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Proof theory of higher-order equations: conservativity, normal forms and term rewriting

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Abstract

We introduce a necessary and sufficient condition for the ω-extensionality rule of higher-order equational logic to be conservative over first-order many-sorted equational logic for ground first-order equations. This gives a precise condition under which computation in the higher-order initial model by term rewriting is possible. The condition is then generalised to characterise a normal form for higher-order equational proofs in which extensionality inferences occur only as the final proof inferences. The main result is based on a notion of observational equivalence between higher-order elements induced by a topology of finite information on such elements. Applied to extensional higher-order algebras with countable first-order carrier sets, the finite information topology is metric and second countable in every type.
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1. Introduction

Higher-order equations have applications in diverse areas of computer science, such as specification languages, logics of programs, and declarative programming languages. Higher-order logic, as a branch of mathematical logic, can be traced back as least as far as Russel [24], a broad and accessible survey is Feferman [7].

The finite-type system of Church [2], which allows the construction of *function types* $(\sigma \to \tau)$ and *product types* $(\sigma \times \tau)$ from a set of basic first-order types (actually a simplified version of Russell's-type system) forms the kernel of many higher-order-type systems currently found in the literature. Syntactically, higher-order equational logic can be viewed as the equational fragment of many-sorted first-order logic with equality where the sorts are type expressions based on Church's

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finite-type system (starting from a collection of basic types such as natural numbers or Booleans). The semantics of higher-order equational logic varies according to how the finite types are interpreted, and also which comprehension (set existence) principles are assumed. For example: (i) models may or may not be *extensional* (though extensionality seems a rather natural requirement); (ii) elements of some function type may be constrained to be *continuous* with respect to some topology. We will consider extensionality and continuity in some detail in this paper. Function definition by lambda abstraction, as in the typed lambda calculus (see e.g. [11]), represents a well-known comprehension principle though much stronger principles are known.

Most semantical models are encompassed inside the generalised semantics for higher-order logic introduced in [10]. Henkin's semantics follows the classical methods of model theory. Today, much research into the semantics of higher-order types also takes place within category theory (see for example [15]). A model theory specifically for higher-order equations and a theory of higher-order algebra based on finite types, following the classical Henkin approach, were considered in [17]. As with first-order logic and first-order universal algebra, a much richer model theory emerges for higher-order structures if one assumes that the only relation on elements is equality. (This is the usual motivation for universal algebra as a separate branch of model theory.)

Deductive calculi for higher-order equations extend the first-order many-sorted equational calculus with additional *extensionality rules*, associated with the function space and product space type constructors. Such rules can either be finitary or infinitary in form, and have different completeness properties with respect to classes of higher-order algebras.

In [17,22] it was shown that every set E of higher-order equations admits an extensional model I which is initial in the class of all minimal extensional models of E. This model is termed the *higher-order initial model* of E. Thus, one can use higher-order equations as an algebraic specification language in an entirely analogous way to first-order equations. (See for example [6].)

Syntactically, higher-order signatures are natural for formalising diverse computing phenomena such as dataflow and stream processing, hardware computations, and programming constructs. Semantically (quantifier free), higher-order equational specifications under higher-order initial semantics have all the expressive power of full first-order logic. (See [14,19].) The syntactic and semantic expressiveness of second-order equational logic for hardware specification and verification is explored in [20], where some of the results of this paper are applied in a more practical context.

Now considering the higher-order initial model I of a set E of higher-order equations, one may pose the following.

Computation Problem. *When can we compute in I using the equations of E oriented as rewrite rules?*

More precisely, given an equation $t = t'$, for ground (i.e. variable free) first-order terms t and t', which is true in I, when can we formally prove $t = t'$ by equational reasoning (and therefore term rewriting) alone, i.e. without using any of the extensionality rules?

This problem arises naturally in various ways, for example in connection with the design of efficient algorithms for executing a higher-order equational specification.

The computation problem can be rephrased as a special case of more general proof theoretic questions about the conservativity properties of the extensionality rules over the underlying first-order many-sorted equational calculus. For both the finitary and infinitary extensionality rules, two conservativity properties are of obvious interest, namely conservativity over equational logic for:

(i) ground first-order equations, and
(ii) all first-order equations (i.e. allowing variables in terms).

The latter conservativity property is very strong, and leads to a normal form for higher-order equational proofs (which we term *eval normal form*) in which each extensionality rule is used at most once, and among the final proof inferences only. Since the higher-order initial model is constructed as a quotient term model using the infinitary ω-extensionality rule, our computation problem above is equivalent to the following proof theoretic problem.

Conservativity Problem. *When is the infinitary ω-extensionality rule conservative over many-sorted first-order equational logic for ground first-order equations?*

The various conservativity properties of the extensionality rules of higher-order equational logic have considerable computational significance. On the one hand, these rules seem to lack any efficient implementation in a computational logic. On the other hand, when the extensionality rules are conservative over equational logic for some class of equations, then such equations can be derived (at least in principle) using the relatively efficient and well understood computational methods of term rewriting.

The main theoretical problem turns out to be characterising the conservativity of the infinitary ω-extensionality rule. Not only does conservativity of this rule relate the higher-order initial model with term rewriting computation, there is also a direct relationship between conservativity of the infinitary and finitary extensionality rules. Therefore, we can apply our main result characterising conservativity for the infinitary rule to characterise conservativity of the finitary extensionality rule as well, and at the same time characterise the existence of eval normal form proofs.

We will show that the conservativity of the infinitary extensionality rule depends upon finitistic properties of the higher-order operators themselves, in particular the property that higher-order operators use just a finite amount of information about their higher-order arguments to determine their values. More precisely, we formulate a notion of *observational equivalence* for elements of higher-order-type based on a topology of *finite information* for higher-order types. This topology appears to be new in the literature. It can be applied to both extensional and non-extensional models of a higher-order signature Σ. Two elements of a Σ algebra A are observationally equivalent if they belong to precisely the same open sets in this topology. Our main result is:

Conservativity Theorem 4.8. *Let Σ be a higher-order signature which contains the homeomorphism operators and let E be a higher-order equational theory which contains the homeomorphism axioms* **Hom**. *Then infinitary higher-order equational logic is conservative over equational logic on E for ground first-order equations if, and only if, observational equivalence $\equiv^{\mathrm{obs}}$ is a congruence on the (first-order) initial model $I(\Sigma, E)$.*

and this theorem is applied to the finitary extensionality rule to yield the following:

Normal Form Theorem 4.12. *Let Σ be a higher-order signature which contains the homeomorphism operators and let E be a higher-order equational theory which contains the homeomorphism axioms* **Hom**. *The following are equivalent*:

(i) *finitary higher-order equational logic is conservative on E for first-order equations*;
(ii) *for every equation $e \in Eqn(\Sigma, X)$ (of any order), if $E \vdash_{\text{eval}} e$ then there is a proof of e which is in eval normal form*;
(iii) *observational equivalence, $\equiv^{\text{obs}}$, is a congruence on the free algebra $T_E(\Sigma, X)$.*

In fact, *continuity* of the initial algebra $I(\Sigma, E)$ and the free algebra $T_E(\Sigma, X)$ with respect to the finite information topology (i.e. continuity of all operations of $I(\Sigma, E)$ and $T_E(\Sigma, X)$) are sufficient (but not necessary) conditions to ensure that observational equivalence is a congruence in both cases. At first sight, it may appear odd that the *first-order* initial model plays a role here, but recall that this is the unique minimal model where a ground equation is true precisely when it is formally provable in the first-order many-sorted equational calculus.

The topology of finite information is well known for first- and second-order types. For first-order types it is (trivially) the discrete topology, while for second-order types it is the product or Tychonoff topology on the function space given the discrete topology on the domain and codomain sets. However, the product topology construction cannot be iterated for higher-order types in such a way that both function currying and function evaluation are continuous. The finite information topology introduced here has strong separation and countability properties. On extensional models with countable first-order carrier sets, it is metric and second countable in every type. The uniformity of these properties in all types imply that the finite information topology is not homeomorphic, for every type, with the well-known Kleene–Kreisel topology on total functionals of finite type (see for example [23]). For example, the Kleene–Kreisel topology is not second countable above second-order types. In the finite information topology, evaluation and currying of functionals are continuous in all types. The topology is constructed using particular types which we term *elementary* or *hereditarily Horn* (following the types as propositions analogy). The elementary types are inductively defined types of the form β, $(\sigma \times \tau)$, and $(\alpha \rightarrow \beta)$ where β is a basic (atomic) type and σ and τ are elementary or hereditarily Horn types. In an extensional model, every space is homeomorphic to a space of elementary type.

The structure of this paper is as follows. In Section 2, we review the finitary and infinitary proof systems for higher-order equations and make precise the conservativity properties of interest. We consider how conservativity leads to the eval normal form for higher-order equational proofs in which extensionality inferences are the final inferences only. In Section 3, we introduce the finite information topology and establish its separation and countability properties. We prove the continuity of evaluation and currying in this topology. In Section 4, we characterise necessary and sufficient conditions for the extensionality rules to be conservative over equational logic for both ground and arbitrary first-order equations by means of the finite information topology. In order to reduce the length of the paper, many of the more tedious but elementary proofs have been left to the reader as an exercise.

The main prerequisites of this paper are a familiarity with first-order many-sorted equational logic (see for example [27] or [21]), term rewriting (see for example [13] or [4]) and point set topology (see for example [5] or [12]). While some familiarity with higher-order universal algebra is useful (a suitable introduction is Meinke [17]), the paper is largely self contained on this subject.

2. Higher-order equational logic

We review some of the fundamental definitions and results of higher-order equational logic, including the proof systems introduced in [17].

By a set S of *sorts* we mean any non-empty set. As usual, S^* denotes the set of all words in the free monoid generated by S. The empty word is denoted by λ and $S^+ = S^* - \{\lambda\}$ denotes the set of all non-empty words over S. An *S-sorted signature* Σ is an $S^* \times S$ indexed family of disjoint sets $\Sigma = \langle \Sigma_{w,s} \mid w \in S^*,\ s \in S \rangle$. For the empty word λ and each sort $s \in S$, each element $c \in \Sigma_{\lambda,s}$ is a constant symbol of sort s. For each non-empty word $w = s(1) \ldots s(n) \in S^+$ and each sort $s \in S$, each element $f \in \Sigma_{w,s}$ is a function symbol of domain type w, codomain type s and arity n. Let S be any sort set, Σ be any S-sorted signature, and let $X = \langle X_s \mid s \in S \rangle$ be any S-indexed family of sets of variable symbols. (We normally assume that the sets $\Sigma_{\lambda,s}$ and X_s are disjoint for each $s \in S$.) We let $T(\Sigma, X)_s$ denote the set of all *terms* over Σ and X of sort $s \in S$.

Let Σ be an S-sorted signature. An *S-sorted Σ algebra* is a pair (A, Σ^A), consisting of an S-indexed family $A = \langle A_s \mid s \in S \rangle$ of sets termed the *carrier sets* of A, and an $S^* \times S$ indexed family.

$$\Sigma^A = \langle \Sigma^A_{w,s} \mid w \in S^*,\ s \in S \rangle$$

of sets of constants and algebraic operations. For each sort $s \in S$, $\Sigma^A_{\lambda,s} = \langle c_A \mid c \in \Sigma_{\lambda,s} \rangle$, where $c_A \in A_s$ is a constant that interprets c in A_s. For each $w = s(1) \ldots s(n) \in S^+$ and each sort $s \in S$, $\Sigma^A_{w,s} = \langle f_A \mid f \in \Sigma_{w,s} \rangle$, where $f_A : A^w \to A_s$ is a function with domain $A^w = A_{s(1)} \times \cdots \times A_{s(n)}$ and codomain A_s which interprets f over A. As usual, we let A denote both a Σ algebra and its S-indexed family of carrier sets. We let $Alg(\Sigma)$ denote the class of all S-sorted Σ algebras. We let $T(\Sigma, X)$ denote the *free term algebra* on the family X of sets of generators, and $T(\Sigma) = T(\Sigma, \emptyset)$ denotes the *absolutely free* or *ground term algebra* on the S-indexed family $\emptyset$ of empty sets of generators. A *homomorphism* ϕ between Σ algebras A and B is an S-indexed family of mappings $\phi = \langle \phi_s : A_s \to B_s \mid s \in S \rangle$ between the carrier sets of A and B that preserves all constants and operations named in Σ. Thus $\phi_s(c_A) = c_B$ for each $s \in S$ and constant symbol $c \in \Sigma_{\lambda,s}$, and also

$$\varphi_s(f_A(a_1, \ldots, a_n)) = f_B(\phi_{s(1)}(a_1), \ldots, \phi_{s(n)}(a_n))$$

for each $w = s(1) \ldots s(n) \in S^+$, each sort $s \in S$ and each operation symbol $f \in \Sigma_{w,s}$.

Recall that $T(\Sigma)$ is *initial* in $Alg(\Sigma)$, i.e. there exists a unique homomorphism from $T(\Sigma)$ to each algebra $A \in Alg(\Sigma)$. A Σ algebra A is *minimal* if, and only if, A has no proper subalgebra, for example $T(\Sigma)$ is minimal. If A is a Σ algebra and $\equiv$ is a congruence on A and $a \in A_s$ is an element of some carrier set of A then we let $[a]$ denote the equivalence class of a in the quotient algebra $A/\equiv$.

Higher-order signatures and algebras are defined over the following system of types, often known as the system of *finite* or *simple types* [2].

Definition 2.1. By a *type basis* B we mean any non-empty set. The (*finite*) *type hierarchy* $H(B)$ generated by a type basis B is the set $H(B) = \bigcup_{n\in\omega} H_n(B)$ defined inductively by

$$H_0(B) = B$$

and

$$H_{n+1}(B) = H_n(B) \bigcup \{(\sigma \times \tau), (\sigma \rightarrow \tau) \mid \sigma, \tau \in H_n(B)\}.$$

Each element $\sigma \in B$ is termed a *basic type*; each element $(\sigma \times \tau) \in H(B)$ is termed a *product type* and each element $(\sigma \rightarrow \tau) \in H(B)$ is termed a *function type*.

We can assign an *order* to each type $\sigma \in H(B)$ as follows. Each basic type $\sigma \in B$ has order 1. If $\sigma, \tau \in H(B)$ have order m and n respectively then $(\sigma \times \tau)$ has order $sup\{m, n\}$ and $(\sigma \rightarrow \tau)$ has order $sup\{m+1, n\}$. A *type structure* S over a type basis B is a subset $S \subseteq H(B)$, which is closed under subtypes in the sense that for any $\sigma, \tau \in H(B)$, if $(\sigma \rightarrow \tau) \in S$ or $(\sigma \times \tau) \in S$ then both $\sigma \in S$ and $\tau \in S$.

A higher-order signature is simply an S-sorted signature Σ in which S is a type structure and Σ contains distinguished operation symbols for the product and function types of S as follows.

Definition 2.2. Let S be a type structure over a type basis B. An *S-typed signature Σ* is an S-sorted signature such that for each product type $(\sigma \times \tau) \in S$ we have *left* and *right projection operation symbols*

$$proj^{(\sigma\times\tau),\sigma} \in \Sigma_{(\sigma\times\tau),\sigma}, \quad proj^{(\sigma\times\tau),\tau} \in \Sigma_{(\sigma\times\tau),\tau},$$

also for each function type $(\sigma \rightarrow \tau) \in S$ we have a binary *evaluation operation symbol*

$$eval^{(\sigma\rightarrow\tau)} \in \Sigma_{(\sigma\rightarrow\tau)\sigma,\tau}.$$

When the types σ and τ are clear, we let $proj^1$ and $proj^2$ denote $proj^{(\sigma\times\tau),\sigma}$ and $proj^{(\sigma\times\tau),\tau}$, and we let *eval* denote $eval^{(\sigma\rightarrow\tau)}$. Furthermore, we will often write terms of the form $eval(t, t')$ using the meta-notation $t(t')$ (applicative form) thereby omitting the evaluation operation symbol which can be inferred from the types of t and t'.

We can now introduce the intended interpretations of an S-typed signature Σ.

Definition 2.3. Let S be a type structure over a type basis B. Let Σ be an S-typed signature and A be an S-sorted Σ algebra. We say that A is an *S-typed Σ algebra* if, and only if, for each product type $(\sigma \times \tau) \in S$ we have $A_{(\sigma\times\tau)} \subseteq A_\sigma \times A_\tau$, and the mappings

$$proj_A^{(\sigma\times\tau),\sigma} : A_{(\sigma\times\tau)} \rightarrow A_\sigma, \qquad proj_A^{(\sigma\times\tau),\tau} : A_{(\sigma\times\tau)} \rightarrow A_\tau$$

are the *left* and *right projection mappings* defined on $A_{(\sigma\times\tau)}$ by

$$proj_A^{(\sigma\times\tau),\sigma}((a_1, a_2)) = a_1, \quad proj_A^{(\sigma\times\tau),\tau}((a_1, a_2)) = a_2,$$

for any pair $(a_1, a_2) \in A_{(\sigma\times\tau)}$. Furthermore, for each function type $(\sigma \rightarrow \tau) \in S$ we have $A_{(\sigma\rightarrow\tau)} \subseteq [A_\sigma \rightarrow A_\tau]$, and the operation $eval_A^{(\sigma\rightarrow\tau)} : A_{(\sigma\rightarrow\tau)} \times A_\sigma \rightarrow A_\tau$ is the *evaluation mapping* on the function space $A_{(\sigma\to\tau)}$ defined by

$$eval_A^{(\sigma\to\tau)}(a,b) = a(b),$$

for each $a \in A_{(\sigma\to\tau)}$ and $b \in A_\sigma$.

Notice that in a higher-order signature Σ we can define a symbol f denoting some function of domain type σ and codomain type τ either as a constant symbol $f \in \Sigma_{\lambda,(\sigma\to\tau)}$ or as a function symbol $f \in \Sigma_{\sigma,\tau}$. These alternative forms are similar but not entirely equivalent. For example, the denotation $f_A \in A_{(\sigma\to\tau)}$ of the constant symbol $f \in \Sigma_{\lambda,(\sigma\to\tau)}$ is an element of a carrier set in every Σ algebra A, but may not be a function from A_σ to A_τ, nor even extensional. If every element of $A_{(\sigma\to\tau)}$ is continuous (with respect to some topology), then such an f_A must also be continuous. On the other hand the denotation $f_A : A_\sigma \to A_\tau$ of the function symbol $f \in \Sigma_{\sigma,\tau}$ need not be an element of any carrier set of A. Therefore, it may not be continuous even if every element of $A_{(\sigma\to\tau)}$ is continuous. However, it must of necessity be a function from A_σ to A_τ, and therefore extensional. On a more practical level, defining such a symbol f as a function symbol rather than as a constant symbol may reduce the number and complexity of the carrier sets used for semantical models. This distinction plays a role for our results, since we address proof theoretic conservativity with respect to proof rules for evaluation, which is a function symbol in our formulation of higher-order equational logic.

