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One-Sided Resonance Problems for Quasilinear Elliptic Operators

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We present some new existence results for a quasilinear elliptic problem with an unbounded driving force. The quasilinear elliptic operator is assumed to be variational and is such that 0 acts like an isolated eigenvalue with a corresponding eigenfunction which does not change sign. The driving force is further assumed to be in one-sided resonance around the eigenvalue 0, and a solvability condition of potential type is imposed. Variational methods are used to obtain existence. Our results significantly improve earlier results of the authors. © 2001 Academic Press

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1. INTRODUCTION

Let Ω be a bounded domain in $\Re^N$, $N \geq 1$, with a continuous boundary, and satisfying the cone property; i.e., there exists a finite cone C such that each point x in Ω is a vertex of a finite cone C_x contained in Ω and congruent to C. Denote by D^α the differential operator

$$\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_N^{\alpha_N}},$$

where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$ is a multi–index consisting of nonnegative integers, and $|\alpha| = \sum_{j=1}^N \alpha_j$ denotes the order of D^α. In order to write nonlinear partial differential operators in a convenient form, we introduce as in [5] the vector space $\Re^{s_m}$ whose elements are of the form $\xi_m = \{\xi_\alpha : |\alpha| \leq m\}$. For each $u \in W^{m,p}(\Omega)$, define $\xi_m(u)(x)$ to be the vector in $\Re^{s_m}$ given by $\{D^\alpha u(x) : |\alpha| \leq m\}$. (Observe that $D^{(0,0,\dots,0)}u = u$.) In this paper we study the $2m$th order quasilinear differential operator in generalized divergence form

$$Q(u) := \sum_{1 \leq |\alpha| \leq m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, \xi'_m(u)),$$

where ξ'_m denotes the vector $\{\xi_\alpha : 1 \leq |\alpha| \leq m\}$ in $\Re^{s_m - 1}$.

We will assume that Q has a variational structure in the sense that there exists a function $F\colon \Omega \times \Re^{s_m-1} \to \Re$ satisfying

(F-1) The map $x \to F(x, \xi'_m)$ is measurable for each $\xi'_m \in \Re^{s_m-1}$, and the map $\xi'_m \to F(x, \xi'_m)$ is continuously differentiable for a.e. $x \in \Omega$.

(F-2) There exist constants p and c_1, with $1 < p < \infty$ and $c_1 > 0$, and a nonnegative function $h \in L^1(\Omega)$ such that

$$|F(x, \xi')| \leq h(x) + c_1 |\xi'_m|^p$$

for a.e. $x \in \Omega$ and all $\xi'_m \in \Re^{s_m-1}$.

(F-3) $F(x, 0) = 0$ for a.e. $x \in \Omega$, and for each α, with $1 \leq |\alpha| \leq m$,

$$\frac{\partial F}{\partial \xi_\alpha}(x, \xi') = A_\alpha(x, \xi') \qquad \text{for} \quad (x, \xi') \in \Omega \times \Re^{s_m-1}.$$

The functions $A_\alpha\colon \Omega \times \Re^{s_m-1} \to \Re$ defined in (F-3) will be assumed to satisfy the following conditions

(A-1) There exists a constant c_2, with $c_2 > 0$, and a nonnegative function $\tilde{h} \in L^{p'}(\Omega)$, where $p' = p/(p-1)$ and p is as in (F-2), such that

$$|A_\alpha(x, \xi'_m)| \leq \tilde{h}(x) + c_2 |\xi'_m|^{p-1}, \qquad 1 \leq |\alpha| \leq m$$

for a.e. $x \in \Omega$ and for all $\xi'_m \in \Re^{s_m-1}$.

(A-2) (*Ellipticity*) There exists a positive constant c_o such that

$$\sum_{1\le|\alpha|\le m} A_\alpha(x, \xi'_m)\xi_\alpha \ge c_o \sum_{1\le|\alpha|\le m} |\xi_\alpha|^p$$

for a.e. $x \in \Omega$ and for all $\xi'_m \in \Re^{s_m-1}$, where p is as in (F-2).

(A-3) (*Monotonicity*) Let $\xi'_m = (\eta'_{m-1}, \zeta_m)$ be the division of ξ'_m into its mth order component and the corresponding $(m-1)$st order terms η'_{m-1}; i.e., $\eta'_{m-1} = \{\xi_\beta : 1 \le |\beta| \le m-1\} \in \Re^{s_{m-1}-1}$, and $\zeta_m = \{\xi_\alpha : |\alpha| = m\}$. Put $A_\alpha(x, \xi'_m) = A_\alpha(x, \eta'_{m-1}, \zeta_m)$. Assume that for a.e. $x \in \Omega$ and each $\eta'_{m-1} \in \Re^{s_{m-1}-1}$,

$$\sum_{|\alpha|=m} [A_\alpha(x, \eta'_{m-1}, \zeta_m) - A_\alpha(x, \eta'_{m-1}, \zeta^*_m)](\zeta_\alpha - \zeta^*_\alpha) > 0$$

for $\zeta_m \neq \zeta^*_m$ and a.e. $x \in \Omega$.

