Large monochromatic components in colorings of complete 3-uniform hypergraphs

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Abstract

Let \( f(n, r) \) be the largest integer \( m \) with the following property: if the edges of the complete 3-uniform hypergraph \( K_3^n \) are colored with \( r \) colors then there is a monochromatic component with at least \( m \) vertices. Here we show that \( f(n, 5) \geq \frac{3}{4} n \) and \( f(n, 6) \geq \frac{5}{6} n \). Both results are sharp under suitable divisibility conditions (namely if \( n \) is divisible by 7, or by 6 respectively).

Keywords:

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1. Introduction

A first exercise in graph theory – in fact an old remark of Erdős and Rado – states that for any graph \( G \), either \( G \) or its complement is connected. The following generalization (and the solution for \( r = 3 \)) was suggested in [3]: suppose that the edges of \( K_n \) are colored with \( r \) colors in any fashion, what is the order of the largest monochromatic connected subgraph? The answer for general \( r \), \( \left\lceil \frac{r + 1}{2} \right\rceil \), was given in [4] (it is sharp if \( r - 1 \) is a prime power and \( n \) is divisible by \( (r - 1)^2 \)). This also follows from a result of Füredi [1] on fractional transversals of hypergraphs. The problem was generalized to hypergraphs in [2]. In the generalization, connectivity and components of hypergraphs are understood as follows. Let \( \mathcal{H} \) be a hypergraph. We say that \( \mathcal{H} \) is connected if the shadow graph of \( \mathcal{H} \), with vertex set \( V(\mathcal{H}) \) and edge set \( \{xy : xy \subseteq e \text{ for some } e \in E(\mathcal{H})\} \), is connected. A component of \( \mathcal{H} \) is a maximal connected subhypergraph. The main result of [2] says that any \( r \)-coloring of the edges of the complete \( t \)-uniform hypergraph on \( n \) vertices contains a connected monochromatic subhypergraph on at least \( \frac{n}{t} \) vertices, where \( q \) is the smallest integer satisfying \( r \leq \sum_{i=0}^{t-1} q^i \). The result is best possible if \( q \) is a prime power and \( n \) is divisible by \( q \).

The case \( t = 2 \) (with \( q = r - 1 \)) gives the graph case discussed above. This paper focuses on \( t = 3 \).

Let \( f(n, r) \) be the largest integer \( m \) with the following property: if the edges of the complete 3-uniform hypergraph \( K_3^n \) are colored with \( r \) colors then there is a monochromatic component with at least \( m \) vertices. Applying the result mentioned above for \( t = 3 \) we get that \( f(n, r) = \frac{n}{q} \) if \( r = q^2 + q + 1 \) with a prime power \( q \) and \( n \) is divisible by \( q^3 \). The case \( q = 2 \) solves \( r = 7 \) and the cases \( r \leq 4 \) are also solved in [2] (\( f(n, 3) = n \) and \( f(n, 4) \geq \frac{3}{4} n \) with equality if \( n \) is divisible by 4). The cases \( r = 5, 6 \) are left open and the purpose of this note is to fill this gap. We apply the proof method of Füredi used first in [1] (see also in [2]) which connects \( f(n, r) \) to fractional transversals of certain hypergraphs.

A hypergraph is \( r \)-partite if its vertices are partitioned into \( r \) classes and each edge intersects each class in exactly one vertex. A hypergraph is 3-wise intersecting if any three edges have nonempty intersection. A fractional transversal is a non-negative weighting of the vertices such that the sum of the weights over any edge is at least 1. The value of a fractional
transversal is the sum of the weights over all vertices of the hypergraph. Finally, \( \tau^*(\mathcal{H}) \) is the minimum of the values over all fractional transversals of \( \mathcal{H} \). We use the following lemma from [2].

**Lemma 1.** Let \( \tau^*(r) \) be defined as the maximum of \( \tau^*(\mathcal{H}) \) over all \( r \)-partite 3-wise intersecting hypergraphs \( \mathcal{H} \). Then \( f(n, r) \geq \frac{n}{r^{(r-1)/r}} \).

