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NOTE

Thābit ibn Qurra and the Pair of Amicable Numbers 17296, 18416

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Two numbers n and m are said to be *amicable* if n is the sum of the proper divisors of m and if, at the same time, m is the sum of the proper divisors of n. The earliest known reference to a pair of amicable numbers is in the commentary by Iamblichus (fl. A.D. 300) to the *Arithmetica* of Nicomachus [Iamblichus 1975, 35:3]. Iamblichus mentions the smallest pair 220, 284. However, because Iamblichus is not known to have been a mathematician of any originality, we can safely assume that the pair was known before his time, possibly even to the later Pythagoreans in the second part of the fifth century B.C.

There is no evidence of further progress in the study of amicable numbers before Thābit ibn Qurra (A.D. 836–901) proved the following general rule:

> If three numbers $p = 3 \cdot 2^{n-1} - 1$, $q = 3 \cdot 2^n - 1$ and $r = 9 \cdot 2^{2n-1} - 1$ are prime, and if p, $q > 2$, then pair $2^n \cdot pq$ and $2^n \cdot r$ (indicated by P_n) is amicable.

This general rule is somewhat similar to a rule proved by Euclid in *Elements* IX:36 [Heath 1956 2, 421–426] to the effect that if $s = 2^{n+1} - 1$ is prime, the number $2^n \cdot s$ is *perfect*, that is to say equal to the sum of its proper divisors. This rule for perfect numbers is also stated without proof by Nicomachus (fl. A.D. 100) in his *Arithmetica* (Greek text in [Nicomachus 1866, 40–43], Arabic translation by Thābit ibn Qurra in [Kutsch 1958, 39–41], English translation in [D'Ooge 1926, 210]), and by Theon of Smyrna (fl. early second century A.D.) in his exposition of the mathematics necessary for understanding the works of Plato [Theon 1878, 45–46]. However, these authors do not mention amicable numbers.

For $n = 2$ Thābit's rule yields the pair $P_2 = 220, 284$, mentioned above.

For $n = 3, 5, 6, 8$, and 9, the conditions of the rule are not satisfied because the numbers p, q, and r are not all prime.

For $n = 4$ and 7, p, q, and r are all prime, and we obtain two more pairs of amicable numbers:

$$P_4: \quad 17296,\ 18416$$
$$P_7: \quad 9363584,\ 9437056.$$

P_4 is usually named after Pierre de Fermat, who was the first to mention it in Western Europe [Dickson 1934 1, 40]. Nevertheless, the earliest known refer-

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ences to P_4 are to be found in Arabic texts of the early 14th century A.D. [Rashed 1983, 120, 121, notes 46, 47]. Generally historians of science have assumed that P_2 is the only pair of amicable numbers that was known before the 14th century. The purpose of this note is to show that Thābit ibn Qurra must have known P_4 and that he probably discovered this pair.

Thābit's *Treatise on the Derivation of the Amicable Numbers in an Easy Way* [Sezgin 1974, 270, No. 13] is extant in full in the Arabic manuscript Paris, Bibliothèque Nationale, Fonds Arabe 2457, 170b–180b. A French translation of the preface and an abstract of the rest of the treatise were published by Woepcke [1852]. The Arabic manuscript Istanbul, Aya Sofya 4830, 110a–121b, contains the text of the treatise but not the preface. The edition of the Arabic text by Saidan [1977] is based on both manuscripts.

In the preface Thābit says that perfect and amicable numbers were studied by the Pythagorean philosophers. Thābit refers to the rule (described above) for perfect numbers, saying that it had been stated by Nicomachus without proof, and that Euclid had proved it in the arithmetical Books of the *Elements*. Thābit then says that neither Euclid nor Nicomachus had mentioned amicable numbers. He adds:

> Since the matter of them (the amicable numbers) has occurred to my mind, and since I have derived a proof for them, I did not wish to write it (i.e. the rule) without proving it, because they have been mentioned this way (i.e. they have been neglected by Euclid and Nicomachus).

(Arabic text in [Saidan 1977, 33:23–24]; ms. Paris, Bibliothèque Nationale, Fonds Arabe 2457, f. 171a:6–7; *awwaluhā* in Saidan's edition should be changed to *amruhā,* as in the manuscript. French translation in [Woepcke 1852, 424].)

