On the Convergence of Waveform Relaxation Methods for Differential-Functional Systems of Equations

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In this paper the convergence of a waveform relaxation method applied to an initial value problem for the Volterra functional-differential system is discussed. It is shown that the method is convergent under the assumption that the splitting function satisfies only the one side Lipschitz condition with respect to some arguments and the Lipschitz condition with respect to the others. The conditions given in the paper also guarantee the existence and uniqueness of the solution to the initial problem discussed in the paper. The convergence of the perturbed continuous time waveform relaxation method is also discussed.

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1. INTRODUCTION

Consider the initial value problem for the Volterra functional-differential system of the form,

\[ x'(t) = f(t, x(t), x(\cdot)), \quad t \in I = [0, T], \]  
\[ x(0) = a, \]  

where \( f : I \times \mathbb{R}^n \times C(I, \mathbb{R}) \to \mathbb{R}^n \) is a given continuous function and \( C(I, \mathbb{R}) \) is equipped with the standard maximum norm. Observe, that the case when the initial condition is of the form

\[ x(t) = g(t), \quad t \in [\alpha, 0], \quad \alpha < 0 \]

can be reduced to that considered above by the corresponding redefinition of the function \( f \) which incorporates the initial function \( g \) into the definition of the new right-hand side function \( f \) and \( a = g(0) \) (for a more detailed discussion of these questions see [4, 5]).

There exists a vast literature devoted to the application of waveform relaxation methods to the numerical solution of initial value problems arising from the modeling of large-scale integrated circuits and the efficiency of these methods (see, for example, [2, 8, 9, 12, 13] and references therein). The applicability of such methods in dynamic process simulation is discussed, for example, in [10, 11]. The implementation aspects of waveform algorithms are described in [2].

In papers of many authors the convergence of relaxation methods for systems of differential equations was discussed under the assumption that the corresponding splitting function satisfies the Lipschitz condition with respect to the functional arguments. The waveform relaxation methods for differential delay equations of neutral type under the Lipschitz condition with respect to the functional arguments were discussed, for instance, in [4, 5].

The aim of the present paper is to give the conditions guaranteeing the convergence of the relaxation methods on a finite interval as well as on the whole half line. We assume that the splitting function satisfies only the one side Lipschitz condition with respect to some arguments and the Lipschitz condition with respect to the other arguments. In the whole paper we assume that the Lipschitz coefficients are time dependent. Similar questions for equations without delays were considered by Bremer [1] and in 't Hout [3].

We assume in Sections 2 and 3 of the paper that the problem (1.1)--(1.2) has a unique solution \( x_0 \) defined in the whole interval \( I \). In Section 4 we prove that under Conditions (A) and (B) assumed in Section 2 the uniqueness and existence of the solution is guaranteed. In Section 3 we
discuss shortly the convergence of the perturbed continuous time waveform relaxation method.

2. THE CONVERGENCE OF THE WAVEFORM RELAXATION METHOD

Let the continuous function $F: I \times \mathbb{R}^n \times \mathbb{R}^n \times C(I, \mathbb{R}^n) \to \mathbb{R}^n$ denote a splitting function for the equation (1.2), i.e.,

$$f(t,x(t),x(\cdot)) = F(t,x(t),x(t),x(\cdot)).$$

(2.1)

We assume that

(A) $\|F(t,x,y,z) - F(t,x,\bar{y},\bar{z})\| \leq K(t)\|y - \bar{y}\| + L(t)\|z - \bar{z}\|$, for all $t \in I$, $x, y, \bar{x}, \bar{y} \in \mathbb{R}^n$, $z, \bar{z} \in C(I, \mathbb{R}^n)$ and some functions $K, L \in C(I, \mathbb{R}_+)$, $R_+ = [0, +\infty)$. Here we mean that $\|x\|^2 = (x, x)$, $(\cdot, \cdot)$ is a scalar product in $\mathbb{R}^n$ and $\|z\|, = \max_{0 \leq t \leq s} \|z(t)\|$,.

(B) $(F(t,x,y,z) - F(t,\bar{x},\bar{y},\bar{z}), x - \bar{x}) \leq m(t)\|x - \bar{x}\|^2$ for all $t \in I$, $x, \bar{x} \in \mathbb{R}^n$, $y, \bar{y} \in \mathbb{R}^n$, $z, \bar{z} \in C(I, \mathbb{R}^n)$ and some function $m \in C(I, \mathbb{R})$.