**Theorem 1.** \( f(n, 5) \geq \frac{5n}{7} \) and this is sharp if \( n \) is divisible by 7.

**Proof.** We start with a construction, showing that \( f(n, 5) \) is not larger than the claimed value if \( n \) is divisible by 7. Let \( n = 7k \) and partition \( [n] = \{1, \ldots, n\} \) into seven \( k \)-element sets, \( X_i \). We define five subsets \( I_j \subset [7] \) as

\[
I_1 = \{1, 4, 5, 6, 7\}, \quad I_2 = \{2, 4, 5, 6, 7\}, \quad I_3 = \{3, 4, 5, 6, 7\}, \quad I_4 = \{2, 3, 6, 7\}, \quad I_5 = \{1, 2, 3, 4, 5\}.
\]

Observe that every triple of \( [7] \) is covered by at least one \( I_j \). Thus every triple \( T \subset [n] \) is covered by at least one of the five sets \( A_j = \{\cup_{i \in I_j} X_i\} \). Color \( T \) with color \( j \) where \( j \) is the smallest index such that \( T \subset A_j \). Clearly each triple of \([n]\) is colored with one of five colors and there is no monochromatic component of size larger than \( 5k = \frac{5n}{7} \).

On the other hand, \( f(n, 5) \geq \frac{5n}{7} \) follows from **Lemma 1** if we show that \( \tau^*(\mathcal{H}) \leq \frac{5}{7} \) holds for every 5-partite 3-wise intersecting hypergraph \( \mathcal{H} \). We shall define only the non-zero weights \( w(x) \) for \( x \in V(\mathcal{H}) \). Let \( A_j \) denote the vertex classes of \( \mathcal{H} \), vertices in \( A_j \) will be indexed with \( j \). Note that if there are two edges \( e, f \in E(\mathcal{H}) \) with \( |e \cap f| = 1 \) then all edges of \( \mathcal{H} \) intersect and \( \tau^*(\mathcal{H}) = 1 \) follows. Thus we may assume that any two edges of \( \mathcal{H} \) intersect in at least two vertices.

Case (i): there exist \( e, f \in E(\mathcal{H}) \) with \( |e \cap f| = 2 \). Assume \( e = \{x_1, x_2, y_3, y_4, y_5\}, f = \{x_1, x_2, z_3, z_4, z_5\} \). Set \( Y = \{y_3, y_4, y_5\}, Z = \{z_3, z_4, z_5\} \). Using that \( \mathcal{H} \) is 3-wise intersecting, it follows that the edge set of \( \mathcal{H} \) can be partitioned into \( E_1, E_2, E_3 \) where

\[
E_1 = \{h \in E(\mathcal{H}) : x_1, x_2, h \in h\}, \quad E_2 = \{h \in E(\mathcal{H}) : x_1, h, x_2 \notin h\}, \quad E_3 = \{h \in E(\mathcal{H}) : x_2, h, x_1 \notin h\}.
\]

We may assume that \( E_1, E_2 \) are both non-empty otherwise – as before – all edges of \( \mathcal{H} \) intersect and \( \tau^*(\mathcal{H}) = 1 \).

Assume first that there is a pair of edges \( e_1 \in E_1, e_2 \in E_2 \) such that \( e_1, e_2 \) intersect on \( A_3 \cup A_4 \cup A_5 \) in a 3-element set \( T = \{t_1, t_2, t_3\} \). Since \( e, e_1, e_2 \) each intersect in at least two vertices, \( T \cap Y, T \cap Z \) are non-empty sets, at least one of them, say \( T \cap Z \) has exactly one element. We may suppose without loss of generality that \( t_3 \neq y_5 \). Since \( E_1 \neq E_2 \), then \( \mathcal{H} \) is 3-wise intersecting T and \( T \cap Z \) is empty, thus assigning \( 1 \) to each element of \( Z \) gives a fractional transversal of value \( \frac{3}{5} < \frac{5n}{7} \) finishing this part of the proof.

Fix \( t_5 = y_5 \) as, in the argument above, the existence of the triple intersections \( e \cap e_1 \cap b, e \cap e_2 \cap a, f \cap e_2 \cap a, f \cap e_1 \cap b \) for \( a \in E_1, b \in E_2 \) imply that all edges of \( E_1 \cup E_2 \) contain \( t_5 \). If there exists an edge \( e_{12} \in E_{12} \) such that neither \( t_3 \) nor \( t_4 \) is in \( e_{12} \) then the existence of the triple intersections \( e_{12} \cap e_1 \cap b, e_{12} \cap e_2 \cap a \) for \( a \in E_1, b \in E_2 \) imply that all edges of \( E_1 \cup E_2 \) contain \( t_3 \) as well. Moreover, then all edges of \( E_{12} \) must also contain \( t_3 \). Now every edge in \( E_1 \cup E_2 \) intersects \( \{x_1, x_2\} \) in one and intersects \( T \) in three elements; every edge of \( E_{12} \) intersects \( \{x_1, x_2\} \) in two and \( T \) in at least one element. Thus the weight assignment \( w(x_1) = w(x_2) = \frac{2}{5}, w(t_3) = w(t_4) = w(t_5) = \frac{1}{5} \) is a fractional transversal of \( \mathcal{H} \) with value \( \frac{7}{5} \).