We define the following semilinear Dirichlet form

$$\mathcal{Q}(u, v) := \sum_{1\le|\alpha|\le m} \int_\Omega A_\alpha(x, \xi'_m(u))D^\alpha v \qquad \forall u, v \in W^{m,p}(\Omega). \tag{1}$$

In view of (A-1) we see that $\mathcal{Q}$ is well defined on $W^{m,p}(\Omega) \times W^{m,p}(\Omega)$. Throughout this paper we will use the norm in $W^{m,p}(\Omega)$ given by

$$\|u\|^p_{m,p} = \sum_{|\alpha|\le m} \|D^\alpha u\|^p_{L^p},$$

where $\|\cdot\|_{L^p}$ denotes the L^p norm. We will also be using the seminorm

$$|u|'_{m,p} = \left\{ \sum_{1\le|\alpha|\le m} \|D^\alpha u\|^p_{L^p} \right\}^{1/p}.$$

Observe that by the definition of $\mathcal{Q}$ in (1) and (A-2) we get

$$\mathcal{Q}(u, u) \ge c_o \int_\Omega \sum_{1\le|\alpha|\le m} |D^\alpha u|^p = c_0(|u|'_{m,p})^p \tag{2}$$

for all $u \in W^{m,p}(\Omega)$, so that $\liminf_{\|u\|_{L^p}\to\infty}(\mathcal{Q}(u, u)/\|u\|^p_{L^p}) \ge 0$. Define as in [16, p. 1821],

$$\lambda_1 := \liminf_{\|u\|_{L^p}\to\infty} \frac{\mathcal{Q}(u, u)}{\|u\|^p_{L^p}}.$$

Since $\mathcal{Q}(u, u) = 0$ for u constant, we see that $\lambda_1 = 0$. On the other hand, for nonconstant $v \in W^{m,p}(\Omega)$ we obtain from (2) that $\mathcal{Q}(v, v) > 0$, so $\lambda_1 = 0$ behaves like a simple eigenvalue with constant normalized eigenfunction $\phi_1 \equiv 1/|\Omega|^{1/p}$ and corresponding eigenspace $W := span\{1\}$.

In this paper we investigate the solvability of the problem

$$Qu = -a(\cdot, u)|u|^{p-2}u^- + g(\cdot, u) + H \qquad \text{a.e. in } \Omega, \quad u \in W^{m,p}(\Omega), \tag{3}$$

where p is as in (F-2), $u^- = \max\{-u, 0\}$ is the negative part of u, $H \in W^{m,p}(\Omega)^*$, the dual of $W^{m,p}(\Omega)$, $g: \Omega \times \Re \to \Re$ is a function satisfying

(g-1) (*Carathéodory conditions*) the map $x \to g(x, s)$ is measurable for all $s \in \Re$, and the map $s \to g(x, s)$ is continuous for a.e. $x \in \Omega$;

(g-2) g is bounded; that is, there exists constant M such that

$$|g(x, s)| \le M \qquad \text{for all } s \in \Re \text{ and a.e. } x \in \Omega;$$

and $a: \Omega \times \Re \to \Re$ is a function satisfying

(a-1) the Carathéodory conditions as in (g-1);

(a-2) There is a $\gamma > 0$ such that $0 \le a(x, s) \le \gamma$ for a.e. $x \in \Omega$ and all $s \in \Re$.

(a-3) $\liminf_{s \to -\infty} a(x, s) = \gamma_1$ uniformly for a.e. $x \in \Omega$ and some $\gamma_1 > 0$. To be explicit, we mean that there exists $E \subset \Omega$ with $|E| = 0$ such that (i) $\liminf\limits_{s \to -\infty} a(x, s) = \gamma_1$ for $x \in \Omega \backslash E$; and (ii) given $\varepsilon > 0$, there exists s_o such that for $s < s_o$, $a(x, s) > \gamma_1 - \varepsilon$ for $x \in \Omega \backslash E$.

By a solution to problem (3) we mean a weak or generalized solution; i.e., a function $u \in W^{m,p}(\Omega)$ satisfying

$$\mathscr{Q}(u, v) = -\int_\Omega a(x, u)|u|^{p-2}u^- v + \int_\Omega g(x, u)v + H(v), \tag{4}$$

for all $v \in W^{m,p}(\Omega)$. There is a connection between problem (3) and the Neumann problem for Q. To see this connection we refer the reader to [17, pp. 365–367].