In the remainder of this section, unless stated otherwise, we let S denote a fixed, but arbitrarily chosen type structure over a type basis B and we let Σ denote a fixed, arbitrarily chosen S-typed signature. We let $X = \langle X_\tau \mid \tau \in S \rangle$ denote an S-indexed family of disjoint, infinite sets X_τ of variable symbols of type τ.

Within the class $Alg(\Sigma)$ of all algebras of signature Σ, it is important to distinguish between those algebras which are *extensional*, and those which are *non-extensional*. We say that a Σ algebra A is *extensional* if, and only if, A satisfies the Σ sentences:

$$\forall x \forall y \quad (proj^1(x) = proj^1(y) \ \&\ proj^2(x) = proj^2(y) \Rightarrow x = y) \tag{2.1}$$

for each product type $(\sigma \times \tau) \in S$, and

$$\forall x \forall y \quad (\forall z (eval(x,z) = eval(y,z)) \Rightarrow x = y) \tag{2.2}$$

for each function type $(\sigma \to \tau) \in S$. We let **Ext** denote the set of all extensionality sentences of the forms (2.1) and (2.2) above, and we let $Alg_{\mathbf{Ext}}(\Sigma)$ denote the class of all extensional Σ algebras.

Clearly, every S-typed Σ algebra is extensional. The significance of the distinction between extensional and non-extensional algebras can be summarised by the following

Collapsing Theorem 2.4 (Shepherdson, Mostowski). *Let A be an S-sorted Σ algebra. Then A is isomorphic to an S-typed Σ algebra if, and only if, A is extensional.*

Proof. See [17]. $\square$

Thus we study the intended models of Σ, up to isomorphism, as the extensional models of Σ.

By a *higher-order equation* over Σ and X *of type* $\tau \in S$ we mean a formula of the form $t = t'$ where $t, t' \in T(\Sigma, X)_\tau$ are terms of type τ. If τ is an nth-order type then we may say that $t = t'$ is an *nth-order equation*. We let $Eqn(\Sigma, X)_\tau$ denote the set of all equations over Σ and X of type τ and $Eqn(\Sigma, X) = \bigcup_{\tau \in S} Eqn(\Sigma, X)_\tau$. An equation $t = t' \in Eqn(\Sigma, X)_\tau$ is said to be *ground* if, and only if, the terms t and t' are *ground terms*, i.e. $t, t' \in T(\Sigma)_\tau$. A *higher-order equational theory* over Σ and X is a set $E \subseteq Eqn(\Sigma, X)$ of higher-order equations. We let $Alg(\Sigma, E)$ denote the class of all Σ algebras which are models of E,

$$Alg(\Sigma, E) = \{A \in Alg(\Sigma) : A \models E\},$$

and we let $Alg_{\mathbf{Ext}}(\Sigma, E)$ denote the class of all extensional Σ algebras which are models of E,

$$Alg_{\mathbf{Ext}}(\Sigma, E) = \{A \in Alg_{\mathbf{Ext}}(\Sigma) : A \models E\}.$$

We can construct a sound and complete calculus for any higher-order equational theory E with respect to the class $Alg_{\mathbf{Ext}}(\Sigma, E)$ of all extensional models of E using just finitary deduction rules. For this we add to the deduction rules of many-sorted equational logic additional rules which incorporate the extensionality axiom schemas (2.1) and (2.2) above, as follows.

Definition 2.5. The *finitary deduction rules of higher-order equational logic* are the following:

(i) For any type $\tau \in S$ and any term $t \in T(\Sigma, X)_\tau$,

$$\frac{}{t = t}$$

is a *reflexivity rule*.

(ii) For any type $\tau \in S$ and any terms $t_0, t_1 \in T(\Sigma, X)_\tau$,

$$\frac{t_0 = t_1}{t_1 = t_0}$$

is a *symmetry rule*.

(iii) For any type $\tau \in S$ and any terms $t_0, t_1, t_2 \in T(\Sigma, X)_\tau$,

$$\frac{t_0 = t_1, \quad t_1 = t_2}{t_0 = t_2}$$

is a *transitivity rule*.

(iv) For each type $\tau \in S$, any terms $t, t' \in T(\Sigma, X)_\sigma$, any type $\sigma \in S$, any variable symbol $x \in X_\sigma$ and any terms $t_0, t_1 \in T(\Sigma, X)_\sigma$,

$$\frac{t = t', \quad t_0 = t_1}{t[x/t_0] = t'[x/t_1]}$$

is a *substitution rule*. (As usual, $t_i[x/t_j]$ denotes the result of substituting the variable x by the term t_j uniformly in t_i when x and t_j have the same sort.)

(v) For each product type $(\sigma \times \tau) \in S$ and any terms $t_0, t_1 \in T(\Sigma, X)_{(\sigma\times\tau)}$,

$$\frac{proj^1(t_0) = proj^1(t_1), \; proj^2(t_0) = proj^2(t_1)}{t_0 = t_1}$$

is a *projection rule*.

(vi) For each function type $(\sigma \to \tau) \in S$, any terms $t_0, t_1 \in T(\Sigma, X)_{(\sigma \to \tau)}$ and any variable symbol $x \in X_\sigma$ not occurring in t_0 or t_1,

$$\frac{eval(t_0, x) = eval(t_1, x)}{t_0 = t_1}$$

is a (*finitary*) *extensionality rule*.

Note that in each of the above deduction rules both the conclusion and each of the premises is an equation. In particular, the calculus is quantifier free.

By a *proof* of an equation $e \in Eqn(\Sigma, X)$ from a set $E \subseteq Eqn(\Sigma, X)$ of equations *using the finitary deduction rules of higher-order equational logic*, we mean a finitely branching rooted tree P of finite depth with each node n labelled by an equation $e_n \in Eqn(\Sigma, X)$ such that the root of P is labelled by e, and for each node n in P, either n has no antecedent nodes and $e_n \in E$ is an axiom, or n has exactly k antecedent nodes $m_1, \ldots, m_k$ for $0 \leqslant k \leqslant 2$ and

$$\frac{e_{m_1}, \ldots, e_{m_k}}{e_n}$$

is a finitary deduction rule of higher-order equational logic. (In the sequel, we also consider infinitary proofs using infinitary deduction rules.)

The finitary deduction rules of higher-order equational logic induce an *inference relation*, denoted by $\vdash_{\mathrm{eval}}$, between equational theories $E \subseteq Eqn(\Sigma, X)$ and equations $e \in Eqn(\Sigma, X)$, defined by $E \vdash_{\mathrm{eval}} e$ if, and only if, there exists a proof of e from E using the finitary deduction rules of higher-order equational logic alone. We shall reserve the symbol $\vdash$ to denote the inference relation induced by the rules of first-order many-sorted equational logic. Thus $E \vdash e$ if, and only if, there exists a proof of e from E (in the above sense) using only rules 2.5(i)–(iv).

The completeness theorem for first-order single-sorted equational logic is due to Birkhoff [1]. The problems of the many-sorted case, i.e. the possibility of an empty carrier set for some sort, and the consequent unsoundness of rules 2.5(i)–(iv) above, were considered in [9]. We can avoid the soundness problems of the many-sorted case here by the simplifying assumption on Σ that we can form a ground Σ term of each type $\tau \in S$. In this case we say that Σ is *non-void*. The completeness theorem for higher-order equational logic is then similar in style and proof to the first-order case.

Completeness Theorem 2.6. *Let $E \subseteq Eqn(\Sigma, X)$ be any higher-order equational theory. If Σ is non-void then for any higher-order equation $e \in Eqn(\Sigma, X)$,*

$$E \vdash_{\mathrm{eval}} e \;\Leftrightarrow\; Alg_{\mathrm{Ext}}(\Sigma, E) \models e.$$

Proof. See [17]. $\square$

Recall that for any equational theory E, the class $Alg(\Sigma, E)$ contains a free algebra $T_E(\Sigma, X)$ generated by an S-indexed family X of sets of generators. In particular, $Alg(\Sigma, E)$ contains an initial or absolutely freely generated algebra $I(\Sigma, E) = T_E(\Sigma, \emptyset)$. Now in general, for a higher-order equational theory E, the class $Alg_{\mathrm{Ext}}(\Sigma, E)$ of all extensional models will not contain an initial algebra. (Consider that this class need not be closed under subalgebras.) However, it is obvious that $Alg_{\mathrm{Ext}}(\Sigma, E)$ contains at least one minimal algebra, namely the unit Σ algebra.

Furthermore, using purely model theoretic constructions, it is possible to show that the non-empty class $Min_{\rm Ext}(\Sigma, E) \subseteq Alg_{\rm Ext}(\Sigma, E)$ of all minimal extensional models of E contains an initial algebra $I_{\rm Ext}(\Sigma, E)$. By initiality, there exists a unique homomorphism from $I_{\rm Ext}(\Sigma, E)$ to each algebra $A \in Min_{\rm Ext}(\Sigma, E)$, and $I_{\rm Ext}(\Sigma, E)$ is unique up to isomorphism. This algebra is termed the *higher-order initial model* of E. For computer science, it provides a suitable minimal model semantics for E viewed as an equational specification of some computational system. Examples of the use of higher-order equations as system specifications may be found in [18,20,25].

The higher-order initial model $I_{\rm Ext}(\Sigma, E)$ cannot usually be constructed as the quotient of the term algebra $T(\Sigma)$ factored by the deductive closure of E using the finitary calculus of Definition 2.5, the recursion theoretic complexity of the model may be too great for this construction to be possible. (See [14,19], which compare the complexity of first and higher-order initial models.) Instead, the following infinitary version of the extensionality rule 2.5(vi) is required to make a quotient term model construction of $I_{\rm Ext}(\Sigma, E)$.

Definition 2.7. For each function type $(\sigma \to \tau) \in S$ and any terms $t_0, t_1 \in T(\Sigma, X)_{(\sigma\to\tau)}$,

$$\frac{\langle\, eval(t_0, t) = eval(t_1, t) \mid t \in T(\Sigma)_\sigma \,\rangle}{t_0 = t_1}$$

is an (*infinitary*) *ω-extensionality rule*. If $\tau \in B$ is a basic type then this rule is also termed a *basic ω-extensionality rule*.

By a *proof* of an equation $e \in Eqn(\Sigma, X)$ from a set $E \subseteq Eqn(\Sigma, X)$ of equations *using the infinitary rules of higher-order equational logic* we mean the obvious generalisation of a finitary proof P allowing the use of rule schemes 2.5(i)–(v) together with infinite branching and the use of the ω-extensionality rule for each function type $(\sigma \to \tau) \in S$ instead of rule scheme 2.5(vi). For use in Section 4, we define here the *degree* $deg(P) \in Ord$ of an infinitary higher-order equational proof P to be the ordinal depth of nesting of ω-extensionality inferences in P. Thus $deg(P) = 0$ for any proof P consisting of a single equational axiom $e \in E$. If the final inference in P uses one of the rule schemes 2.5(i)–(v) applied to subproofs $P_1, \ldots, P_i$ (for $i = 0, 1, 2$) then $deg(P) = sup\{deg(P_1), \ldots, deg(P_i)\}$. If the final inference in P uses an ω-extensionality rule applied to a family $\langle\, P(t_0) \mid t_0 \in T(\Sigma)_\tau \,\rangle$ of subproofs, then

$$deg(P) = 1 + sup\{deg(P(t_0)) \mid t_0 \in T(\Sigma)_\tau\}.$$

We define the *inference relation* $\vdash_\omega$ between higher-order equational theories $E \subseteq Eqn(\Sigma, X)$ and higher-order equations $e \in Eqn(\Sigma, X)$ by $E \vdash_\omega e$ if, and only if, there exists a proof of e from E using the infinitary rules of higher-order equational logic. If $E \vdash_\omega e$ and there exists a proof P of e of degree $\alpha \in Ord$, then we may write $E \vdash_{\omega,\alpha} e$.

Define the S-indexed family $\equiv^{E,\omega} = \langle \equiv^{E,\omega}_\tau \mid \tau \in S \rangle$ of binary relations $\equiv^{E,\omega}_\tau$ on terms in $T(\Sigma)_\tau$ by

$$t \equiv^{E,\omega}_\tau t' \;\Leftrightarrow\; E \vdash_\omega t = t',$$

for each type $\tau \in S$ and any terms $t, t' \in T(\Sigma)_\tau$. Clearly, rules 2.5(i)–(iv) ensure that $\equiv^{E,\omega}$ is a congruence on the ground term algebra $T(\Sigma)$. Then $\equiv^{E,\omega}$ gives the following quotient term model construction.

Lemma 2.8. *Let $E \subseteq Eqn(\Sigma, X)$ be any higher-order equational theory. The quotient term algebra $T(\Sigma)/\equiv^{E,\omega}$ is initial in $Min_{\mathrm{Ext}}(\Sigma, E)$.*

Proof. See [17]. □

In the sequel, we let $I_{\mathrm{Ext}}(\Sigma, E)$ denote the quotient term algebra $T(\Sigma)/\equiv^{E,\omega}$. The higher-order initial model can be used to obtain the following completeness result for infinitary higher-order equational logic.

Completeness Theorem 2.9. *Let $E \subseteq Eqn(\Sigma, X)$ be any higher-order equational theory. If Σ is non-void then for any ground equation $e \in Eqn(\Sigma, X)$,*

$$E \vdash_{\omega} e \;\Leftrightarrow\; Min_{\mathrm{Ext}}(\Sigma, E) \models e.$$

Proof. Exercise. □

Note that Theorem 2.9 does not hold if variables are allowed in equations. To obtain completeness in this case the full ω-rule of equational logic is required. (See for example [21].)

Having presented the finitary and infinitary calculi for higher-order equations, and their completeness properties, we can now make precise the conservativity properties of interest in this paper.

Definition 2.10. Let $E \subseteq Eqn(\Sigma, X)$ be a higher-order equational theory and let $K \subseteq Eqn(\Sigma, X)$ be any class of equations.

(i) We say that finitary higher-order equational logic is *conservative* (*over equational logic*) *on E for K equations* if, and only if, for every equation $e \in K$,

$$E \vdash_{\mathrm{eval}} e \;\Rightarrow\; E \vdash e.$$

(ii) We say that infinitary higher-order equational logic is *conservative* (*over equational logic*) *on E for K equations* if, and only if, for every equation $e \in K$,

$$E \vdash_{\omega} e \;\Rightarrow\; E \vdash e.$$

We are interested in conservativity in the cases where K is:

(i) the class of all ground first-order Σ equations, and
(ii) the class of all first-order Σ equations.

Obviously, if $E \vdash_{\mathrm{eval}} e$ then $E \vdash_{\omega} e$, although the converse need not hold. Thus if infinitary higher-order equational logic is conservative on E for K equations then finitary higher-order equational logic is also conservative on E for K equations, but the converse need not hold. We are primarily interested in the conservativity of infinitary higher-order equational logic for classes of equations. This property is intimately connected with term rewriting computation and equational theorem proving in the higher-order initial model by virtue of the following proposition.

Proposition 2.11. *Let $E \subseteq Eqn(\Sigma, X)$ be any higher-order equational theory and suppose that Σ is non-void. Then the following are equivalent*:

(i) *infinitary higher-order equational logic is conservative on E for ground first-order equations*,
(ii) *for any ground first-order equation $e \in Eqn(\Sigma, X)$*,

$$I_{\mathrm{Ext}}(\Sigma, E) \models e \;\Leftrightarrow\; E \models e.$$

Proof. Immediate from Lemma 2.8 and Definition 2.10. $\square$

Let us consider examples of higher-order equational theories for which infinitary higher-order equational logic is conservative over equational logic for various types of equations.

Example 2.12. (i.a) Let Σ be any S-typed signature over a type basis B, and let A be any minimal extensional Σ algebra. Let $Eqn_A(\Sigma)^1$ be the *ground first-order equational theory* of A,

$$Eqn_A(\Sigma)^1 = \{t = t' \mid \tau \in B \text{ and } t, t' \in T(\Sigma)_\tau \text{ and } A \models t = t'\}.$$

Then clearly infinitary higher-order equational logic is conservative on $Eqn_A(\Sigma)^1$ for ground first-order equations. In fact, by induction on the complexity of types, it is easily established that

$$Eqn_A(\Sigma)^1 \vdash_\omega t = t' \;\Leftrightarrow\; A \models t = t'$$

for every ground equation $t = t'$, and thus $A \cong I_{\mathrm{Ext}}(\Sigma, Eqn_A(\Sigma)^1)$.

(i.b) Similarly, if $Eqn_A(\Sigma, X)^1$ is the full first-order equational theory of A (where X_τ is a countably infinite set of variables for each $\tau \in S$) then infinitary higher-order equational logic is conservative on $Eqn_A(\Sigma, X)^1$ for all first-order equations. However, conservativity may not hold for second, or higher-order equations.

(ii) Consider the second-order-type structure $S = \{nat, (nat \to nat)\}$. Define the S-typed signature Σ^1, where

$$\Sigma^1_{\lambda,nat} = \{0\}, \quad \Sigma^1_{nat,nat} = \{succ\},$$

$$\Sigma^1_{\lambda,(nat\to nat)} = \{\bar{0}, \overline{zero}\}, \quad \Sigma_{(nat\to nat)nat,nat} = \{eval\},$$

$$\Sigma^1_{(nat\to nat),(nat\to nat)} = \{f\}.$$

Define the equational theory E_1 to be the set of equations

$$\bar{0}(x) = 0, \tag{2.3}$$

$$\bar{0}(x) = \overline{zero}(x), \tag{2.4}$$

$$f(\bar{0})(x) = 0. \tag{2.5}$$

Then $E_1 \vdash_\omega f\,(\overline{zero})(0) = 0$, but $E_1 \nvdash f(\overline{zero})(0) = 0$ since $I(\Sigma^1, E_1) \nvDash f(\overline{zero})(0) = 0$. So infinitary higher-order equational logic is not conservative on E_1 for ground first-order equations.

