, $A_7^{(3)} = B_1$, $A_7^{(3)} = B_3$, $A_{10}^{(3)} = B_1$, $A_{10}^{(3)} = B_3$ are premultiplied by $\mathbf{P}_i$, then it follows that

$$P_{(2u;j)} = P_{(2u+1;i)} \quad \text{and} \quad P_{(2u+1;j)} = P_{(2u+2;i)} \tag{2.20}$$

in the first situation and in the further five, respectively. To exhaust the list of conditions involving u, it is to be noted that if the equality $A_{10}^{(3)} = B_2$ is combined with $A_{10}^{(3)}\mathbf{P}_i = B_2\mathbf{P}_i$, then

$$P_{(2u+1;j)} = P_{(2u+2;j)}. \tag{2.21}$$

Continuing this line of argumentation, it is seen that if $c_2 \neq 0$, then with $A_5^{(3)} = B_1$ and $A_5^{(3)} = B_3$ premultiplied by $\mathbf{P}_i$ and with $A_7^{(3)} = B_4$ postmultiplied by $\mathbf{P}_j$ the procedure, which is permanently applied in the present section, leads to (2.12) in the first two cases and to the middle equality in (2.13) in the third one. The final observation in this part of the proof is that if $c_1 \neq 0$, then with $A_5^{(3)} = B_2$ and $A_6^{(3)} = B_3$ postmultiplied by $\mathbf{P}_i$ and with $A_8^{(3)} = B_4$ premultiplied by $\mathbf{P}_j$ the procedure yields

$$P_{(2s;i)} = P_{(2s+1;i)} \quad \text{and} \quad P_{(2s;i)} = P_{(2s+1;j)} \tag{2.22}$$

in the first two cases and in the third one, respectively. Consequently, on account of conditions (2.12), (2.13), and (2.19)–(2.22), in all 16 cases discussed above the commutativity condition (1.2) again follows immediately from Theorem 1.

The proof concerning the last two cases, namely $A_5^{(3)} = B_4$ and $A_7^{(3)} = B_2$, utilizes an additional tool, which is the trace of a matrix, denoted by $\operatorname{tr}(\cdot)$. Premultiplying $A_5^{(3)} = B_4$ by $\mathbf{P}_i$ results in

$$c_1 P_{(2s;i)} + c_2 P_{(2t+1;i)} + c_3 P_{(2u+1;i)} = P_{(2v+2;i)},$$

and combining this equality with the original one leads to

$$c_2(P_{(2t;j)} - P_{(2t+1;i)}) = P_{(2v+1;j)} - P_{(2v+2;i)}. \tag{2.23}$$

It is clear that

$$\operatorname{tr}(P_{(2t+1;i)}) = \operatorname{tr}(\mathbf{P}_i P_{(2t;j)}) = \operatorname{tr}(P_{(2t;j)}\mathbf{P}_i) = \operatorname{tr}(P_{(2t;j)}), \tag{2.24}$$

and hence (2.23) entails

$$\operatorname{tr}(P_{(2v+1;j)} - P_{(2v+2;i)}) = 0. \tag{2.25}$$

Since $P_{(2v+1;j)} - P_{(2v+2;i)} = (\mathbf{I}_n - \mathbf{P}_i)P_{(2v+1;j)}$ and $P_{(2v+1;j)} = P_{(v+1;j)}P^*_{(v+1;j)}$, where $\mathbf{I}_n$ denotes the identity matrix of order n, it follows that the difference of matrices in (2.25) is a product of two nonnegative definite Hermitian matrices, and therefore has its trace equal to zero if and only if it is the zero matrix, which coincides with (2.8).

The proof concerning $A_7^{(3)} = B_2$ is analogous. Premultiplying by $\mathbf{P}_i$ and combining the result obtained with the original equality leads to

$$c_3(P_{(2u+1;j)} - P_{(2u+2;i)}) = P_{(2v;j)} - P_{(2v+1;i)}. \tag{2.26}$$

On account of the chain (2.24) of trace identities with t replaced by v and the assumption $c_3 \neq 0$, it is seen from (2.26) that

$$\operatorname{tr}(P_{(2u+1;j)} - P_{(2u+2;i)}) = 0. \tag{2.27}$$

Since (2.27) corresponds to (2.25) with u in place of v, a consequence is that $P_{(2u+1;j)} = P_{(2u+2;i)}$, which coincides with the second equality in (2.20). Referring once again to Theorem 1 completes the proof of Theorem 3.

From the discussion presented in this section hitherto it is clear that similar analysis is possible also when the number of components in an affine combination occurring in the equality under considerations exceeds three. It should be pointed out, however, that together with the increase of the number of components, the number of cases to be analyzed grows quite rapidly. For instance, instead of 40 situations arising in the case of three components, there are 76 analogous situations when the number of components is four. Similarly as in (2.1) and (2.17), all possibilities can be described by the equalities

$$A_\alpha^{(4)} = B_\beta, \quad \alpha = 1, \ldots, 19; \quad \beta = 1, \ldots, 4,$$

where $B_1, \ldots, B_4$ are as specified in the proof of Theorem 2, while $A_1^{(4)}, \ldots, A_{19}^{(4)}$ form an appropriately composed list starting for instance from the situation where all matrix products in the affine combination in question have even number of factors and $\mathbf{P}_i$ as their first factor and finishing with the situation where all matrix products have odd number of factors, with the first two having $\mathbf{P}_i$ and the remaining two having $\mathbf{P}_j$ as their first factors.

It is of course of interest to identify general situations in which equalities of the form $A_\alpha^{(l)} = B_\beta$ do not imply commutativity property (1.2) independently of the value of integer $l > 1$. Clearly, one of such situations is when all $l + 1$ products involved in a given equality begin with the same factor and end with the same factor. On the other hand, if in a given equality one of the products begins (or ends) with a different factor than the remaining l factors, then commutativity follows. An analysis of this problem led to the following observation. A necessary condition to ensure that for given α, β, and l, the equality $A_\alpha^{(l)} = B_\beta$ does not imply (1.2) is that at least two of $l + 1$ products involved in it begin with the same factor, say $\mathbf{P}_a$, $a \in \{1, 2\}$, and if there are any products not beginning with $\mathbf{P}_a$, then there must be at least two of them; simultaneously, at least two of $l + 1$ products involved in the equality under consideration must end with the same factor, say $\mathbf{P}_b$, $b \in \{1, 2\}$, and if there are any products not ending with $\mathbf{P}_b$, then there must be at least two of them. Unfortunately, this relatively simple characterization does not provide sufficient conditions, what is seen, for example, from the analysis of equalities $A_5^{(3)} = B_4$ and $A_7^{(3)} = B_2$.

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References

[1] J.K. Baksalary, Algebraic characterizations and statistical implications of the commutativity of orthogonal projectors, in: T. Pukkila, S. Puntanen (Eds.), Proceedings of the Second International Tampere Conference in Statistics, University of Tampere, Tampere, Finland, 1987, pp. 113–142.

[2] J.K. Baksalary, O.M. Baksalary, Commutativity of projectors, Linear Algebra Appl. 341 (2002) 129–142.

[3] J.K. Baksalary, O.M. Baksalary, T. Szulc, A property of orthogonal projectors, Linear Algebra Appl. 354 (2002) 35–39.

[4] P.R. Halmos, Finite-Dimensional Vector Spaces, Van Nostrand, Princeton, NJ, 1958.