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Spatial ceasing and decay of solutions to nonlinear hyperbolic equations with nonlinear boundary conditions[☆]

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Abstract

In this work we study the spatial behavior of solutions to some nonlinear hyperbolic equations with nonlinear boundary conditions. Under suitable conditions, by using the weighted energy method, we prove that the solutions either cease to exist for a finite value of the spatial variable or decay algebraically in the spatial variable.

Keywords: Hyperbolic equations; Nonlinear boundary condition; Spatial ceasing; Decay estimate; Weighted energy method

1. Introduction

Recently, much attention has been devoted to the investigation of the spatial behavior of solutions to partial differential equations and systems. The authors of paper [1] studied the spatial decay of solutions for a class of diffusion–reaction equations with Dirichlet boundary conditions on a semi-infinite cylinder. The authors of paper [2] investigated the spatial decay and growth estimates for solutions to the initial–boundary value problem for the linear wave equation with the damping term under nonlinear boundary conditions. The spatial decay bounds for a class of quasilinear parabolic equations have been established in paper [4]. In paper [7], the authors investigated the spatial behavior of several nonlinear parabolic equations with nonlinear boundary conditions. They proved that, under suitable conditions, the solutions cease to exist for a finite value of the spatial variable or decay algebraically in the spatial variable.

Motivated by the ideas of [7], in this short work, we extend the results of [7] to a class of hyperbolic equations with nonlinear boundary conditions. For more results related to this problem, we refer the reader to [3,5,6,8–10]. The main method that will be used here is weighted energy integration.

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Let D be a two-dimensional bounded smooth domain. Throughout this work, we use the following notation:

$$R = \{(x_1, x_2, x_3) | x_1 > 0, (x_2, x_3) \in D\},$$
$$R(z) = \{(x_1, x_2, x_3) | x_1 > z, (x_2, x_3) \in D\},$$
$$D(z) = \{(x_1, x_2, x_3) | x_1 = z, (x_2, x_3) \in D\},$$
$$\Sigma(z) = \{(x_1, x_2, x_3) | x_1 > z, (x_2, x_3) \in \partial D\}.$$

This work is concerned with the following equation:

$$s(x, u_t)u_{tt} = \operatorname{div}\left(\rho(x, q^2)\nabla u\right) + E(x, u_t), \quad q^2 = |\nabla u|^2, x \in R, t > 0 \tag{1.1}$$

with the boundary condition

$$\rho \frac{\partial u}{\partial n} + f(u) = 0, \quad x \in \Sigma(0),\ t > 0, \tag{1.2}$$

and the initial condition

$$u(x, 0) = 0, \quad u_t(x, 0) = 0,\ x \in R, \tag{1.3}$$

where $\partial/\partial n$ denotes the outward normal derivative on $\Sigma(0)$ and, s, ρ, E and f are given functions. Obviously, if $f(u) \equiv 0$, then this case corresponds to the homogeneous Neumann condition. In general, considering the well-posedness of the problem, the boundary conditions should be imposed on the finite end ($x_1 = 0$) of the cylinder. Since our main attention is on studying the conditions under which the solutions cease to exist or decay algebraically, we do not mention the explicit boundary conditions on the finite end of the cylinder.

2. Results and proof

Assume that there exists a positive constant ω such that

$$S(x, p) \geq 0, \quad \omega S(x, p) - E(x, p)p \geq \gamma |p|^{2\alpha}, \tag{2.1}$$

where γ is a positive constant, $\alpha > 1$ and $S(x, p)$ is defined by the conditions

$$\frac{\partial S(x, p)}{\partial p} = s(x, p)p, \qquad S(x, 0) = 0. \tag{2.2}$$

We also assume that $f(u)$ satisfies

$$f(u) = F'(u), \qquad F(u) \geq 0, \qquad F(0) = 0 \tag{2.3}$$

and

$$\rho(x, p) > 0, \qquad W(q^2) = \frac{1}{2}\int_0^{q^2} \rho(x, p)\mathrm{d}p, \qquad \rho^2 q^2 \leq C W(q^2), \tag{2.4}$$

where C is a positive constant.

Throughout this work, we assume that the problem (1.1)–(1.3) admits a classical solution. Multiplying (1.1) by $\mathrm{e}^{-\omega t} u_t$ and integrating by parts over $[0, t] \times [z_0, z] \times D(z)$, we obtain