(iii) Given S as in (ii) above, let Σ^2 be the S-typed signature obtained by deleting the operation symbol f from Σ^1 in (ii) above. Let E_2 be the equational theory obtained by adding to Eq. (2.3) in (ii) above the recursion equations

$$\overline{zero}(0) = 0, \tag{2.6}$$

$$\overline{zero}(succ(x)) = \overline{zero}(x). \tag{2.7}$$

Then $E_2 \vdash_\omega \overline{zero}(x) = \bar{0}(x)$ but $E_2 \nvdash \overline{zero}(x) = \bar{0}(x)$ since in the free model $T_{E_2}(\Sigma^2, X)$ on infinitely many variables of each type we have $T_{E_2}(\Sigma^2, X) \nvDash \overline{zero}(x) = \bar{0}(x)$. Thus infinitary higher-order equational logic is not conservative on E_2 for first-order equations. However, note that it is conservative on E_2 for ground first-order equations.

One aspect of conservativity for higher-order equational logics which is of particular interest is the existence of *normal forms* for higher-order equational proofs. For the finitary higher-order equational calculus, the following class of normal form proofs can be identified.

Definition 2.13. Let $E \subseteq Eqn(\Sigma, X)$ be any higher-order equational theory. For each type $\tau \in S$, we define the set of all finitary higher-order equational proofs from E of type τ in *eval normal form*, by induction on the complexity of τ.

(i) For each basic type $\tau \in B$ and any terms $t, t' \in T(\Sigma, X)_\tau$, if P is a proof of $t = t'$ from E using only rules (i)–(iv) of Definition 2.5 (i.e. first-order many-sorted equational logic) then P is in eval normal form

(ii) For each product type $(\sigma \times \tau) \in S$ and any terms $t, t' \in T(\Sigma, X)_{(\sigma \times \tau)}$, if P_1 and P_2 are proofs of $proj^1(t) = proj^1(t')$ and $proj^2(t) = proj^2(t')$, respectively, from E in eval normal form, then

$$\frac{P_1 \quad P_2}{t = t'}$$

is a proof of $t = t'$ from E in eval normal form.

(iii) For each function type $(\sigma \to \tau) \in S$ and any terms $t, t' \in T(\Sigma, X)_{(\sigma \to \tau)}$ and any variable $x \in X_\sigma$ not occurring in t or t', if P is a proof of $eval(t, x) = eval(t', x)$ from E in eval normal form, then

$$\frac{P}{t = t'}$$

is a proof of $t = t'$ from E in eval normal form.

Clearly, a finitary proof is in eval normal form when every extensionality or projection rule of each type is used at most once, and occurs at the end of the proof in the manner indicated. That is to say, the proof can be divided into an initial section, consisting of purely equational reasoning, and a final section, where only extensionality and projection inferences are used. A similar concept of eval normal form can be introduced for infinitary higher-order equational proofs. The definition is left as an exercise for the reader.

The existence of eval normal form proofs is of importance for automated reasoning with finitary higher-order equational logic. In particular, it provides a uniform way to reduce the problem of constructing a finitary higher-order equational proof to the problem of constructing a first-order equational proof (for example by term rewriting). This is significant, since there appears to be no obvious way to implement efficiently the finitary extensionality rule in a computational logic. The existence of eval normal forms is equivalent to the following conservativity property in higher-order equational logic.

Proposition 2.14. *Let* $E \subseteq Eqn(\Sigma, X)$ *be any higher-order equational theory. The following are equivalent*:

(i) *finitary higher-order equational logic is conservative on E for first-order equations*;
(ii) *for any equation* $e \in Eqn(\Sigma, X)$ (*of any order*), *if*

$$E \vdash_{\text{eval}} e$$

then there is a finitary proof P of *e which is in eval normal form.*

Proof. (i) $\Rightarrow$ (ii) By induction on the complexity of τ.
(ii) $\Rightarrow$ (i) Immediate from Definition 2.13. □

By Proposition 2.14, any characterisation of conservativity for the finitary extensionality rule simultaneously characterises the existence of eval normal form proofs.

3. A topology of finite information

In this section we introduce a topology on higher-order algebras which will be used in Section 4 to characterise conservativity of the ω-extensionality rule over equational logic. The intuition for this topology is that a basic open set (in any type) contains all elements which share the same specific and finite amount of information. Thus the topology is termed a topology of finite information.

The finite information topology can be defined on both extensional and non-extensional algebras. (This fact is important for Section 4.) It is constructed by induction on the complexity of types in such a way that, for extensional algebras, all spaces are homeomorphic with certain spaces of distinguished type which are termed the *elementary* or *hereditarily Horn types*. On extensional algebras, the topology has strong separation and countability properties: in particular, it is metric and second countable in every type if every carrier set of basic type is countable (unlike the Kleene–Kreisel topology, see [23]). On every algebra, the projection, pairing, currying and evaluation mappings are continuous in the finite information topology.

It will be helpful if the reader already has some familiarity with the fundamentals of topology. All topological prerequisites may be found in any standard reference such as Dugundji [5] or Kelley [12]. However, to keep this paper as self contained as possible, we recall some of the essential definitions which will be needed later.

Definition 3.1. Let X be a set. A *topology* in X is a family T of subsets of X that satisfies: (i) each union of members of T is also a member of T; (ii) each finite intersection of members of T is also a member of T; (iii) the empty set $\emptyset$ and X are members of T.

A member of X is termed a *point*, while a member of T is termed an *open set*. The relative complement $X - U$ of an open set in $U \in T$ is termed a *closed set*. The pair (X, T) is termed a *topological space*. Given a point $x \in X$, a *neighbourhood* of x is any open set $u \in T$ containing x. We let $Nbd(x)$ denote the set of all neighbourhoods of a point x. A point y is *adherent* to a subset $Y \subseteq X$ if every neighbourhood of y has a non-empty intersection with Y; the *topological closure* of Y, denoted $cl(Y)$ is the the set of all points in X adherent to Y.

Given topological spaces (X, T_X) and (Y, T_Y), a mapping $f : X \to Y$ is said to be *open* if the image of every open set under f is an open set, i.e. for every open set $U \in T_X$,

$$f(U) = \{ f(x) \mid x \in U \} \in T_Y.$$

We say that f is *continuous* if, and only if, the inverse image of every open set under f is an open set, i.e. for every open set $V \in T_Y$,

$$f^{-1}(V) = \{ x \in X \mid f(x) \in V \} \in T_X.$$

A continuous open bijection $f : X \to Y$ between two topological spaces (X, T_X) and (Y, T_Y) is termed a *homeomorphism* and the two spaces are said to be *homeomorphic*.

Continuity generalises naturally from functions to algebras. Let Σ be an S-sorted signature. Let A be a Σ algebra and let $T = \langle T_s \mid s \in S \rangle$ be an S-indexed family of sets, where T_s is a topology in the carrier set A_s for each $s \in S$. We say that A is *continuous in* T if, and only if, for each $w = s(1), \ldots, s(n) \in S^+$, each $s \in S$ and each function symbol $f \in \Sigma_{w,s}$ the operation

$$f_A : A^w \to A_s$$

is continuous. (We assume that A^w is given the product topology induced by $T_{s(1)}, \ldots, T_{s(n)}$.)

Given a topological space (X, T_X), a family $B \subseteq T_X$ is said to be a *basis* for T_X if, and only if, each open set $U \in T_X$ is the union of members of B. Also, $B \subseteq T_X$ is said to be a *subbasis* for T_X if, and only if, each open set $U \in T_X$ is the union of finite intersections of members of B.

Intuitively, an open set is a generalisation from analysis of the concept of an open ball around a point in euclidian space. It thus captures abstractly the concept of points being "near" one another. This intuition is less clear in topological spaces which are not metric spaces (i.e. cannot be defined using a distance measure between points). However, the spaces we will study turn out to be metric in the important case of extensional algebras. Two topological spaces which are homeomorphic have the "same" topological structure up to a renaming of their points. Homeomorphism is an equivalence relation on topological spaces, and topology concerns the study of topological spaces up to homeomorphic equivalence only. (Cf the concept of isomorphism in abstract algebra.) The concepts of basis and subbasis allow us to compactly define a topological space starting from a set of generating members.

In the sequel we will actually study indexed families of topological spaces, where the indexing set is a set of higher-order types. This generalisation, which we could term "many-sorted topology" presents no technical difficulties however. One should simply bear in mind that in general different types will be associated with topological spaces that are not homeomorphic.

In order to define the finite information topology, it is necessary to be able to form homeomorphic images of open sets in certain types. For this we introduce a collection of names for distinguished homeomorphism operations which will then be assumed to be present in any higher-order signature.

Definition 3.2. Let Σ be an $H(B)$-typed signature over a type basis B. We say that Σ *contains t he homeomorphism operators*, if, and only, if, Σ includes the following families of function symbols.

(i) For each $\sigma, \tau, \delta \in H(B)$, *currying* and *uncurrying* function symbols,

$$cu \in \Sigma_{((\sigma\times\tau)\to\delta),(\sigma\to(\tau\to\delta))}, \quad uc \in \Sigma_{(\sigma\to(\tau\to\delta)),((\sigma\times\tau)\to\delta)}.$$

(ii) For each $\sigma, \tau, \delta, \varepsilon, \in H(B)$, *generalised currying* and *inverse generalised currying* function symbols,

$$gcu \in \Sigma_{(\varepsilon\to((\sigma\times\tau)\to\delta)),(\varepsilon\to(\sigma\to(\tau\to\delta)))}, \quad gcu^{-1} \in \Sigma_{(\varepsilon\to(\sigma\to(\tau\to\delta))),(\varepsilon\to((\sigma\times\tau)\to\delta))}.$$

(iii) For each $\sigma, \tau, \delta \in H(B)$, *function-pairing* and *inverse function-pairing* function symbols,

$$fp \in \Sigma_{((\sigma\to\tau)\times(\sigma\to\delta)),(\sigma\to(\tau\times\delta))}, \quad fp^{-1} \in \Sigma_{(\sigma\to(\tau\times\delta)),((\sigma\to\tau)\times(\sigma\to\delta))}.$$

(iv) For each $\sigma, \tau, \delta, \varepsilon, \in H(B)$, *generalised function-pairing* and *inverse generalised function-pairing* function symbols,

$$gfp \in \Sigma_{(\varepsilon\to((\sigma\to\tau)\times(\sigma\to\delta))),(\varepsilon\to(\sigma\to(\tau\times\delta)))}, \quad gfp^{-1} \in \Sigma_{(\varepsilon\to(\sigma\to(\tau\times\delta))),(\varepsilon\to((\sigma\to\tau)\times(\sigma\to\delta)))}.$$

(v) For each $\sigma, \tau, \delta, \gamma \in H(B)$, *right-pairing* and *inverse right-pairing* function symbols,

$$rp \in \Sigma_{(((\sigma\times\tau)\times\delta)\to\gamma),((\sigma\times(\tau\times\delta))\to\gamma)}, \quad rp^{-1} \in \Sigma_{((\sigma\times(\tau\times\delta))\to\gamma),(((\sigma\times\tau)\times\delta)\to\gamma)}.$$

(vi) For each $\sigma, \tau, \delta, \varepsilon, \gamma \in H(B)$, *generalised right-pairing* and *inverse generalised right-pairing* function symbols,

$$grp \in \Sigma_{((((\sigma\times\tau)\times\delta)\times\varepsilon)\to\gamma),(((\sigma\times(\tau\times\delta))\times\varepsilon)\to\gamma)}, \quad grp^{-1} \in \Sigma_{(((\sigma\times(\tau\times\delta))\times\varepsilon)\to\gamma),((((\sigma\times\tau)\times\delta)\times\varepsilon)\to\gamma)}.$$

(vii) For each $\sigma, \tau, \delta \in H(B)$, *left-bracketing* and *inverse left-bracketing* function symbols,

$$lb \in \Sigma_{(\sigma\times(\tau\times\delta)),((\sigma\times\tau)\times\delta)}, \quad lb^{-1} \in \Sigma_{((\sigma\times\tau)\times\delta),(\sigma\times(\tau\times\delta))}.$$

(viii) For each $\sigma, \tau, \delta, \varepsilon \in H(B)$, *generalised left-bracketing* and *inverse generalised left-bracketing* function symbols,

$$glb \in \Sigma_{((\sigma\times(\tau\times\delta))\times\varepsilon),(((\sigma\times\tau)\times\delta)\times\varepsilon)}, \quad glb^{-1} \in \Sigma_{(((\sigma\times\tau)\times\delta)\times\varepsilon),((\sigma\times(\tau\times\delta))\times\varepsilon)}.$$

(ix) For each σ, $\tau \in H(B)$, *pairing*

$$\langle .,. \rangle \in \Sigma_{\sigma\tau,(\sigma\times\tau)}.$$

The generalised operations of parts (ii), (iv), (vi) and (viii) of Definition 3.2 are best understood in terms of a normal form for types that will be introduced in Definition 3.14. (The so-called elementary or hereditarily Horn types.) Definition 3.2 introduces a set of operations, which will be equationally axiomatised in Definition 3.4. Then in any model of these axioms, the operations can be used to define a homeomorphism between any type and its normal form. Operations such as uncurrying and inverse function pairing will provide a homeomorphism between some types and their normal forms. However, they cannot be applied to all types since we have no operation corresponding to lambda abstraction. (To see this, compare the domain type of each operation in 3.2 with the domain type of its generalised form.) The generalised operations of (ii), (iv), (vi) and (viii) are needed to deal with all cases that would otherwise be dealt with by the special operations (i), (iii), (v) and (vii) and lambda abstraction. Interestingly, these generalised operations can be used to express equational relationships between the specialised operations which are not often considered, and these equations play a crucial role, e.g. in proofs of continuity of all homeomorphisms.

In the sequel, we will assume that Σ is a fixed, but arbitrarily chosen $H(B)$-typed signature, over a type basis B, and that Σ contains the homeomorphism operators. Then we can use operations named by the distinguished homeomorphism operation symbols, in any algebra A, to simultaneously define collections of *subbasic open sets* and the *continuous elements* in A.

Definition 3.3. For each type $\tau \in H(B)$, we define

(i) the collection of all *subbasic open subsets* of A_τ; and,
(ii) the set $C(A)_\tau$ of all *continuous elements* of A_τ,

by induction on the complexity of τ. Then for such τ as usual the *basic open subsets* of A_τ are precisely the finite intersections of subbasic open subsets of A_τ and the *open subsets* of A_τ are precisely the unions of basic open subsets of A_τ.

(i) *Basis*: Consider any basic type $\tau \in B$. A subset $U \subseteq A_\tau$ is subbasic open if, and only if, U is a singleton set. Every element of A_τ is continuous, i.e. $C(A)_\tau = A_\tau$.

(ii) *Induction step*: Consider any product type $(\sigma \times \tau) \in H(B)$. A subset $U \subseteq A_{(\sigma\times\tau)}$ is subbasic open if, and only if, U has the form

$$\langle V_1, C(A)_\tau \rangle$$

for $V_1 \subseteq A_\sigma$ an open subset of A_σ, or

$$\langle C(A)_\sigma, V_2 \rangle$$

for $V_2 \subseteq A_\tau$ an open subset of A_τ. For any $a \in A_{(\sigma\times\tau)}$, a is continuous if, and only if, $proj_A^1(a)$ and $proj_A^2(a)$ are continuous.

(iii) Consider any function type $(\sigma \to \tau) \in H(B)$. We proceed by a subinduction on the complexity of τ.

(a) *Subbasis*: Suppose $\tau \in B$ is a basic type. A subset $U \subseteq A_{(\sigma\to\tau)}$ is subbasic open if, and only if, U has the form

$$O_{V,b} = \{a \in A_{(\sigma\to\tau)} \mid \textit{for all } a_0 \in V,\ eval_A(a, a_0) = b\}$$

for $V \subseteq A_\sigma$ a basic open subset and $b \in A_\tau$. For any $a \in A_{(\sigma\to\tau)}$, a is continuous if, and only if, for any open subset $U \subseteq A_\tau$,

$$a^{-1}(U) = \{b \in A_\sigma \mid eval_A(a, b) \in U\}$$

is open.

(b) *Subinduction step*: Suppose $\tau = (\tau_1 \times \tau_2)$ is a product type. A subset $U \subseteq A_{(\sigma\to(\tau_1\times\tau_2))}$ is subbasic open if, and only if, $fp_A^{-1}(U)$ is subbasic open. For any $a \in A_{(\sigma\to(\tau_1\times\tau_2))}$, a is continuous if, and only if, $fp_A^{-1}(a)$ is continuous.

(c) Suppose $\tau = (\tau_1 \to \tau_2)$ is a function type, then we proceed by a subinduction on the complexity of τ_2.

(c.i) *Subbasis*: Suppose $\tau_2 \in B$ is a basic type. A subset $U \subseteq A_{(\sigma\to(\tau_1\to\tau_2))}$ is subbasic open if, and only if, $uc_A(U)$ is subbasic open. For any $a \in A_{(\sigma\to(\tau_1\to\tau_2))}$, a is continuous if, and only if, $uc_A(a)$ is continuous.

(c.ii) *Subinduction step*: Suppose $\tau_2 = (\delta_1 \times \delta_2)$ is a product type. A subset $U \subseteq A_{(\sigma\to(\tau_1\to(\delta_1\times\delta_2)))}$ is subbasic open if, and only if, $gfp_A^{-1}(U)$ is subbasic open. For any $a \in A_{(\sigma\to(\tau_1\to(\delta_1\times\delta_2)))}$, a is continuous if, and only if, $gfp_A^{-1}(a)$ is continuous.

(c.iii) Suppose $\tau_2 = (\delta_1 \to \delta_2)$ is a function type. A subset $U \subseteq A_{(\sigma\to(\tau_1\to(\delta_1\to\delta_2)))}$ is subbasic open if, and only if, $gcu_A^{-1}(U)$ is subbasic open. For any $a \in A_{(\sigma\to(\tau_1\to(\delta_1\to\delta_2)))}$, a is continuous if, and only if, $gcu_A^{-1}(a)$ is continuous.