(C) the sequence $\langle x_k \rangle$, $x_k \in C^2(I, \mathbb{R}^n)$ is defined by the continuous-time waveform relaxation method $(\text{WRM})$ in short,

$$x_{k+1}(t) = F(t,x_{k+1}(t),x_k(t),x_k(\cdot)), \quad t \in I,$$

$$x_{k+1}(0) = a,$$

(2.2) 

(2.3)

with some initial function $x_0 \in C^2(I, \mathbb{R}^n)$ such that $x_0(0) = a$. It is obvious that for the Euclidean norm the following holds

$$\frac{d}{dt}\|x(t)\|^2 = 2(x'(t), x(t))$$

and

$$\frac{d}{dt}\|x(t)\|^2 = 2\|x(t)\| \cdot \frac{d}{dt}\|x(t)\|.$$ 

Hence,

$$(x'(t), x(t)) = \|x(t)\| \frac{d}{dt}\|x(t)\|,$$

(2.4)

for any $x \in C^2(I, \mathbb{R}^n)$, $x(t) \neq 0$. 

Now, from the definitions of \( x_* \) and \( x_{k+1} \) and the conditions (A)--(C) we find

\[
(x'_{k+1}(t) - x'_*(t), x_{k+1}(t) - x_*(t))
\]

\[
= (F(t, x_{k+1}(t), x_*(t)), x'_k(\cdot))
\]

\[
- F(t, x_*(t), x_*(t), x'_*(\cdot)), x_{k+1}(t) - x_*(t))
\]

\[
= (F(t, x_{k+1}(t), x_*(t)), x'_k(\cdot))
\]

\[
- F(t, x_*(t), x'_k(\cdot), x_{k+1}(t) - x_*(t))
\]

\[
+ (F(t, x_*(t), x'_k(t), x'_k(\cdot))
\]

\[
- F(t, x_*(t), x_*(t), x'_*(\cdot)), x_{k+1}(t) - x_*(t))
\]

\[
\leq m(t)||x_{k+1}(t) - x_*(t)||^2
\]

\[
+ K(t)||x_k(t) - x_*(t)|| ||x_{k+1}(t) - x_*(t)||
\]

\[
+ L(t)||x_k - x_*|| ||x_{k+1} - x_*(t)||
\]

According to Eq. (2.4) we have

\[
\|x_{k+1}(t) - x_*(t)\| - \|x_{k+1}(t) - x_*(t)\|
\]

\[
\leq m(t)||x_{k+1}(t) - x_*(t)||^2
\]

\[
+ K(t)||x_k(t) - x_*(t)|| ||x_{k+1}(t) - x_*(t)||
\]

\[
+ L(t)||x_k - x_*|| ||x_{k+1} - x_*(t)||
\]

After dividing both sides of this inequality by \( ||x_{k+1}(t) - x_*(t)|| \) and introducing the notation,

\[
u_k(t) = ||x_k(t) - x_*(t)||, \quad |u_k|_{\cdot} = \max_{0 \leq s \leq t} u_k(s),
\]

\[
h(t) = K(t)u_k(t) + L(t)|u_k|_{\cdot}
\]

we get

\[
u'_{k+1}(t) \leq m(t)u_{k+1}(t) + h(t),
\]

\[
u_{k+1}(0) = 0.
\]

From inequality (2.5) we find

\[
u_{k+1}(t) \leq \int_0^t h(s) \exp \left( \int_s^t m(\tau) \, d\tau \right) ds.
\]
Using the inequality,

\[ u_k(s) \leq \|x_k - x_*\|_s, \]

and (2.6) we get

\[
u_{k+1}(t) \leq \int_0^t \exp \left( \int_s^t m(\tau) \, d\tau \right) [K(s) + L(s)] \|x_k - x_*\|_s \, ds \tag{2.7}
\]
as well as

\[
u_{k+1}(t) \leq |u_k| \int_0^t [K(s) + L(s)] \exp \left( \int_s^t m(\tau) \, d\tau \right) \, ds.
\]

Now we can write

\[
u_{k+1}(t) \leq |u_k| \int_0^t \frac{K(s) + L(s)}{-m(s)} (-m(s)) \exp \left( \int_s^t m(\tau) \, d\tau \right) \, ds.
\]

Now, let us assume that \( m(t) < 0 \) for \( t \in [0, T] \). Thus we have

\[
u_{k+1}(t) \leq |u_k| \max_{0 \leq s \leq t} \left[ \frac{K(s) + L(s)}{-m(s)} \int_0^t (-m(s)) \exp \left( \int_s^t m(\tau) \, d\tau \right) \, ds \right] = |u_k| \max_{0 \leq s \leq t} \left[ \frac{K(s) + L(s)}{-m(s)} \left(1 - \exp \left( \int_0^t m(\tau) \, d\tau \right) \right) \right].
\]

This implies the final inequality,

\[ |u_{k+1}| \leq P(t)|u_k|, \]

with

\[
P(t) = \max_{0 \leq s \leq t} \left[ \frac{K(s) + L(s)}{-m(s)} \left(1 - \exp \left( \int_0^t m(\tau) \, d\tau \right) \right) \right]. \tag{2.8}
\]

We see that

\[ |u_k| \leq \|u_0\| \leq \|x_0 - x_*\|, \]

which means that

\[ \|x_k - x_*\| \leq \|x_0 - x_*\|, \tag{2.9} \]
for \( t \in I \). Taking \( t = T \) we get
\[
\|x_k - x_\ast\|_T \leq [P(T)]^k \cdot \|x_0 - x_\ast\|_T, \quad k = 0, 1, \ldots, \tag{2.10}
\]
and the uniform convergence in \( I \) of the sequence \( \langle x_k \rangle \) to \( x_\ast \) holds whenever
\[
P(T) < 1. \tag{2.11}
\]
So, we have proven the following.