$$\begin{aligned}
&\int_{z_0}^{z}\int_{D(z)} \mathrm{e}^{-\omega t} S \mathrm{d}x + \omega \int_0^t \int_{z_0}^{z}\int_{D(z)} \mathrm{e}^{-\omega t} S \mathrm{d}x\mathrm{d}\tau \\
&= \int_0^t \int_{D(z)} \mathrm{e}^{-\omega t} \rho u_{x_1} u_t \mathrm{d}\sigma\mathrm{d}\tau - \int_0^t \int_{D(z_0)} \mathrm{e}^{-\omega t} \rho u_{x_1} u_t \mathrm{d}\sigma\mathrm{d}\tau \\
&\quad - \int_{z_0}^{z}\int_{\partial D(z)} \mathrm{e}^{-\omega t} F(u)\mathrm{d}\sigma - \omega \int_0^t \int_{z_0}^{z}\int_{\partial D(z)} \mathrm{e}^{-\omega t} F(u)\mathrm{d}\sigma\mathrm{d}\tau \\
&\quad - \int_{z_0}^{z}\int_{D(z)} \mathrm{e}^{-\omega t} W \mathrm{d}x - \omega \int_0^t \int_{z_0}^{z}\int_{D(z)} \mathrm{e}^{-\omega t} W \mathrm{d}x\mathrm{d}\tau + \int_0^t \int_{z_0}^{z}\int_{D(z)} \mathrm{e}^{-\omega t} E u_t \mathrm{d}x\mathrm{d}\tau.
\end{aligned} \tag{2.5}$$

Let

$$\Phi(z,t)=-\int_0^t\int_{D(z)}\mathrm{e}^{-\omega t}\rho u_{x_1}u_t\mathrm{d}\sigma\mathrm{d}\tau. \tag{2.6}$$

Then we have by (2.1)

$$\begin{aligned}\Phi(z,t)=&\ \Phi(z_0,t)-\int_0^t\int_{z_0}^z\int_{D(z)}\mathrm{e}^{-\omega t}(\omega W+P)\mathrm{d}x\mathrm{d}\tau-\omega\int_0^t\int_{z_0}^z\int_{\partial D(z)}\mathrm{e}^{-\omega t}F(u)\mathrm{d}\sigma\mathrm{d}\tau\\&-\int_{z_0}^z\int_{\partial D(z)}\mathrm{e}^{-\omega t}F(u)\mathrm{d}\sigma-\int_{z_0}^z\int_{D(z)}\mathrm{e}^{-\omega t}S\mathrm{d}x-\int_{z_0}^z\int_{D(z)}\mathrm{e}^{-\omega t}W\mathrm{d}x,\end{aligned} \tag{2.7}$$

where $P(x,u_t)=\omega S-E(x,u_t)u_t$.

If $\Phi(z,t)\to 0$ as $z\to\infty$, it follows from (2.7) that

$$\begin{aligned}\Phi(z,t)=&\int_0^t\int_{R(z)}\mathrm{e}^{-\omega t}(\omega W+P)\mathrm{d}x\mathrm{d}\tau+\omega\int_0^t\int_{\Sigma(z)}\mathrm{e}^{-\omega t}F(u)\mathrm{d}\sigma\mathrm{d}\tau\\&+\int_{\Sigma(z)}\mathrm{e}^{-\omega t}F(u)\mathrm{d}\sigma+\int_{R(z)}\mathrm{e}^{-\omega t}S\mathrm{d}x+\int_{R(z)}\mathrm{e}^{-\omega t}W\mathrm{d}x.\end{aligned} \tag{2.8}$$

Differentiating (2.7) with respect to z yields

$$\begin{aligned}\frac{\partial\Phi(z,t)}{\partial z}=&-\int_0^t\int_{D(z)}\mathrm{e}^{-\omega t}(\omega W+P)\mathrm{d}\sigma\mathrm{d}\tau-\omega\int_0^t\int_{\partial D(z)}\mathrm{e}^{-\omega t}F(u)\mathrm{d}l\mathrm{d}\tau\\&-\int_{\partial D(z)}\mathrm{e}^{-\omega t}F(u)\mathrm{d}l-\int_{D(z)}\mathrm{e}^{-\omega t}S\mathrm{d}\sigma-\int_{D(z)}\mathrm{e}^{-\omega t}W\mathrm{d}\sigma.\end{aligned} \tag{2.9}$$

Now we estimate $|\Phi(z,t)|$. From (2.6), by using Hölder's inequality we have

$$\begin{aligned}|\Phi(z,t)|&\le\left(\int_0^t\int_{D(z)}\mathrm{e}^{-\omega t}\rho^2q^2\mathrm{d}\sigma\mathrm{d}\tau\right)^{\frac12}\left(\int_0^t\int_{D(z)}\mathrm{e}^{-\omega t}u_t^2\mathrm{d}\sigma\mathrm{d}\tau\right)^{\frac12}\\&\le C_1(t)\left(\int_0^t\int_{D(z)}\mathrm{e}^{-\omega t}W\mathrm{d}\sigma\mathrm{d}\tau\right)^{\frac12}\left(\int_0^t\int_{D(z)}\mathrm{e}^{-\omega t}|u_t|^{2\alpha}\mathrm{d}\sigma\mathrm{d}\tau\right)^{\frac{1}{2\alpha}},\end{aligned} \tag{2.10}$$

where

$$C_1(t)=C^{\frac12}\left(\int_0^t\int_{D(z)}\mathrm{e}^{-\omega t}\mathrm{d}\sigma\mathrm{d}\tau\right)^{\frac{\alpha-1}{2\alpha}}=C^{\frac12}\left[\frac{1-\mathrm{e}^{-\omega t}}{\omega}|D(z)|\right]^{\frac{\alpha-1}{2\alpha}}.$$