We will prove the following result:

THEOREM 1.1. *Let $1 < p < \infty$ and let Ω be an open bounded connected set with continuous boundary and satisfying the cone property. Assume* (F-1)–(F-3), *and suppose that $\mathscr{Q}(u, v)$ is given by* (1) *where $A_\alpha(x, \xi'_m)$ satisfies* (A-1)–(A-3) *for $1 \le |\alpha| \le m$. Let $g(x, s)$ satisfy* (g-1)–(g-2), *$a(x, s)$ satisfy* (a-1)–(a-3), *and $H \in W^{m,p}(\Omega)^*$, the dual of $W^{m,p}(\Omega)$. If*

$$\lim_{t \to \infty} \left\{ \int_\Omega G(x, t) + H(t) \right\} = +\infty, \tag{5}$$

where $G(x, s) := \int_0^s g(x, t)\, dt$, then problem (3) *has at least one solution.*

As in [15] it is straightforward to show that condition (5) is also necessary for solvability when we consider a specialized class of functions $g(x, s)$.

THEOREM 1.2. *In addition to the hypotheses of Theorem* 1, *assume that* $g(x, s) = b(x)f(s)$ *for* $(x, s) \in \Omega \times \Re$, *where* $b \in C(\Omega) \cap L^\infty(\Omega)$, $f: \Re \to \Re$ *is continuous, and* $b(x) > 0$ *for all* $x \in \Omega$. *If* $\lim_{s\to\infty} f(s) = f_+$ *exists, and*

$$f(s) < f_+ \qquad \text{for all } s \in \Re,$$

then condition (5) *is necessary and sufficient for the solvability of problem* (3).

Our existence proof will rely on a variation of the well-known saddle point theorem in [13]. Rather than decomposing the Banach space $W^{m,p}(\Omega)$ into complementary linear subspaces, as was done in [15], we will decompose $W^{m,p}(\Omega)$ into a linear subspace and a complementary cone. This approach has been used by other authors. For example, see the nonresonance results in [9, 10].

Remark 1.1. If we let

$$g_1(x, s) = -a(x, s)|s|^{p-2}s^- + g(x, s)$$

for a.e. $x \in \Omega$ and all $s \in \Re$, conditions (g-2) and (a-2) imply that

$$0 \le \liminf_{|s|\to\infty} \frac{g_1(x, s)}{|s|^{p-2}s} \le \limsup_{|s|\to\infty} \frac{g_1(x, s)}{|s|^{p-2}s} \le \gamma \qquad \text{for a.e. } x \in \Omega.$$

Thus the nonlinear term *interacts* with the eigenvalue $\lambda_1 = 0$ from above and we are well motivated to call problem (3) a one-sided resonance problem. In problems such as this it is often helpful to use the growth condition (a-2) to prevent *interaction* with other eigenvalues. This requires a reasonable definition of the next eigenvalue, λ_2, and an added restriction such as $\gamma < \lambda_2$. This approach was used in [15], where

$$\lambda_2 := \liminf_{\|v\|_{L^p}\to\infty} \frac{\mathcal{Q}(v, v)}{\|v\|_{L^p}^p} \qquad v \in V := \left\{ v \in W^{m,p}(\Omega) : \int_\Omega v = 0 \right\}.$$

Our results improve upon those in [15] by allowing an arbitrary choice of the constant γ. They key differences in the proofs are centered on how to *split* the Banach space $W^{m,p}(\Omega)$. In [15] the arguments and estimates are all relative to the linear splitting $W^{m,p}(\Omega) = W \oplus V$. In this paper our arguments and estimates are relative to the *nonlinear splitting* $W^{m,p}(\Omega) = W \oplus V_\epsilon$, where V_ϵ is a certain cone centered on $W^+ := \{w \in W : w \ge 0\}$.

Remark 1.2. Perhaps a better context for understanding this work involves the Fučik spectrum of the operator Q, which can be defined as the set of pairs (α, β) such that the problem

$$Qu = \alpha|u|^{p-2}u^+ - \beta|u|^{p-2}u^-, \qquad \text{a.e. in } \Omega,$$

$$u \in W^{m,p}(\Omega),$$

has a nontrivial solution. It is clear that the sets $\{0\} \times \Re$ and $\Re \times \{0\}$ are part of the Fučik spectrum, and cross at the principal eigenvalue where $\alpha = \beta = \lambda_1 = 0$. These sets are often referred to as the principal, or trivial, branch of the Fučik spectrum. The given growth conditions on $a(x, u)$ and $g(x, u)$ allow us to view problem (3) as a perturbation of

$$Qu = -\gamma_1 |u|^{p-2} u^-, \qquad \text{a.e. in } \Omega,$$
$$u \in W^{m,p}(\Omega),$$

It was this point of view that suggested that the upper bound on γ in [15] was an artifact of the proof and not of the problem.

For more details on the Fučik spectrum of the Laplacian and the p-Laplacian, respectively, see [10, 8]. These papers also contain general nonresonance results where the forcing term lies asymptotically in the gap between the trivial and first nontrivial Fučik curves. A detailed description of the Fučik spectrum for the general class of operators studied here remains a topic for further research.