**Theorem 2.1.** If Assumptions (A) and (B) are satisfied with \( m(t) < 0 \), \( t \in I \), and \( P(T) < 1 \) for the function \( P \) defined by the relation (2.8) then the sequence \( \langle x_k \rangle \) defined by Condition (C) converges uniformly in \( I \) to the solution \( x_\ast \) of problem (1.1)—(1.2).

**Remark 1.** One can easily see that the result of Theorem 2.1 holds when the interval \( I \) is replaced by the interval \( I_u = [0, \infty) \) and the condition
\[
\sup_{0 \leq s < \infty} \left[ \frac{K(s) + L(s)}{-m(s)} \right] \left[ 1 - \exp \left( \int_0^\infty m(\tau) \, d\tau \right) \right] < 1
\]
holds.

Now, the following question arises: what about the convergence of the sequence \( \langle x_k \rangle \) when the condition (2.11) does not hold?

To answer this question we go back to inequality (2.7), which we rewrite in the form,
\[
u_{k+1}(t) \leq \int_0^t \exp \left( \int_s^t m(\tau) \, d\tau \right) Q(s) |u_k|, \quad ds, \tag{2.12}
\]
where
\[
Q(s) = K(s) + L(s).
\]
Because \( m(t) < 0 \), from inequality (2.12) we have
\[
|u_{k+1}| \leq \int_0^t Q(s) |u_k|, \quad ds
\]
and consequently,
\[
|u_k| \leq \left( \int_0^t Q(s) \, ds \right)^k |u_0|, \quad t \in I, \tag{2.13}
\]
which implies the factorial convergence of the sequence \( \langle x_k \rangle \).
But from (2.12) we can obtain another estimation. Namely, it implies
\[ u_{k+1}(t) \leq P_*(t) \exp \left( \int_0^t m(\tau) \, d\tau \right) \int_0^t (-m(s)) \exp \left( -\int_0^{s} m(\tau) \, d\tau \right) |u_k|, \, ds, \]
with \( P_*(t) = \max_{0 \leq s \leq t} \left( [K(s) + L(s)]/[\exp(-m(s))] \right) \). An induction argument and the fact that the functions
\[
H_0(t) = 1,
\]
\[
H_k(t) = \left[ 1 - \left( \sum_{i=0}^{k-1} \frac{1}{i!} \left( \int_0^{i} (-m(s)) \, ds \right) \right) \exp \left( \int_0^{i} m(\tau) \, d\tau \right) \right],
\]
\[ k = 1, 2, \ldots, \, t \in I \]
are increasing with respect to \( t \) for each \( k \) (their derivatives are positive) give
\[
|u_k| \leq P_*^k(t) |u_0|, H_k(t). \tag{2.14}
\]
One can also observe that the sequence of functions \( \langle H_k \rangle \), \( k = 0, 1, \ldots \) is decreasing.

From the last inequality and
\[
\left| \exp(z) - \sum_{s=0}^{k-1} \frac{z^s}{s!} \right| \leq \exp(z) \frac{z^k}{k!}, \quad \text{for} \ z \geq 0,
\]
we have
\[
u_k(t) \leq P_*^k(t) |u_0|, \exp \left( \int_0^{t} m(\tau) \, d\tau \right) \exp \left( -\int_0^{t} m(\tau) \, d\tau \right)
\times \left( \int_0^{t} (-m(\tau)) \, d\tau \right)^k \frac{k!}{k!}, \tag{2.15}
\]
i.e.,
\[
u_k(t) \leq |u_0| \frac{\left( -P_*^k(t) \int_0^{t} m(\tau) \, d\tau \right)^k}{k!}. \tag{2.16}
\]

Remark 2. From the properties of the sequence \( \langle H_k \rangle \) and the inequality (2.14) it follows the geometrical (and uniform on \( I \)) convergence to zero of the sequence \( \langle u_k \rangle \) when \( P_*(T) < 1 \) and only the factorial (and uniform on \( I \)) convergence of that sequence to zero when \( P_*(T) \geq 1 \). The
same holds for $T = +\infty$ with the loss of the uniform convergence in the last case.