Thābit's proof is entirely in the style of the arithmetical Books of Euclid's *Elements:* thus Thābit represents numbers by line segments, which are designated by letters. We quote Thābit's final theorem, which contains his rule. (All explanatory additions of my own are in square brackets; the symbols (*) and (**) indicate minor interpolations in the manuscript Aya Sofya 4830, to be mentioned below.)

> We wish to find as many amicable numbers as we please. Thus we set out numbers in continued double proportion, beginning with the unit; the unit precedes them (and is taken) together with them. Let them be numbers A, B, G, D, E and W. We add them successively, including the unit, as we do in the derivation of the perfect numbers. Let the sum of A, B, G, D and E be the number Z. We add to the number Z the last one of the numbers that were added, namely number E. Let the sum $[E + Z]$ be number H. We subtract from number Z the number preceding number E, that is number D. Let the remainder $[Z - D]$ be number T.
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> If each of the numbers (*) H, T is a prime number other than the number two, then this is what we want. If not, then we proceed with the (series of) numbers that were added [i.e., A, B, G, . . .] until we arrive at some number such that these numbers which are derived from it are prime. Thus let the numbers H and T be prime, and let neither of them be the number two. We multiply one of them by the other, let the result be number K. We multiply number K by the last one of the numbers that were added, that is E. Let the result [i.e., K times E] be number L. Then this is one number [of the pair of amicable numbers], so stop here and keep it in mind.

> Again, we add the number which follows number E in the [series of] numbers in double proportion—that is number W—to the second number preceding the last one of the numbers that were added—that is number G. Let their sum [$W + G$] be number M, and let the product of number M and number W be number N. We subtract one from it, let the remainder be number S.
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> If S is a prime number, then this is what we want. If not, then we continue with the [series of] numbers that were added until we arrive at a (number) such that this number [i.e., corresponding to S] (**) is prime. Thus let S be a prime number. We multiply it by number E, let the result be number O. I say that the numbers L and O are amicable.

(Arabic text in [Saidan 1977, 50:20–51:8] and in the manuscripts Aya Sofya 4830, 119b:18–120a:11, and Paris, Bibliothèque Nationale, Fonds Arabe 2457, f. 178b:18–179a:9, on which the translation in [Woepcke 1852, 428–429] is based. I have corrected trivial scribal errors without notice.) The Aya Sofya manuscript and Saidan's edition contain the interpolations "Z" at (*) and "and the numbers [i.e., H, T and the erroneously interpolated number Z] we mentioned above" at (**). I omit Thābit's (correct) proof.

In the manner of Euclid in *Elements* IX:36, Thābit proves the rule in a special case, but the proof is general, again in the manner of Euclid. The quoted passage clearly shows that Thābit intended to convey a general theorem and proof. Thābit wishes to find not one but "as many amicable numbers as we please." He sets out a series of unspecified length, consisting of numbers in "continued double proportion." Lacking a suitable notation for a general proof, he then chooses a series of six numbers to work with. However, these numbers are not represented by their numerical values 1, 2, 4, 8, 16, 32, but by letters A, B, G, D, E, W. This notation emphasizes the generality of the argument. Thus modern historians have correctly interpreted the passage quoted above as follows (using the notations p, q, r as above):

$A, B, \ldots, E, W$ is a series of arbitrary length, consisting of $n + 2$ numbers beginning with 1 and increasing in geometric progression, such that the ratio of one term to the next is $1:2$. Thus we can put $A = 1, \ldots, E = 2^n$, $W = 2^{n+1}$, for integer n.

We have $Z = 1 + 2 + \ldots + 2^n = 2^{n+1} - 1$; $H = Z + E = 3 \cdot 2^n - 1 = q$; $T = Z - D = 3 \cdot 2^{n-1} = p$; $L = E \cdot H \cdot T = 2^n \cdot pq$.

Further, $W = 2^{n+1}$, $G = 2^{n-2}$, $M = W + G = 9 \cdot 2^{n-2}$; $N = M \cdot W = 9 \cdot 2^{2n-1}$; $S = N - 1 = 9 \cdot 2^{2n-1} - 1 = r$, $O = E \cdot S = 2^n \cdot r$.

Thus Thābit states (and proves) that $2^n \cdot pq$ and $2^n \cdot r$ are amicable if p, q, and r are prime, and $p, q > 2$.