If every edge in \( E_{12} \) intersects \( \{t_3, t_4\} \) then every edge in \( E_1 \cup E_2 \) intersects \( S = \{x_1, x_2, t_3, t_4\} \) in at least three elements thus assigning \( \frac{1}{5} \) to each element of \( S \) gives a fractional transversal of value \( \frac{3}{5} < \frac{5n}{7} \) finishing this part of the proof.

Now we may assume that any pair of edges \( e \in E_1, e \in E_2 \) intersect on \( A_3 \cup A_4 \cup A_5 \) in a set of at most two elements. Fix \( e_1 \in E_1, e_2 \in E_2 \). In fact – since the triple intersections \( e_1 \cap e_2 \cap a, e_1 \cap e_2 \cap b \) exist – \( e_1 \) and \( e_2 \) intersect on \( A_3 \cup A_4 \cup A_5 \) in a two-element set \( T = \{t_3, t_4\} \) say \( t_3 = y_3, t_4 = z_4 \). Since \( e_1, e_2 \) do not intersect on \( A_5 \), \( t_3, t_4 \) \( e_1 \cap e_2 \) the triple intersections \( e_1 \cap e_2 \cap a \) or \( e_1 \cap e_2 \cap b \) imply \( t_3 \in E_1, t_4 \in E_2 \). Since each intersection \( a \cap b \) for \( a \in E_1, b \in E_2 \) has at least two elements, one of \( t_3, t_4 \), say \( t_3 \) is in all edges of \( E_1 \cup E_2 \). Moreover each \( e_{12} \in E_{12} \) must intersect \( \{t_3, t_4\} \) because of the triple intersection \( e_{12} \cap e_1 \cap e_2 \). If \( t_4 \) is also in all edges of \( E_1 \cup E_2 \) then \( \{x_1, x_2, t_3, t_4\} \) intersects every edge of \( E_1 \cup E_2 \) in at least three elements, implying a fractional transversal of value \( \frac{4}{5} \). Otherwise \( E' = \{b \in E_2 : t_4 \cap b \neq \emptyset\} \neq \emptyset \). In this case, since \( |b \cap f| \geq 2 \) we see that each \( b \in E' \) contains \( z_5 \). Looking at \( b \cap a \) for \( a \in E_1, b \in E' \) tells us that \( z_5 \in a \) as well. Finally, \( b \cap e_1 \cap e_{12} \)
shows us that if \( t_3 \notin e_{12} \) for some \( e_{12} \in E_{12} \) then \( z_5 \in e_{12} \) (and we know that \( t_3 \notin e_{12} \) implies \( t_4 \in e_{12} \)). Summing up, we find that for each \( a \in E_1, x_1, t_3, t_4, z_5 \in a, \) and for each \( b \in E_2, x_2, t_3 \in b \) and \((t_4 \cup z_5) \cap b \) is nonempty. For each \( e_{12} \in E_{12}, x_1, x_2 \in e_{12} \) and either \( t_3 \in e_{12} \) or \( \{t_4, z_5\} \subseteq e_{12} \). Now the weighting \( w(t_3) = w(x_2) = \frac{1}{2}, w(x_1) = w(t_4) = w(z_5) = \frac{1}{2} \) gives the required fractional transversal.