One can observe that the error evaluation (2.13) is better than (2.16) because

$$\int_0^t Q(s) \, ds \leq -\int_0^t (P_\tau) \, m(\tau) \, d\tau, \quad t \in I.$$ 

When $m$ changes the sign in the interval $I$ then from inequality (2.12) we find

$$|u_{k+1}| \leq \int_0^t Q^* \cdot |u_k| \, ds$$

with

$$Q^* = \max_{t \in I} \max_{0 \leq s \leq t} Q(s) \exp\left(\int_s^t m(\tau) \, d\tau\right).$$

The last inequality again implies the factorial convergence with the following error evaluation,

$$|u_k| \leq \frac{(Q^* t)^k}{k!} |u_0| T, \quad t \in I.$$

However, in this case another approach is possible.

Let

$$\|u_k\|_q = \max_{0 \leq s \leq T} |u_k| \exp\left(-\int_0^s q(\tau) \, d\tau\right)$$

for a given function $q \in C(I, R)$. Now, for $q(t) > m(t), \ t \in I$, from inequality (2.12) we have

$$u_{k+1}(t) \leq \int_0^t \exp\left(\int_s^t m(\tau) \, d\tau\right) Q(s) \exp\left(-\int_0^s q(\tau) \, d\tau\right) |u_k| \, ds$$

$$\times \exp\left(-\int_0^s q(\tau) \, d\tau\right) \, ds$$

$$\leq \|u_k\|_q \int_0^t \exp\left(\int_s^t m(\tau) \, d\tau\right) Q(s) \exp\left(-\int_0^s m(\tau) \, d\tau\right)$$

$$\times \exp\left(-\int_0^s q(\tau) \, d\tau\right) \, ds.$$
\[ u_{k+1} \leq \|u_k\|_q \exp\left(\int_0^t m(\tau) \, d\tau\right) S(t) \]
\[ \times \left[ \exp\left(\int_0^t q(\tau) \, d\tau\right) \exp\left(-\int_0^t m(\tau) \, d\tau\right) - 1 \right]. \]

This implies that
\[ u_{k+1} \leq \|u_k\|_q S(t) \exp\left(\int_0^t q(\tau) \, d\tau\right). \]
and
\[ |u_{k+1}|, \exp\left(-\int_0^t q(\tau) \, d\tau \right) \leq S(t)\|u_k\|_q. \]

Taking maximum with respect to \( t, t \in I \) on both sides of the last inequality we get
\[ \|u_{k+1}\|_q \leq S(T)\|u_k\|_q. \] (2.20)

This means that the sequence \( \{x_k\} \) converges to \( x_* \) when
\[ S(T) < 1. \] (2.21)

In this case we have
\[ \|u_k\|_q \leq (S(T))^k\|u_0\|_q, \quad k = 0, 1, \ldots \] (2.22)

Thus we have proven the following.

**Theorem 2.2.** If Assumptions (A) and (B) are satisfied and the function \( q \in C(I, R), q(t) \geq 0, q(t) > m(t), t \in I, \) is such that
\[ S(T) < 1, \] (2.23)

then the sequence \( \{x_k\} \) defined by Condition (C) converges uniformly in \( I \) to the solution \( x^* \) of the problem (1.1)–(1.2).

**Remark 3.** From the definition of \( S(t) \) it is obvious that condition (2.21) holds when
\[ \frac{Q(t)}{q(t) - m(t)} < 1, \] (2.24)
i.e., when
\[ K(t) + L(t) + m(t) < q(t). \] (2.25)

Observe that now we do not have to assume that \( m \) is negative.

Moreover, from (2.22) we have
\[ \|x_k - x_*(t)\| \leq |u_k| \leq (S(T))^k\|u_0\|_q \exp\left(\int_0^t q(s) \, ds \right), \quad t \in I, \quad k = 0, 1, \ldots \]

**Remark 4.** When \( m \) is negative and the condition
\[ \frac{K(t) + L(t)}{-m(t)} < 1 \] (2.26)
is satisfied then for $q(t) = 0$, $t \in I$, the condition (2.21) holds and the following error evaluation

$$
\|x_k(t) - x_*(t)\| \leq |u_k|, \quad t \in I, \quad t, k = 0, 1, \ldots
$$

is true.