We say that A has *continuous carrier sets* if, and only if, for each type $\tau \in S$, $A_\tau = C(A)_\tau$, i.e. every element of every carrier set is a continuous element. We define the *finite information topology* $FI(A)_\tau$ on A_τ to be the set of all open subsets of A_τ,

$$FI(A)_\tau = \{U \subseteq A_\tau \mid U \textit{ is open}\}.$$

Clearly, the subbasic open subsets of A form a subbasis for a topology, and hence $FI(A)_\tau$ is a well-defined topology on A, if, and only if, A has continuous carrier sets. Otherwise, if some element a of A is not continuous then there is no subbasic open subset of A which contains a.

The homeomorphisms named in Definition 3.2 are associated with certain equational axioms, which must be satisfied in any Σ algebra A in order to ensure that these operations really are homeomorphisms. We collect together these equational axioms as follows.

Definition 3.4. The *homeomorphism axioms* **Hom** consist of the following set of equations, for all types σ, τ, δ, $\varepsilon \in H(B)$:

(i) *Currying*: For $x \in X_{(\sigma\to(\tau\to\delta))}$, $x' \in X_{((\sigma\times\tau)\to\delta)}$, $y \in X_\sigma$ and $z \in X_\tau$,

$$cu(uc(x)) = x, \qquad uc(cu(x')) = x', \tag{3.1a, b}$$

$$eval(uc(x), \langle y, z\rangle) = eval(eval(x, y), z),$$
$$eval(eval(cu(x'), y), z) = eval(x', \langle y, z\rangle). \tag{3.1c, d}$$

(ii) *Generalised currying*: For all σ, τ, δ, $\varepsilon \in H(B)$ and for $x \in X_{(\varepsilon \to (\sigma \to (\tau \to \delta)))}$, $x' \in X_{(\varepsilon \to ((\sigma \times \tau) \to \delta))}$ and $y \in X_\varepsilon$,

$$gcu(gcu^{-1}(x)) = x, \qquad gcu^{-1}(gcu(x')) = x', \tag{3.2a, b}$$

$$eval(gcu^{-1}(x), y) = uc(eval(x, y)), \qquad eval(gcu(x'), y) = cu(eval(x', y)), \tag{3.2c, d}$$

$$gcu^{-1}(x) = cu(rp(uc(uc(x)))), \qquad gcu(x') = cu(cu(rp^{-1}(uc(x')))). \tag{3.2e, f}$$

(iii) *Function-pairing*: For $x \in X_{((\sigma \to \tau) \times (\sigma \to \delta))}$, $x' \in X_{(\sigma \to (\tau \times \delta))}$ and $y \in X_\sigma$,

$$fp^{-1}(fp(x)) = x, \qquad fp(fp^{-1}(x')) = x', \tag{3.3a, b}$$

$$\langle eval(proj^1(fp^{-1}(x')), y), eval(proj^2(fp^{-1}(x')), y) \rangle = eval(x', y), \tag{3.3c}$$

$$eval(fp(x), y) = \langle eval((proj^1(x), y), eval((proj^2(x), y) \rangle. \tag{3.3d}$$

(iv) *Generalised function-pairing*: For $x \in X_{(\varepsilon \to ((\sigma \to \tau) \times (\sigma \to \delta)))}$, $x' \in X_{(\varepsilon \to (\sigma \to (\tau \times \delta)))}$ and $y \in X_\sigma$,

$$gfp(gfp^{-1}(x')) = x', \qquad gfp^{-1}(gfp(x)) = x, \tag{3.4a, b}$$

$$eval(gfp^{-1}(x'), y) = fp^{-1}(eval(x', y)), \qquad eval(gfp(x), y) = fp(eval(x, y)), \tag{3.4c, d}$$

$$gfp^{-1}(x') = fp(\langle cu(proj^1(fp^{-1}(uc(x')))), cu(proj^2(fp^{-1}(uc(x')))) \rangle), \tag{3.4e}$$

$$gfp(x) = cu(fp(\langle uc(proj^1(fp^{-1}(x))), uc(proj^2(fp^{-1}(x))) \rangle)). \tag{3.4f}$$

(v) *Right-pairing*: For $x \in X_{(((\sigma \times \tau) \times \delta) \to \gamma)}$, $x' \in X_{((\sigma \times (\tau \times \delta)) \to \gamma)}$ and $y \in X_\sigma$,

$$rp(rp^{-1}(x')) = x', \qquad rp^{-1}(rp(x)) = x. \tag{3.5a, b}$$

If γ is a basic type $\gamma \in B$ then we have

$$eval(rp(x), y) = eval(x, lb(y)), \qquad eval(rp^{-1}(x'), y) = eval(x', lb^{-1}(y)), \tag{3.5c, d}$$

if γ is a product type $\gamma = (\gamma_1 \times \gamma_2)$ then we have

$$rp(x) = fp(\langle rp(proj^1(fp^{-1}(x))), rp(proj^2(fp^{-1}(x))) \rangle), \tag{3.5e}$$

$$rp^{-1}(x') = fp(\langle rp^{-1}(proj^1(fp^{-1}(x'))), rp^{-1}(proj^2(fp^{-1}(x'))) \rangle), \tag{3.5f}$$

and if γ is a function type $\gamma = (\gamma_1 \to \gamma_2)$ then we have

$$rp(x) = cu(grp(uc(x))), \quad rp^{-1}(x') = cu(grp^{-1}(uc(x'))). \tag{3.5g, h}$$

(vi) *Generalised right-pairing*: For $x \in X_{((((\sigma \times \tau) \times \varepsilon) \times \delta) \to \gamma)}$, $x' \in X_{(((\sigma \times (\tau \times \delta)) \times \varepsilon) \to \gamma)}$ and $y \in X_\sigma$,

$$grp(grp^{-1}(x')) = x', \qquad grp^{-1}(grp(x)) = x. \tag{3.6a, b}$$

If γ is a basic type $\gamma \in B$ then we have

$$eval(grp(x), y) = eval(x, glb(y)), \qquad eval(grp^{-1}(x'), y) = eval(x', glb^{-1}(y)), \tag{3.6c, d}$$

if γ is a product type $\gamma = (\gamma_1 \times \gamma_2)$ then we have

$$grp(x) = fp(\langle grp(proj^1(fp^{-1}(x))), grp(proj^2(fp^{-1}(x))) \rangle), \tag{3.6e}$$

$$grp^{-1}(x') = fp(\langle grp^{-1}(proj^1(fp^{-1}(x'))), grp^{-1}(proj^2(fp^{-1}(x'))) \rangle), \tag{3.6f}$$

and if γ is a function type $\gamma = (\gamma_1 \to \gamma_2)$ then we have

$$grp(x) = cu(rp^{-1}(grp(rp(uc(x))))),$$

$$grp^{-1}(x) = cu(rp^{-1}(grp^{-1}(rp(uc(x))))). \tag{3.6g, h}$$

(vii) *Left-bracketing*: For $x \in X_{(\sigma\times(\tau\times\delta))}$ and $x' \in X_{((\sigma\times\tau)\times\delta)}$,

$$lb(lb^{-1}(x')) = x', \qquad lb^{-1}(lb(x)) = x, \tag{3.7a, b}$$

$$lb(x) = \langle\langle proj^1(x), proj^1(proj^2(x))\rangle, proj^2(proj^2(x))\rangle, \tag{3.7c}$$

$$lb^{-1}(x) = \langle proj^1(proj^1(x)), \langle proj^2(proj^1(x)), proj^2(x)\rangle\rangle, \tag{3.7d}$$

(viii) *Generalised left-bracketing*: For $x \in X_{((\sigma\times(\tau\times\delta))\times\varepsilon)}$ and $x' \in X_{(((\sigma\times\tau)\times\delta)\times\varepsilon)}$,

$$glb(glb^{-1}(x')) = x', \qquad glb^{-1}(glb(x)) = x, \tag{3.8a, b}$$

$$glb(x) = \langle lb(proj^1(x)), proj^2(x)\rangle, \qquad glb^{-1}(x') = \langle lb^{-1}(proj^1(x')), proj^2(x')\rangle. \tag{3.8c, d}$$

(ix) *Pairing*: For any $x \in X_{(\sigma\times\tau)}$ and $y \in X_\sigma$ and $z \in X_\tau$,

$$x = \langle proj^1(x), proj^2(x)\rangle, \tag{3.9a}$$

$$y = proj^1(\langle y, z\rangle), \qquad z = proj^2(\langle y, z\rangle). \tag{3.9b, c}$$

With the exception of axioms (3.9a–c), the homeomorphism axioms introduced above correspond with well-known-type isomorphisms for higher-order-type systems based on $\times$ and $\to$. For example, in [3], these isomorphisms are formulated as a first-order equational theory $Th^1_{\times T}$. Clearly, the operations *cu*, *gcu fp*, *gfp*, *rp*, *grp*, *lb* and *glb* and their inverses are bijective functions, and form isomorphisms in the sense of category theory, by virtue of Eqs. (3.1a,b), (3.2a,b), …, (3.8a,b). Then the ordered pair consisting of the domain and codomain types of each such isomorphism can be interpreted as an equation that is trivially a consequence of the theory $Th^1_{\times T}$.

In the sequel, we let $A \in Alg(\Sigma, \mathbf{Hom})$ denote a fixed, but arbitrarily chosen Σ algebra which satisfies the homeomorphism axioms **Hom** and has continuous carrier sets. (Note that A is not necessarily extensional.) Then the finite information topology $FI(A)$ is well defined for A. We will show that all the homeomorphism operations named in Σ are continuous open mappings when interpreted in A. First we list some basic technical facts.

Proposition 3.5. *For any types* σ, $\tau \in H(B)$, $A_{(\sigma\times\tau)} = \langle A_\sigma, A_\tau\rangle_A$.

Proposition 3.6. *For any types* σ, τ, $\delta \in H(B)$, *and any open sets* $V_1 \in FI(A)_\sigma$, $V_2 \in FI(A)_\tau$ *and* $V_3 \in FI(A)_\delta$:

(i) $lb_A(\langle V_1, A_{(\tau\times\delta)}\rangle_A)$,
(ii) $lb_A(\langle A_\sigma, \langle V_2, A_\delta\rangle\rangle_A)$,
(iii) $lb_A(\langle A_\sigma, \langle A_\tau, V_3\rangle\rangle_A)$,

are all subbasic open sets.

Proof. Follows from Proposition 3.5. □

Proposition 3.7. *Let σ, $\tau \in H(B)$ be any types.*

(i) *For any open sets $V_1 \in FI(A)_\sigma$ and $V_2 \in FI(A)_\tau$,*

$$\langle V_1, V_2 \rangle_A \in FI(A)_{(\sigma\times\tau)}.$$

(ii) *For any open set $U \in FI(A)_{(\sigma\times\tau)}$ and $V_1 \in FI(A)_\sigma$ and $V_2 \in FI(A)_\tau$,*

$$\text{(a)} \quad U \bigcap \langle V_1, A_\tau \rangle_A = \langle proj^1_A(U) \bigcap V_1, proj^2_A(U) \bigcap A_\tau \rangle_A$$

and

$$\text{(b)} \quad U \bigcap \langle A_\sigma, V_2 \rangle_A = \langle proj^1_A(U) \bigcap A_\sigma, proj^2_A(U) \bigcap V_2 \rangle_A.$$

(iii) *For any open set $U \in FI(A)_{(\sigma\times\tau)}$,*

$$proj^1_A(U) \in FI(A)_\sigma, \qquad proj^2_A(U) \in FI(A)_\tau.$$

Proof. Exercise. □

Proposition 3.8. *For any types $\sigma, \tau, \delta \in H(B)$ and any open sets $V_1 \in FI(A)_\sigma$, $V_2 \in FI(A)_\tau$, $V_3 \in FI(A)_\delta$ and $U \in FI(A)_{(\sigma\times(\tau\times\delta))}$:*

(i) $lb_A(\langle V_1, A_{(\tau\times\delta)} \rangle_A \bigcap U) = lb_A(\langle V_1, A_{(\tau\times\delta)} \rangle_A) \bigcap lb_A(U)$;
(ii) $lb_A(\langle A_\sigma, \langle V_2, A_\delta \rangle_A \rangle_A \bigcap U) = lb_A(\langle A_\sigma, \langle V_2, A_\delta \rangle_A \rangle_A) \bigcap lb_A(U)$;
(iii) $lb_A(\langle A_\sigma, \langle A_\tau, V_3 \rangle_A \rangle_A \bigcap U) = lb_A(\langle A_\sigma, \langle A_\tau, V_3 \rangle_A \rangle_A) \bigcap lb_A(U)$.

Proof. Exercise. □

Proposition 3.9. *For any types $\sigma, \tau, \delta \in H(B)$ and any open set $U \in FI(A)_{(\tau\times\delta)}$,*

$$\langle A_\sigma, U \rangle_A = \langle A_\sigma, \langle proj^1_A(U), A_\delta \rangle_A \rangle_A \bigcap \langle A_\sigma, \langle A_\tau, proj^2_A(U) \rangle_A \rangle_A.$$

Proof. Follows from Proposition 3.7(ii.b). □

Lemma 3.10. *Let $\sigma, \tau, \delta, \varepsilon \in H(B)$ be any types. The operations*

$$lb_A : A_{(\sigma\times(\tau\times\delta))} \to A_{((\sigma\times\tau)\times\delta)}, \qquad lb^{-1}_A : A_{((\sigma\times\tau)\times\delta)} \to A_{(\sigma\times(\tau\times\delta))},$$

$$glb_A : A_{((\sigma\times(\tau\times\delta))\times\varepsilon)} \to A_{(((\sigma\times\tau)\times\delta)\times\varepsilon)}, \qquad glb^{-1}_A : A_{(((\sigma\times\tau)\times\delta)\times\varepsilon)} \to A_{((\sigma\times(\tau\times\delta))\times\varepsilon)},$$

all preserve basic open sets.

Proof. (i) We prove that

$$lb_A : A_{(\sigma\times(\tau\times\delta))} \to A_{((\sigma\times\tau)\times\delta)}$$

preserves basic open sets as follows. Consider any basic open set $U \subseteq A_{(\sigma\times(\tau\times\delta))}$. For some $m \geqslant 1$, by Proposition 3.9, there exist subbasic open sets $W_1, \ldots, W_m$ with

$$U = W_1 \cap \cdots \cap W_m,$$

and for each $1 \leqslant i \leqslant m$, either

$$W_i = \langle W_i^1, \langle A_\tau, A_\delta \rangle_A \rangle_A$$

or

$$W_i = \langle A_\sigma, \langle W_i^2, A_\delta \rangle_A \rangle_A$$

or

$$W_i = \langle A_\sigma, \langle A_\tau, W_i^3 \rangle_A \rangle_A,$$

for some open sets $W_i^1 \in FI(A)_\sigma$, $W_i^2 \in FI(A)_\tau$ and $W_i^3 \in FI(A)_\delta$. By Proposition 3.8,

$$lb_A(U) = lb_A(W_1) \cap \cdots \cap lb_A(W_m).$$

By Proposition 3.6, for each $1 \leqslant i \leqslant m$, $lb_A(W_i)$ is subbasic open, and so $lb_A(U)$ is basic open.

The proofs that lb^{-1}, glb and glb^{-1} preserve basic open sets are obtained similarly by modifying Proposition 3.8 appropriately. $\square$

The continuity and openness of some of the mappings named in Definition 3.2 will be used to prove the continuity and openness of others. Thus one must prove the following three (apparently similar) theorems in the specific order they appear, and using slightly different methods in each case.

Theorem 3.11. *For any types $\sigma, \tau, \delta, \varepsilon, \gamma \in H(B)$, the following operations are continuous open mappings*:

(i) *generalised function-pairing*: $gfp_A : A_{(\sigma\to((\tau\to\delta)\times(\tau\to\varepsilon)))} \to A_{(\sigma\to(\tau\to(\delta\times\varepsilon)))}$,
(ii) *generalised currying*: $gcu_A : A_{(\sigma\to((\tau\times\delta)\to\varepsilon))} \to A_{(\sigma\to(\tau\to(\delta\to\varepsilon)))}$,
(iii) *left and right projection*: $proj_A^1 : A_{(\sigma\times\tau)} \to A_\sigma$, $proj_A^2 : A_{(\sigma\times\tau)} \to A_\tau$,
(iv) *pairing* $\langle .,. \rangle_A : A_\sigma \times A_\tau \to A_{(\sigma\times\tau)}$,
(iv) *left-bracketing*: $lb_A : A_{(\sigma\times(\tau\times\delta))} \to A_{((\sigma\times\tau)\times\delta)}$,
(vi) *generalised left-bracketing*: $glb_A : A_{((\sigma\times(\tau\times\delta))\times\varepsilon)} \to A_{(((\sigma\times\tau)\times\delta)\times\varepsilon)}$,
(vii) *function-pairing*: $fp_A : A_{((\sigma\to\tau)\times(\sigma\to\delta))} \to A_{(\sigma\to(\tau\times\delta))}$.

Proof. Exercise. $\square$

Theorem 3.12. *For any types $\sigma, \tau, \delta, \varepsilon, \gamma \in H(B)$, the following operations are continuous open mappings*:

(i) *currying*: $cu_A : A_{((\sigma\times\tau)\to\gamma)} \to A_{(\sigma\to(\tau\to\gamma))}$,
(ii) *right-pairing*: $rp_A : A_{(((\sigma\times\tau)\times\delta)\to\gamma)} \to A_{((\sigma\times(\tau\times\delta))\to\gamma)}$,
(iii) *generalised right-pairing*: $grp_A : A_{((((\sigma\times\tau)\times\delta)\times\varepsilon)\to\gamma)} \to A_{(((\sigma\times(\tau\times\delta))\times\varepsilon)\to\gamma)}$.

Proof. Exercise. □

Theorem 3.13. *For any types* $\sigma, \tau \in H(B)$, *the evaluation mapping*

$$eval_A : A_{(\sigma\to\tau)} \times A_\sigma \to A_\tau$$

is continuous.

Proof. We prove the result by induction on the complexity of τ.

