Bernstein Inequalities and Applications to Analytic Geometry and Differential Equations

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We introduce new methods of complex analysis (inequalities of Bernstein type) to study projections of analytic sets. These techniques are applied to problems of bifurcations of periodic orbits of differential equations such as the local Hilbert’s 16th problem.

I. INTRODUCTION

This article relates to two different fields: codimension one analytic hypersurfaces defined on a polydisc and limit cycles of polynomial differential equations of the plane.

Methods of analytic Geometry are quite efficient to get bounds for the number of limit cycles (i.e., periodic orbits which are isolated among periodic orbits). The reader is directed, for instance, to Francoise–Pugh [F–P], Denkowska [D], and Il’ashenko–Yakovenko [I–Y]. In [F–P], Gabrielov’s theorem was used to prove that the number of limit cycles of period less than $T$ (given any $T$) is uniformly bounded.

For germs of analytic hypersurface (codimension 1), Gabrielov’s theorem can be derived in an elementary way using Rolle’s lemma and the property that the ring of germs of analytic functions is Noetherian (cf. Francoise–Pugh [F–P], Roussarie [R], and Yakovenko [Ya]).

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We adopt in this article the following viewpoint: we do not consider a germ but a hypersurface $\Sigma$ defined on a small fixed open polydisc $A_p \subset C^{n+1}$ by an equation

$$f(x_1, \ldots, x_n; z) = \sum_{k=0}^{\infty} f_k(x_1, \ldots, x_n) z^k = 0.$$  \hspace{1cm} (1)

The problem is to find an explicit bound for the number of complex solutions in $z$ to Eq. (1) on a fixed polydisc $A_p \subset A_{p'}$, with an explicit $p' = p'(p)$.

The first part of this article is devoted to this purpose. We show that the bound (and $p'$) can be read on the Newton diagram of the ideal $I$ generated by the coefficients $f_k(x_1, \ldots, x_n)$ ($k = 0, 1, \ldots$). The estimates involve a constant $C$ that can be computed from the matrix given by the change of system of generators of the ideal $I$ from generators chosen among the coefficients to generators of a Gröbner basis.

The size of the polydisc and the bound are explicitly deduced from some results on zeroes of analytic functions by inverting, in some sense, the classical Bernstein inequality.

The second part of the article is concerned with the application of the first part to differential equations. We obtain for quadratic vector fields a bound for the number of complex limit cycles in a fixed neighborhood of the origin. This new result must be compared with the famous article [B] on the subject, where a bound to the number of real limit cycles (without control of the size of the neighborhood) was provided.

The first part is organized as follows. We first remind the reader of the elementary proof of Gabrielov's theorem (codimension 1 case) based on the Rolle property. We propose to adapt this proof to get an explicit bound on a fixed complex polydisc. We replace in some sense the Rolle lemma by (kind of) inversion of Bernstein inequality.

In the second part, we briefly remind the reader what Hilbert's 16th problem and Bautin's theorem are.

Given a quadratic vector field of the plane

$$X = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} + \sum_{i+j=2} \left( a_{ij} x^i y^j \frac{\partial}{\partial x} + b_{ij} x^i y^j \frac{\partial}{\partial y} \right),$$

Bautin proved that $X$ has less than 2 real limit cycles in some neighborhood of the origin. (If the linear part is $y(x(\partial/\partial x) - x(\partial/\partial y)) + x(x(\partial/\partial x) + y(\partial/\partial y))$ instead, the bound is 3. The assumption $x = 0$ in our article is not at all essential but justified only by aesthetic motivations.)

This major contribution to the subject gave the main motivation to Petrowski and Landis [P-L] to write an attempt for solving Hilbert's 16th
problem. Petrowski and Landis proposed to complexify the foliation defined by the vector field $X$ and to bound the number of complex limit cycles of some type.

Our main result here is that the number of complex limit cycles on a fixed polydisc which contains the origin is less than 4 (versus 2).

We hope that this article might be of interest both for specialists of differential equations and of analytic geometry. This is why sometimes we included proofs that can be considered as folklores of one of the two subjects.

II. ANALYTIC GEOMETRY

In this article, we consider a complex analytic hypersurface $\Sigma$ defined on a fixed open polydisc $\mathcal{D} \subset \mathbb{C}^{n+1}$ by an equation

$$f(x_1, \ldots, x_n; z) = \sum_{k=0}^{\infty} z^k f_k(x_1, \ldots, x_n) = 0.$$  

We wish to bound explicitly the number of solutions (in the variable $z$) of $f(x_1, \ldots, x_n; z) = 0$ uniformly for all $(x_1, \ldots, x_n)$ which belongs to a fixed neighborhood of the origin. It turns out that we can do it for only some types of polydiscs. The bound and the estimate of the radius of the polydisc as well as the shape of the polydisc can be read on the Newton diagram of the ideal generated by the coefficients $f_k(x_1, \ldots, x_n)$ ($k = 0, 1, \ldots$).

2.1. Projection of Germs of Real Analytic Hypersurfaces

The relevance of this ideal appears already at the germ level to bound the number of real zeroes. This fact is well known to specialists of Hilbert's 16th problem [B, F–P, R].

We consider the ascending chain of ideals $I_k$ generated in the ring $\mathbb{R}\{x_1, \ldots, x_n\}$ by the $k$ first coefficients $f_j(x_1, \ldots, x_n)$ ($j = 0, \ldots, k - 1$). This ascending chain becomes stationary; we call Bautin ideal and we denote $I = \lim(I_k)$ the limit of this chain. Let $f_{k_1}(x_1, \ldots, x_n), f_{k_2}(x_1, \ldots, x_n), \ldots, f_{k_s}(x_1, \ldots, x_n)$ be a system of generators of the Bautin ideal. We can write

$$f(x_1, \ldots, x_n; z) = f_{k_1}(x_1, \ldots, x_n) z^{k_1} F_1((x, z)) + \cdots + f_{k_s}(x_1, \ldots, x_n) z^{k_s} F_s((x, z)), \quad (3)$$

with

$$F_1((x), 0) = \cdots = F_s((x), 0) = 1.$$
We prove inductively that the number of solutions of \( f(x_1, \ldots, x_n; z) = 0 \) is bounded in some neighborhood of the origin by a formula as follows.

The function \( F_1(y, z) \) does not vanish on some neighborhood of the origin. We do not change the number of zeroes of \( f(x_1, \ldots, x_n; z) \) by dividing it by \( z^{k_1} F_1(y, z) \).

