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Motion by Mean Curvature as the Singular Limit of Ginzburg–Landau Dynamics

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1. INTRODUCTION

We consider the differential equation

$$u_t - \Delta u + \varepsilon^{-2}(u^3 - u) = 0 \tag{1.1}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, with appropriate initial and boundary data. Our interest is in the limiting behavior of $u = u^\varepsilon$ as $\varepsilon \to 0$. Formal analysis suggests the following picture: u^ε separates Ω into two regions, where $u^\varepsilon \approx +1$ and $u^\varepsilon \approx -1$, respectively, and the interface between them moves with normal velocity equal to the sum of its principal curvatures. Our goal here is to present two rigorous results which tend to confirm this picture. The first is a compactness theorem: we show that as $\varepsilon \to 0$, the solutions of (1.1) are in a certain sense compact as functions of space-time (see Theorem 2.3 and Remark 2.5). Thus it makes sense to discuss the limiting behavior. Our second result is a verification of the picture for certain radial solutions: we prove that $\lim_{\varepsilon \to 0} u^\varepsilon$ exists and has the expected form if Ω is a ball, u^ε is radial with one transition sphere, and the boundary condition is of Dirichlet type (see Theorem 3.1).

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The scaling of (1.1) has been chosen so that the associated motion by mean curvature takes place on a time scale of order one. Studying (1.1) is equivalent to considering the solution of

$$w_s - \varepsilon^2 \Delta w + w^3 - w = 0 \tag{1.2}$$

on a time scale of order ε^{-2}, since $w(x, s)$ solves (1.2) if and only if $u(x, t) = w(x, \varepsilon^{-2} t)$ solves (1.1).

The link between (1.1) and motion by mean curvature was first (to our knowledge) observed by Allen and Cahn [1] on the basis of a formal analysis. A much more systematic treatment, still formal in character, was given by Rubinstein, Sternberg, and Keller in [36]. The behavior of (1.1) in one space dimension is somewhat different, since then the transitions are at points rather than along surfaces, so curvature plays no role; this problem has been treated in [7, 10, 11, 20, 21]. Prior to our work the only rigorous result in a multidimensional context was due to Freidlin and Gärtner [19, 23]; when applied to (1.2) their analysis shows that $w(x, s)$ evolves on a time scale longer than ε^{-1}. Section two of this paper establishes conclusively that the proper time scale for the evolution of w is in fact $s \sim \varepsilon^{-2}$, see especially Remark 2.4. Since the completion of our work de Mottoni and Schatzman have proved a result similar to our Theorem 3.1 without the hypothesis of radial symmetry [16]. Although our result is more limited in scope than that of [16], we feel that it remains of independent interest, because the proof is totally different.

The study of (1.1) is of value both for understanding phase transitions, and as a tool for discussing motion by mean curvature. In the former direction, we note that this equation was proposed in [1] as a model for the motion of antiphase boundaries in crystalline solids. It is a non-conservative or "type A" Ginzburg–Landau equation, in the terminology of [26]. There are a number of other situations where Ginzburg–Landau type dynamics leads through a singular limit to a geometric model for phase boundary motion, see, e.g., [8, 9, 35]. We hope that the tools developed here may help shed some light on these more difficult problems as well. As for geometric motivation, we note that motion by mean curvature has been the object of much recent interest among geometers, e.g., [2, 3, 5, 22, 25, 40]. Local-in-time existence and uniqueness of a classical solution was proved in [27]. If the initial data are suitably restricted, then the solution remains a classical one until it shrinks to a point [22, 25, 28]; however, in general the surface can develop singularities, beyond which the meaning of "motion by mean curvature" is unclear. The first attempt to define a generalized solution was due to Brakke [5]. He proved the global-in-time existence of a weak solution in the class of codimension-one varifolds; the uniqueness of such a solution is unfortunately not known. A more successful notion of weak solution has recently been developed independently by

Chen *et al.* [12] and Evans and Spruck [17]. They prove the existence of a unique viscosity solution of a certain degenerate parabolic equation; where the solution is smooth, its level sets execute "flow by mean curvature." The singular limit (1.1) provides a third alternative. We conjecture that it gives an example of a weak solution in the sense of [5]. It should be noted however that, like Brakke, we lack a uniqueness theorem: for all we know, different sequences $\varepsilon_j \to 0$ might produce different limits. We hope that the model of motion by mean curvature obtained through (1.1) might be the same as that studied in [12, 17]; if so, this would lend enhanced credibility to both approaches.‡