(i) *Basis*: Suppose $\tau \in B$ is a basic type. Consider any subbasic open set $\{c\} \in FI(A)_\tau$ for $c \in A_\tau$. We must show that

$$eval_A^{-1}(\{c\}) = \{(a,b) \in A_{(\sigma\to\tau)} \times A_\sigma \mid eval_A(a,b) = c\}$$

is open. So consider any $a \in A_{(\sigma\to\tau)}$ and $b \in A_\sigma$ such that $eval_A(a,b) = c$. Since a is continuous, $a^{-1}(\{c\})$ is open and $b \in a^{-1}(\{c\})$. So there exists a basic open set $U_b^a \in FI(A)_\sigma$ with $U_b^a \subseteq a^{-1}(\{c\})$ and $b \in U_b^a$ and for all $b' \in U_b^a$,

$$eval_A(a,b') = eval_A(a,b) = c.$$

Consider the subbasic open set $O_{U_b^a,c} \in FI(A)_{(\sigma\to\tau)}$. Then

$$(a,b) \in O_{U_b^a,c} \times U_b^a$$

and

$$O_{U_b^a,c} \times U_b^a \subseteq eval_A^{-1}(\{c\}).$$

Thus

$$\bigcup_{(a,b)\in eval_A^{-1}(\{c\})} O_{U_b^a,c} \times U_b^a = eval_A^{-1}(\{c\}),$$

and since c was arbitrarily chosen that $eval_A$ is continuous.

(ii) *Induction step*. Suppose $\tau = (\tau_1 \times \tau_2)$ is a product type.

Consider any subbasic open set $\langle V_1, C(A)_{\tau_1} \rangle_A = \langle V_1, A_{\tau_1} \rangle_A$, for open $V_1 \subset FI(A)_{\tau_1}$. By the induction hypothesis $eval_A : A_{(\sigma\to\tau_1)} \times A_\sigma \to A_{\tau_1}$ is continuous. So

$$U_1 = eval_A^{-1}(V_1)$$

is open. Now for some indexing set I,

$$U_1 = \bigcup_{i\in I} W_{i,1} \times W_{i,2},$$

where for each $i \in I$, $W_{i,1} \in FI(A)_{(\sigma\to\tau_1)}$ and $W_{i,2} \in FI(A)_\sigma$ are basic open sets. For each $i \in I$, define

$$Y_{i,1} = fp_A(\langle W_{i,1}, A_{(\sigma\to\tau_2)} \rangle_A), \quad Y_{i,2} = W_{i,2}.$$

Then $Y_{i,1} \in FI(A)_{(\sigma\to(\tau_1\times\tau_2))}$ by Proposition 3.7(i) and Theorem 3.11(vii) above and $Y_{i,2} \in FI(A)_\sigma$. Let

$$Y = \bigcup_{i\in I} Y_{i,1} \times Y_{i,2}.$$

We need only show that

$$Y = eval_A^{-1}(\langle V_1, C(A)_{\tau_1} \rangle_A). \tag{3.9}$$

Now

$$(a,b) \in eval_A^{-1}(\langle V_1, C(A)_{\tau_1} \rangle_A)$$
$$\Leftrightarrow eval_A(a,b) \in \langle V_1, C(A)_{\tau_1} \rangle_A$$
$$\Leftrightarrow proj_A^1(eval_A(a,b)) \in V_1 \quad \text{and} \quad proj_A^2(eval_A(a,b)) \in A_{\tau_1}$$
$$\Leftrightarrow eval_A(proj_A^1(fp_A^{-1}(a),b) \in V_1 \quad \text{and} \quad eval_A(proj_A^2(fp_A^{-1}(a),b) \in A_{\tau_1}$$

by Equation (3.3c),

$$(proj_A^1(fp_A^{-1}(a)),b) \in W_{i,1} \times W_{i,2} \quad \text{and} \quad (proj_A^2(fp_A^{-1}(a)),b) \in A_{(\sigma \to \tau_2)} \times A_\sigma$$

for some $i \in I$,

$$\Leftrightarrow fp_A(\langle proj_A^1(fp_A^{-1}(a)), proj_A^2(fp_A^{-1}(a)) \rangle_A) \in Y_{i,1} \quad \text{and} \quad b \in Y_{i,2}$$

for some $i \in I$,

$$\Leftrightarrow a \in Y_{i,1} \text{ and } b \in Y_{i,2} \;\Leftrightarrow\; (a,b) \in Y.$$

Thus (3.9) holds.

Similarly, for any subbasic open set $\langle C(A)_\sigma, V_2 \rangle_A$ for open $V_2 \in FI(A)_{\tau_2}$,

$$eval_A^{-1}(\langle C(A)_\sigma, V_2 \rangle_A)$$

is open. It follows that $eval_A$ is continuous.

(iii) Suppose that $\tau = (\tau_1 \to \tau_2)$ is a function type. We proceed by subinduction on the complexity of τ_2.

(iii.a) Subbasis: Suppose that $\tau_2 \in B$ is a basic type. Consider any subbasic open set $O_{U,c} \in FI(A)_{(\tau_1 \to \tau_2)}$ and any $a \in A_{(\sigma \to (\tau_1 \to \tau_2))}$. Now by assumption a is continuous, so

$$V^a = a^{-1}(O_{U,c}) \in FI(A)_\sigma.$$

Thus by Proposition 3.7(i),

$$O_{\langle V^a, U \rangle_A, c} \in FI(A)_{((\sigma \times \tau_1) \to \tau_2)}$$

is subbasic open. Then by Theorem 3.12(i), (continuity of uc_A),

$$W^a = cu_A(O_{\langle V^a, U \rangle_A, c}) \in FI(A)_{(\sigma \to (\tau_1 \to \tau_2))}.$$

Let

$$Y = \bigcup_{a \to A_{(\sigma \to (\tau_1 \to \tau_2))}} W^a \times V^a,$$

then we need only show that

$$Y = eval_A^{-1}(O_{U,c}).$$

Consider any a, b such that $eval_A(a,b) \in O_{U,c}$. Then $b \in V^a$, and for any $b' \in V^a$, $eval_A(a,b') \in O_{U,c}$. So for any $b' \in V^a$ and $a_0 \in U$,

$$eval_A(eval_A(a,b'),a_0) = c,$$

and so

$$uc_A(a) \in O_{\langle V^a, U \rangle_A, c},$$

and hence $a \in W^a$. Thus $(a,b) \in Y$, and since a and b were arbitrarily chosen then $eval_A^{-1}(O_{U,c}) \subseteq Y$.

Conversely, consider any a, b such that $(a,b) \in Y$. Then for some $a_i \in A_{(\sigma \to (\tau_1 \to \tau_2))}$, $a \in W^{a_i}$ and $b \in V^{a_i} = a_i^{-1}(O_{U,c})$. So

$$a \in cu_A(O_{\langle V^{a_i}, U \rangle_A, c}),$$

and hence for some $a_j \in O_{\langle V^{a_i}, U \rangle_A, c}, uc_A(a) = a_j$. Now for any $a_0 \in V^{a_i}$ and $a_1 \in U$,

$$\begin{aligned} eval_A(a_j, \langle a_0, a_1 \rangle_A) &= eval_A(uc_A(a), \langle a_0, a_1 \rangle_A) \\ &= eval_A(eval_A(a, a_0), a_1) = c. \end{aligned}$$

So $eval_A(a,b) \in O_{U,c}$, and since a and b were arbitrarily chosen then $Y \subseteq eval_A^{-1}(O_{U,c})$. Since $O_{U,c}$ was arbitrarily chosen then $eval_A$ is continuous.

(iii.b) *Subinduction step*: Suppose that $\tau_2 = (\delta_1 \times \delta_2)$ is a product type. Consider any subbasic open set $U \in FI(A)_{(\tau_1 \to (\delta_1 \times \delta_2))}$. Then by Definition 2.2,

$$fp_A^{-1}(U) \in FI(A)_{((\tau_1 \to \delta_1) \times (\tau_1 \to \delta_2))}.$$

So either

$$fp_A^{-1}(U) = \langle V_1, A_{(\tau_1 \to \delta_1)} \rangle_A, \tag{3.10}$$

for some open $V_1 \in FI(A)_{(\tau_1 \to \delta_1)}$, or

$$fp_A^{-1}(U) = \langle A_{(\tau_1 \to \delta_1)}, V_2 \rangle_A, \tag{3.11}$$

for some open $V_2 \in FI(A)_{(\tau_1 \to \delta_2)}$. Suppose that (3.10) holds. Then by the subinduction hypothesis,

$$eval_A : A_{(\sigma \to (\tau_1 \to \delta_1))} \times A_\sigma \to A_{(\tau_1 \to \delta_1)}$$

is continuous, and so $eval_A(proj_A^1(fp_A^{-1}(U)))$ is open. Hence for some set I,

$$eval_A(proj_A^1(fp_A^{-1}(U))) = \bigcup_{i \in I} W_{i,1} \times W_{i,2},$$

where for each $i \in I$, $W_{i,1} \in FI(A)_{(\sigma \to (\tau_1 \to \delta_1))}$ and $W_{i,2} \in FI(A)_\sigma$. For each $i \in I$, define

$$Y_{i,1} = gfp_A(fp_A(\langle W_{i,1}, A_{(\sigma \to (\tau_1 \to \delta_1))} \rangle_A))$$

and

$$Y_{i,2} = W_{i,2}.$$

Then by Theorem 3.11(i) and (vii) (openness of gfp_A and fp_A), for each $i \in I$,

$$Y_{i,1} \in FI(A)_{(\sigma \to (\tau_1 \to (\delta_1 \times \delta_2)))}$$

and $Y_{i,2} \in FI(A)_\sigma$. Let

$$Y = \bigcup_{i \in I} Y_{i,1} \times Y_{i,2},$$

then we need only show that $Y = eval_A^{-1}(U)$.

Now for any a and b, $(a, b) \in Y$ if, and only if, for some $i \in I$,

$$a \in Y_{i,1} \quad \text{and} \quad b \in Y_{i,2} \Leftrightarrow$$

for some $i \in I$

$$proj_A^1(fp_A^{-1}(gfp_A^{-1}(a))) \in W_{i,1} \text{ and } proj_A^2(fp_A^{-1}(gfp_A^{-1}(a))) \in A_{(\sigma \to (\tau_1 \to \delta_2))} \text{ and } b \in W_{i,2}$$
$$\Leftrightarrow eval_A(proj_A^1(fp_A^{-1}(gfp_A^{-1}(a))), b) \in proj_A^1(fp_A^{-1}(U))$$

and

$$eval_A(proj_A^2(fp_A^{-1}(gfp_A^{-1}(a))), b) \in proj_A^2(fp_A^{-1}(U))$$
$$\Leftrightarrow \langle eval_A(proj_A^1(fp_A^{-1}(gfp_A^{-1}(a))), b), eval_A(proj_A^2(fp_A^{-1}(gfp_A^{-1}(a))), b) \rangle_A$$
$$\in \langle proj_A^1(fp_A^{-1}(U)), proj_A^2(fp_A^{-1}(U)) \rangle_A$$
$$\Leftrightarrow eval_A(gfp_A^{-1}(a), b) \in fp_A^{-1}(U)$$
$$\Leftrightarrow fp_A(eval_A(gfp_A^{-1}(a), b)) \in U \quad \Leftrightarrow \quad eval_A(a, b) \in U.$$

Similarly, if (3.11) holds then $eval_A^{-1}(U)$ is open. Since U was arbitrarily chosen, the $eval_A$ is continuous.

(iii.c) Suppose $\tau_2 = (\delta_1 \to \delta_2)$ is a function type. Consider any subbasic open set

$$U \in FI(A)_{(\tau_1 \to (\delta_1 \to \delta_2))}.$$

By Theorem 3.12(i) above (continuity of cu_A), $uc_A(U) \in FI(A)_{((\tau_1 \times \delta_1) \to \delta_2)}$ is open. By the subinduction hypothesis,

$$eval_A : A_{(\sigma \to ((\tau_1 \times \delta_1) \to \delta_2))} \times A_\sigma \to A_{((\tau_1 \times \delta_1) \to \delta_2)}$$

is continuous. So

$$eval_A^{-1}(uc_A(U))$$

is open. Hence for some set I,

$$eval_A^{-1}(uc_A(U)) = \bigcup_{i \in I} W_{i,1} \times W_{i,2},$$

where for each $i \in I$, $W_{i,1} \in FI(A)_{(\sigma \to ((\tau_1 \times \delta_1) \to \delta_2))}$ and $W_{i,2} \in FI(A)_\sigma$. For each $i \in I$, define

$$Y_{i,1} = gcu_A(W_{i,1}), \quad Y_{i,2} = W_{W,2}.$$

Then by Theorem 3.11(ii), above (openness of gcu_A), for each $i \in I$,

$$Y_{i,1} \in FI(A)_{(\sigma \to (\tau_1 \to (\delta_1 \to \delta_2)))}, \quad Y_{i,2} \in FI(A)_\sigma.$$

Let

$$Y = \bigcup_{i \in I} Y_{i,1} \times Y_{i,2},$$

then we need only show that

$$Y = eval_A^{-1}(U).$$

Now for any a and b, $(a,b) \in Y$ if, and only if, for some $i \in I$,

$$\begin{aligned} & a \in Y_{i,1} \quad \text{and} \quad b \in Y_{i,2} \\ & \quad \Leftrightarrow eval_A(gcu_A^{-1}(a), b) \in uc_A(U) \quad \text{and} \quad b \in W_{i,2} \\ & \quad \Leftrightarrow cu_A(eval_A(gcu_A^{-1}(a), b)) \in U \quad \text{and} \quad b \in W_{i,2} \ \Leftrightarrow \ eval_A(a,b) \in U. \end{aligned}$$

Since U was arbitrarily chosen then $eval_A$ is continuous. □

We conclude this section with a brief summary of some of the basic properties of the finite information topology. First, under the assumption of extensionality, we can observe that every space of type τ is homeomorphic with some space of distinguished type. These types can be defined inductively as follows.

Definition 3.14. Let B be a type basis. For each $n \geqslant 1$ we define the set $E_n(B) \subseteq H(B)$ of all *elementary* or *hereditarily Horn types* of order n inductively as follows:

(i) $E_1 = B$.

(ii) For any $n \geqslant 1$ and elementary type $\sigma \in E_n(B)$ and any basic type $\beta \in B$,

$$(\sigma \to \beta) \in E_{n+1}(B).$$

(iii) For any m, $n \geqslant 1$ and any elementary types $\tau \in E_m$ and $\sigma \in E_n$, if $k = sup\{m, n\}$ then

$$(\sigma \times \tau) \in E_k(B).$$

We let $E(B) = \bigcup_{n \in \omega} E_n(B)$.

It is easily verified that if $\tau \in E_n(B)$ then τ is a type of order n. The motivation for the term hereditarily Horn type comes from the Curry Howard correspondence between propositions and types. Under this correspondence basic types can be interpreted as prepositional variables, $\to$ as logical implication and $\times$ as logical conjunction. (See for example [8].)

Proposition 3.15. *If A is extensional then for every type $\tau \in S$ there exists an elementary type $n(\tau) \in E(B)$ such that $FI(A)_\tau$ is homeomorphic with $FI(A)_{n(\tau)}$.*

Proof. By induction on the complexity of types using Theorems 3.11 and 3.12. □

An important theme in topology is to characterise the so-called separation and countability properties of topological spaces in general.

Separation properties concern the possibility to distinguish between distinct points and closed sets by means of open sets. An important case is the *Hausdorff* or T_2 space where any two distinct points have non-intersecting neighbourhoods. A topological space is *regular* if every point x and every closed set not containing x (a closed set can be regarded as a generalisation of a point) have disjoint neighbourhoods. A Hausdorff space which is regular is termed a T_3 space. A Hausdorff space is *separable* if it contains a countable dense subset, i.e. a countable subset, the topological closure of which contains the whole space.

Countability properties concern abstractly the "size" of the basis for a topology. An important case is a *second countable space*, which is a space that has a countable basis. The famous *Urysohn Metrization Lemma* gives that for every second countable space, the T_3 condition is equivalent to metrizability. A topological space is *metrizable* if, and only if, it has a basis consisting of open balls $O(x,r)$ where the open ball $O(x,r)$ around a point $x \in X$ of radius $r \in \mathbf{R}^+$ is given by

$$O(x,r) = \{y \in X \mid D(x,y) < r\}$$

and $D : X \times X \to \mathbf{R}^+$ is a *distance* or metric function from pairs of points to positive reals satisfying: (i) $D(x,x) = 0$, (ii) symmetry $D(x,y) = D(y,x)$ and (iii) the triangle inequality $D(x,z) \leqslant D(x,y) + D(y,z)$. For metrizable spaces, the concept of closesness between points is quite intuitive, though different metric functions can give rise to the same topology.

Recall from Definition 3.1 the concepts of closed set and the closure operation on a set. In order to investigate separation properties, we require the following fact.

Proposition 3.16. *For any types $\sigma, \tau \in S$ and any closed set $U \in FI(A)_{(\sigma\times\tau)}$, $proj_A^1(U)$ and $proj_A^2(U)$ are closed.*

Proof. Consider any $\sigma, \tau \in S$ and closed $U \in FI(A)_{(\sigma\times\tau)}$. It is a routine exercise to show that for any $V_1 \subseteq A_\sigma$ and $V_2 \subseteq A_\tau$,

$$cl(\langle V_1, V_2 \rangle_A) = \langle cl(V_1), cl(V_2) \rangle_A,$$

from which the result follows. □

Now we can consider the separation and countability properties of the finite information topology. When A is an extensional algebra these properties are quite strong.

Theorem 3.17. (i) *For each type $\tau \in S$, the topology $FI(A)_\tau$ is regular.*

(ii) *If for each basic type $\tau \in B$ the set A_τ is countable, then for each type $\tau \in S$, the topology $FI(A)_\tau$ is second countable.*

(iii) *If A is extensional then for each type $\tau \in S$, the topology $FI(A)_\tau$ is Hausdorff.*

Proof. By induction on the complexity of types.

Basis: Consider any basic type $\tau \in B$.

(i) By definition, $FI(A)_\tau = P(A_\tau)$ is regular.
(ii) By assumption, A_τ is countable, so clearly $FI(A)_\tau = P(A_\tau)$ is second countable.
(iii) Clearly $FI(A)_\tau = P(A_\tau)$ is Hausdorff.

Induction step: Consider any product type $(\sigma \times \tau) \in S$.