This yields to

\[
 f(x_1, \ldots, x_n; z)/z^{k_1} F_1(y, z) = f_{k_1}(x_1, \ldots, x_n) + \cdots + f_{k_n}(x_1, \ldots, x_n) z^{k_n - k_1} F(y, z)/F_1(y, z).
\]

By Rolle's lemma, this expression has at most one zero more than its derivatives. Now, its derivatives relative to \( z \) can be written as

\[
 f_{k_1}(x_1, \ldots, x_n) z^{k_1 - k_n} G_2(y, z) + \cdots + f_{k_n}(x_1, \ldots, x_n) z^{k_n - k_1} G_n(y, z).
\]

This expression (4) is similar to (3) with \( s - 1 \) terms. We keep repeating the process (division–derivation) and conclude that the number of zeroes of (3) is bounded by the number \( s \) of generators of the Bautin ideal \( I \).

Note that we could bound the number of positive zeroes of an analytic function using the same method. This is precisely what we do in the applications to differential equations (Bautin theorem).

In order to replace the process above by an explicit control of zeroes on a fixed polydisc, we need first to remind a technique of division by an ideal.

### 2.2. Hironaka Theorem on Division by an Ideal

The Hironaka theorem on division by an ideal is well known by specialists of analytic geometry. We include a short introduction to this subject which is particularly adapted to our purpose (see Hironaka [H], Lejeune–Jalabert [L], Bierstone–Milman [B–M], Maltsiniotis [M], Galligo [Ga], and Briançon [Br]).

We fix a total ordering \( < \) on \( \mathbb{N}^n \) so that

\[
 \alpha \in \mathbb{N}^n, \quad \beta \in \mathbb{N}^n, \quad \beta \neq 0, \quad \alpha < \alpha + \beta
\]

\[
 \alpha \in \mathbb{N}^n, \quad \beta \in \mathbb{N}^n, \quad \gamma \in \mathbb{N}^n, \quad \alpha < \gamma \text{ then } \alpha + \beta < \gamma + \beta.
\]

There are several possible choices for such an ordering. For instance, we can choose \( < \) as follows: Let \( C = \sum_{i=1}^n c_i X_i, c_i \geq 0 \), be a linear form on \( \mathbb{R}^n \), \( \alpha < \beta \) if \( C(\alpha) < C(\beta) \) or if \( C(\alpha) = C(\beta) \) and if there is \( k, 1 \leq k \leq n, \alpha_j = \beta_j, \)

\( j < k \) and \( x_k < \beta_k \).

**Definition 2.2.1.** Given a polynomial \( f = \sum_{A \in \mathbb{N}^n} f_A x^A \), and a total ordering on \( \mathbb{N}^n \), we denote \( \exp(f) \) the largest exponent \( A \) such that \( f_A \neq 0 \).
Let \( \{ f_1, \ldots, f_k \} \) be \( k \) polynomials, we define a partition of \( \mathbb{N}^n \) as
\[
A_1 = \exp f_1 + \mathbb{N}^n, \quad A_2 = \exp f_2 + \mathbb{N}^n - A_1, \ldots
\]
\[
A_k = \exp f_k + \mathbb{N}^n - (A_1 \cup \cdots \cup A_{k-1}), \quad \mathcal{A} = \mathbb{N}^n \bigcup (\exp f_i + \mathbb{N}^n).
\]

**Theorem 2.2.2.** For all polynomials \( f \), there exists a unique decomposition (called division of \( f \) by \( \{ f_1, \ldots, f_k \} \)):
\[
f = p_1 f_1 + \cdots + p_k f_k + p,
\]
so that

(i) for \( i = 1, \ldots, k \), if \( p_i = \sum_A p_{i,A} x^A \), \( p_{i,A} \neq 0 \), then \( \exp f_i + A \in \mathcal{A} \),

(ii) if \( p = \sum_A p_A x^A \), \( p_A \neq 0 \), then \( A \in \mathcal{A} \).

Given now an ideal \( I \) generated by the polynomials \( \{ f_1, \ldots, f_k \} \), and \( f \) an element of the ideal \( I \), if we divide \( f \) by \( \{ f_1, \ldots, f_k \} \), the remainder \( p \) is not necessarily zero. This entails a special type of system of generators of an ideal called a standard basis (or Gröbner basis).

**Proposition 2.2.3.** Let \( I \) be an ideal. Let \( \exp(I) = \{ A \in \mathbb{N}^n, A = \exp f, f \in I \} \). There is a unique minimal set \( \mathcal{E} = \{ E_1, E_2, \ldots, E_d \} \) such that for all elements \( \beta \) of \( \exp(I) \), there is an element \( \varepsilon \) in \( \mathcal{E} \) and an element \( \alpha \) in \( \mathbb{N}^n \) such that \( \beta = \varepsilon + \alpha \).

**Definition 2.2.4.** We call standard basis (or Gröbner basis) (for a given ordering of the multi-indices) of the ideal \( I \), a set of elements \( \{ h_1, \ldots, h_d \} \) of \( I \) such that \( \exp h_i = E_i \). We denote as previously
\[
A_1 = E_1 + \mathbb{N}^n, \quad A_2 = E_2 + \mathbb{N}^n - A_1, \ldots
\]
\[
A_d = E_d + \mathbb{N}^n - (A_1 \cup \cdots \cup A_{d-1}), \quad \mathcal{A} = \mathbb{N}^n \bigcup (E_i + \mathbb{N}^n).
\]
This yields to the following result:

**Theorem 2.2.5.** Given an ideal \( I \), a standard basis \( \{ h_1, \ldots, h_d \} \) of \( I \) (for a total ordering of the multi-indices) for all elements \( f \) of the ideal \( I \), there is a unique decomposition:
\[
f = \phi_1 h_1 + \cdots + \phi_k h_k,
\]
so that for \( i = 1, \ldots, k \), if
\[
\phi_i = \sum_A \phi_{i,A} x^A, \quad \phi_{i,A} \neq 0,
\]
then
\[ \exp h_j + A \in A_j. \]

This entails that a standard basis is a system of generators of the ideal.

Let \( \rho = (\rho_1, \ldots, \rho_n) \in \mathbb{R}^n_+ \), we denote by \( A_\rho \) the polydisc \( A_\rho = \{ x \in \mathbb{C}^n \mid |x_i| < \rho_i, \ i = 1, \ldots, n \} \). We introduce the Banach algebra \( \mathcal{C}(\rho) = \{ f : A_\rho \to \mathbb{C} \mid f = \sum_{A \in \mathbb{N}^n} f_A x^A, \ |f_A|_\rho = \sum_{A \in \mathbb{N}^n} |f_A| < +\infty \} \).