Viewed from the perspective of (1.2) we are studying an example of *dynamical metastability*, i.e., a pattern which persists for a long time though it eventually disappears. It is well known that as $s \to \infty$ any solution of (1.2) is asymptotically stationary [31]; for generic initial data w tends to a local minimum of the "energy"

$$\int_\Omega \left[\frac{\varepsilon^2}{2} |\nabla w|^2 + \frac{1}{4}(w^2-1)^2\right] dx. \tag{1.3}$$

The solutions of (1.2) under consideration here are in no sense near to critical points of the energy, but they nevertheless evolve slowly, on a time scale $s \sim \varepsilon^{-2}$. Such a phenomenon is to be expected when an evolution equation has a Liapunov function with a small parameter ε, if there are more (local) minima at $\varepsilon = 0$ than at $\varepsilon > 0$. This is the case for (1.3): when $\varepsilon = 0$ any measurable w taking the values ± 1 is a minimizer, while for $\varepsilon > 0$ the perimeter of the transition interface becomes important, see, e.g., [32, 38].

The analysis of de Mottoni and Schatzman is based on the use of an ansatz for the form of u^ε, and on estimates for the linearization of (1.1) about this ansatz. Our method is totally different, much closer in spirit to recent studies of the stationary problem which makes use of the notion of Γ-convergence [4, 18, 29, 30, 32, 33, 38, 39]. It is convenient to normalize the energy so as to keep it positive and finite as $\varepsilon \to 0$; we therefore set

$$E_\varepsilon[u] = \int_\Omega \left[\frac{\varepsilon}{2} |\nabla u|^2 + \frac{\varepsilon^{-1}}{4}(u^2-1)^2\right] dx. \tag{1.4}$$

It is easy to verify that the solution $u = u^\varepsilon$ of (1.1) with a suitable boundary condition satisfies

$$E_\varepsilon[u^\varepsilon](0) - E_\varepsilon[u^\varepsilon](T) = \varepsilon \int_0^T \int_\Omega (u_t^\varepsilon)^2 \, dx\, dt$$

‡ A result of this type has recently been announced by L. C. Evans, M. Soner, and P. Souganides.

for any $T>0$. Thus control over the energy of the initial data gives control over the energy at any time, and also an estimate for the space-time integral of u_t^2. Modica showed in [32] how a bound on E_ε leads to compactness at any fixed time; our Theorem 2.3 is proved similarly, making use of the extra estimate on u_t^2 to obtain compactness in space-time.

Our analysis of the radial problem begins by changing variables into a moving coordinate system with respect to which u^ε should be asymptotically stationary. If the initial data have their transition at radius r_0, then the expected radius of the transition at time t is $r=\rho(t)$, where

$$\dot{\rho}=\frac{-(n-1)}{\rho(t)}, \qquad \rho(0)=r_0.$$

Setting $v^\varepsilon(R,\tau)=u^\varepsilon(\rho(\tau)+R,\tau)$, we see that u^ε has its transition at $r=\rho(t)$ exactly if v^ε has its transition at $R=0$. One computes that Eq. (1.1) for u is equivalent to

$$v_\tau - v_{RR} + (n-1)\frac{R}{\rho(\tau)(R+\rho(\tau))}v_R + \varepsilon^{-2}(v^3-v)=0, \tag{1.5}$$

which differs from the one-dimensional version of (1.1) only by the presence of a first order term. We used an energy-based argument in [7] to study (1.1) in one space dimension; the conclusion there was that the transitions move slower than any power of ε. Our analysis of (1.5) is similar, though considerably more complicated due to the presence of the first order term. A central role is played by the weighted "energy"

$$\int \phi(R,\tau)\left[\frac{\varepsilon}{2}v_R^2+\frac{\varepsilon^{-1}}{4}(v^2-1)^2\right]dR \tag{1.6}$$

with

$$\phi(R,\tau)=e^{-(n-1)(R/\rho(\tau))}\left(1+\frac{R}{\rho(\tau)}\right)^{n-1},$$

which turns out to be a Liapunov functional for (1.5), see Proposition 3.2. When transformed back to the original variables, (1.6) becomes

$$\tilde{E}_\varepsilon[u](t)=\int \psi(r,t)\left[\frac{\varepsilon}{2}u_r^2+\frac{\varepsilon^{-1}}{4}(u^2-1)^2\right]r^{n-1}\,dr \tag{1.7}$$

with

$$\psi(r,t)=\rho(t)^{-(n-1)}e^{-(n-1)[r/\rho(t)-1]}.$$

Thus while our compactness theorem makes use of the "natural" energy E_ε, our radial theorem relies instead on the somewhat bizarre Liapunov functional (1.7).