(i) Consider any $a \in A_{(\sigma\times\tau)}$ and closed subset $U \in FI(A)_{(\sigma\times\tau)}$ with $a \notin U$. Then either

$$proj_A^1(a) \notin proj_A^1(U) \tag{3.12}$$

or

$$proj_A^2(a) \notin proj_A^2(U). \tag{3.13}$$

Suppose that (3.12) holds. By the induction hypothesis $FI(A)_\sigma$ is regular and by Proposition 3.16, $proj_A^1(U)$ is closed. So there exist open $V, W \in FI(A)_\sigma$ with $proj_A^1(a) \in V$ and $proj_A^1(U) \subseteq W$ and

$$V \cap W = \emptyset.$$

So $a \in \langle V, A_\tau \rangle_A$, $U \subseteq \langle W, A_\tau \rangle_A$ and

$$\langle V, A_\tau \rangle_A \cap \langle W, A_\tau \rangle_A = \emptyset.$$

Also $\langle V, A_\tau \rangle_A$ and $\langle W, A_\tau \rangle$ are subbasic open. Similarly, if (3.13) holds then there exist open $V, W \in FI(A)_\tau$ with $a \in \langle A_\sigma, V \rangle_A$ $U \subseteq \langle A_\sigma, W \rangle_A$ and

$$\langle A_\sigma, V \rangle_A \cap \langle A_\sigma, W \rangle_A = \emptyset.$$

Since a and U were arbitrarily chosen, $FI(A)_{(\sigma\times\tau)}$ is regular.

(ii) By the induction hypothesis $FI(A)_\sigma$ and $FI(A)_\tau$ are second countable, i.e. have countable bases B_σ and B_τ. So

$$\{ \langle V_1, V_2 \rangle_A \mid V_1 \in B_\sigma, V_2 \in B_\tau \}$$

is a countable basis for $FI(A)_{(\sigma\times\tau)}$.

(iii) By the induction hypothesis, $FI(A)_\sigma$ and $FI(A)_\tau$ are Hausdorff. Now consider any $a, b \in A_{(\sigma\times\tau)}$ and suppose that $a \neq b$. By assumption A is extensional, so either

$$proj_A^1(a) \neq proj_A^1(b) \tag{3.13}$$

or

$$proj_A^2(a) \neq proj_A^2(b). \tag{3.14}$$

Suppose that (3.13) holds. Then since $FI(A)_\sigma$ is Hausdorff, there exist neighbourhoods $U_a \in Nbd(proj_A^1(a))$ and $U_b \in Nbd(proj_A^1(b))$ such that

$$U_a \cap U_b = 0.$$

So $\langle U_a, A_\tau \rangle_A$ and $\langle U_b, A_\tau \rangle_A$, are disjoint neighbourhoods of a and b, respectively. Similarly if (3.14) holds then a and b have disjoint neighbourhoods. Thus $FI(A)_{(\sigma\times\tau)}$ is Hausdorff.

Consider any function type $(\sigma \to \tau) \in S$. Then we prove the result by a subinduction on the complexity of τ.

Subbasis: Suppose that $\tau \in B$ is a basic type.

(i) To show that $FI(A)_{(\sigma\to\tau)}$ is regular, consider any $a \in A_{(\sigma\to\tau)}$ and closed subset $U \subseteq A_{(\sigma\to\tau)}$ such that $a \notin U$. Now for some open set $V \in FI(A)_{(\sigma\to\tau)}$, $U = cl(V)$ and since $a \notin cl(V)$ then for some open neighbourhood $Y \in FI(A)_{(\sigma\to\tau)}$ of a,

$$Y \bigcap V = \emptyset.$$

So for some basic open neighbourhood $W \in FI(A)_{(\sigma \to \tau)}$ of a,

$$W \bigcap V = \emptyset.$$

Then for any $b \in cl(V)$, $b \notin W$, for if $b \in W$ then W is an open neighbourhood of b which is disjoint from V contradicting the fact that b is adherent to V. Hence

$$W \bigcap cl(V) = \emptyset.$$

We construct an open set $X \in FI(A)_{(\sigma \to \tau)}$ such that

$$cl(V) \subseteq X \quad \text{and} \quad W \bigcap X = \emptyset.$$

Now for some $m \geqslant 1$ and basic open $Y_1, \ldots, Y_m \in FI(A)_\sigma$ and $a_1, \ldots, a_m \in A_\tau$,

$$W = O_{Y_1, a_1} \bigcap \cdots \bigcap O_{Y_m, a_m}.$$

Consider any $b \in cl(V)$. Then $b \notin W$ and so for some $1 \leqslant i(b) \leqslant m$,

$$b \notin O_{Y_{i(b)}, a_{i(b)}}.$$

Now b is continuous. So $b^{-1}(\{a_{i(b)}\})$ is open. Hence

$$Y_{i(b)} \nsubseteq b^{-1}(\{a_{i(b)}\}).$$

So for some $y_{i(b)} \in Y_{i(b)}$,

$$eval_A(b, y_{i(b)}) \neq a_{i(b)}.$$

Let $c_{i(b)} = eval_A(b, y_{i(b)})$ and consider the open set

$$b^{-1}(\{c_{i(b)}\}).$$

Then $y_{i(b)} \in b^{-1}(\{c_{i(b)}\})$ and so for some basic open $X_{i(b)} \in FI(A)_\sigma$ we have $y_{i(b)} \in X_{i(b)}$ and

$$X_{i(b)} \subseteq b^{-1}(\{c_{i(b)}\})$$

So $b \in O_{X_{i(b)}, c_{i(b)}}$ Now for any $a' \in O_{X_{i(b)}, c_{i(b)}}$,

$$eval_A(a', y_{i(b)}) \neq a_{i(b)}$$

and so $a' \notin O_{Y_{i(b)}, a_{i(b)}}$. Hence $a' \notin W$. Thus

$$O_{X_{i(b)}, c_{i(b)}} \cap W = \emptyset.$$

Then letting $X = \bigcup_{b \in cl(V)} O_{X_{i(b)}, c_{i(b)}}$ we have that X is open and $cl(V) \subseteq X$ and $X \cap W = \emptyset$. Since A and U were arbitrarily chosen then $FI(A)_{(\sigma \to \tau)}$ is regular.

(ii) By the induction hypothesis, $FI(A)_\sigma$ has a countable basis B_σ, and by assumption A_τ is countable. So there are at most countably many subbasic open sets $O_{U,a} \in FI(A)_{(\sigma \to \tau)}$ for $U \in B_\sigma$ and $a \in A_\tau$. Thus $FI(A)_{(\sigma \to \tau)}$ has a countable basis.

(iii) By the induction hypothesis, $FI(A)_\sigma$ is Hausdorff. Consider any $a, b \in A_{(\sigma \to \tau)}$ and suppose that $a \neq b$. Since A is extensional then for some $a_0 \in A_\sigma$, $eval_A(a, a_0) \neq eval_A(b, a_0)$. Since A has continuous data sets then a and b are continuous. So there exist basic open neighbourhoods

U_a, $U_b \in Nbd(a_0)$ such that for all $a' \in U_a$,

$$eval_A(a, a') = eval_A(a, a_0)$$

and for all $a' \in U_b$,

$$eval_A(b, a') = eval_A(b, a_0).$$

So $a \in O_{U_a, eval_A(a,a_0)}$ and $b \in O_{U_b, eval_A(b,a_0)}$ and

$$O_{U_a, eval_A(a,a_0)} \cap O_{U_b, eval_A(b,a_0)} = \emptyset.$$

Thus $FI(A)_{(\sigma \to \tau)}$ is Hausdorff.

Subinduction step: Suppose that τ is a product type or a function type. Then in each case we establish the result using the induction hypothesis and one of the homeomorphisms $fp_A^{-1}, uc_A, gfp_A^{-1}$ and gcu_A^{-1}. $\square$

Corollary 3.18. *Suppose that for each basic type $\tau \in B$, A_τ is a countable set.*

(i) *For every type $\tau \in S$, the topology $FI(A)_\tau$ is separable.*
(ii) *If A is extensional then for each type $\tau \in S$, the topology $FI(A)_\tau$ is a metric space.*

Proof. (i) Immediate, (ii) By Theorem 3.17(i) and (iii), for each $\tau \in S, FI(A)_\tau$ is a T_3 space. So by Theorem 3.17(ii) and the Urysohn Metrisation Lemma (see for example [5]), $FI(A)_\tau$ is a metric space. $\square$

An interesting open problem is to find a simple and explicit metric which gives our finite information topology, in the case that A is extensional. The usual proof of the Urysohn Metrisation Lemma does not provide any simple construction.

On the other hand, the separation properties of the finite information topology on nonextensional models must of necessity be weak, since equality between higher-order elements will no longer be extensional equality.

Lemma 3.19. *Suppose that A is not extensional, i.e. $A \nvDash$* **Ext**.

(i) *For some basic type $\tau \in B$ and some type $\sigma \in S$ there exist $a, b \in A_{(\sigma \to \tau)}$ such that $a \neq b$ but for all $a_0 \in A_\sigma$,*

$$eval_A(a, a_0) = eval_A(b, a_0).$$

(ii) *For some type $\tau \in S$, the topology $FI(A)_\tau$ is not a T_0 space.*

(iii) *For some basic type $\tau \in B$ and some type $\sigma \in S$ there exists $a \in A_{(\sigma \to \tau)}$ such that $\{a\}$ is not closed in $FI(A)_{(\sigma \to \tau)}$.*

Proof. (i) It suffices to show that for any types $\sigma, \tau \in S$, if there exist $a, b \in A_{(\sigma \to \tau)}$ such that $a \neq b$, but for all $a_0 \in A_\sigma$,

$$eval_A(a, a_0) = eval_A(b, a_0),$$

then there exists $\sigma' \in S$ and basic $\tau' \in B$ and $a', b' \in A_{(\sigma\to\tau)}$ such that $a' \neq b'$ but for all $a_0 \in A_\sigma$,

$$eval_A(a', a_0) = eval_A(b', a_0).$$

This can be proved by induction on the complexity of τ.

(ii) By (i), for some basic type $\tau \in B$ and $\sigma \in S$ there exist $a, b \in A_{(\sigma\to\tau)}$ such that $a \neq b$ but for all $a_0 \in A_\sigma$,

$$eval_A(a, a_0) = eval_A(b, a_0).$$

So for any basic open set $U \in FI(A)_\sigma$ and $a_0 \in A_\tau$,

$$a \in O_{U,a_0} \Leftrightarrow b \in O_{U,a_0}.$$

Since $a \neq b$ then $FI(A)_{(\sigma\to\tau)}$ is not a T_0 space.

(iii) By (i), for some basic type $\tau \in B$ and $\sigma \in S$ there exist $a, b \in A_{(\sigma\to\tau)}$ such that $a \neq b$ but for all $a_0 \in A_\sigma$,

$$eval_A(a, a_0) = eval_A(b, a_0).$$

Thus b is adherent to $\{a\}$ but $b \notin \{a\}$. So $\{a\}$ is not closed. □

We conclude this section with one proposition regarding discontinuous elements in the first-order initial model of a higher-order equational theory E. This model always exists (since E is equational), but may not be extensional. Nevertheless, the initial model characterises many properties of E and is therefore useful in the study of all models of E including extensional models, as we shall see in the next section.

Proposition 3.20. *Let $t, t' \in T(\Sigma)_{(\sigma\to\tau)}$ be any ground terms over a higher-order signature Σ containing the homeomorphism operators. Let E be any equational theory over Σ which contains the homeomorphism axioms* **Hom** *and let $I = T(\Sigma)/\equiv_E$ be the first-order initial model of E. If*

$$E \vdash eval(t, t_0) = eval(t', t_0)$$

for all $t_0 \in T(\Sigma)_{(\sigma)}$ and t_I is a discontinuous element in I then t'_I is also discontinuous in I.

Proof. By induction on the complexity of τ using the homeomorphism axioms **Hom**. □

4. Conservativity, normal forms and term-rewriting

In this section we establish the main result of this paper, which gives a necessary and sufficient condition for infinitary higher-order equational logic to be conservative over first-order equational logic for ground first-order equations. We then apply this result to characterise the conservativity of finitary higher-order equational logic over first-order equational logic for arbitrary first-order equations. For these results we introduce a notion of *observational equivalence* on the elements of a higher-order algebra A. Two elements $a, b \in A_\tau$ of type $\tau \in S$ are observationally equivalent when they cannot be distinguished by any basic open set U, i.e. $a \in U \Leftrightarrow b \in U$. Recall that basic open sets in the finite information topology are sets of elements containing the same finite information. Thus this notion of equivalence can with some justification be claimed to be observable. We will prove that conservativity arises precisely when observational equivalence forms a congruence on the initial model $I(\Sigma, E)$ of a higher-order equational theory $E \subseteq Eqn(\Sigma, X)$.

We begin by showing how the structure of infinitary higher-order equational proofs can be simplified, both by eliminating use of the projection rules, and by limiting the use of ω-extensionality rules to certain structurally simple types. Then we show how the structure of such proofs can be further simplified by translating them into *term rewriting proofs with ω-rewrite steps*, which generalise first-order term rewriting proofs to the higher-order case. Finally we prove our main results.

Our first lemma asserts that in the presence of the pairing operation and its axioms, the projection rule for product types can be eliminated from higher-order equational proofs.

Lemma 4.1. *Let $E \subseteq Eqn(\Sigma, X)$ be any higher-order equational theory containing the homeomorphism axioms* **Hom**. *For any type $\tau \in S$ and any terms $t, t' \in T(\Sigma, X)_\tau$, if*

$$E \vdash_\omega t = t'$$

then there exists a proof P of $t = t'$ from E using the rules of infinitary higher-order equational logic without the projection rules 2.5(v).

Proof. By induction on the complexity of proofs, using Eq. (3.9a). $\square$

Next, we show that in the presence of currying, uncurrying, function-pairing and inverse function-pairing and their axioms, all instances of the ω-extensionality rule in a proof can be replaced by instances of basic ω-extensionality rules. (Recall that the ω-extensionality rule for a type $(\sigma \to \tau)$ is basic, if, and only if, τ is a basic type, $\tau \in B$.)

Lemma 4.2. *Let $E \subseteq Eqn(\Sigma, X)$ be any higher-order equational theory containing the homeomorphism axioms* **Hom**. *For any type $\tau \in S$ and any terms $t, t' \in T(\Sigma, X)_\tau$, if*

$$E \vdash_\omega t = t',$$

then there exists a proof P of $t = t'$ from E using the rules of infinitary higher-order equational logic with instances of basic ω-extensionality rules only.

Proof. We prove the result in two stages. Firstly, we establish that for any higher-order equational theory $E \subseteq Eqn(\Sigma, X)$ containing **Hom**, and for any types $\sigma, \tau \in S$ and any terms $t, t' \in T(\Sigma, X)_{(\sigma \to \tau)}$, if there exists a proof P of $t = t'$ from E in which the final inference uses the ω-extensionality rule and any other instances of this rule are basic, then there exists a proof P' of $t = t'$ in which all instances of the ω-extensionality rule are basic.

We prove this result by induction on the complexity of τ.

(i) *Basis*: Suppose that $\tau \in B$ is a basic type, then the result holds trivially.

(ii) *Induction step*: Suppose that τ is a product type, $\tau = (\tau_1 \times \tau_2)$. Let P be a proof of $t = t'$ from E in which the final step uses the ω-extensionality rule and all other uses are basic. Then P has the form

$$\frac{\langle P(t_0) \mid t_0 \in T(\Sigma)_\sigma \rangle}{t = t'},$$

where for each $t_0 \in T(\Sigma)_\sigma$, $P(t_0)$ is a proof of

$$eval(t, t_0) = eval(t', t_0)$$

using basic ω-extensionality rules only.

Consider any $t_0 \in T(\Sigma)_\sigma$. Then, using $P(t_0)$ and Eqs. 3.4.(3.c), (9.b) and (9.c), we can derive

$$eval(proj^1(fp^{-1}(t)), t_0) = eval(proj^1(fp^{-1}(t')), t_0)$$

and

$$eval(proj^2(fp^{-1}(t)), t_0) = eval(proj^2(fp^{-1}(t')), t_0).$$

Since t_0 was arbitrarily chosen then we obtain proofs of

$$proj^1(fp^{-1}(t)) = proj^1(fp^{-1}(t')), \quad proj^2(fp^{-1}(t)) = proj^2(fp^{-1}(t'))$$

in which the final inference uses the ω-extensionality rule for the types $(\sigma \to \tau_1)$ and $(\sigma \to \tau_2)$ respectively. So by the induction hypothesis there exist proofs P_1 and P_2 of these equations using basic ω-extensionality rules only. Using such P_1 and P_2 and substitution we can derive

$$\langle proj^1(fp^{-1}(t)), proj^2(fp^{-1}(t)) \rangle = \langle proj^1(fp^{-1}(t')), proj^2(fp^{-1}(t')) \rangle,$$

and then, using Eq. (3.9a),

$$fp^{-1}(t) = fp^{-1}(t'),$$

and by substitution

$$fp(fp^{-1}(t)) = fp(fp^{-1}(t')).$$

Finally, using Eq. (3.3a), we can derive the equation $t = t'$.

(iii) Suppose that τ is a function type $\tau = (\tau_1 \to \tau_2)$. Let P be a proof of $t = t'$ from E in which the final inference uses the ω-extensionality rule and all other instances are basic. Then P has the form

$$\frac{\langle P(t_0) \mid t_0 \in T(\Sigma)_\sigma \rangle}{t = t'},$$

where for each $t_0 \in T(\Sigma)_\sigma$, $P(t_0)$ is a proof of

$$eval(t, t_0) = eval(t', t_0)$$

using basic ω-extensionality rules only.

Consider any term $t_0 \in T(\Sigma)_{(\sigma \times \tau_1)}$. Then using the proof $P(proj^1(t_0))$ we can derive

$$eval(t, proj^1(t_0)) = eval(t', proj^1(t_0)),$$

and hence

$$eval(eval(t, proj^1(t_0)), proj^2(t_0)) = eval(eval(t', proj^1(t_0)), proj^2(t_0)).$$

Then using Eq. (3.1c) we can derive

$$eval(uc(t), \langle proj^1(t_0), proj^2(t_0)\rangle) = eval(uc(t'), \langle proj^1(t_0), proj^2(t_0)\rangle)$$

and then using Eq. (3.9a),

$$eval(uc(t), t_0) = eval(uc(t'), t_0).$$

Since t_0 was arbitrarily chosen we can derive

$$uc(t) = uc(t')$$

from E, where the final inference uses the ω-extensionality rule for the type $((\sigma \times \tau_1) \to \tau_2)$ and all other instances of this rule are basic. Thus by the induction hypothesis there exists a proof P' of $uc(t) = uc(t')$ using basic ω-extensionality rules only. Using such P' and substitution we can derive

$$cu(uc(t)) = cu(uc(t'))$$

and hence by Eq. (3.1a),

$$t = t'.$$

This completes the induction.