Given \( d \) multi-indices \( E_1, \ldots, E_d \in \mathbb{N}^n \), we introduce \( A_1 = E_1 + \mathbb{N}^n, \ A_2 = E_2 + \mathbb{N}^n - A_1, \ldots, A_d = E_d + \mathbb{N}^n - (A_1 \cup \cdots \cup A_{d-1}), \ A = \mathbb{N}^n - \bigcup (E_j + \mathbb{N}^n) \).

Any element \( f \in \mathcal{C}(\rho) \), can be written
\[
f = \sum_{A \in \mathbb{N}^n} f_A x^A = \sum_{i=1}^{d} \sum_{A \in A_i} f_A x^A + \sum_{A \in A_d} f_A x^A,
\]
\[
f = \sum_{i=1}^{d} g_i x^{E_i} + g_0,
\]
where
\[
g_i = \sum_{A \in A_i} f_A x^{E_i}, \quad g_0 = \sum_{A \in A_d} f_A x^A.
\]

We get
\[
|f|_\rho = \sum_{i=1}^{d} |g_i|_\rho \rho^{E_i} + |g_0|_\rho
\]
and hence,
\[
|g_i|_\rho \leq \rho^{-E_i} |f|_\rho, \quad |g_0|_\rho \leq |f|_\rho.
\]

Observe now that \((g_i, g_0)\) determines mappings
\[
\Phi_i : \mathcal{C}(\rho) \ni f \mapsto g_i \in \mathcal{C}(\rho),
\]
\[
\Phi_0 : \mathcal{C}(\rho) \ni f \mapsto g_0 \in \mathcal{C}(\rho).
\]

Given an ideal \( I \), the minimal set \( E = \bigcup E_j, \ A_1 = E_1 + \mathbb{N}^n, \ A_2 = E_2 + \mathbb{N}^n - A_1, \ldots, A_d = E_d + \mathbb{N}^n - (A_1 \cup \cdots \cup A_{d-1}), \ A = \mathbb{N}^n - \bigcup (E_j + \mathbb{N}^n) \) and a standard basis \( \{ h_1, \ldots, h_d \} \) of \( I \) (for a total ordering of the multi-indices), we introduce the mapping \( \Phi \)
\[
\Phi : \mathcal{C}(\rho') \ni f \mapsto \sum_{i=1}^{d} h_i \Phi_i(f) + \Phi_0(f) \in \mathcal{C}(\rho).
\]
**Theorem 2.2.6.** There is a $\rho' = \rho'(\rho)$ such that, $A_\rho \supset A_{\rho'}$, $\Phi^{-1} : \mathbb{C}(\rho) \to \mathbb{C}(\rho')$ exists and is continuous.

**Proof.** Write $\Phi = I + (\Phi - I)$. Given $\rho'$ such that $A_\rho \supset A_{\rho'}$, we have

$$|\Phi - I(f)|_{\rho'} \leq \sum_{i=1}^d |h_i - x^{E_i}|_{\rho'} \rho^{-E_i} |f|_{\rho'}.$$  

Choose for instance $\rho' = (\rho_1', ..., \rho_n') = (\lambda^d \rho_1, ..., \lambda^d \rho_n)$ for some $\lambda$, $0 < \lambda < 1$, and appropriate exponents $(\delta_1, ..., \delta_n)$ such that $|h_i - x^{E_i}|_{\rho'} \rho^{-E_i} \leq 1/2d$. This yields the existence of the inverse $\Phi^{-1} : A_{\rho'} \to A_{\rho}$ and its continuity $|\Phi^{-1}(f)|_{\rho} \leq 2$. Denote $g = \Phi^{-1}(f)$, we have

$$f = \Phi(g) = \sum_{i=1}^d h_i \Phi_i(g) + \Phi_0(g), \quad \text{with } \Phi_0(g) = \sum_{A \in A} g_A x^A. \quad (12)$$  

The definition of a standard basis entails:

**Theorem 2.2.7.** Given an ideal $I$, an ordering on the multi-indices, a corresponding standard basis $\{h_1, ..., h_A\}$ of $I$, and a polydisc $A_{\rho'}$, there is $\rho' = \rho'(\rho)$ such that, $A_\rho \supset A_{\rho'}$, and $C > 0$ such that for all element $f$ of $I$

$$f = \sum_{i=1}^d h_i \phi_i \quad \text{and } |\phi_i|_{\rho'} \leq C |f|_{\rho'}.$$

The same kind of division theorem is valid for analytic functions instead of polynomials. The definition of a standard basis should be revised accordingly, replacing the highest exponent by the smallest one.

2.3. Uniform Bounds, Bernstein Inequality, and the Number of Zeroes

We first give two definitions (cf. [R-Y]). $A_R$ denotes the open disk of radius $R$, centered at 0.

**Definition 2.3.1.** Let $R > 0$, $0 < \kappa < 1$ and $K > 0$ be given and let $f$ be holomorphic in a neighborhood of $A_R$. We say that $f$ belongs to the Bernstein class $B_{R, \kappa, K}$ if

$$\max(|f(z)|, z \in A_R)/\max(|f(z)|, z \in A_{R\kappa}) \leq K.$$  

**Remark.** The name “Bernstein class” is justified by the fact that, according to one of the classical Bernstein inequalities, any polynomial of degree $d$ belongs to $B_{1, 1, K}$, $K = (1/\kappa)^d$ for any $R$ and $\kappa$. 

**BERNSTEIN INEQUALITY**
Definition 2.3.2. Let a natural $N$, $R > 0$ and $C > 0$ be given, and let $f(z) = \sum_{k=0}^{\infty} f_k z^k$ be an analytic function in a neighborhood of $0 \in \mathbb{C}$. We say that $f$ belongs to the Bernstein class $B_{N,R,C}^2$, if

$$|f_j| R^j \leq C \max(|f_i| R^i, i = 0, \ldots, N), \quad j \geq N + 1.$$ 

The two classes $B^1$ and $B^2$ essentially coincide. More precisely, we have the following:

**Lemma 2.3.3.** Let $f$ be an element of $B_{N,R,C}^2$. Then $f$ is analytic in the open disk $A_R$ and for any $R' < R$ and $0 < \alpha < 1$ and $K = (1/\alpha)^N (1 + (1 - \alpha^N) (\alpha/1 - \alpha) + C(\beta/1 - \beta))$, $\beta = R'/R$, $f$ belongs to $B_{R',\alpha,K}^1$.

**Proof.** The convergence of $f(z) = \sum_{k=0}^{\infty} f_k z^k$ on $A_R$ is immediate. Let $m = \max(|f(z)|, z \in A_R)$. Then by the Cauchy formula

$$|f_i| \leq m(\alpha R')^j$$

for any $i$.