In the current literature most resonance results relative to the Fučik spectrum have been restricted to linear or homogeneous quasilinear ordinary differential operators. One excellent example of a resonance result for the PDE case is found in [7], where the boundary value problem

$$-\Delta u = \alpha u^+ - \beta u^- + g(x, u) \qquad \text{in } \Omega,$$
$$u|_{\partial\Omega} = 0,$$

is examined assuming that $(\alpha, \beta) \in C_2$, the first nontrivial branch of the spectrum, and g and its primitive G satisfy the conditions that g is a Caratheodory function with subcritical growth, $\lim_{|s|\to\infty}(2G(x, s)/s^2) = 0$ uniformly, and $\lim_{|s|\to\infty}((sg(x, s) - 2G(x, s))/s^2) = \pm\infty$ uniformly.

Remark 1.3. The solvability condition (5) in Theorem 1 was first introduced by Ahmad et al. [3] to deal with resonance problems for bounded nonlinear perturbations of linear second order self–adjoint elliptic operators with Dirichlet boundary condition. A similar condition was used by Castro and Lazer (see [6, Theorem 3, p. 148]) to deal with nonlinear perturbations of the Neumann problem, in the case $p = 2$, in which the nonlinearity is assumed to be differentiable with derivative bounded from above. The results in [15] treat this kind of condition allowing a very general class of quasilinear elliptic operators, an unbounded driving force, and no boundedness assumption on the derivative of the driving force.

Remark 1.4. For the case where Q is a second order linear elliptic operator our results are complementary to Theorem 4 of [4], where similar growth and solvability conditions are assumed. The primary difference between our results and theirs, other than the fact that we allow a more general class of differential operators, is that they impose conditions on g and use degree theoretic arguments, whereas we impose conditions on the primitive G and use variational arguments.

Remark 1.5. It is a straightforward exercise to construct examples to illustrate Theorem 1.1. Let $Q = -\Delta_p$, the p-Laplacian, or some well-behaved nonhomogeneous perturbation of the p-Laplacian. Let $a(x,u) = c + \frac{c}{2}\sin(u)$ for some $c > 0$. Let $g(x,u) = \arctan(u) + k$ for some constant k. In this case condition (5) reduces to the well known Landesman–Lazer condition of [12]. For a more detailed discussion of examples see [15].

2. THE VARIATIONAL SETTING

For $a(x,s)$ satisfying (a-1) and (a-2), set

$$A(x,s) := \begin{cases} \int_0^s a(x,t)|t|^{p-2}t\,dt & \text{for } s \le 0 \\ 0 & \text{for } s > 0, \end{cases}$$

and for g satisfying (g-1) and (g-2), put

$$G(x,s) := \int_0^s g(x,t)\,dt \qquad \text{for all } s \in \Re.$$

Let F be as given in (F-1)–(F-3). Define a functional $\mathcal{J}\colon W^{m,p}(\Omega) \to \Re$ by

$$\mathcal{J}(u) := \int_\Omega F(x, \xi'_m(u)) - \int_\Omega A(x,u) - \int_\Omega G(x,u) - H(u)$$

for all $u \in W^{m,p}(\Omega)$. By virtue of (F-1)–(F-2), (g-1)–(g-2), and (a-1)–(a-2) we see that $\mathcal{J}$ is well defined and continuous. Observe also that if (F-2), (g-2), and (a-2) hold then $\mathcal{J}$ maps bounded sets in $W^{m,p}(\Omega)$ to bounded sets in $\Re$. Moreover, using (F-3), (A-1), (g-2), and (a-2) we can show, as in [5, p. 35], that in fact, $\mathcal{J} \in C^1(W^{m,p}(\Omega), \Re)$, and that its Fréchet derivative is given by

$$\mathcal{J}'(u)v = \mathcal{Q}(u,v) + \int_\Omega a(x,u)|u|^{p-2}u^- v - \int_\Omega g(x,u)v - H(v) \tag{6}$$

for all $u, v \in W^{m,p}(\Omega)$. Observe that using (F-3) and Fubini's theorem, whose use in this case is justified by (A-1), we can write

$$\mathcal{J}(u) = \int_0^1 \mathcal{Q}(tu,u)\,dt - \int_\Omega A(x,u) - \int_\Omega G(x,u) - H(u) \tag{7}$$

for all $u \in W^{m,p}(\Omega)$.

A critical point of $\mathcal{J}$ is a function $u \in W^{m,p}(\Omega)$ for which $\mathcal{J}'(u)v = 0$ for all $v \in W^{m,p}(\Omega)$. In view of (6) above, this corresponds precisely to the definition (4) of a solution for problem (3). Therefore, in order to prove Theorem 1.1 it suffices to show that $\mathcal{J}$ possesses at least one critical point. We will establish this fact by means of a saddle point theorem over linked sets. For the reader's convenience we state the relevant definitions and theorem below. The definition of the Palais–Smale condition is standard. For more details on linking and the saddle point theorem see [18, Definition 8.1 and Theorem 8.4].