We now prove the main lemma using the above fact, by induction on the complexity of proofs.

(i) *Basis*: Consider any proof P of $t = t'$ from E consisting of a single axiom or a single use of the reflexivity rule. Then the result holds trivially.

(ii) *Induction step*: Consider any proof P of $t = t'$ from E in which the final inference uses one of the rules of symmetry, transitivity, substitution or projection. Then the result follows trivially from the induction hypothesis.

(iii) Consider any proof P of $t = t'$ from E in which the final inference uses the ω-extensionality rule for a type $(\sigma \to \tau)$. Then P has the form

$$\frac{\langle P(t_0) \mid t_0 \in T(\Sigma)_\sigma \rangle}{t = t'},$$

where for each $t_0 \in T(\Sigma)_\sigma$, $P(t_0)$ is a proof of

$$eval(t, t_0) = eval(t', t_0).$$

By the induction hypothesis, for each $t_0 \in T(\Sigma)_\sigma$ there exists a proof $P'(t_0)$ of

$$eval(t, t_0) = eval(t', t_0)$$

from E using basic ω-extensionality rules only. So

$$\frac{\langle P'(t_0) \mid t_0 \in T(\Sigma)_\sigma \rangle}{t = t'}$$

is a proof of $t = t'$ in which the final inference uses the ω-extensionality rule and all other instances of this rule are basic. Thus by the above result there exists a proof Q of $t = t'$ in which all instances of the ω-extensionality rule are basic. $\square$

To simplify the structure of infinitary higher-order equational proofs even further, we introduce the notion of a *term rewriting proof with ω-rewrite steps*. This proof system extends the correspondence between many-sorted first-order equational logic and many-sorted term rewriting to the infinitary higher-order case in an obvious way. However, the structure of a term rewriting proof imposes a normal form on the use of the transitivity rule that is useful for proving our main theorem.

We assume the reader is familiar with the definitions of an *occurrence* $\bar{i} \in \mathbf{N}^*$ in a term $t \in T(\Sigma, X)$ and the set $Occ(t)$ of all occurrences in t, the *subterm of t at the occurrence* $\bar{i} \in Occ(t)$, denoted by $t(\bar{i})$, and the *substitution* of a term t' at $\bar{i}$ in t, denoted by $t(\bar{i}/t')$. The reader may consult any introductory text on term rewriting, for example [16].

Definition 4.3. Let $E \subseteq Eqn(\Sigma, X)$ be any higher-order equational theory.

(i) A *rewrite step s from E* is a five tuple

$$s = (t, \bar{i}, t_l = t_r, \alpha, t'),$$

where for some types $\sigma, \tau \in S$, terms $t, t' \in T(\Sigma, X)_\tau$, occurrence $\bar{i} \in Occ(t)_\sigma$, terms $t_l, t_r \in T(\Sigma, X)_\sigma$ and assignment $\alpha : X \to T(\Sigma, X)$, either: (a) t_l is t_r, or (b) $t_l = t_r \in E$, or (c) $t_r = t_l \in E$, and in each case $t(\bar{i}) = \bar{\alpha}(t_l)$ and $t' = t(\bar{i}/\bar{\alpha}(t_r))$. We define the *degree* of s to be $deg(s) = 0$.

(ii) An *ω-rewrite step s from E* is a six tuple

$$s = (t, \bar{i}, t_l = t_r, \bar{s}, \alpha, t'),$$

where for some type $\tau \in S$, $t, t' \in T(\Sigma, X)_\tau$ are terms and for some function type $(\sigma \to \delta) \in S$ and occurrence $\bar{i} \in Occ(t)_{(\sigma \to \delta)}$, and terms $t_l, t_r \in T(\Sigma, X)_{(\sigma \to \delta)}$

$$\bar{s} = \langle \bar{s}(t_0) \mid t_0 \in T(\Sigma)_\sigma \rangle$$

is a family of rewrite proofs, $\bar{s}(t_0)$ being a rewrite proof of $eval(t_1, t_0) = eval(tr, t_0)$ from E with ω-rewrite steps for each $t_0 \in T(\Sigma)_\sigma$, and $\alpha : X \to T(\Sigma, X)$ is an assignment such that $t(\bar{i}) = \bar{\alpha}(t_l)$ and $t' = t(\bar{i}/\bar{\alpha}(t_r))$. We define the *degree* of s to be

$$deg(s) = 1 + sup\{deg(\bar{s}(t_0)) \mid t_0 \in T(\Sigma)_\sigma\}.$$

We say that s is *basic ω-rewrite step* if, and only if, $\delta \in B$ is a basic type.

(iii) A *rewrite proof P from E with ω-rewrite steps* is a nonempty finite sequence

$$P = s_1, \ldots, s_n,$$

where for each $1 \leqslant j \leqslant n$, either:

(a) $s_j = (t_j, \bar{i}_j, t_l^j = t_r^j, \alpha_j, t_j')$, is a rewrite step from E, or

(b) $s_j = (t_j, \bar{i}_j, t_l^j = t_r^j, \bar{s}_j, \alpha_j, t_j')$ is an ω-rewrite step from E,

and for each $1 \leqslant j \leqslant n-1$, t_j' is t_{j+1}. We say that P is a rewrite proof of $t_1 = t_n'$ from E with ω-rewrite steps. We define the *degree* of P to be

$$deg(P) = sup\{deg(s_1), \ldots, deg(s_n)\}.$$

If there exists a rewrite proof P of $t = t'$ from E with ω-rewrite steps of degree $\alpha \in Ord$ then we write

$$t \xrightarrow{E, \omega, \alpha} t',$$

or simply, if the order of the rewrite proof is irrelevant,

$$t \xrightarrow{E,\omega} t'.$$

If P has degree 0 then we may write

$$t \xrightarrow{E} t'$$

following the conventional notation, and say that P is a *rewrite proof from* E of $t = t'$.

The well-known correspondence between term rewriting proofs and derivations in first-order equational logic extends to rewrite proofs with ω-rewrite steps and infinitary higher-order equational logic in the obvious way.

Correspondence Theorem 4.4. *Let $E \subseteq Eqn(\Sigma, X)$ be any higher-order equational theory. For any type $\tau \in S$ and any terms $t, t' \in T(\Sigma, X)_\tau$:*

(i) $E \vdash t = t' \Leftrightarrow t \xrightarrow{E} t'$,

(ii) *for any* $\alpha \in Ord$, $E \vdash_{\omega,\alpha} t = t' \Leftrightarrow t \xrightarrow{E,\omega,\alpha} t'$,

(iii) *$t = t'$ is provable from E using the infinitary rules of higher-order equational logic and basic ω-extensionality rules only if, and only if, there exists a rewrite proof of $t = t'$ from E using basic ω-rewrite steps only.*

Proof. (i) See for example [6]. (ii) By induction on the degree of infinitary proofs and rewrite proofs (using (i) as the induction basis), and Lemma 4.1. (iii) By inspection the proof of (ii). $\square$

We will require an elementary fact about rewrite proofs (without ω-rewrite steps) of ground equations.

Proposition 4.5. *Let $E \subseteq Eqn(\Sigma, X)$ be any higher-order equational theory. For any type $\tau \in S$ and any ground terms $t, t' \in T(\Sigma)_\tau$ and any $n \geqslant 1$, if there exists a rewrite proof of $t = t'$ of length n from E (without ω-rewrite steps) then there exists a rewrite proof of $t = t'$ of length n where all intermediary terms are ground.*

Proof. By induction on n. $\square$

Next, we introduce our notion of observational equivalence on elements of a higher-order algebra.

Definition 4.6. Let A be any Σ algebra (not necessarily extensional). For each $\tau \in S$, we define the relation $\equiv_\tau^{\text{obs}} \subseteq A_\tau^2$ of *observational equivalence* by

$$a \equiv_\tau^{\text{obs}} b \;\Leftrightarrow\; \text{for all subbasic open } U \in FI(A)_\tau, a \in U \Leftrightarrow b \in U.$$

By definition $a, b \in A_\tau$ are observationally equivalent if, and only if, no open set, in particular no basic open set (i.e. finite observation) can separate them. Note that since the subbasic open sets of the finite information topology can be defined for any algebra A of signature Σ (irrespective of whether A is extensional or has continuous carrier sets) then the notion of observational equivalence can also be defined for any Σ algebra A. In particular, observational equivalence can always be defined on elements of the first-order initial algebra $I(\Sigma, E)$ of an equational theory $E \subseteq Eqn(\Sigma, X)$, even if this algebra is not extensional. Furthermore, for any algebra A of signature Σ, all discontinuous elements of a particular carrier set A_τ will belong to the same equivalence class.

We note at this point that the term observational equivalence has been used elsewhere in the literature on algebraic structures in data type theory. (See for example [26] and the references cited there.) Our use of the term is specific to the topology introduced in this paper, and there appears to be no connection to the concept previously known in data type theory.

First we note that observational equivalence is a finer equivalence relation on terms than provable equivalence using infinitary higher-order equational logic.

Lemma 4.7. *Let $E \subseteq Eqn(\Sigma, X)$ be any equational theory. For any $\tau \in S$, and any ground terms $t, t' \in T(\Sigma)_\tau$,*

$$t \equiv_\tau^{\text{obs}} t' \Rightarrow E \vdash_\omega t = t'.$$

Proof. By induction on the complexity of τ. $\square$

Now we can establish our main result concerning conservativity.

Conservativity Theorem 4.8. Let *$E \subseteq Eqn(\Sigma, X)$ be any higher-order equational theory which contains the homeomorphism axioms* **Hom**. *Then infinitary higher-order equational logic is conservative over equational logic on E for ground first-order equations if, and only if, observational equivalence $\equiv^{\text{obs}}$ is a congruence on the (first-order) initial model $I(\Sigma, E)$.*

Proof. ($\Rightarrow$) We prove the contrapositive by showing that for any type $\tau \in S$ and any term $t(x_1, \ldots, x_n) \in T(\Sigma, X)_\tau$ with n free variables $x_i \in X_{\tau(i)}$ for $1 \leqslant i \leqslant n$, and for any ground terms $t_i, t_i' \in T(\Sigma)_{\tau(i)}$ for $1 \leqslant i \leqslant n$, if

$$[t_i] \equiv_{\tau(i)}^{\text{obs}} [t_i']$$

for each $1 \leqslant i \leqslant n$, but

$$[t(t_1, \ldots, t_n)] \not\equiv_\tau^{\text{obs}} [t(t_1' \ldots, t_n')],$$

then there exists a basic type $\tau' \in B$ and ground terms $t_0, t_0' \in T(\Sigma)_\tau$ such that

$$E \vdash_\omega t_0 = t_0'$$

but

$$E \nvdash t_0 = t_0'.$$

We prove this by induction on the complexity of τ.

(i) *Basis*: Suppose that $\tau \in B$ is a basic type. By assumption $[t_i] \equiv^{\text{obs}}_{\tau(i)} [t'_i]$ for $1 \leqslant i \leqslant n$, so by Lemma 4.7, $E \vdash_\omega t_i = t'_i$ for each $1 \leqslant i \leqslant n$. So by substitution,

$$E \vdash_\omega t(t_1, \ldots, t_n) = t(t'_1, \ldots, t'_n).$$

But also by assumption

$$[t(t_1, \ldots, t_n)] \not\equiv^{\text{obs}}_{\tau} [t(t'_1, \ldots, t'_n)]$$

so

$$E \nvdash t(t_1, \ldots, t_n) = t(t'_1, \ldots, t'_n)$$

since τ is basic. Thus taking $\tau' = \tau$, $t_0 = t(t_1, \ldots, t_n)$ and $t'_0 = t(t'_1, \ldots, t'_n)$ then the result follows.

(ii) *Induction step*: Consider any product type $\tau = (\sigma \times \delta)$ and suppose

$$[t(t_1, \ldots, t_n)] \not\equiv^{\text{obs}}_{(\sigma \times \delta)} [t(t'_1, \ldots, t'_n)],$$

but $[t_i] \equiv^{\text{obs}}_{\tau(i)} [t'_i]$ for $1 \leqslant i \leqslant n$. Then there exists a subbasic open set $U \in FI(I(\Sigma, E))_{(\sigma \times \delta)}$ such that, using symmetry,

$$[t(t_1, \ldots, t_n)] \in U \quad \text{and} \quad [t(t'_1, \ldots, t'_n)] \notin U.$$

Suppose that U has the form $\langle V_1, C(I(\Sigma, E))_\delta \rangle_{I(\Sigma,E)}$ for some open $V_1 \in FI(I(\Sigma, E))_\sigma$. Then

$$[proj^1(t(t_1, \ldots, t_n))] \in V_1 \quad \text{and} \quad [proj^1(t(t'_1, \ldots, t'_n))] \notin V_1.$$

Thus

$$[proj^1(t(t_1, \ldots, t_n))] \not\equiv^{\text{obs}}_{\sigma} [proj^1(t(t'_1, \ldots, t'_n))].$$

Thus by the induction hypothesis there exists $\tau' \in B$ and $t_0, t'_0 \in T(\Sigma)_{\tau'}$ such that

$$E \vdash_\omega t_0 = t'_0$$

but $E \nvdash t_0 = t'_0$. Similarly the result holds if U has the form $\langle C(I(\Sigma, E))_\delta, V_2 \rangle_{I(\Sigma,E)}$ for some open $V_2 \in FI(I(\Sigma, E))_\delta$.

(iii) Consider any function type $\tau = (\sigma \to \delta)$. We prove the result by subinduction on the complexity of δ.

(iii.a) *Subbasis*: Suppose $\delta \in B$ is a basic type and $[t_i] \equiv^{\text{obs}}_{\tau(i)} [t'_i]$ for $1 \leqslant i \leqslant n$ but

$$t(t_1, \ldots, t_n) \not\equiv^{\text{obs}}_{(\sigma \to \delta)} t(t'_1, \ldots, t'_n).$$

Then there exists a subbasic open set $U \in FI(I(\Sigma, E))_{(\sigma \to \delta)}$ such that, using symmetry,

$$[t(t_1, \ldots, t_n)] \in U \quad \text{and} \quad [t(t'_1, \ldots, t'_n)] \notin U$$

Then for some non-empty open set $V \in FI(I(\Sigma, E))_\sigma$ and element $a \in I(\Sigma, E)_\delta$ we have

$$[t(t_1, \ldots, t_n)] \in O_{V,\{a\}} \quad \text{and} \quad [t(t'_1, \ldots, t'_n)] \notin O_{V,\{a\}}.$$

So for some $t' \in T(\Sigma)_\sigma$ with $[t'] \in V$ we have

$$eval_{I(\Sigma,E)}(t(t_1, \ldots, t_n)], [t']) = a \quad \text{and} \quad eval_{I(\Sigma,E)}([t(t'_1, \ldots, t'_n)], [t']) \neq a.$$

So

$$[eval(t(t_1, \ldots, t_n), t')] \neq [eval(t(t'_1, \ldots, t'_n), t')],$$

and hence

$$[eval(t(t_1, \ldots, t_n), t')] \not\equiv_\delta^{\rm obs} [eval(t(t'_1, \ldots, t'_n), t')],$$

since δ is basic. By assumption $[t_i] \equiv_{\tau(i)}^{\rm obs} [t'_i]$ for $1 \leqslant i \leqslant n$, so the result follows from the induction basis.

(iii.b) *Subinduction step*: Suppose that $\delta = (\delta_1 \times \delta_2)$ is a product type, and

$$[t_i] \equiv_{\tau(i)}^{\rm obs} [t'_i]$$

for $1 \leqslant i \leqslant n$, but

$$[t(t_1, \ldots, t_n)] \not\equiv_{(\sigma \to (\delta_1, \times \delta_2))}^{\rm obs} [t(t'_1, \ldots, t'_n)].$$

Then there exists a subbasic open set $U \in FI(I(\Sigma, E))_{(\sigma \to (\delta_1 \times \delta_2))}$ such that, using symmetry,

$$[t(t_1, \ldots, t_n)] \in U \quad \text{and} \quad [t(t'_1, \ldots, t'_n)] \notin U.$$

Therefore

$$[fp^{-1}(t(t_1, \ldots, t_n))] \in fp^{-1}_{I(\Sigma,E)}(U) \quad \text{and} \quad [fp^{-1}(t(t'_1 \ldots, t'_n))] \notin fp^{-1}_{I(\Sigma,E)}(U),$$

and thus

$$[fp^{-1}(t(t_1, \ldots, t_n))] \not\equiv_{((\sigma \to \delta_1) \times (\sigma \to \delta_2))}^{\rm obs} [fp^{-1}(t(t'_1, \ldots, t'_n))].$$

So by the subinduction hypothesis there exists $\tau' \in B$ and $t_0, t'_0 \in T(\Sigma)_{\tau'}$ such that

$$E \vdash_\omega t_0 = t'_0$$

but

$$E \nvdash t_0 = t'_0.$$

(iii.c) Suppose that $\delta = (\delta_1 \to \delta_2)$ is a function type. We prove the result by a subinduction on the complexity of δ_2, the proof in each of the three cases is similar to (iii.b) above using the appropriate operator ($uc_{I(\Sigma,E)}$, $gfp^{-1}_{I(\Sigma,E)}$ and $gcu^{-1}_{I(\Sigma,E)}$, respectively) and is omitted.

($\Leftarrow$) We prove the contrapositive, i.e. we show that if infinitary higher-order equational logic is not conservative over equational logic on E for ground first-order equations then $\equiv^{\rm obs}$ is not a congruence on $I(\Sigma, E)$.

Suppose for some basic type $\tau \in B$ and ground first-order terms $t, t' \in T(\Sigma)_\tau$ that

$$E \vdash_\omega t = t'$$

but

$$E \nvdash t = t'.$$

Then by Lemma 4.2 and Correspondence Theorem 4.4(iii) there exists a rewrite proof of $t = t'$ from E using basic ω-rewrite steps only,

$$t \xrightarrow{E,\omega,\beta} t'$$

for some $\beta \in Ord$ with $\beta > 0$, but

$$t \overset{E}{\not\mapsto} t'.$$

We prove the result by transfinite induction on the degree of rewrite proofs using basic ω-rewrite steps only.