In particular,

$$|f_i| R^j \leq m(\alpha R') R^j \leq m(\alpha R/R)^N \quad \text{for } i = 0, \ldots, N.$$ 

Hence

$$|f_j| R^j \leq C m(\alpha R'/R)^N \quad \text{for } j \geq N + 1.$$ 

Now we can bound $|f|$ on $A_R$ as

$$\max(|f(z)|, z \in A_R) \leq \sum_{k=0}^{N} |f_k| R^k + \sum_{k=N+1}^{\infty} |f_k| R^k \leq m \sum_{k=0}^{N} (\alpha R')^k R^k + (C m(\alpha R'/R)^N) \sum_{N+1}^{\infty} (R'/R)^k = m(1/\alpha)^N (1 + (1 - \alpha^N)(\alpha/1 - \alpha) + C(\beta/1 - \beta)).$$ 

**Remark.** The constant $K$ in Lemma 2.3.3 can be chosen as $(1/\alpha)^N (1 + C(\alpha/1 - \alpha) + C(\beta/1 - \beta))$ which in some cases gives a better estimate.

Conversely, if $f$ belongs to $B_{R,\alpha,K}^1$, then it belongs to $B_{N,R,C}^2$ with $N = \log_\alpha K$ and $C$ given explicitly through $R$, $\alpha$, $K$ (cf. [R-Y, Ha]).

A relevance of Bernstein classes to our purpose is explained by the following lemma (which is well known in different forms in various fields of complex analysis (cf. [L-G]; we give a version, obtained by M. Waldschmidt [W] in relation to transcendental numbers theory):
Lemma 2.3.4. Let \( R > 0 \), and \( 0 < \alpha < 1 \) be given and let \( f \) be holomorphic in a neighborhood of \( A_R \). Then the number of zeroes of \( f \) in \( A_{aR} \) does not exceed

\[
\log(\max(|f(z)|, z \in A_R) / \max(|f(z)|, z \in A_{aR}))/\log(1 + \alpha^2)/2\alpha).
\]

In other words, for an element \( f \) of \( B^{1}_{R, a, K} \), \( \# \{ f^{-1}(0) \} \cap A_{aR} \leq \log K / \log(1 + \alpha^2)/2\alpha) \).

Frequently, in the theory of differential equations, one deals with analytic developments \( f(z) = \sum_{k=0}^{\infty} f_k z^k \), where \( f_k \) is defined inductively by an expression which involves the preceding coefficients. So it can often be shown that \( f \) belongs to a certain Bernstein class \( B^2 \). Combining the results above, we can estimate the number of zeroes of the functions in \( B^2 \) as follows:

Proposition 2.3.5. Let \( f \) be an element of \( B^{2}_{N, R, C} \). Then for any \( R^* < R \), the number of zeroes of \( f \) in \( A_{R^*} \) does not exceed

\[
N \cdot \min[(1 + \log(1 + (1 - \alpha^N)(\alpha/1 - \alpha) + C(\gamma/1 - \gamma))/
(1 + \log((1 + \alpha^2)/\log(1/\alpha)); \alpha, (R^*/R) < \alpha < 1)],
\]

where \( \gamma = (R^*/aR) < 1 \).

Proof. For any \( \alpha, (R^*/R) < \alpha < 1 \), let \( R' = R^*/\alpha \). Then by Lemma 2.3.3, \( f \) belongs to \( B^{2}_{R', a, a'K} \), with \( K = (1/\alpha)^N (1 + (1 - \alpha^N)(\alpha/1 - \alpha) + C(\gamma/1 - \gamma)) \), where \( \gamma = (R'/aR) = R'/R \). Hence, by Lemma 2.3.4, the number of zeroes of \( f \) on \( A_{R'} = A_{aR^*} \) is bounded by

\[
\log((1 + (1 - \alpha^N)(\alpha/1 - \alpha) + C(\gamma/1 - \gamma))/\log(1 + \alpha^2)/2\alpha))
\]

Since the value of \( \alpha \) between \( (R^*/R) \) and 1 or, equivalently, the value of \( R' \), \( R > R' > R^* \) can be chosen arbitrarily, the proposition follows.

Corollary 2.3.6. Let \( f \) be an element of \( B^{2}_{N, R, C} \). Then

1. For \( R^* = R/4 \), the number of zeroes of \( f \) on \( A_{R^*} \) does not exceed \( N \log_2(4 + 2C) \).
2. For \( R^* = R/2 \max(C, 2) \), the number of zeroes of \( f \) on \( A_{R^*} \) is at most \( 20N \).
3. For \( R^* = Re^{-1/2}(\log(C, 2)) \), this number is at most \( N \).
Proof. To prove 1, let \( \alpha = (1/2) \). Then \( \gamma = (1/2) \) and \( \# \{ f^{-1}(0) \} \cap \mathcal{A}_{\mathcal{H}^d} \leq N. (1 + \log (1 + (1 - (1/2)^N) + C/\log(2))/(1 + \log (5/8)/\log(2))) \leq N \log \, \text{d}(4 + 2C) \).

In Case 2 we also choose \( \alpha \) to be \( (1/2) \). Then \( \gamma = 1/\text{max}(C, 2) \), and we get
\[
\# \{ f^{-1}(0) \} \cap \mathcal{A}_{\mathcal{H}^d} \leq N. (1 + \log (1 + (1 - (1/2)^N) + 2)/(1 + \log (5/8)/\log(2))) \leq 20N.
\]

Finally, for \( R^* = \text{Re}^{-10N + 2}/\text{max}(C, 2) \), we put \( \alpha = e^{-10N} \); we get \( \gamma = 1/4 \, \text{max}(C, 2) \), and \( \# \{ f^{-1}(0) \} \cap \mathcal{A}_{\mathcal{H}^d} \leq N. (1 + (2/3N)) \). Since the number of zeroes is an integer, this yields
\[
\# \{ f^{-1}(0) \} \cap \mathcal{A}_{\mathcal{H}^d} \leq N.
\]

Remark. Taking into account Bernstein inequality, the last conclusion is strong enough to prove that the number of zeroes of a polynomial of degree \( d \) does not exceed \( d \).

We introduce now the Banach algebra \( \mathbb{C}(R, \rho) \) as
\[
f(x, z) = f_0 = f(x_1, \ldots, x_n; z) = \sum_{k=0}^{\infty} z^k f_k(x_1, \ldots, x_n)
\]
belongs to \( \mathbb{C}(R, \rho) \) if all its coefficients \( f_k(x_1, \ldots, x_n) \) belong to \( \mathbb{C}(\rho) \) and if
\[
\sum_{k=0}^{\infty} R^k |f_k|_\rho < \infty.
\]

We denote \( I \) the Bautin ideal generated by the coefficients \( f_k(x_1, \ldots, x_n) \) in the ring \( \mathbb{C}[x_1, \ldots, x_n] \) of germs of holomorphic functions of \( 0 \in \mathbb{C}^n \). Let \( f_0, \ldots, f_N \) generate this ideal (in fact, \( f_k(x_1, \ldots, x_n) \) are holomorphic on \( \mathcal{A}_{\mathcal{H}^d} \)).