DEFINITION 2.1. Let X be a Banach space. Let Y be a closed subset of X, and let Z be a submanifold of X with relative boundary ∂Z. We say that Y and ∂Z link if

(1) $Y \bigcap \partial Z = \emptyset$, and

(2) for any map $h \in C^0(X, X)$ such that $h|_{\partial Z} = id$ there holds $h(Z) \bigcap Y \neq \emptyset$.

DEFINITION 2.2. Let X be a Banach space and let $J \in C^1(X, \Re)$. J satisfies the Palais–Smale condition, (PS), if any sequence $\{u_n\} \subset X$ such that

(1) $\{J(u_n)\}$ is bounded, and

(2) $J'(u_n) \to 0$ in X^*,

has a strongly converging subsequence.

THEOREM 2.1. *Suppose that X is a Banach space and $J \in C^1(X, \Re)$ satisfies* (PS). *Consider a closed subset $Y \subset X$ and a submanifold $Z \subset X$ with relative boundary ∂Z. Suppose that*

(1) *Y and ∂Z link,*

(2) $\inf_{u \in Y} J(u) > \sup_{u \in \partial Z} J(u)$.

Let $\Gamma := \{h \in C^0(X, X) : h|_{\partial Z} = id\}$. Then the number

$$\beta := \inf_{h \in \Gamma} \sup_{u \in Z} J(h(u))$$

defines a critical value of J.

Theorem 1.1 will be proved as an application of Theorem 2.1. Assuming the conditions of Theorem 1.1, we will verify the geometric and topological conditions for $\mathcal{J}$ in Section 3 and will show that $\mathcal{J}$ satisfies (PS) in Section 4.

3. THE SADDLE GEOMETRY OVER LINKED SETS

In this section we establish the saddle geometry and linking properties that are necessary for the application of Theorem 2.1. Throughout the section we assume (F-1)–(F-3), (A-1)–(A-2), (g-1)–(g-2), (a-1)–(a-3) and let $\mathcal{J}$ be given by (7).

Let $W := span\{1\}$, $W_R := \{w \in W : -R \leq w \leq R\}$, $V := \{v \in W^{m,p}(\Omega) : \int_\Omega v = 0\}$, and $V_\epsilon := \{v = t(1 + \epsilon\tilde{v}) : t \geq 0, \tilde{v} \in V, \int_\Omega |\tilde{v}|^p = 1\}$. Notice that $\partial W_R = \{\pm R\}$.

LEMMA 3.1. *If condition* (5) *holds*, *then*

$$\mathcal{J}(w) \to -\infty \qquad \text{as } \|w\|_{m,p} \to \infty \text{ in } W.$$

Proof. For $w \in W$ we have $\mathcal{Q}(tw, w) \equiv 0$. Also, if $w > 0$ we have $A(x, w) \equiv 0$. Thus $\mathcal{J}(w) = -\int_\Omega G(x, w) - Hw$ for $w \in W$ with $w > 0$. Condition (5) immediately implies that $\lim_{w\to\infty} \mathcal{J}(w) = -\infty$.

For $w \in W$ with $w < 0$ we have $\mathcal{J}(w) = -\int_\Omega A(x, w) - \int_\Omega G(x, w) - Hw$. Clearly,

$$\left| \int_\Omega G(x, w) + Hw \right| \leq K|w|$$

for some constant $K > 0$. Using (a-3) let $s_0 < 0$ such that $a(x, s) \geq \gamma_1/2$ for all $s < s_0$ and all $x \in \Omega\backslash E$. Then for $w < s_0$ and $x \in \Omega\backslash E$ we get

$$\begin{aligned} A(x, w) &= \int_0^{s_0} a(x, t)|t|^{p-2}t\, dt + \int_{s_0}^{w} a(x, t)|t|^{p-2}t\, dt \\ &\geq \frac{\gamma_1}{2p}(|w|^p - |s_0|^p). \end{aligned}$$

It follows that

$$\mathcal{J}(w) \leq -|\Omega| \frac{\gamma_1}{2}(|w|^p - |s_0|^p) + K|w|,$$

so $\lim_{w\to-\infty} \mathcal{J}(w) = -\infty$. ∎

The following lemma provides a Poincarè type inequality on the set V_ϵ. The proof uses an idea found in [8, Lemma 2.4].

LEMMA 3.2. *Given any constant $k > 0$ there is an $\epsilon > 0$ and a $\delta > 0$ such that $\int_\Omega |\nabla v|^p \geq \delta \int_\Omega |v^+|^p + k \int_\Omega |v^-|^p$ for all $v \in V_\epsilon$.*

Proof. First, we show that there is an $\epsilon > 0$ such that $\int_\Omega |\nabla v^-|^p \geq k \int_\Omega |v^-|^p$ for all $v \in V_\epsilon$. If not, then there are sequences $\{\epsilon_n\} \subset \Re^+$ and $\tilde{v}_n \subset V$ such that $\epsilon_n \to 0$, $\int_\Omega |\tilde{v}_n|^p = 1$, and $\int_\Omega |\nabla v_n^-|^p \leq k \int_\Omega |v_n^-|^p$ for all n where $v_n = 1 + \epsilon_n \tilde{v}_n$. (Note that we have set $t_n = 1$ by a simple rescaling.) Clearly, $v_n \to 1$ in $L^p(\Omega)$, so $|\{x : v_n(x) < 0\}| \to 0$.