Case (ii): Any two distinct \( e, f \in \mathcal{H} \) intersect in at least three vertices. Assume first that there is a pair \( e, f \in \mathcal{H} \) intersecting in three elements, \( e = \{x_1, x_2, x_3, x_4, x_5\}, f = \{x_1, x_2, x_3, y_4, y_5\} \). Observe then that every edge must intersect \( \{x_1, x_2, x_3\} \) in at least two elements. Again, if the set of edges \( E_i \) that intersect \( \{x_1, x_2, x_3\} \) in \( \{x_i, x_j\} \) is empty for some pair \( i, j \in [3] \) then, for \( k = [3] \setminus \{i, j\} \), all edges of \( \mathcal{H} \) contain \( x_k \) and \( \tau^*(\mathcal{H}) = 1 \). Thus these sets \( E_i \) are non-empty. Selecting \( e_{12} \in E_{12}, e_{13} \in E_{13}, e_{23} \in E_{23} \), the assumptions on the intersection sizes imply that for each of the three pairs of indices \( e_{ij} \cap (A_k \cup A_j) \) is the same pair, say \( \{x_4, y_5\} \). Any edge \( e_{23} \) that contains all of \( \{x_1, x_2, x_3\} \) must also intersect \( \{x_4, y_5\} \), otherwise \( |e_{23} \cap e_{12}| \leq 2 \). Now assigning \( w(x_1) = w(x_2) = w(x_3) = \frac{1}{2}, w(x_4) = w(y_5) = \frac{1}{2} \) we have a fractional transversal of \( \mathcal{H} \) with value \( \frac{5}{2} \).

Finally, if each pair of edges of \( \mathcal{H} \) intersect in at least four elements, we can assign weight \( \frac{1}{2} \) to vertices of any fixed edge. This gives a fractional transversal of \( \mathcal{H} \) with value \( \frac{5}{4} < \frac{7}{2} \). \( \square \)

**Theorem 2.** \( f(n, 6) \geq \frac{2n}{3} \) and this is sharp if \( n \) is divisible by 6.

**Proof.** To show that \( f(n, 6) \) is not larger than claimed value if \( n \) is divisible by 6, let \( n = 6k \) and partition \([n]\) into six \( k\)-element sets, \( X_i \). We define six subsets \( I_j \subseteq [6] \) as

\[
I_1 = \{3, 4, 5, 6\}, \quad I_2 = \{1, 4, 5, 6\}, \quad I_3 = \{2, 4, 5, 6\}, \\
I_4 = \{1, 2, 3, 6\}, \quad I_5 = \{1, 2, 3, 4\}, \quad I_6 = \{1, 2, 3, 5\}.
\]

Observe that every triple of \([6]\) is covered by at least one \( I_j \). Thus every triple \( T \subseteq [n] \) is covered by at least one of the six sets \( A_I = \{\cup_{j \in I} X_i\} \). Color \( T \) with color \( j \) where \( j \) is the smallest index such that \( T \subseteq A_j \). Clearly each triple of \([n]\) is colored with one of six colors and there is no monochromatic component of size larger than \( 4k = \frac{2n}{3} \).

As in the proof of Theorem 1, \( f(n, 6) \geq \frac{2n}{3} \) follows from Lemma 1 if we show that \( \tau^*(\mathcal{H}) \leq \frac{3}{2} \) holds for every 6-partite 3-wise intersecting hypergraph \( \mathcal{H} \). To see that, let \( A_I \) denote the vertex classes of \( \mathcal{H} \). Note that if there are two edges \( e, f \in E(\mathcal{H}) \) with \(|e \cap f| = 1\) then all edges of \( \mathcal{H} \) intersect and \( \tau^*(\mathcal{H}) = 1 \) follows. Thus we may assume that any two edges of \( \mathcal{H} \) intersect in at least two vertices. We basically follow the argument of the proof of Theorem 1.

Case (i): There exist \( e, f \in E(\mathcal{H}) \) with \(|e \cap f| = 2\). Set \( e \cap f = \{x_1, x_2\} \) and define

\[
E_{12} = \{h \in E(\mathcal{H}) : x_1, x_2 \in h\}, \quad E_1 = \{h \in E(\mathcal{H}) : x_1 \in h, x_2 \notin h\}, \quad E_2 = \{h \in E(\mathcal{H}) : x_2 \in h, x_1 \notin h\}.
\]

Then as before \( \mathcal{H} = E_1 \cup E_2 \cup E_{12} \).

Let \( E_1 = \{a_1, a_2, \ldots, a_1\}, E_2 = \{b_1, b_2, \ldots, b_1\} \). We may assume that \( E_1, E_2 \) are both nonempty, otherwise – as before – all edges of \( \mathcal{H} \) intersect and \( \tau^*(\mathcal{H}) = 1 \). Notice that \( a_i \cap b_j \subseteq \cup_{k=1}^6 A_i \) for any \( a_i \in E_1, b_j \in E_2 \).