**Theorem 2.3.7.** There exist \( \rho' \) and \( C > 0 \), depending on \( f \), such that for any \( x \) in \( \mathcal{A}_{\mathcal{H}^d} \), the function \( f_x(z) \) belongs to \( B^N_{\mathcal{H}_{\rho'}, \mathcal{C}} \).

Proof. Let \( h_1(x_1, \ldots, x_n), \ldots, h_d(x_1, \ldots, x_n) \) be a Gröbner basis of the ideal \( I \). By its construction, we can assume \( h_j \) to be analytic on \( \mathcal{A}_{\mathcal{H}^d} \). Moreover, since \( f_0, \ldots, f_N \) generate \( I \), we have
\[
h_j = \sum_{i=0}^{N} \phi_i^j f_i,
\]
where the functions \( \phi_i^j \) can be also chosen analytic on \( \mathcal{A}_{\mathcal{H}^d} \). Let \( C_1 = \max(|\phi_i^j|_\rho; i, j) \).
By Theorem 2.2.7 above, there are \( \rho' \) and \( C_2 > 0 \) such that for any \( h \) element of the ideal \( I, h \) analytic on \( A_{\rho'} \), \( h = \sum_{j=1}^{\rho'} g_j h_j \), with \( g_j \) analytic on \( A_{\rho'} \) and \( |g_j|_{\rho'} \leq C_2 |h|_{\rho'} \). Hence, \( h = \sum_{i=0}^{N} g'_i f_i \), with \( g'_i \) analytic on \( A_{\rho'} \) and \( |g'_i|_{\rho'} \leq dC_1 C_2 |h|_{\rho'} \).

In particular, for any \( j \geq N + 1 \), we have
\[
f_j = \sum_{i=0}^{N} g'_i f_i,
\]
with
\[
|g'_i|_{\rho'} \leq mC_1 C_2 |f_i|_{\rho'} \leq dC_1 C_2 (1/R)^i,
\]
for any element \( f(x, z) = \sum_{k=0}^{\infty} z^k f_k(x_1, ..., x_n) \) of \( \mathbb{C}(R, \rho) \), there is a constant \( C_3 \) such that \( |f_j|_{\rho'} \leq C_3 (1/R)^j \). Denote \( dC_1 C_2 C_3 \) by \( C_4 \). Now let us fix \( x \) in the polydisc \( A_{\rho'} \). Substituting \( x \) into (13) we get
\[
|f_j(x)| \leq \sum_{i=0}^{N} |g'_i(x)||f_i(x)| \leq C_4 (1/R)^j \sum_{i=0}^{N} |f_i(x)|.
\]
Hence, \( |f_j(x)| R^j \leq C_4 (N + 1) \max(|f_i(x)|, i = 0, ..., N) \)
\[
\leq C_5 \max(|f_j(x)| R^j, i = 0, ..., N), \text{ where } C_5 = C_4 \max(1/R, 1).
\]
This proves Theorem 2.3.7 with \( C = C_5 \).

**Theorem 2.3.8.** Let \( f \) be an element of \( \mathbb{C}(R, \rho) \), \( N, \rho', C \) be as above. Let \( R' = (1/4) R, R'' = R/2 \max(C, 2), R^* = \Re^{-10N + 2}/\max(C, 2) \). Then for any \( x \) in the polydisc \( A_{\rho'} \), the function \( f(x) \) can have on the disks \( A_{R'}, A_{R''}, A_{R^*} \), at most \( N \log_{5/4}(4 + 2C), 20N, \text{ and } N \) zeroes, respectively.

Now let us assume additionally, that each \( f_k(x) \) is a homogeneous polynomial in \( x \) of degree \( k \). Thus we have
\[
f_{\lambda}(x) = f_k(x \lambda)
\]
for any complex number \( \lambda \).

Hence in this case we get:

**Theorem 2.3.9.** Let \( R, \rho, \rho', C, R', R'', R^* \) be as above (thus we consider \( f_k(x) \) just as elements of \( \mathbb{C}(\rho) \)). Then for any \( x \) in \( \mathbb{C}^n \), the number of zeroes of \( f_k(x) \) on the disks \( A_{R'/|x|_{\rho}}, A_{R''/|x|_{\rho}}, A_{R^*/|x|_{\rho}} \), is at most \( N \log_{5/4}(4 + 2C), 20N \) and \( N \), respectively, where \( |x|_{\rho} = \max(|x_k|/|\rho_k|; k) \).
III. APPLICATION TO DIFFERENTIAL EQUATIONS

Let $X$ be a polynomial vector field of the plane of the type

$$X = \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} + \sum_{2 \leq i + j \leq d} \left( a_{ij} x^i y^j \frac{\partial}{\partial x} + b_{ij} x^i y^j \frac{\partial}{\partial y} \right).$$  \hspace{1cm} (17)

The flow of $X$ at time $t$ is a solution to the system

$$\frac{dx}{dt} = y + \sum_{2 \leq i + j \leq d} a_{ij} x^i y^j$$

$$\frac{dy}{dt} = -x + \sum_{2 \leq i + j \leq d} b_{ij} x^i y^j.$$  \hspace{1cm} (17a)

The vector field $X$ defines a foliation (with singularities) of the plane. This foliation can be equivalently defined by using the differential 1-form

$$\omega = i_X dx \wedge dy = dH + \sum_{2 \leq i + j \leq d} \left( a_{ij} x^i y^j dy - b_{ij} x^i y^j dx \right),$$

with

$$H(x, y) = \frac{1}{2} (x^2 + y^2).$$

Further, we will preferably use the complex coordinates

$$z = \frac{1}{\sqrt{2}} (x + \sqrt{-1} y), \quad \bar{z} = \frac{1}{\sqrt{2}} (x - \sqrt{-1} y)$$

and

$$\omega = d\bar{H} + \sum_{2 \leq i + j \leq d} \left( A_{ij} z^i \bar{z}^j dz + \bar{A}_{ij} z^i \bar{z}^j d\bar{z} \right),$$  \hspace{1cm} (18)

with $H(z, \bar{z}) = z\bar{z}$.

Assume first that the coefficients of $X$ $(a_{ij}, b_{ij}; \ i, j = 1, \ldots, d; \ 2 \leq i, j \leq d)$ vary in a fixed ball $B(0, R) \subset \mathbb{R}^d$, $(D = d^2 + 3d - 4)$.