One can see that (2.18) implies that

$$
u_{k+1}(t) \leq \|u_k\| S(t) \left[ \exp \left( \int_0^t q(\tau) \, d\tau \right) - \exp \left( \int_0^t m(\tau) \, d\tau \right) \right].
$$

For $q(t) \geq 0$ and $q(t) \geq m(t)$, $t \in I$, one has

$$
\|u_{k+1}\| \leq \|u_k\| S(t) \left[ \exp \left( \int_0^t q(\tau) \, d\tau \right) - \exp \left( \int_0^t m(\tau) \, d\tau \right) \right],
$$

which implies that

$$
\|u_{k+1}\| \leq \|u_k\| \max_{0 \leq t \leq T} \left( S(t) \left[ 1 - \exp \left( - \int_0^t [q(\tau) - m(\tau)] \, d\tau \right) \right] \right).
$$

This means that the assertion of Theorem 2.1 holds if condition (2.23) is replaced by the following one:

$$
S(t) \left[ 1 - \exp \left( - \int_0^t [q(\tau) - m(\tau)] \, d\tau \right) \right] < 1.
$$

Observe, that in the case when $m$ is negative the result of Theorem 2.1 is a consequence of the result of Theorem 2.2 with $q(t) = 0$ for $t \in I$.

Finally, let us consider the case when there is no delay in Eq. (1.1), which means that $L(t) = 0$, $t \in I$. Now the weighted norm $\|u_k\|_q$ can be defined by the formula,

$$
\|u_k\|_q = \max_{0 \leq s \leq T} \left\{ u_k(s) \exp \left( - \int_0^s q(\tau) \, d\tau \right) \right\}.
$$

In this case from the inequality (2.6) we have

$$
u_{k+1}(t) \leq \int_0^t K(s) u_k(s) \exp \left( \int_s^t m(\tau) \, d\tau \right) \, ds,
$$

where $K(s) = \exp \left( - \int_0^s q(\tau) \, d\tau \right)$.
and
\[
\begin{align*}
    u_{k+1}(t) \leq & \int_0^t K(s)u_k(s)\exp\left(-\int_0^s q(\tau)\,d\tau\right)\exp\left(\int_0^s m(\tau)\,d\tau\right) \\
    & \times \exp\left(\int_0^s (q(\tau) - m(\tau))\,d\tau\right)\,ds.
\end{align*}
\]

(2.34)

This implies that
\[
\begin{align*}
    u_{k+1}(t) \leq & \|u_k\|_q \exp\left(\int_0^t m(\tau)\,d\tau\right) \int_0^t \frac{K(s)}{q(s) - m(s)} (q(s) - m(s)) \\
    & \times \exp\left(\int_0^s (q(\tau) - m(\tau))\,d\tau\right)\,ds,
\end{align*}
\]

(2.35)

and for \(q(t) \geq m(t)\) we get
\[
\begin{align*}
    u_{k+1}(t) \leq & \|u_k\|_q \exp\left(\int_0^t m(\tau)\,d\tau\right) \max_{0 \leq s \leq t} \frac{K(s)}{q(s) - m(s)} \\
    & \times \exp\left(\int_0^s (q(\tau) - m(\tau))\,d\tau\right)\,ds.
\end{align*}
\]

(2.36)

This means that
\[
\begin{align*}
    u_{k+1}(t) \leq & \|u_k\|_q S^*(t) \left[\exp\left(\int_0^t (q(\tau) - m(\tau))\,d\tau - 1\right)\right] \exp\left(\int_0^t m(\tau)\,d\tau\right),
\end{align*}
\]

(2.37)

and
\[
\|u_{k+1}\|_q \leq \|u_k\|_q S^*(T) \left[1 - \exp\left(-\int_0^T (q(\tau) - m(\tau))\,d\tau\right)\right].
\]

(2.38)

with
\[
S^*(t) = \max_{0 \leq s \leq t} \frac{K(s)}{q(s) - m(s)}.
\]

It is important to notice that in the case discussed we do not need to assume that \(q(t)\) is nonnegative and again the convergence of the sequence \(x_k\) is guaranteed when the condition
\[
P^*(T) = S^*(T) \left[1 - \exp\left(-\int_0^T (q(\tau) - m(\tau))\,d\tau\right)\right] < 1
\]

(2.39)

holds with \(S^*(t)\) defined above.
If this condition holds for some negative $q$ (what only may happen when $m$ is negative) then we have

$$\|x_k(t) - x_*(t)\| = u_k(t) \leq (P^*(T))^k \|u_0\|_q \exp\left(\int_0^t q(\tau) \, d\tau\right), \quad (2.40)$$

which means that for a fixed $k$ the error of the approximation $x_k$ to $x_*$ is a decreasing function of $t$. Moreover, in the case when $T$ is replaced with $+\infty$ this error tends to zero when

$$\sup_{0 \leq t < +\infty} P^*(t) < 1,$$

and

$$\int_0^t q(\tau) \, d\tau \to -\infty,$$

with $t \to +\infty$.