If all edges of \( E_1 \cup E_2 \) have a common vertex \( v \) then assigning weight \( \frac{1}{2} \) to the vertices in \( \{x_1, x_2, v\} \) we have a fractional transversal of value \( \frac{1}{2} \) and the proof is finished. Thus we may suppose that

\[
\bigcap_{i \in [6]} a_i \cap \bigcap_{j \in [6]} b_j = \emptyset. \quad (1)
\]

**Lemma 2.** Suppose there exist distinct edges \( a_1, a_2 \in E_1, b_1, b_2 \in E_2 \) such that \( a_1 \cap a_2 \cap b_1 \cap b_2 = \emptyset \). Then \( \tau^*(\mathcal{H}) \leq \frac{3}{2} \).

**Proof.** Observe that the four triple intersections among these edges are all disjoint (and nonempty). Let \( U \) denote the union over all four triple intersections, so \(|U| \geq 4\). Note that if \( x, x' \in U \) then one of (in fact, at least two of) \( a_1, a_2, b_1, b_2 \) contain both \( x \) and \( x' \). Thus we cannot have distinct \( x, x' \) in the same partite class \( A_i \). Therefore \( U = \{x_3, x_4, x_5, x_6\} \) for some \( x_i \in A_i \) for \( i = 3, 4, 5, 6 \), and we may assume without loss of generality that

\[
x_3 \in (a_1 \cap b_1 \cap b_2) \setminus a_2, x_4 \in (a_2 \cap b_1 \cap b_2) \setminus a_1, \\
x_5 \in (a_1 \cap a_2 \cap b_1) \setminus b_2, x_6 \in (a_1 \cap a_2 \cap b_2) \setminus b_1. \quad (2)
\]

We observe that – apart from the exceptional case when \( a_1 \cap U = \{x_3, x_4\} \) – each edge \( a_1 \in E_1 \) intersects \( U \) in at least three vertices. Indeed, if \( a_1 \cap U \subseteq \{x_3, x_4\} \) then the triple intersection \( a_1 \cap a_2 \cap b_2 \) is missing. If \( a_1 \cap U \subseteq \{x_4, x_5\} \) then \( a_1 \cap a_1 \cap b_1 \) is missing. Similarly, \( a_1 \cap U \subseteq \{x_3, x_5\} \) and \( \{x_4, x_5\} \) and \( \{x_5, x_6\} \) in turn imply the missing intersections \( a_1 \cap a_2 \cap b_1, a_1 \cap a_1 \cap b_2, a_1 \cap b_1 \cap b_2 \).

(The argument in the exceptional case would require missing \( a_1 \cap a_1 \cap a_2 \) but that intersection is present at \( x_1 \).)

Similarly, apart from the exceptional case when \( b_1 \cap U = \{x_5, x_6\} \), each edge of \( b_1 \in E_2 \) intersects \( U \) in at least three vertices. Finally, observe that any \( e_{12} \in E_{12} \) intersects \( U \) in at least two vertices. Indeed, \( e_{12} \cap U \subseteq \{x_i\} \) for some \( i \in \{3, 4, 5, 6\} \).
would contradict the existence of the triple intersection $e_{12} \cap a_i \cap b_j$ where $i, j \in \{2\}$ such that one of $a_i$, $b_j$ does not contain $x_i$. Consider $e_{12} \in E_{12}$ exceptional if $e_{12} \cap \{x_3, x_4\}$ or $e_{12} \cap \{x_5, x_6\}$.

Based on the above observations we can define the required fractional transversal as follows. If no edge in $E_1 \cup E_2$ is exceptional, $w(x_i) = \frac{1}{4}$ for $i = 1, 2, \ldots, 6$ is suitable. If there exists an exceptional edge in $E_1 \cup E_2$, say $a_i$, then no $b_j \in E_2$ can be exceptional (otherwise $a_i \cap b_j$ cannot exist) — in fact the following stronger statement is true for any $b_j$: if $\{x_3, x_5\} \subseteq b_j$ then $U \subseteq b_j$. Indeed, $U \cap b_j = \{x_3, x_5, x_6\}$ (or $\{x_3, x_5, x_6\}$) contradicts the existence of $a_i \cap b_j \cap a_1(a_i \cap b_j \cap a_2)$. Moreover no $e_{12} \in E_{12}$ is exceptional with $e_{12} \cap \{x_3, x_5\}$ otherwise $e_{12} \cap a_i \cap b_j$ cannot exist. These properties ensure that $w(x_1) = w(x_3) = w(x_4) = \frac{1}{2}$, $w(x_2) = w(x_5) = w(x_6) = \frac{1}{6}$ is a suitable fractional transversal. □

By Lemma 2, from now on we may suppose that
\[ a_1 \cap a_i \cap b_j \cap b_i \neq \emptyset \]
for every choice of the indices (if $i = j$ or $k = l$ the 3-wise intersecting property ensures it).