For some positive constant ν, there holds

$$|\Phi(z,t)|\le C_1(t)\left[\left(\nu^\alpha\int_0^t\int_{D(z)}\mathrm{e}^{-\omega t}|u_t|^{2\alpha}\mathrm{d}\sigma\mathrm{d}\tau\right)^{\frac{1}{\alpha+1}}\left(\nu^{-1}\int_0^t\int_{D(z)}\mathrm{e}^{-\omega t}W\mathrm{d}\sigma\mathrm{d}\tau\right)^{\frac{\alpha}{\alpha+1}}\right]^{\frac{\alpha+1}{2\alpha}}. \tag{2.11}$$

With the help of Young's inequality, we can obtain

$$|\Phi(z,t)|\le C_2(t)\left[\nu^\alpha\int_0^t\int_{D(z)}\mathrm{e}^{-\omega t}|u_t|^{2\alpha}\mathrm{d}\sigma\mathrm{d}\tau+\alpha\nu^{-1}\int_0^t\int_{D(z)}\mathrm{e}^{-\omega t}W\mathrm{d}\sigma\mathrm{d}\tau\right]^{\frac{\alpha+1}{2\alpha}}, \tag{2.12}$$

where $C_2(t)=\frac{C_1(t)}{\alpha+1}$. Taking $\nu=(\frac{\alpha\gamma}{\omega})^{\frac{1}{\alpha+1}}$ in (2.12), it follows that

$$|\Phi(z,t)|\le C_3(t)\left[\int_0^t\int_{D(z)}\mathrm{e}^{-\omega t}(\omega W+P)\mathrm{d}\sigma\mathrm{d}\tau\right]^{\frac{\alpha+1}{2\alpha}}\le C_3(t)\left[-\frac{\partial\Phi(z,t)}{\partial z}\right]^{\frac{\alpha+1}{2\alpha}}, \tag{2.13}$$

where $C_3(t) = C_2(t)(\frac{\alpha}{\omega})^{\frac{1}{2}}\gamma^{-\frac{\alpha}{2}}$. By (2.9), we see that $\frac{\partial\Phi(z,t)}{\partial z} \le 0$ for all z. If we assume that $\Phi(z_0, t) < 0$ for fixed t, then we have $\Phi(z, t) < 0$ for all $z \ge z_0$. Therefore, we conclude that

$$-\frac{\partial\Phi(z,t)}{\partial z} \ge \left(-\frac{\Phi(z,t)}{C_3(t)}\right)^{\frac{2\alpha}{\alpha+1}}. \tag{2.14}$$

This implies

$$(-\Phi(z,t))^{\frac{1-\alpha}{1+\alpha}} \le (-\Phi(z_0,t))^{\frac{1-\alpha}{1+\alpha}} - \frac{\alpha-1}{\alpha+1}C_3(t)^{-\frac{2\alpha}{\alpha+1}}(z-z_0). \tag{2.15}$$

The inequality (2.15) shows that the solutions cease to exist for a finite value of z. Thus, we have proved the following result:

Lemma 2.1. *Let u be a solution of the initial–boundary value problem (1.1)–(1.3), where s, E and ρ satisfy Eqs. (2.1)–(2.4). Assume that there exists $z_0 \ge 0$ such that $\Phi(z_0, t) < 0$; then the solution ceases to exist for a finite value of z.*

On the other hand, if we suppose that $\Phi(z, t) \ge 0$ for all z and fixed t, from (2.13) we can get

$$-\frac{\partial\Phi(z,t)}{\partial z} \ge \left(\frac{\Phi(z,t)}{C_3(t)}\right)^{\frac{2\alpha}{\alpha+1}}, \tag{2.16}$$

which implies

$$\Phi(z,t) \le \left[\Phi(0,t)^{\frac{1-\alpha}{1+\alpha}} + \frac{\alpha-1}{\alpha+1}C_3(t)^{-\frac{2\alpha}{\alpha+1}}z\right]^{-\frac{\alpha+1}{\alpha-1}}. \tag{2.17}$$

The inequality (2.17) gives a characterization of the spatial decay of the solution whenever it exists for all $z \ge 0$. We can state now the main result of this note:

Theorem 2.1. *Let u be a solution of the initial–boundary value problem (1.1)–(1.3), where s, E and ρ satisfy (2.1)–(2.4). Then, either the solution ceases to exist for a finite value of the spatial variable z or the solution satisfies the decay estimate (2.17).*

Remark 1. In a similar way, we can prove that the above results still hold for the case of homogeneous Dirichlet boundary conditions.

It is worth noting that there are some typical examples of equations satisfying the conditions proposed in this work. One paradigm is the following equation:

$$|u_t|^\epsilon u_{tt} = k\triangle u + \lambda|u_t|^\epsilon u_t,$$

where ϵ, k are positive constants and λ is an arbitrary parameter.

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