For $\delta$ small enough, the flow defined by $X$, for all $(a_{ij}, b_{ij}) \in B(0, R)$, has a first return mapping $L$:

With $r = (\frac{1}{2} x^2)^{1/2}$, $\Sigma = \{(x, 0) / 0 \leq x \leq \delta \}$, write $L : \Sigma \rightarrow \Sigma$

$L : r \mapsto r + L_1(a, b) r^2 + L_2(a, b) r^3 + \cdots$
A priori, the Taylor series \( L(r) \) has analytic coefficients \( L_k(a, b) \). In fact, these coefficients are polynomials in \( (a, b) = (a_i, b_j) \); and the degree of \( L_k(a, b) \) is less than \( k + 1 \).

This cannot be seen as a consequence of a general argument. We include a proof of this fact together with an estimate of the growth of these coefficients.

### 3.1. Growth of the Coefficients of the First Return Mapping

We write \( \omega \) using polar coordinates,

\[
\begin{align*}
  x &= r \cos \varphi, \\
  y &= r \sin \varphi.
\end{align*}
\]

We obtain

\[
\frac{dr}{d\varphi} = \sum_{i=2}^{d} r^i P_i / 1 + \sum_{i=1}^{d-1} r^i Q_i,
\]

\[
\frac{dr}{d\varphi} = r^2 R_2 + r^3 R_3 + \cdots + r^k R_k + \cdots. \tag{19}
\]

Where the coefficients \( R_k \) are polynomials in \( (\sin \varphi, \cos \varphi) \), and in the parameters \( (a_i, b_j) \).

Let \( r_0 \) be the minimum of the moduli of the complex zeroes of \( 1 + \sum_{i=1}^{d-1} r^i Q_i \). The Taylor series (19) is convergent on the complex disc \( D(0, r_0) \), and we get a growth condition \( |R_k| \leq C' \) for some constant \( C \).

We obtain also that the coefficients \( R_k \) are polynomials in \( (a_i, b_j) \) and in \( (\sin \varphi, \cos \varphi) \) of degree less than \( k - 1 \).

We look for a solution \( r = r(\varphi) \) given as a Taylor series

\[
r = r_0 + r_0^2 v_2(\varphi, (a, b)) + \cdots + r_0^k v_k(\varphi, (a, b)) + \cdots
\]

with initial data \( r_0 = r(0) \).

The \( v_i \) are thus obtained inductively by integration of a differential system

\[
\begin{align*}
  \frac{dv_2}{d\varphi} &= R_2 \\
  \frac{dv_3}{d\varphi} &= 2v_2 R_2 + R_3, \tag{20} \\
  \frac{dv_k}{d\varphi} &= F_k(v_1, ..., v_{k-1}; R_1, ..., R_k)
\end{align*}
\]

with initial condition \( v_1 = 1, v_i(0) = 0 \), for \( i > 1 \).
We get easily that the coefficients $v_k$ are polynomials in $(a_{ij}, b_{ij})$ and in $(\sin \phi, \cos \phi)$ of degree less than $k + 1$.

Next, we apply a majorizing series technique to bound the growth of the coefficients. We first obtain

$$|v_2| \leq 2\pi |R_2|,$$

then define $V_2 = 2\pi |R_2|$.

We get next

$$|v_3| \leq 2\pi (2V_2 |R_2| + |R_3|),$$

and define $V_3 = 2\pi (2V_2 |R_2| + |R_3|)$.

We build up a majorizing series $V(r)$ solution of the analytic equation

$$V(r) = r + 2\pi(V(r)^2 |R_2| + V(r)^3 |R_3| + \cdots + V(r)^k |R_k| + \cdots).$$

A control on the growth of the coefficients $V_k$ can be thus obtained: the function $V(r)$ is analytic on the domain $D(0, \rho)$ where $\rho$ is the minimum of the modulii of the complex zeroes $z$ of the analytic equation

$$1 - 2\pi(2z |R_2| + 3z^2 |R_3| + \cdots + k z^{k-1} |R_k| + \cdots) = 0.$$  

3.2. Successive Derivatives of the First Return Mapping

Given the 1-parameter family of vector fields

$$X_{\varepsilon} = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} + \varepsilon \sum_{2 \leq i + j \leq d} \left( a_{ij} x^i y^j \frac{\partial}{\partial x} + b_{ij} x^i y^j \frac{\partial}{\partial y} \right),$$

or equivalently the 1-parameter family of differential 1 forms:

$$\omega_{\varepsilon} = dH + \varepsilon \sum_{2 \leq i + j \leq d} \left( A_{ij} z^i \overline{z}^j dz + \overline{A}_{ij} z^i \overline{z}^j d\overline{z} \right),$$

there is a corresponding first return mapping $r \mapsto L(R, \varepsilon)$

$$L(r, \varepsilon) = r + \varepsilon L_1(A, r) + \cdots + \varepsilon^k L_k(A, r) + O(\varepsilon^{k+1}).$$

In [F], a formula was given for $L_k(A, r)$ if all the preceding coefficients $L_1(A, r), \ldots, L_{k-1}(A, r)$ vanish. Here, we give a more precise result: we can build in a completely algorithmic manner the ideal $I_k$ generated by the $k$ first coefficients of the first return mapping.
Lemma 3.2.1. Let $\omega$ be a polynomial (resp. holomorphic) 1 form, there exists a unique decomposition

$$\omega = \Psi(H)(z \, d\bar{z} - \bar{z} \, dz)/2 + g \, dH + dR,$$

where $\Psi(H)$ is a polynomial (resp. holomorphic) in $H = z\bar{z}$, $g$ and $R$ are polynomials (resp. holomorphic).

Proof. Consider first $d\omega = F(z, \bar{z}) \, dz \wedge d\bar{z}$, then write

$$F(z, \bar{z}) = \sum_{i\neq j} F_{ij} \bar{z}^i z^j + \sum_{i} F_i H^i.$$

Let $\phi(H) = \sum_i F_i H^i$. There is a unique analytical function $\psi(t)$ such that $t\psi'(t) + \psi(t) = \phi(t)$.

We set $g = \sum_{i\neq j} (F_{ij}/i - j) \, z^i \bar{z}^j$, and thus we get

$$d\omega = dg \wedge dH + d(\psi(H)(z \, d\bar{z} - \bar{z} \, dz)/2).$$

We then conclude that there is a polynomial (resp. holomorphic) $R$ so that we have the decomposition that we wish.