However, $v_n^-/(\int_\Omega |v_n^-|^p)^{1/p}$ is bounded in $W^{1,p}(\Omega)$, and so, without loss of generality, converges in L^p to some $\bar{v} \geq 0$ with $\int_\Omega |\bar{v}|^p = 1$. Thus $|\{x : (v_n(x)/(\int_\Omega |v_n^-|^p)^{1/p}) < 0\}| \nrightarrow 0$, a contradiction.

Choose ϵ as in the previous paragraph. Using an argument by contradiction, similar to the argument above, it is straightforward to show that there is a $\delta > 0$ such that $\int_\Omega |\nabla v^+|^p \geq \delta \int_\Omega |v^+|^p$ for all $v \in V_\epsilon$. Hence the lemma is proved. ∎

COROLLARY 3.1. *There is an $\epsilon > 0$ and $d > 0$ such that $\mathcal{Q}(v, v) \geq d\|v\|_{m,p}^p + \gamma \int_\Omega |v^-|^p$ for all $v \in V_\epsilon$.*

Proof. Recall that $\mathcal{Q}(v, v) \geq c_0(|v|'_{m,p})^p \geq c_0 \int_\Omega |\nabla v|^p$ by (A-2). Choose $\epsilon > 0$ as in Lemma 3.2 such that

$$\int_\Omega |\nabla v|^p \geq \delta \int_\Omega |v^+|^p + \frac{4\gamma}{c_0} \int_\Omega |v^-|^p$$

for all $v \in V_\epsilon$. Then

$$\begin{aligned}\mathcal{Q}(v, v) &= \tfrac{1}{2}\mathcal{Q}(v, v) + \tfrac{1}{2}\mathcal{Q}(v, v)\\ &\geq \tfrac{c_0}{2}(|v|'_{m,p})^p + \tfrac{c_0}{2}\int_\Omega |\nabla v|^p\\ &\geq \tfrac{c_0}{2}(|v|'_{m,p})^p + \tfrac{c_0\delta}{2}\int_\Omega |v^+|^p + 2\gamma \int_\Omega |v^-|^p\\ &\geq d\|v\|_{m,p}^p + \gamma \int_\Omega |v^-|^p,\end{aligned}$$

where $d := \min\{c_0/2, c_0\delta/2, \gamma\}$. ∎

COROLLARY 3.2. *There is an $\epsilon > 0$ and $d > 0$ such that $\int_0^1 \mathcal{Q}(tv, v)\, dt \geq \frac{d}{p}\|v\|_{m,p}^p + \frac{\gamma}{p}\int_\Omega |v^-|^p$ for all $v \in V_\epsilon$.*

Proof. Select $\epsilon > 0$ as in Corollary 3.1 and use the fact that

$$Q(tv, v) = \frac{1}{t}Q(tv, tv) \geq t^{p-1}\left(d\|v\|_{m,p}^p + \gamma \int_\Omega |v^-|^p\right)$$

for all $v \in V_\epsilon$. ∎

LEMMA 3.3. *There is an $\epsilon > 0$ such that*

$$\mathcal{J}(v) \to \infty \qquad \text{as } \|v\|_{m,p} \to \infty \text{ in } V_\epsilon.$$

Proof. Observe that for some $K > 0$ we have

$$\int_\Omega G(x, u) + Hu \leq K\|u\|_{m,p}$$

for all $u \in W^{m,p}(\Omega)$. Also, by (a-2),

$$\int_\Omega A(x, u) \leq \frac{\gamma}{p}\int_\Omega |u^-|^p$$

for all $u \in W^{m,p}(\Omega)$. Choose $\epsilon > 0$ as in Corollaries 3.1 and 3.2. Combining inequalities we see that

$$\mathcal{J}(v) \geq \frac{d}{p}\|v\|_{m,p}^p - K\|v\|_{m,p}$$

for all $v \in V_\epsilon$, so the proof is done. ∎

The estimates above immediately imply the following.

LEMMA 3.4. *Let $\epsilon > 0$ be chosen as in Corollaries* 3.1 *and* 3.2. *Then there is an $R > 0$ such that* $\inf_{u\in V_\epsilon} \mathcal{J}(u) > \sup_{u\in\partial W_R} \mathcal{J}(u)$.