A word is in order about the boundary and initial data. The boundary conditions satisfied by u^ε enter our analysis only in verifying that $E_\varepsilon[u]$ and $\tilde{E}_\varepsilon[u]$ are indeed Liapunov functionals. For the former u^ε may satisfy a homogeneous Neumann condition

$$\frac{\partial u^\varepsilon}{\partial n}=0 \qquad \text{at } \partial\Omega, \tag{1.8}$$

or else a Dirichlet condition

$$u^\varepsilon(x,t)=U^\varepsilon(x), \qquad x\in\partial\Omega. \tag{1.9}$$

For the latter, which is relevant only in the radial case, our analysis works only for the Dirichlet condition (1.9): it seems that (1.6) is not a Liapunov functional when the boundary condition is of Neumann type. (See Remark 3.4.) As for the initial condition, we assume that its "energy" is controlled. More specifically, Section 2 requires that

$$E_\varepsilon[u^\varepsilon](0)\leqslant M<\infty. \tag{1.10}$$

Section 3 requires in addition the stronger hypothesis

$$\tilde{E}_\varepsilon[u^\varepsilon](0)\leqslant c_0+C\varepsilon^{1/2}, \tag{1.11}$$

where $c_0=2\sqrt{2}/3$. Both conditions demand that the initial data make the transition from -1 to $+1$ reasonably efficiently. To explain (1.11), we remark that the $\tilde{E}_\varepsilon[u^\varepsilon](0)$ must be *at least* $c_0-C\varepsilon^{1/2}$, according to Proposition 3.6; thus (1.11) asserts that the initial data waste no more than $O(\varepsilon^{1/2})$ energy in making their transition. It is easy to construct functions satisfying (1.10) with a transition along any smooth, closed, orientable hypersurface in Ω. The construction of radial functions satisfying (1.11) is not difficult either, by proceeding for example as in [38]. The analysis of de Mottoni and Schatzman requires much stronger hypotheses on the initial data. It should be noted, however, that like them we are obliged to consider initial values which depend on ε. Only the probabilistic method of Freidlin and Gartner [19, 23] is free of this deficiency.

Our attention is focussed on the specific Eq. (1.1) only for the sake of simplicity. In fact our results extend to the more general equation

$$u_t-\Delta u+\varepsilon^{-2}F'(u)=0, \tag{1.12}$$

where F is a bistable potential with both wells of equal depth. We suppose that the case of vector-valued u could be treated similarly, albeit with more effort, by using the methods of [4, 18, 39]. The situation is totally different, however, if F achieves its minimum value on a continuum rather than at isolated points. If, for example, $F(u) = \frac{1}{4}(|u|^2 - 1)^2$ with $u \in \mathbb{R}^m$ then (1.12) becomes

$$u_t - \Delta u + \varepsilon^{-2}(|u|^2 - 1)u = 0,$$

and u^ε converges to a solution of the evolution equation associated to the harmonic map functional for maps from Ω to S^{m-1} [13, 14, 37].

We wish to highlight some of the questions that remain open concerning (1.1) and related equations. First, what is the limiting behavior of (1.1) when the initial partition is as in Fig. 1? In other words, what does it mean for a stratified set to move by mean curvature? The appendix of [5] presents some similarity solutions which may be relevant to the motion of the "corners." Second, is there a general relation between Γ-convergence of functionals and convergence of solutions of the associated parabolic evolution equations? This question was first raised by de Giorgi [15]. Our example shows that one must allow for a *change of scale in time.* Indeed, the functionals E_ε Γ-converge to a perimeter problem, for which the evolution equation is flow by mean curvature. However, the parabolic equation associated to E_ε is $u_t - \varepsilon \Delta u + \varepsilon^{-1}(u^3 - u) = 0$, not (1.1). Finally, what about the numerical calculation of flow by mean curvature? The method of [12, 17] had previously been introduced as a numerical method by Osher and Sethian [34, 40]. It is reminiscent of (1.1), in that the moving surface is represented as the level set of a function; unlike (1.1), however, the evolution law of [12, 17, 34, 40] contains no small parameter. Surely there must be some relation between these two evolutions. Can the method of

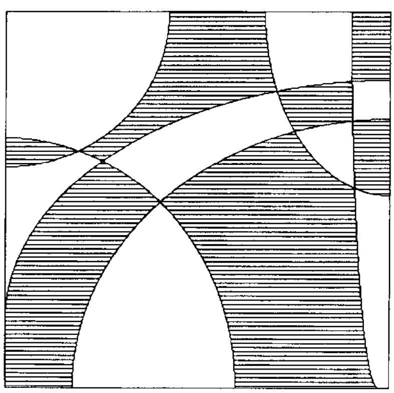

FIG. 1. $u^\varepsilon \approx +1$ in the white region, $u^\varepsilon \approx -1$ in the shaded region. What does it mean for this interface to move by mean curvature?