To simplify our notations further, we set $\eta = (z \, d\bar{z} - \bar{z} \, dz)/2$. This decomposition is a special case of the Brieskorn–Sebastiani decomposition (cf. [S], see also [Gu]). We now come back to our data:

$$\omega_1 = \sum_{2 \leq i+j \leq d} (A_{ij} z^i \bar{z}^j + \bar{A}_{ij} z^j \bar{z}^i) \, d\bar{z}^i z^j, \quad \text{(cf. (26))}. \quad (28)$$

We consider then the following sequence of polynomial 1-forms generated by the preceding decomposition.

$$\omega_1 = \Psi_1(H)(z \, d\bar{z} - \bar{z} \, dz)/2 + g_1 \, dH + dR_1,$$

$$g_1 \omega_1 = \Psi_2(H)(z \, d\bar{z} - \bar{z} \, dz)/2 + g_2 \, dH + dR_2,$$

$$\vdots$$

$$g_k \omega_1 = \Psi_{k+1}(H)(z \, d\bar{z} - \bar{z} \, dz)/2 + g_{k+1} \, dH + dR_{k+1}.$$

We observe that the coefficients of the polynomials $\Psi_i(H)$ are themselves polynomial in the parameters $(A_{ij}, \bar{A}_{ij})$ $(i, j = 1, \ldots, d; \ 2 \leq i + j \leq d)$ of degree less than $k + 1$.

Lemma 3.2.2. Assume that $\omega_1$ is homogeneous of degree $d$.

(i) If $d$ is odd, $d = 2p + 1$, then:

$$\Psi_1(H) = \psi_1(A) \, H^{2p}. $$
If \( d \) is even, \( d = 2p \), then: \( \Psi_k(H) = 0 \) if \( k \) is odd, and
\[
\Psi_k(H) = \psi_k(A) H^{2p-1} k, \quad k = 2q.
\]

The polynomials \( \psi_k(A) \) and \( g_k \) can be easily computed from their definition.

**Example: The Quadratic Case.** This case is of special interest for us in this article and we give explicitly the first six coefficients \( \psi_k(A) \) (\( k = 1, \ldots, 6 \)).

\[
\psi_1(A) = 0, \\
\psi_2(A) = \text{Im} \, \Phi_2(A), \; \Phi_2(A) = -2A_{20}A_{11}, \\
\psi_3(A) = 0, \\
\psi_4(A) = \text{Im} \, \Phi_4(A), \; \Phi_4(A) = 30A_{20}^2A_{21}A_{20} - 2A_{20}^2A_{02}A_{11} \\
- 24A_{20}^2A_{11} - 3A_{20}A_{02}A_{21} - 4A_{20}A_{11}A_{02}A_{20} \\
+ 15A_{20}A_{11}A_{11} - 2A_{02}A_{11}, \\
\psi_5(A) = 0, \\
\psi_6(A) = \text{Im} \, \Phi_6(A), \; \Phi_6(A) = 516A_{20}^3A_{11}A_{20} - (116/3) A_{40}^2A_{02}A_{11} \\
- (389/9) A_{20}A_{02}A_{11}^3 - 146A_{20}A_{21}A_{11}^3 + 882A_{20}A_{21}A_{20} \\
- 468A_{11}A_{21}A_{11} - 402A_{20}A_{11}^3 + (127/6) A_{02}A_{11}A_{11} \\
+ (17/2) A_{02}A_{11}A_{02} + (1252/9) A_{11}A_{02}A_{20}A_{20} \\
- (53/6) A_{20}A_{11}A_{02}A_{11} - 125A_{11}A_{02}A_{02}A_{20} \\
- (307/9) A_{11}A_{02}A_{11}A_{20} + (242/3) A_{02}A_{20}A_{11}A_{20} \\
- 840A_{11}A_{20}A_{11}A_{20} + (241/9) A_{02}A_{20}A_{11}A_{20} \\
+ (541/9) A_{11}A_{02}A_{11}A_{20} + (602 - 9) A_{02}A_{11}A_{20}A_{11} \\
- 120A_{02}A_{20}A_{11}A_{20} - (20/3) A_{02}A_{02}A_{20}A_{11} \\
- 14A_{20}A_{11}A_{11}A_{20} + 6A_{02}A_{02}A_{20}A_{11}.
\]

**Lemma 3.2.3. (Control of the Norms on a Polydisc).** There is a universal constant \( K \) such that for all \( \alpha \in \mathbb{H} \), we have
\[
|\Psi(H)|_\rho \leqslant K |\alpha|_\rho, \quad |g|_\rho \leqslant K |\alpha|_\rho, \quad |R|_\rho \leqslant K |\alpha|_\rho.
\]
Proof. We can first control $|F|$ and then $|Ψ(H)|$, because $Φ(H)$ is obtained as part of the Taylor series of $F$. The analytic series $g$ is obtained from $F$ by dividing the corresponding coefficient by $i−j$. Finally, the series $R$ can be controlled by Poincaré’s formula.

We now remark that for homogeneous $ω₁$, the successive derivatives of $L$ relative to $r$ (for $r=0$) are essentially the same as the derivatives of $L$ relative to the parameter $ε$.

Lemma 3.2.4. Write $L(r, ε, A) = r + εL₁(r, A) + ε²L₂(r, A) + \cdots + ε^kL_k(r, A) + \cdots$, then if $ω₁$ is homogeneous of degree $d$, there exist polynomials $L_k(A)$ such that: $L_k(r, A) = L_k(A) r^{(d−1)+i}$.  

Proof. We use the same notations found in Section 3.1. Following Bautin, we look for a solution $r = r(φ, ε)$ of the flow of the vector field

$$X = X₀ + εX₁ = \frac{d}{dφ} + ε \left( P(φ) \frac{d}{dr} + Q(φ) \frac{d^{d−1}}{dφ} \right) . (29)$$

From the definition of a flow, we get ($X^k$ means the $k$th iterated action of $X$ as a derivation)

$$r(φ, ε) = r + X₁r + X₂r/2! + \cdots + X^kr/k! + \cdots ,$$

and so, from (29), we get that $X₁^kr$ is of the form $w_k(φ, A) r^{(k−1)(d−1)+d}$, where $w_k(φ, A)$ is homogeneous of degree $k$ in the parameters $A$ and is defined inductively by

$$w_{k+1}(φ, A) = \left( (k(d−1)+d) Pw_k(φ, A) + Qw_k(φ, A) \right) / (k+1). (30)$$

The rest of the proposition follows easily from the fact that the derivative relatively to the angle $φ$ preserves the form of the coefficients.

We now give the main result of this paragraph, which is an improvement of $[F]$. 