Because of (1) we can select a minimal nonintersecting subfamily of $E_1 \cup E_2$, that is $S \subseteq [s], T \subseteq [t]$ such that
\[ \bigcap_{i \in S} a_i \cap \bigcap_{j \in T} b_j = \emptyset \]
but for any proper subset $S_1 \cup T_1 \subset S \cup T$

\[ \bigcap_{i \in S_1} a_i \cap \bigcap_{j \in T_1} b_j \neq \emptyset. \tag{3} \]

Since $A = \bigcap_{l \in [s]} a_l \cap \bigcap_{j \in [t]} b_j$ are both nonempty ($x_1 \in A, x_2 \in B$), it follows that $S, T$ are nonempty. Moreover $|S \cup T| \geq 4$ because $\mathcal{H}$ is 3-wise intersecting. Set $u = |S \cup T|$. Then by choice of $S \cup T$, all $(u - 1)$-wise intersections of elements of $S \cup T$ are disjoint and nonempty, so their union $U$ has size at least $u$, and as in the proof of Lemma 2 no two vertices in $U$ are in the same partite class $A_i$. Thus if $|S|, |T| \geq 2$ then $U \subseteq \bigcup_{k=3}^6 A_k$, implying that $u = 4$. But then the assumptions of Lemma 2 hold, so the proof is done in this case.

Thus we may assume that one of $S, T$ has one element only, say $T = \{1\}$. In this case $x_1 \in U$ and $x_2 \notin U$, so $U \subseteq \{x_1\} \cup \{x_3, x_5\}$ $\cup \{x_4, x_6\}$, implying that $u = 4$ or $u = 5$. In both cases, without loss of generality we may select three vertices $X = \{x_3, x_4, x_6\}$ from $U$ with $x_i \in A_i$ for $i = 3, 4, 5$ as follows:

\[ x_3 \in (a_1 \cap a_2 \cap b_1) \setminus a_3, x_4 \in (a_1 \cap a_3 \cap b_1) \setminus a_2, x_5 \in (a_2 \cap a_3 \cap b_1) \setminus a_1. \tag{4} \]

Lemma 3. Suppose there exists $a_i \in E_1$ such that $|a_i \cap X| \leq 1$. Then $\tau^+(\mathcal{H}) \leq \frac{3}{2}$.

Proof. Suppose without loss of generality that $a_1 \cap X \times \{x_3, x_4\} = \emptyset$. Then, for each $b_j \in E_2$, the (nonempty) quadruple intersection $a_1 \cap a_i \cap b_1 \cap b_j$ must in $A_6$. This is possible only if all $b_j$’s intersect on $A_6$, say in a vertex $x_6 \in a_3 \cap a_i \cap b_1$. Because of (1) the set $K = \{k \in \{x \mid x \in [s] \setminus \{a_k\}\}$ is nonempty. For every $k \in K, j \in [t]$ the quadruple intersection $a_k \cap a_i \cap b_1 \cap b_j$ contains $x_3$. This implies $x_3 \in B \cap (\bigcap_{k \in K} a_k)$.

Reversing the argument, $L = \{l \in [s] \mid x_l \notin a_i\}$ is nonempty implying that for every $l \in L, j \in [t]$ the quadruple intersection $a_l \cap a_i \cap b_1 \cap b_j$ contains $x_6$, implying $x_3 \in B \cap (\bigcap_{k \in K} a_k)$. Thus each edge in $E_1$ contains $x_3$ and at least one vertex of $\{x_3, x_4\}$. Every edge in $E_2$ contains both $x_3, x_5$ and every $e_{12} \in E_{12}$ contains $x_3$ and also at least one vertex of $\{x_3, x_4\}$ because the triple intersection $e_{12} \cap a_i \cap b_i$ is nonempty. Therefore $w(x_1) = w(x_3) = w(x_6) = \frac{1}{2}$ is a required fractional transversal. □

By Lemma 3 we may suppose from now on that every edge $a_i \in E_1$ meets $X$ in at least two elements.