LEMMA 3.5. *Given any $R > 0$ and $\epsilon > 0$, ∂W_R and V_ϵ link.*

Proof. It suffices to consider a continuous $h : [-1, 1] \to W^{m,p}(\Omega)$ such that $h(\pm 1) = \pm R$ and show that $h([-1, 1]) \bigcap V_\epsilon \neq \emptyset$. Note that if $h(s) \equiv 0$ for any s, then we immediately have $h(s) \in V_\epsilon$; so for the remainder of the proof we consider the case where $h(s)$ is nontrivial for all s. Let $t(s) := \frac{1}{|\Omega|}\int_\Omega h(s)$, let $a := \max\{s : -1 < s < 1, t(s) = 0\}$, and let $\varepsilon(s) := (\|h(s) - t(s)\|_{m,p})/t(s)$ for $s \in (a, 1]$. Note that $t(\pm 1) = \pm R$, $t(a) = 0$, and $\varepsilon(1) = 0$. Since $h(a)$ is nontrivial, $\lim_{s\to a^+} \|h(s) - t(s)\|_{m,p} = \|h(a)\|_{m,p} \neq 0$, so $\lim_{s\to a^+} \varepsilon(s) = \infty$. Thus, from the Intermediate Value Theorem, we have that $\varepsilon(s^*) = \epsilon$ for some $s^* \in (a, 1)$. It follows that $h(s^*) \in V_\epsilon$. ∎

Thus, assuming the hypotheses of Theorem 1.1, and letting $X = W^{m,p}(\Omega)$, $Y = V_\epsilon$, $Z = W_R$, and $J = \mathcal{J}$, we have shown that conditions (1) and (2) of Theorem 2.1 hold. Therefore Theorem 1.1 will be proved if we can show that $\mathcal{J}$ satisfies (PS).

4. THE PALAIS–SMALE CONDITION

In this section we prove that if condition (5) is satisfied then $\mathcal{J}$ satisfies (PS). Throughout the section we assume (F-1)–(F-3), (A-1)–(A-2), (g-1)–(g-2), (a-1)–(a-3) and let $\mathcal{J}$ be given by (7). Also, we assume that $\epsilon > 0$ is chosen as in Corollary 3.1.

LEMMA 4.1. *Assume that* $\mathcal{J}$ *satisfies* (5). *If* $\{u_n\} \subset W^{m,p}(\Omega)$ *and* $K > 0$ *such that*

(1) $|\mathcal{J}(u_n)| \le K$ *for all* n, *and*
(2) $\mathcal{J}'(u_n) \to 0$ *in* $(W^{m,p}(\Omega))^*$,

then $\{u_n\}$ *is bounded.*

Proof. We will suppose that $\{u_n\}$ is unbounded and derive a contradiction. Without loss of generality we may assume that $\|u_n\|_{m,p} \to \infty$. Let $v_n := u_n/\|u_n\|_{m,p}$. Without loss of generality we have that $v_n \to v$ in $L^p(\Omega)$, with pointwise convergence a.e., and $v_n \rightharpoonup v$ in $W^{m,p}(\Omega)$.

Our first step towards proving a contradiction is to show that $v \equiv 1/|\Omega|^{1/p}$. Consider

$$J'(u_n) \cdot 1 = \int_\Omega a(x, u_n)|u_n|^{p-2}u_n^- - \int_\Omega g(x, u_n) - H(1).$$

Dividing through by $\|u_n\|_{m,p}^{p-1}$ and using the boundedness of g and H as well as $\mathcal{J}'(u_n) \to 0$ we get

$$0 = \lim_{n\to\infty} \int_\Omega a(x, u_n)\frac{|u_n|^{p-2}u_n^-}{\|u_n\|_{m,p}^{p-1}}.$$

Clearly, $|u_n|^{p-2}u_n^-/\|u_n\|_{m,p}^{p-1} \to |v^-|^{p-1}$, and the integrand above is nonnegative so Fatou's Lemma can be applied to get

$$0 \ge \gamma_1 \int_\Omega |v^-|^p.$$

Hence $v^- \equiv 0$. Now consider

$$\begin{aligned}\mathcal{J}'(u_n)u_n &= \mathcal{Q}(u_n, u_n) + \int_\Omega a(x, u_n)|u_n^-|^p - \int_\Omega g(x, u_n)u_n - H(u_n)\\ &\ge c_0(|u_n|'_{m,p})^p + \int_\Omega a(x, u_n)|u_n^-|^p - \int_\Omega g(x, u_n)u_n - H(u_n),\end{aligned}$$

by (A-2). Divide through by $\|u\|_{m,p}^p$ and let $n \to \infty$ to get

$$0 \ge \lim_{n\to\infty} \frac{(|u_n|'_{m,p})^p}{\|u_n\|_{m,p}^p},$$

where we have used the facts that g is bounded, H is bounded, and $v_n^- \to 0$. Hence $|v|'_{m,p} = 0$ and v is a nonnegative constant function. Since $\lim_{n\to\infty}((|u_n|'_{m,p})^p/\|u_n\|_{m,p}^p) = 0$ it must be that $\lim_{n\to\infty}(\|u_n\|_{L^p}/\|u_n\|_{m,p}) = 1$. Thus $v = \lim_{n\to\infty}(u_n/\|u_n\|_{m,p}) = \lim_{n\to\infty}(u_n/\|u_n\|_{L^p})$, and so $\|v\|_{L^p} = 1$. It follows that $v \equiv 1/|\Omega|^{1/p}$.