Basis: Suppose that $\beta = 1$ and that

$$t \xrightarrow{E,\omega,1} t'$$

but

$$t \overset{E}{\not\mapsto} t'.$$

Let $\bar{s} = s_1, \ldots, s_k$ be a rewrite proof of $t = t'$ with basic ω-rewrite steps of degree 1. Then we can choose $1 \leqslant j \leqslant k$ such that s_j is a basic ω-rewrite step of degree 1,

$$s_j = (t_j, \overline{i_j}, t_l^j = t_r^j, \overline{s_j}, \alpha_j, t_j'),$$

where for some function type $(\sigma \to \delta) \in S$ with $\delta \in B$, we have $t_l^j, t_r^j \in T(\Sigma, X)_{(\sigma \to \delta)}$ and

$$\overline{s_j} = \langle \overline{s_j}(t_0) \mid t_0 \in T(\Sigma)_\sigma \rangle$$

is a family of rewrite proofs such that for each $t_0 \in T(\Sigma)_\sigma$,

$$\overline{s_j}(t_0) = \overline{s_j}(t_0)_1, \ldots, \overline{s_j}(t_0)_{k_j(t_0)}$$

is a rewrite proof of length $k_j(t_0)$ of degree 0 (i.e. without ω-rewrite steps) of the equation

$$eval(t_l^j, t_0) = eval(t_r^j, t_0).$$

Furthermore since $t \overset{E}{\not\mapsto} t'$ it follows that for such j,

$$t_j \overset{E}{\not\mapsto} t_j'. \tag{4.1}$$

By Proposition 4.5, $\bar{s}$ can be chosen so that t_j and t_j' are ground; therefore $\overline{\alpha_j}(t_l^j)$ and $\overline{\alpha_j}(t_r^j)$ are ground. Now by the Correspondence Theorem 4.4(i), for each ground term $t_0 \in T(\Sigma)_\sigma$,

$$E \vdash eval(t_l^j, t_0) = eval(t_r^j, t_0).$$

So for each $t_0 \in T(\Sigma)_\sigma$ by substitution,

$$E \vdash eval(\overline{\alpha_j}(t_l^j), t_0) = eval(\overline{\alpha_j}(t_r^j), t_0). \tag{4.2}$$

Suppose there is no subbasic open $U \in FI(I(\Sigma, E))_{(\sigma \to \delta)}$ with $[\overline{\alpha_j}(t_l^j)] \in U$. Then clearly $[\overline{\alpha_j}(t_l^j)]$ is discontinuous. So by (4.2) and Proposition 3.20, $[\overline{\alpha_j}(t_r^j)]$ is discontinuous, and so there is no subbasic open $U \in FI(I(\Sigma, E))_{(\sigma \to \delta)}$ with $[\overline{\alpha_j}(t_r^j)] \in U$. Therefore

$$[\overline{\alpha_j}(t_r^j)] \equiv^{\mathrm{obs}}_{(\sigma \to \delta)} [\overline{\alpha_j}(t_l^j)].$$

Suppose there is a subbasic open set $U \in FI(I(\Sigma, E))_{(\sigma \to \delta)}$ with $[\overline{\alpha_j}(t_l^j)] \in U$. Consider any such set U. Then by definition for some open set $V \in FI(I(\Sigma, E))_\sigma$ and $t' \in T(\Sigma)_\delta$, $U = O_{V,\{[t']\}}$.

Consider any $[t_0] \in V$. Then by definition,

$$eval_{I(\Sigma,E)}([\overline{\alpha_j}(t_l^j)], [t_0]) = [t'].$$

So $E \vdash eval(\overline{\alpha_j}(t_l^j), t_0) = t'$. Thus by (4.2), $E \vdash eval(\overline{\alpha_j}(t_r^j), t_0) = t'$. So

$$eval_{I(\Sigma,E)}([\overline{\alpha_j}(t_r^j)], [t_0]) = [t'].$$

Since $[t_0]$ was arbitrarily chosen then $[\overline{\alpha_j}(t_r^j)] \in U$. Since U was arbitrarily chosen, by symmetry of t_l^j and t_r^j,

$$[\overline{\alpha_j}(t_r^j)] \equiv^{\mathrm{obs}}_{(\sigma \to \delta)} [\overline{\alpha_j}(t_l^j)].$$

Suppose for a contradiction that $\equiv^{\mathrm{obs}}$ is a congruence. Then

$$[t_j] \equiv^{\mathrm{obs}}_{\tau} [t_j(\overline{i_j}/\overline{\alpha_j}(t_r^j))]$$

but since τ is a basic type it follows that

$$[t_j] = [t_j(\overline{i_j}/\overline{\alpha_j}(t_r^j))]$$

i.e.

$$E \vdash t_j = t_j(\overline{i_j}/\overline{\alpha_j}(t_r^j)),$$

which contradicts (4.1) above. Thus $\equiv^{\mathrm{obs}}$ is not a congruence.

Induction step: Suppose that $\beta > 1$ and that

$$t \xrightarrow{E,\omega,\beta} t'$$

with only basic ω-rewrite steps, but

$$t \overset{E}{\not\mapsto} t'.$$

Let $\bar{s} = s_1, \ldots, s_k$ be a rewrite proof with basic ω-steps of the equation $t = t'$ with degree β. Then for some $1 \leqslant j \leqslant k$, s_j is a basic ω-rewrite step of degree $\beta_j \leqslant \beta$,

$$s_j = (t_j, \overline{i_j}, t_l^j = t_r^j, \overline{s_j}, \alpha_j, t_j'),$$

where for some function type $(\sigma \to \delta) \in S$ with $\delta \in B$ we have $t_l^j, t_r^j \in T(\Sigma, X)_{(\sigma \to \delta)}$ and

$$\overline{s_j} = \langle \overline{s_j}(t_0) \mid t_0 \in T(\Sigma)_\sigma \rangle$$

is a family of rewrite proofs, such that for each $t_0 \in T(\Sigma)_\sigma$,

$$\overline{s_j}(t_0) = \overline{s_j}(t_0)_1, \ldots, \overline{s_j}(t_0)_{k_j(t_0)}$$

is a rewrite proof with basic ω-rewrite steps of the equation

$$eval(t_l^j, t_0) = eval(t_r^j, t_0)$$

of degree $\beta_j(t_0) < \beta_j$. Also

$$t_j \overset{E}{\not\mapsto} t_j'. \tag{4.3}$$

Again by Proposition 4.5, $\bar{s}$ can be chosen so that t_j and t'_j are ground. Therefore $\overline{\alpha_j}(t_l^j)$ and $\overline{\alpha_j}(t_r^j)$ are ground.

By the Correspondence Theorem 4.4(iii), for each $t_0 \in T(\Sigma)_\sigma$

$$E \vdash_{\omega,\beta_j(t_0)} eval(t_l^j, t_0) = eval(t_r^j, t_0)$$

using only basic ω-evaluation rules, and so by substitution, for each $t_0 \in T(\Sigma)_\sigma$,

$$E \vdash_{\omega,\beta_j(t_0)} eval(\overline{\alpha_j}(t_l^j), t_0) = eval(\overline{\alpha_j}(t_r^j), t_0).$$

Thus by the Correspondence Theorem 4.4(iii), for each $t_0 \in T(\Sigma)_\sigma$,

$$eval(\overline{\alpha_j}(t_l^j), t_0) \xrightarrow{E,\omega,\beta_j(t_0)} eval(\overline{\alpha_j}(t_r^j), t_0) \tag{4.4}$$

using basic ω-rewrite steps only.

Suppose that for some $t_0 \in T(\Sigma)_\sigma$,

$$eval(\overline{\alpha_j}(t_l^j), t_0) \overset{E}{\not\mapsto} eval(\overline{\alpha_j}(t_r^j), t_0).$$

Then by (4.4) and the induction hypothesis, $\equiv^{\text{obs}}$ is not a congruence. Suppose that for each $t_0 \in T(\Sigma)_\sigma$,

$$eval(\overline{\alpha_j}(t_l^j), t_0) \xrightarrow{E} eval(\overline{\alpha_j}(t_r^j), t_0).$$

Then again as for the induction basis we have $\overline{\alpha_j}(t_l^j) \equiv^{\text{obs}}_{(\sigma\to\delta)} \overline{\alpha_j}(t_r^j)$. So the assumption that $\equiv^{\text{obs}}$ is a congruence contradicts (4.3). $\square$

Since our notion of observational equivalence is somewhat unfamiliar, it is natural to seek stronger conditions which imply the congruence property and which themselves seem more natural or easier to verify. One such natural condition on a higher-order algebra is continuity, and this condition is indeed strong enough to imply the congruence property. Recall from Definition 3.1 what it means for an S-sorted Σ algebra A to be continuous in an S-indexed family T of topologies on the carriers of A, namely that each operation of A is continuous. In the case that A is a higher-order algebra and T is the finite information topology, the fact that A has continuous carrier sets does not imply that all operations of A are continuous, since these may not be elements of the carriers of A. (Recall the discussion following Definition 2.3.)

Lemma 4.9. *Let Σ be a higher-order signature containing the homeomorphism operators. Let A be a Σ algebra A with continuous carrier sets. If A is continuous in the finite information topology then $\equiv^{\text{obs}}$ is a congruence on A.*

Proof. Consider any $n \geqslant 1$, any $w = \tau(1)\ldots\tau(n) \in S^+$ and any function symbol $f \in \Sigma_{w,\tau}$. Consider any $a_i, b_i \in A_{\tau(i)}$ for $1 \leqslant i \leqslant n$ and suppose

$$a_i \equiv^{\text{obs}}_{\tau(i)} b_i,$$

for $1 \leqslant i \leqslant n$. We must show that

$$f_A(a_1, \ldots, a_n) \equiv_{\tau}^{\text{obs}} f_A(b_1, \ldots, b_n). \tag{4.5}$$

Now consider any subbasic open set $U \in FI(A)_{\tau}$ and suppose that

$$f_A(a_1, \ldots, a_n) \in U.$$

Since A has continuous carrier sets and is continuous in the finite information topology there exists $V_i \in Nbd(a_i)$ for each $1 \leqslant i \leqslant n$ such that for all $v_i \in V_i$ for $1 \leqslant i \leqslant n$,

$$f_A(v_1, \ldots, v_n) \in U.$$

Now since $a_i \equiv_{\tau(i)}^{\text{obs}} b_i$ for each $1 \leqslant i \leqslant n$ then $b_i \in V_i$ for each $1 \leqslant i \leqslant n$. Thus

$$f_A(b_1, \ldots, b_n) \in U.$$

Since U was arbitrarily chosen, then (4.5) holds by the symmetry of a_i and b_i. $\square$

Lemma 4.9 is particularly useful in studying second-order algebras, including algebras of stream transformations used to model digital hardware at different levels of time and data abstraction. In this context, a second-order algebra A trivially has continuous carrier sets. Continuity of the operations of A on first and second-order carrier sets is precisely continuity in the product or Tychonoff topology. As is well known from higher-order recursion theory (see [23]) all computable second-order operations are continuous in this topology. Thus the assumption of Lemma 4.9 can often be satisfied in this setting. An application of Lemma 4.9 to digital hardware can be found in [20].

Note that the converse of Lemma 4.9 does not hold, by virtue of Example 2.12(i.a). To see this, let A be a discontinuous Σ algebra and E be the ground first-order equational theory of A as defined in 2.12. Since E is the theory of A, the additional infinitary rules add no new ground first-order equations to E and thus infinitary higher-order equational logic is conservative over equational logic on E for ground first-order equations. Thus, by the Conservativity Theorem 4.8, observational equivalence $\equiv^{\text{obs}}$ is a congruence.

Corollary 4.10. *Let $E \subseteq Eqn(\Sigma, X)$ be any higher-order equational theory which contains the homeomorphism axioms* **Hom**. *If $I(\Sigma, E)$ is continuous in the finite information topology then infinitary higher-order equational logic is conservative on E for ground first-order equations.*

Proof. Immediate from Theorem 4.8 and Lemma 4.9. $\square$

Next, we turn to the characterisation of conservativity for finitary higher-order equational logic and the existence of normal form proofs. We begin by considering the well-known folk theorem of first-order equational logic (sometimes called the Theorem on Constants) which states that variable symbols can be exchanged for (or simply reinterpreted as) fresh constant symbols in a signature without altering the provability of formulas. If Σ is any S-sorted signature and $X = \langle X_s \mid s \in S \rangle$ is an S-indexed family of sets of variable symbols (with X_s disjoint from $\Sigma_{\lambda,s}$ for each $s \in S$) then we can define the S-sorted signature $\Sigma \cup X$, where for each $w \in S^*$

and $s \in S$:

$$(\Sigma \cup X)_{w,s} = \begin{cases} \Sigma_{\lambda,s} \cup X_s & \text{if } w = \lambda, \\ \Sigma_{w,s} & \text{otherwise.} \end{cases}$$

Then every equation $e \in Eqn(\Sigma, X)$ is also a ground equation over $\Sigma \cup X$, and every equational theory $E \subseteq Eqn(\Sigma, X)$ is also a ground equational theory over $\Sigma \cup X$. To express the Theorem on Constants precisely, as well as its generalisation to the higher-order case, it is necessary to make explicit two further parameters of the inference relations. Thus, in the sequel, we let $\vdash^{\Sigma,X}$ (respectively $\vdash^{\Sigma,X}_{\text{eval}}$, $\vdash^{\Sigma,X}_{\omega}$) denote provability in first-order many-sorted equational logic (respectively finitary higher-order equational logic, infinitary higher-order equational logic) with respect to the signature Σ and family X of sets of variables.

Theorem on Constants 4.11. *Let $Y = \langle Y_\tau \mid \tau \in S \rangle$ be any S-indexed family of infinite sets of variable symbols disjoint from X and Σ, (i.e. $Y_\tau \cap (\Sigma \cup X)_{\lambda,\tau} = \emptyset$ for each $\tau \in S$). Then for any higher-order equational theory $E \subseteq Eqn(\Sigma, X)$ and any equation $e \in Eqn(\Sigma, X)$:*

(i) $E \vdash^{\Sigma,X}_{\text{eval}} e \Leftrightarrow E \vdash^{\Sigma \cup X, Y}_{\omega} e$,
(ii) $E \vdash^{\Sigma,X} e \Leftrightarrow E \vdash^{\Sigma \cup X, Y} e$.

Proof. The proofs of (i) and (ii) are entirely similar, (i) Follows from Completeness Theorems 2.6 and 2.9. (ii) Follows from the Completeness Theorem for first-order many-sorted equational logic (see for example [21]). □

The Theorem on Constants for higher-order equational logic (4.11(i)) is quite different to the first-order result (4.11(ii)), by virtue of the form of the extensionality and ω-extensionality rules. The free variable in the premise of the finitary extensionality rule, associated with implicit universal quantification, becomes a fresh constant symbol when variables are re-interpreted as constants. Thus the Theorem on Constants for higher-order equations turns out to relate finitary and infinitary higher-order equational logic to each other. In particular, it allows us to relate the conservativity properties of these two logics over first-order equational logic. Thus we obtain the following characterisation of conservativity for finitary higher-order equational logic, and simultaneously a characterisation of the existence of eval normal form proofs for this logic. (Recall Proposition 2.14.)

Normal Form Theorem. *Let $E \subseteq Eqn(\Sigma, X)$ be any higher-order equational theory which contains the homeomorphism axioms* **Hom**. *The following are equivalent*:

(i) *finitary higher-order equational logic is conservative on E for first-order equations*;
(ii) *for every equation $e \in Eqn(\Sigma, X)$ (of any order), if $E \vdash_{\text{eval}} e$ then there is a finitary proof P of e which is in eval normal form*;
(iii) *observational equivalence, $\equiv^{\text{obs}}$, is a congruence on the free algebra $T_E(\Sigma, X)$.*

Proof. The equivalence of (i) and (ii) is simply Proposition 2.14. We need only consider the equivalence of (i) and (iii). Now by the Theorem on Constants, 4.11, finitary higher-order equational logic is conservative on first-order equations $e \in Eqn(\Sigma, X)$ if, and only if, infinitary higher-order equational logic is conservative on ground first-order equations $e \in Eqn(\Sigma \cup X, Y)$, where $Y = \langle Y_\tau \mid \tau \in S \rangle$ is an S-indexed family of sets of variable symbols disjoint from X and Σ, (i.e. $Y_\tau \cap (\Sigma_{\lambda,\tau} \cup X_\tau) = \emptyset$ for each $\tau \in S$). Recall that the free algebra $T_E(\Sigma, X)$ can be concretely constructed as the quotient term algebra $T(\Sigma, X)/\equiv^E$, where $\equiv^E$ is the congruence on terms induced by provable equivalence using first-order many-sorted equational logic. Noting that $\equiv^{\text{obs}}$ is a congruence on $T(\Sigma \cup X)/\equiv^E$ if and only if, $\equiv^{\text{obs}}$ is a congruence on $T(\Sigma, X)/\equiv^E$ then the result follows from the Conservativity Theorem 4.8. □

Corollary 4.13. *Let $E \subseteq Eqn(\Sigma, X)$ be any higher-order equational theory which contains the homeomorphism axioms* **Hom**. *If the free algebra $T_E(\Sigma, X)$ is continuous in the finite information topology then*:

(i) *finitary higher-order equational logic is conservative on E for first-order equations*, *and*,
(ii) *for every equation $e \in Eqn(\Sigma, X)$ (of any order), if $E \vdash_{\text{eval}} e$ then there is a finitary proof P of e which is in eval normal form*;

Proof. Immediate from Lemma 4.9 and Theorem 4.12. □

5. Conclusions

In this paper we have characterised necessary and sufficient conditions under which the extensionality rules of higher-order equational logic can be eliminated from higher-order equational proofs. These results provide information on: (i) when higher-order equational specifications can be implemented by first-order term rewriting, and (ii) when higher-order equational logic can be efficiently automated through the existence of normal form proofs, which introduce the extensionality inference only as a final trivial step. Both of these phenomena have been observed by us in carrying out practical specification work. (See [20].)

An interesting open problem is to give an explicit metric for our finite information topology on extensional higher-order algebras. The Urysohn Metrization Lemma used in Section 3 makes use of an embedding into the Hilbert cube, but this method is somewhat opaque. We even conjecture that this topology is an ultrametric space on extensional higher-order algebras. (The conjecture is obviously valid up to second-order types.)

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