3. THE CONVERGENCE OF THE PERTURBED CONTINUOUS TIME WAVEFORM RELAXATION METHOD

Assume that the sequence $(x_{k, h})$, $x_{k, h} \in C^3_0(I, R^n)$, where $C^3_0(I, R^n)$ stands for the class of piecewise continuous functions defined on $I$ with values in $R^n$, is defined by the perturbed continuous-time waveform relaxation method,

$$x_{k+1, h}(t) = F_h(t, x_{k+1, h}(t), x_{k, h}(t), x_{k, h}(')), \quad t \in I_h, \quad (3.1)$$

$$x_{k+1, h}(0) = a_h, \quad (3.2)$$

with some initial function $x_{0, h} \in C_0(I, R^n)$ and $h \in (0, h_0] = H$ for some $h_0 > 0$, where $I_h$ denotes the set of all points in $I$ except for a finite number of them, which depends on $h$; $F_h: I \times R^n \times R^n \times C^3_0(I, R^n) \to R^n$ is continuous and satisfies the condition,

$$\|F_h(t, x, y, z) - F(t, x, y, z)\| \leq \eta(h), \quad (3.3)$$

for all $t, x, y, z, h \in H$ and some continuous function $\eta: H \to R_+\,$ with the property $\eta(h) \to 0$ when $h \to 0^+$. Now we want to know what is the error

$$\|x_{k, h}(t) - x_*(t)\|.$$
It is clear that we have
\[ \|x_{k,h}(t) - x_\ast(t)\| \leq \|x_{k,h}(t) - x_k(t)\| + \|x_k(t) - x_\ast(t)\|. \]

Under the conditions considered in the previous section the second term of the right-hand side of the above inequality has the estimate,
\[ \|x_k(t) - x_\ast(t)\| \leq A \alpha^k, \tag{3.4} \]
with some \( \alpha < 1 \).

For the first one we can get
\[ \|x_{k,h}(t) - x_k(t)\| \leq P_k h^r \tag{3.5} \]
when an appropriate discretization method is used.

So, to get the total error estimation for a given \( \varepsilon > 0 \) we must solve one inequality with two unknowns \( h \) and \( k \),
\[ P_k h^r + A \alpha^k < \varepsilon. \tag{3.6} \]

Another possible approach consists in the following: Assume that the sequence \( \langle \tilde{x}_k \rangle, \tilde{x}_k \in C^1(I, \mathbb{R}^n) \) is defined by the relations,
\[
\begin{align*}
\tilde{x}_{k+1}(t) &= F(t, \tilde{x}_{k+1}(t), \tilde{x}_k(t), \tilde{x}_k(\cdot)) + \rho_k(t), \tag{3.7} \\
\tilde{x}_{k+1}(0) &= a_{k+1}, \tag{3.8}
\end{align*}
\]
with some initial function \( \tilde{x}_0 \in C^1(I, \mathbb{R}^n) \) such that \( \tilde{x}_0(0) = a_0 = a \) for given sequences \( \langle \tilde{\rho}_k \rangle, \tilde{\rho}_k \in C(I, \mathbb{R}^n) \) and \( \langle a_k \rangle, a_k \in \mathbb{R}^n \).

Now, using the conditions (A) and (B), we have
\[
(\tilde{x}_{k+1}'(t) - x_\ast'(t), \tilde{x}_{k+1}(t) - x_\ast(t))
= (F(t, \tilde{x}_{k+1}(t), \tilde{x}_k(t), \tilde{x}_k(\cdot)) \\
- F(t, x_\ast(t), x_\ast(t), x_\ast(\cdot)), \tilde{x}_{k+1}(t) - x_\ast(t))
+ (\rho_k(t), \tilde{x}_{k+1}(t) - x_\ast(t))
\leq m(t)\|\tilde{x}_{k+1}(t) - x_\ast(t)\|^2
+ K(t)\|\tilde{x}_k(t) - x_\ast(t)\|\|\tilde{x}_{k+1}(t) - x_\ast(t)\|
+ L(t)\|\tilde{x}_k - x_\ast\|\|\tilde{x}_{k+1}(t) - x_\ast(t)\|
+ \|\rho_k(t)\|\|\tilde{x}_{k+1}(t) - x_\ast(t)\|.
\]
According to Eq. (2.4) we have

\[
\|\tilde{x}_{k+1}(t) - x_*(t)\| \frac{d}{dt}\|\tilde{x}_{k+1}(t) - x_*(t)\| \\
\leq m(t)\|\tilde{x}_{k+1}(t) - x_*(t)\|^2 \\
+ K(t)\|\tilde{x}_k(t) - x_*(t)\|\|\tilde{x}_{k+1}(t) - x_*(t)\| \\
+ L(t)\|\tilde{x}_k - x_*\|\|\tilde{x}_{k+1}(t) - x_*(t)\| \\
+ \|\rho_k(t)\|\|\tilde{x}_{k+1}(t) - x_*(t)\|. 
\]

After dividing both sides of this inequality by \(\|\tilde{x}_{k+1}(t) - x_*(t)\|\) and introducing the notation,

\[
\tilde{u}_k(t) = \|\tilde{x}_k(t) - x_*(t)\|, \quad |\tilde{u}_k| = \max_{0 \leq s \leq t} \tilde{u}_k(s),
\]

we get

\[
\tilde{u}_{k+1}'(t) \leq m(t)\tilde{u}_{k+1}(t) + K(t)\tilde{u}_k(t) + L(t)|\tilde{u}_k| + \|\rho_k(t)\|. \quad (3.9)
\]

with \(\tilde{u}_{k+1}(0) = \|a_{k+1} - a\|\) for \(k = 0, 1, \ldots\).