Our second step towards deriving a contradiction is to obtain a more precise description of how $v_n \to 1/|\Omega|^{1/p}$. It will help to decompose the elements of the sequence into components in W and V_ϵ. For any $u \in W^{m,p}(\Omega)$ it is clear that there is a unique constant c such that $u - c \in V_\epsilon$. Thus we can write $u_n = c_n + \tilde{u}_n$, where $\tilde{u}_n \in V_\epsilon$. What happens to these components when we divide through by $\|u_n\|_{m,p}$ and let $n \to \infty$? The following arguments will show that $\{\tilde{u}_n\}$ is bounded. If $|c_n|/\|u_n\|_{m,p} \to \infty$, or if some subsequence does, then $u_n/c_n \to 0$ and so $\tilde{u}_n/c_n \to -1$. But V_ϵ is closed and $-1 \notin V_\epsilon$, a contradiction. Thus $c_n/\|u_n\|_{m,p}$ is bounded, and, without loss of generality, converges to some constant c. It follows that $\tilde{u}_n/\|u_n\|_{m,p} \to 1/|\Omega|^{1/p} - c$. Once again, since V_ϵ is closed, and 0 is the only constant in V_ϵ, we must have $c = 1/|\Omega|^{1/p}$. Thus $\tilde{u}_n/\|u_n\|_{m,p} \to 0$. Now consider

$$\mathcal{J}'(u_n)\tilde{u}_n = \mathcal{Q}(u_n, \tilde{u}_n) + \int_\Omega a(x, u_n)|u_n|^{p-2}u_n^-\tilde{u}_n$$
$$- \int_\Omega g(x, u_n)\tilde{u}_n - H(\tilde{u}_n).$$

Clearly,

$$\left|\int_\Omega g(x, u_n)\tilde{u}_n + H \cdot \tilde{u}_n\right| \le K_1\|\tilde{u}\|_{m,p},$$

for some positive constant K_1. Also, since $u_n = c_n + \tilde{u}_n$, where c_n is a positive constant we have that $\tilde{u}_n < u_n < 0$ on $\{x : u_n < 0\}$. Thus

$$\left|\int_\Omega a(x, u_n)|u_n|^{p-2}u_n^-\tilde{u}_n\right| \le \gamma \int_\Omega |\tilde{u}_n^-|^p.$$

Since u_n and $\tilde{u}_n$ differ by a constant we have $\mathcal{Q}(u_n, \tilde{u}_n) = \mathcal{Q}(\tilde{u}_n, \tilde{u}_n)$, and we can use Corollary 3.1 to show that

$$\mathcal{J}'(u_n)\tilde{u}_n \ge d\|\tilde{u}_n\|_{m,p}^p - K_1\|\tilde{u}_n\|,$$

which leads to a contradiction of the fact that $\mathcal{J}'(u_n) \to 0$. Hence $\{\tilde{u_n}\}$ is bounded.

The third step in deriving a contradiction will be to show that condition (5) forces $\mathcal{J}(u_n)$ to be unbounded. This is now possible because our previous estimates have shown that u_n behaves very much like a sequence of positive constants diverging to infinity. We begin by observing that $\int_0^1 \mathcal{Q}(tu_n, u_n)\,dt = \int_0^1 \mathcal{Q}(t\tilde{u}_n, \tilde{u}_n)\,dt$, which we now know is bounded. Similarly, $|\int_\Omega A(x, u_n)| \le \frac{\gamma}{p}\int_\Omega |\tilde{u}_n^-|^p$, which is bounded. Thus $\mathcal{J}(u_n)$ is unbounded if and only if its last two terms are unbounded. Notice that for a.e. x we have $|G(x, u_n(x)) - G(x, c_n)| \le M|\tilde{u}_n|$, so

$$\int_\Omega G(x, u_n) + Hu_n \ge \int_\Omega G(x, c_n) + H(c_n) - K_1,$$

for some $K_1 > 0$. Since $c_n \to \infty$ we can now apply condition (5) to conclude that $\lim_{n\to\infty} \int_\Omega G(x, u_n) + H(u_n) = \infty$, and thus, since all other terms are bounded, $\lim_{n\to\infty} \mathcal{J}(u_n) = -\infty$. We have arrived at the desired contradiction, so the proof is done. ∎

LEMMA 4.2. *If $\mathcal{J}$ satisfies* (5), *then $\mathcal{J}$ satisfies* (*PS*).

Proof. This is a direct consequence of Lemma 4.1. See Lemma 4 in [15] for details. ∎

The proof of Theorem 1.1 is now finished.

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