Claim: Either $X \subseteq B$ or $B \cap A_6 \neq \emptyset$. Indeed, if an element of $X$, say $x_3 \notin B$, for some $i \in [t]$ then the quadruple intersection $a_i \cap a_3 \cap b_1 \cap b_m$ is in $A_6$ for all $m \in [t]$. This implies that $B \cap A_6 \neq \emptyset$. The argument works similarly if $x_4$ or $x_5$ plays the role of $x_3$ considering $a_1 \cap a_3 \cap b_1 \cap b_m$ or $a_2 \cap a_3 \cap b_1 \cap b_m$, proving the claim.

We look at the two cases of the claim. If $X \subseteq B$ holds then $w(x_1) = \frac{1}{2}, w(x_2) = w(x_3) = w(x_4) = w(x_5) = \frac{1}{4}$ is a required fractional transversal. Indeed, each $a_i \in E_1$ contains $x_1$ and at least two elements of $X$, each $b_j \in E_2$ contains $x_3$ and all $X$. Each $e_{12} \in E_{12}$ contains $x_1, x_2$ and at least one element of $X$ otherwise — considering the triple intersections $e_{12} \cap a_i \cap b_i$ — all $a_i, b_i$ should intersect in $A_6$, contradicting (1). Thus we may assume that $X \subseteq B$ does not hold.

Select $x_6 \in A_6 \cap B$. By definition of $S$, at least one $a_i$ with $j \notin S$ does not contain $x_6$, say $x_6 \notin a_1$. We show that $\{x_4, x_5\} \subseteq B$. Indeed, if $x_4 \notin a_k \cap x_5 \notin a_k$ then the quadruple intersection $a_k \cap a_3 \cap b_1 \cap b_j(a_2 \cap a_3 \cap b_1 \cap b_j)$ does not exist.

Therefore since $X \subseteq B$ does not hold, we know $x_3 \notin b_j$ for some $j \in [t]$. Define $K = \{k \in \{s\} \mid x_k \notin a_k\}$ as before. We show that for each $k \in K, \{x_3, x_4\} \subseteq a_k$. Indeed, if $x_4 \notin a_k \cap x_5 \notin a_k$ for some $k \in K$ then $a_1 \cap a_3 \cap b_1 \cap b_j(a_2 \cap a_3 \cap b_1 \cap b_j)$ does not exist.

Now we finish the proof by showing that $w(x_1) = \frac{1}{2}, w(x_2) = w(x_4) = w(x_5) = w(x_6) = \frac{1}{4}$ is a required fractional transversal. Notice that for every $a_i \in E_1 \text{ either } x_6 \in a_i \text{ or } i \in K \text{ and } \{x_4, x_5\} \subseteq a_i$. This property and that every $a_i$ contains at least one of $x_4, x_5$ ensures that the weight of $a_i$ is at least one. The weighting is also good for every $b_j \in E_2$ since $\{x_2, x_4, x_5, x_6\} \subseteq B$. Finally, each $e_{12} \in E_{12}$ contains $x_1, x_2$ and at least one vertex of $\{x_3, x_4, x_5\}$ because $e_{12} \cap a_3 \cap b_1 \neq \emptyset$. Thus the weighting is a required fractional transversal. □

Case (ii): $|e \cap f| \geq 3$ for each $e, f \in E(\mathcal{H})$. In this case let us first suppose that there exist $e$ and $f$ such that $e \cap f = M$ where $|M| = 3$. Then we define a fractional transversal by giving weight $\frac{1}{2}$ to each vertex in $M$. This is valid because every other edge $g$ must intersect $M$ in at least two vertices — otherwise either $|g \cap e| \leq 2$ or $|g \cap f| \leq 2$, contradicting the assumption
for Case (ii). Thus we have a fractional transversal of value $\frac{3}{2}$. Thus we may suppose that every pair of edges intersects in at least four vertices. Let $e$ and $f$ be an arbitrary pair and let $M \subseteq e \cap f$ be a set of size four. Define a fractional transversal by weighting each vertex of $M$ with $\frac{1}{3}$. Now every other edge $g$ intersects $M$ in at least three vertices — otherwise either $|g \cap e| \leq 3$ or $|g \cap f| \leq 3$, contradicting our assumption. Now we get a fractional transversal of value $\frac{4}{3} < \frac{3}{2}$. □

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References