Theorem 3.2.5. Let $X₁$ be the vector field considered above which corresponds to the 1-form $ω₁ = dH + εω₁$ with $ω₁$ homogeneous of degree $d$. Let $L(r, A)$ be the first return mapping of $X₁ = X₁$ in a neighborhood of the origin:

$$L(r, A) = r + L₁(A) r² + \cdots + L_{k−1}(A) r^k + \cdots ,$$

then the ideal $I_k$ generated in the ring $R[A, A]$ (in fact $Q[A, A]$) by $L₁(A), \ldots, L_k(A)$ is the same as the ideal generated by $ψ₁(A), \ldots, ψₖ(A)$ for
all $k$, and we have an explicit control of the norm on a polydisc of the
elements of the matrix $T$ (triangular, with diagonal elements equal to 1) so that

$$
\psi_k(A) = \sum_j T_{jk}(A) L_j(A).
$$

**Proof.** The proof is made by induction on the integer $k$. For $k = 1$, we get $\psi_1(A) = L_1(A)$ as a consequence of the Andronov–Poincaré formula

$$
L_1(A) = -\int_{H-r} \omega_1.
$$

We now compute modulo $\varepsilon^{k+1}$ ($k > 1$)

$$
\int_{\gamma_r} (1 - \varepsilon g_1 + \varepsilon^2 g_2 + \cdots + (-1)^k \varepsilon^k g_k)(dH + \varepsilon \omega),
$$

where $\gamma_r$ is the part of the orbit of $X_\varepsilon$ between the points $(0, r)$ and

$$(0, L(r, \varepsilon)).$$

We get

$$
\int_{\gamma_r} dH + \sum_{i=1}^{k-1} \psi_i(A) \frac{\partial^{k-i}}{\partial \varepsilon^{k-i}} \left[ \int_{\gamma_r} r^{i+1} \eta \right]
$$

$$
+ \int_{\gamma_r} (\varepsilon dR_1 + \varepsilon^2 dR_2 + \cdots + (-1)^{k+1} \varepsilon^k dR_k)
$$

$$
- \varepsilon^{k+1} \psi_{k+1}(A) = 0.
$$

We identify each term which contributes to the order $k+1$ in $\varepsilon$. From

$$
\int_{\gamma_r} dH,$
$$

we get exactly $\varepsilon^{k+1} L_{k+1}(A)$. From

$$
\sum_{i=1}^{k-1} \psi_i(A) \frac{\partial^{k-i}}{\partial \varepsilon^{k-i}} \left[ \int_{\gamma_r} r^{i+1} \eta \right],
$$

we get a term which is a linear combination of $\psi_2(A), \ldots, \psi_k(A)$ with poly-
nomial coefficients. These coefficients $\left[ \int_{\gamma_r} r^{i+1} \eta \right]$ can be obtained by the integration of the terms of the series

$$
\int_{\gamma_r} r^{i+1} \eta = \int_0^{2\pi} (r(\phi))^{i+1} r(\phi)^2 d\phi,
$$

and thus we control their growth (cf. Section 3.1). The term

$$
\int_{\gamma_r} (\varepsilon dR_1 + \varepsilon^2 dR_2 + \cdots + (-1)^{k+1} \varepsilon^k dR_k)
$$

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gives rise to an element of the ideal generated by the coefficients $L_2(A), \ldots, L_d(A)$. More precisely, we obtain

$$\sum_{i=1}^{k-1} \frac{\partial R_i}{\partial y}(0, r) L_{k-i-1}(A).$$

Its polynomial coefficients can easily be controlled with Lemma 3.2.3.

### 3.3. The Construction of the Standard Basis

In this paragraph, we focus on the special case of quadratic vector fields ($d=2$). Nevertheless, all of what we do here is valid for the more general case of an arbitrary homogeneous perturbation of arbitrary degree $d$. We refrain to do it for degree $d=3$. The computations become much harder for the case $d>3$ but the same method we use remains in principle effective.

Once we have obtained the control of the coefficients of the matrix which takes the collection of the $k$ first coefficients of the first return mapping to the $k$ first polynomials $\psi_i(A)$, we have next to find an explicit change of systems of generators to produce a standard basis. At this point, we follow the usual Buchberger algorithm. The elements of the standard basis are obtained by performing Euclidean divisions and adding eventually the remainders to the list of generators (cf. [Bu, L]). This transition from a system of generators to the standard basis does not involve any hard problem to control the size of the coefficients. We have the choice of the order of the variables. We propose to choose the following order: $A_{20} > A_{11} > A_{02}$ (and reverse order for the complex conjugated variable). After this completely algorithmic and effective step is done using computational algebra (Maple), we obtain as standard basis for the ideal $I$

$$h_1 = \Psi_2(A) = \text{Im } \Phi_3(A), \Phi_2(A) = -2A_{20}A_{11},$$

$$h_2 = \text{Im}(2A_{11}A_{20}^2A_{02} + 3A_{11}^2A_{02}A_{20} - 2A_{02}A_{11}^2),$$

$$h_3 = 4A_{20}^2A_{20}A_{11} - 4A_{20}A_{11}A_{02}A_{20} + 13A_{11}^2A_{02}A_{20}A_{11} - 6A_{11}A_{02}A_{11} - 13A_{20}A_{02}A_{11} + 6A_{02}A_{11},$$

$$h_4 = A_{11}^2A_{02}A_{20} - A_{11}A_{02}A_{20}^2 - 2A_{11}A_{02}A_{20}A_{11} + A_{11}A_{02}A_{11} - A_{20}A_{02}A_{11} + A_{02}A_{11} + 2A_{02}A_{11}A_{02}.$$  

We now apply our main result (Corollaries 2.3.8 and 2.3.9) to the problem of differential equations and we obtain:
Theorem 3.3.1. Let $X$ be a polynomial vector field of the type:

$$X = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} + \sum_{i+j=1} \left( a_{ij} x^i y^j + b_{ij} x^i y^j \right),$$

and its corresponding first return mapping $L$. There is a fixed polydisc whose radius can be effectively determined on which the number of complex zeroes of $L$ is less than 4.

Proof. We apply all of the previous calculations. The coefficients $L_2(A)$, $L_4(A)$ and $L_6(A)$ generate the Bautin ideal $I$. The equation for the real fixed points of the first return mapping is

$$r = r + L_2(A) r^1(1 + \cdots) + L_4(A) r^1(1 + \cdots) + L_6(A) r^1(1 + \cdots),$$

or

$$L_2(A)(1 + \cdots) + L_4(A) r^1(1 + \cdots) + L_6(A) r^1(1 + \cdots) = 0.$$ 

Following the method explained in Section 2.1, this equation has less than two real and positive solutions. From our main result (Corollary 2.3.8), this equation has less than four complex solutions. Note that this last result is sharp because, by symmetry, the analytic extension of the first return mapping has at least two positive real solutions and two negative real solutions.

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Bernstein Inequality