However, if we ask ourselves what the numbers A, B, G, D, E, etc., in Thābit's example really are, we obtain $A = 1$, $B = 2$, $G = 4$, $D = 8$, $E = 16$, $W = 32$, $Z = 31$, $H = 47$, $T = 23$, $L = 17296$; $M = 36$, $N = 1152$, $S = 1151$, $O = 18416$. Since 23, 47, and 1151 are prime, L and O are amicable. Thus Thābit stated his general reasoning in the case $n = 4$, where his rule actually yields the pair of amicable numbers P_4.

It is highly unlikely that this is accidental. Thābit's proof is, for the most part, so carefully structured that he must have made sure that the rule was correct for the special case he used in his proof. In this respect he also followed Euclid. Euclid, in the arithmetical Books of the *Elements*, proved general theorems and did not give numerical examples. However, as Becker [1936, 540–542] pointed out, the general proof for the rule for perfect numbers in *Elements* IX:36 is stated in a case where the rule actually yields a perfect number (496). Thus it is likely that Thābit, following a Euclidean model, omitted any discussion of examples but deliberately chose a case where his rule yields a pair of amicable numbers (P_4). We conclude, therefore, that Thābit knew P_4.

This conclusion is further supported by other considerations. In the beginning of the quoted passage Thābit states his intention "to find as many amicable numbers as we please." Thābit may have been overly optimistic; according to Guy [1981, 31–32], most present-day experts in number theory believe that there are infinitely many pairs of amicable numbers, but in 1981 this had not yet been proved. The recent volumes of the *Mathematical Reviews* and the *Zentralblatt für Mathematik* contain no evidence that this situation has changed (November 12, 1984). But it is plausible that Thābit's expectation was based on the experience that his rule could provide at least two pairs.

In Proposition 5 of the treatise on amicable numbers, Thābit proves Euclid's rule for perfect numbers in essentially the same way in which it is proved in *Elements* IX:36. Thābit also states the proof in the case where the rule yields the perfect number 496, but again he does not mention any numerical example. However, he obviously knew the perfect numbers 6, 28, 496, 8128, because these perfect numbers are mentioned in the *Arithmetica* of Nicomachus, which Thābit translated into Arabic. (The numbers occur in the Arabic translation; see [Kutsch 1958, 39, lines 4–7].) Thus it is plausible that Thābit knew P_4 but did not mention it in the last proposition in his treatise on amicable numbers.

It is of course conceivable that Thābit mentioned P_4 elsewhere and that the pair was, somehow, transmitted to the 14th century. We know little of the history of amicable numbers between the 9th and the 14th century, since most of the relevant treatises have not yet come to light (see [Rashed 1983, 120, 121, notes 46, 47]). It is also perfectly possible that P_4 was "rediscovered" by one or several scholiasts, who simply computed the numbers A, B, etc., in Thābit's reasoning, in exactly the same way as above.

Finally the question arises whether P_4 was known *before* Thābit ibn Qurra, that is to say in antiquity. The available evidence suggests that this was not the case. First, there is no ancient reference to any pair of amicable numbers other than P_2. Second, in view of the magnitude of the numbers that are involved, it is hardly conceivable that P_4 could have been found by trial and error in antiquity or the middle ages. We note that Fermat and Descartes had rediscovered the rule of Thābit before they found P_4 [Dickson 1934 1, 40]. We also note that the first new pair of amicable numbers one would find by trial and error would not be P_4, but the pair 1184, 1210, first obtained by Paganini in 1866 [Dickson 1934 1, 47]. Thus, if we suppose that P_4 was known in antiquity, we must assume that Thābit's general rule was known as well. Because this general rule is similar to the venerated rule for perfect numbers, it would have been a subject of the highest interest to all ancient students of number theory. In view of the popularity of elementary number theory among the Neo-Pythagoreans, it would be very difficult to explain the total absence of references to it in the extant works of Nichomachus, etc. We also note that Thābit did not find his rule in any of the literature available in his day, because he says specifically in a passage quoted above that "the matter of them (the amicable numbers) has occurred to my mind (Arabic: khaṭara bi-bālī)," and "I have derived a proof for them." This should be compared with the end of the proof of the rule for perfect numbers, where Thābit adds a reference to Euclid: "and this is what Euclid proved" ([Saidan 1977, 42:9–10], Ms. Paris, B. N., Fonds Arabe, 2457, f. 174b:17; Ms. Aya Sofya 4830, f. 114b:21–22).

We conclude, therefore, that Thābit's rule for amicable numbers was probably unknown in antiquity, and that in all likelihood Thābit was the first to discover this rule as well as the pair of amicable numbers 17296, 18416.

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