From this inequality we find

\[
\tilde{u}_{k+1}(t) \leq \|a_{k+1} - a\|\exp\left(\int_0^t m(s) \, ds\right) \\
+ \int_0^t \exp\left(\int_s^t m(\tau) \, d\tau\right) [K(s) + L(s)]|\tilde{u}_k| \, ds \\
+ \int_0^t \exp\left(\int_s^t m(\tau) \, d\tau\right) \|\rho_k(s)\| \, ds. \quad (3.10)
\]

Now, for \(m(t) < 0\) we have

\[
\tilde{u}_{k+1}(t) \leq \|a_{k+1} - a\| \\
+ \int_0^t \frac{K(s) + L(s)}{-m(s)} \exp\left(\int_s^t m(\tau) \, d\tau\right) ds|\tilde{u}_k|, \\
+ \int_0^t \left(\frac{1}{-m(s)}\right) \exp\left(\int_s^t m(\tau) \, d\tau\right) ds \max_{0 \leq s \leq t} \|\rho_k(s)\|,
\]
and
\[
\bar{u}_{k+1}(t) \leq \|a_{k+1} - a\|
+ \max_{0 \leq s \leq t} K(s) + L(s) \left( 1 - \exp \left( \int_0^t m(\tau) \ d\tau \right) \right) |\bar{u}_k|,
+ \max_{0 \leq s \leq t} \frac{1}{-m(s)} \max_{0 \leq s \leq t} \|\rho_k(s)\| \left( 1 - \exp \left( \int_0^t m(\tau) \ d\tau \right) \right),
\]
which gives the inequality,
\[
|\bar{u}_{k+1}| \leq \|a_{k+1} - a\| + \max_{0 \leq s \leq t} \frac{1}{-m(s)} \max_{0 \leq s \leq t} \|\rho_k(s)\|
\times \left( 1 - \exp \left( \int_0^t m(\tau) \ d\tau \right) \right) + \max_{0 \leq s \leq t} \frac{K(s) + L(s)}{-m(s)} |\bar{u}_k|.
\]

The last inequality means that
\[
\|\tilde{x}_{k+1} - x_*\| \leq \delta_k(t) + P(t)\|\tilde{x}_k - x_*\|, \tag{3.11}
\]
with
\[
\delta_k(t) = \|a_{k+1} - a\| + \max_{0 \leq t \leq t} \max_{0 \leq t \leq t} \left( \frac{1}{-m(s)} \max_{0 \leq s \leq t} \|\rho_k(s)\| \right)
\times \left( 1 - \exp \left( \int_0^t m(\tau) \ d\tau \right) \right),
\]
and
\[
P(t) = \max_{0 \leq t \leq t} \left( \max_{0 \leq t \leq t} \frac{K(s) + L(s)}{-m(s)} \left( 1 - \exp \left( \int_0^t m(\tau) \ d\tau \right) \right) \right)
\]
(one can see that we can drop the term \(1 - \exp(\int_0^t m(\tau) \ d\tau)\) for simplicity).

Solving the inequality (3.11) we find
\[
\|\tilde{x}_k - x_*\| \leq (P(t))^k \|\tilde{x}_0 - x_*\| + \sum_{i=0}^{k-1} (P(t))^{k-1-i} \delta_i(t).
\]

It is known that the right-hand side of this inequality tends to zero when \(P(t) < 1\) and \(\delta_k(t) \to 0\) as \(k \to \infty\) (see [7]). We see that \(\delta_k(t) \to 0\) as \(k \to \infty\) when \(a_k \to a\) and \(\rho_k(t) \to 0\) with \(k \to \infty\).
If one assumes only that $\delta(t) \leq \delta$, $\delta$-constant, then the following estimation

$$\|\tilde{x}_k - x_\ast\| \leq (P(t))^k \|\tilde{x}_0 - x_\ast\| + \frac{\delta}{1 - P(t)}$$

holds.

4. THE EXISTENCE AND UNIQUENESS OF THE SOLUTION ON THE WHOLE INTERVAL $I$

Under the Assumptions (A) and (B) there exists a unique solution of problem (1.1)–(1.2) defined on the whole interval $I$.

Uniqueness: assume that there exists another solution of the problem mentioned above, which we denote by $x_{\ast\ast}$. Then we have

$$(x_{\ast\ast}^\prime(t) - x_{\ast\ast}^\prime(t), x_{\ast}(t) - x_{\ast\ast}(t))
= (F(t, x_{\ast}(t), x_{\ast}(t), x_{\ast}(\cdot)), x_{\ast}(t) - x_{\ast\ast}(t))
\leq m(t)\|x_{\ast}(t) - x_{\ast\ast}(t)\|^2 + K(t)\|x_{\ast}(t) - x_{\ast\ast}(t)\|^2
+ L(t)\|x_{\ast} - x_{\ast\ast}\|\|x_{\ast}(t) - x_{\ast\ast}(t)\|.$$}

Next, according to (2.4), we have

$$\|x_{\ast}(t) - x_{\ast\ast}(t)\| \frac{d}{dt}\|x_{\ast}(t) - x_{\ast\ast}(t)\|
\leq m(t)\|x_{\ast}(t) - x_{\ast\ast}(t)\|^2 + K(t)\|x_{\ast}(t) - x_{\ast\ast}(t)\|^2
+ L(t)\|x_{\ast} - x_{\ast\ast}\|\|x_{\ast}(t) - x_{\ast\ast}(t)\|.$$}

Hence,

$$\frac{d}{dt}\|x_{\ast}(t) - x_{\ast\ast}(t)\|
\leq m(t)\|x_{\ast}(t) - x_{\ast\ast}(t)\|
+ K(t)\|x_{\ast}(t) - x_{\ast\ast}(t)\| + L(t)\|x_{\ast} - x_{\ast\ast}\|,
\leq (K(t) + L(t))\|x_{\ast} - x_{\ast\ast}\| + m(t)\|x_{\ast}(t) - x_{\ast\ast}(t)\|,$$

and

$$\|x_{\ast}(t) - x_{\ast\ast}(t)\| \leq \int_0^t \exp\left(\int_s^t m(\tau) \, d\tau\right) (K(s) + L(s))\|x_{\ast} - x_{\ast\ast}\| \, ds.$$
This implies that
\[ \|x_\ast - x_{\ast\ast}\|_t = 0, \]
according to the well-known Gronwall's inequality (or lemma).

**Existence:** the Cauchy–Peano theorem implies the local existence of the solution of problem (1.1)–(1.2). What we need to prove is the fact that it can be prolonged onto the whole interval \( I \). If \( x_\ast \) is the solution of the problem then we have
\[
(x'_\ast(t) - 0, x_\ast(t) - 0) = (F(t, x_\ast(t), x_\ast(t), x_\ast(\cdot)) - F(t, 0, 0, 0), x_\ast(t) - 0)
+ (F(t, 0, 0, 0), x_\ast(t) - 0)
\leq m(t)\|x_\ast(t)\|^2 + K(t)\|x_\ast(t)\|^2
+ L(t)\|x_\ast\|, \|x_\ast(t)\| + \|F(t, 0, 0, 0)\| \|x_\ast(t)\|,
\]
and
\[
\|x_\ast(t)\| \frac{d}{dt}\|x_\ast(t)\| \leq m(t)\|x_\ast(t)\|^2 + K(t)\|x_\ast(t)\|^2
+ L(t)\|x_\ast\|, \|x_\ast(t)\| + \|F(t, 0, 0, 0)\| \|x_\ast(t)\|.
\]

Hence,
\[
\frac{d}{dt}\|x_\ast(t)\| \leq m(t)\|x_\ast(t)\| + K(t)\|x_\ast(t)\| + L(t)\|x_\ast\|, + \|F(t, 0, 0, 0)\|,
\]
and
\[
\|x_\ast(t)\| \leq a \exp\left(\int_0^t m(\tau) \, d\tau \right)
+ \int_0^t (K(s)\|x_\ast(s)\| + L(s)\|x_\ast\|) \exp\left(\int_s^t m(\tau) \, d\tau \right) \, ds
+ \int_0^t \exp\left(\int_s^t m(\tau) \, d\tau \right) \|F(s, 0, 0, 0)\| \, ds.
\]
This implies
\[
\|x_\ast\|_t \leq M\left[ a + \int_0^t [K(s) + L(s)]\|x_\ast\|_s + \int_0^t \|F(s, 0, 0, 0)\| \, ds \right],
\]
and
\[
\|x_\ast\|_s \leq M\left( a + \int_0^s \|F(s, 0, 0, 0)\| \, ds \right) \exp\left( M\int_0^s [K(s) + L(s)] \, ds \right),
\]
for some

\[ M \geq \max_{0 \leq s \leq T} \max_{0 \leq t \leq t} \exp \left( \int_s^t m(\tau) \, d\tau \right), \]

which in turn implies that the solution can be prolonged onto whole interval \( I \).

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REFERENCES