[34, 40] be used to compute the motion of a stratified set by its mean curvature (cf. Fig. 1)? If so, does it give the same result as (1.1)?

The work presented here was strongly influenced by discussions with S. Luckhaus and R. Pego during the fall of 1987. It is a pleasure to acknowledge their significant role in the development of our ideas.

2. A Compactness Theorem

This section presents a compactness theorem for solutions $u = u^\varepsilon(x, t)$ of (1.1). The domain $\Omega \subset \mathbb{R}^n$ must be a bounded, Lipschitz domain. The boundary condition may be of either Dirichlet or Neumann type, i.e., (1.8) or (1.9). The initial data are assumed to satisfy

$$E_\varepsilon[u^\varepsilon](0) \leqslant M, \tag{2.1}$$

where E_ε is defined by (1.4) and M is independent of ε. We also assume that the initial data converge in L^1 to a limit $v_0(x)$:

$$\lim_{\varepsilon \to 0} \int_\Omega |u^\varepsilon(x, 0) - v_0(x)| \, dx = 0. \tag{2.2}$$

Our goal is to show that for a subsequence $\varepsilon_j \to 0$ the limit

$$\lim_{\varepsilon_j \to 0} u^{\varepsilon_j}(x, t) = v(x, t)$$

exists almost everywhere on $\Omega \times (0, \infty)$. We shall show moreover that v is in a certain sense Hölder continuous in time, and that $v(x, 0) = v_0(x)$. As discussed in the introduction, the interface where v makes its transition from $+1$ to -1 is expected to describe the motion by mean curvature of the interface associated to v_0.

We begin with some energy estimates associated to the existence of the Liapunov functional E_ε.

Proposition 2.1. *Let u^ε solve* (1.1) *with boundary conditions and initial data as discussed above. Then*

$$\frac{d}{dt} E_\varepsilon[u^\varepsilon](t) = -\varepsilon \int_\Omega (u_t^\varepsilon)^2 \, dx, \tag{2.3}$$

from which it follows that

$$\sup_{t \geqslant 0} E_\varepsilon[u^\varepsilon](t) \leqslant M \tag{2.4}$$

$$\sup_{t \geqslant 0} \int_\Omega ((u^\varepsilon)^2 - 1)^2 \, dx \leqslant 4M\varepsilon \tag{2.5}$$

and, for $0 \leqslant t_1 < t_2 < \infty$,

$$\varepsilon \int_{t_1}^{t_2} \int_\Omega (u_t^\varepsilon)^2 \, dx \, dt = E_\varepsilon[u^\varepsilon](t_1) - E_\varepsilon[u^\varepsilon](t_2) \leqslant M. \tag{2.6}$$

Proof. We multiply the equation by u_t, integrate, and integrate by parts to obtain (2.3). Relations (2.4)–(2.6) follow easily from (2.3) and (2.1).

As in the analysis of the stationary case, the essential a priori estimates leading to compactness involve not u^ε but rather $g(u^\varepsilon)$, where

$$g(s) = \frac{1}{\sqrt{2}} \int_{-1}^{s} |\sigma^2 - 1| \, d\sigma. \tag{2.7}$$

PROPOSITION 2.2. *Let* u^ε *solve* (1.1) *with boundary conditions and initial data as discussed above. Then*

$$\sup_{t \geqslant 0} \int_{\Omega \times \{t\}} |\nabla g(u^\varepsilon)| \, dx \leqslant M \tag{2.8}$$

and, for $0 \leqslant t_1 < t_2$,

$$\int_{t_1}^{t_2} \int_\Omega |\partial_t g(u^\varepsilon)| \, dx \, dt \leqslant \sqrt{2} M (t_2 - t_1)^{1/2}. \tag{2.9}$$

Proof. The fact that (2.4) implies (2.8) was first noted in [33]; we repeat the proof here for the sake of completeness. For any function u, one has

$$\begin{aligned} E_\varepsilon[u] &= \int_\Omega \left[\frac{\varepsilon}{2} |\nabla u|^2 + \frac{\varepsilon^{-1}}{4} (u^2 - 1)^2 \right] dx \\ &\geqslant \int_\Omega \frac{1}{\sqrt{2}} |u^2 - 1| \, |\nabla u| \, dx \\ &= \int_\Omega |g'(u)| \, |\nabla u| \, dx \\ &= \int_\Omega |\nabla g(u)| \, dx. \end{aligned}$$

Taking $u = u^\varepsilon$ at any fixed time, we obtain (2.8) as a consequence of (2.4).

The proof of (2.9) is only slightly different. We have

$$\begin{aligned} \int_{t_1}^{t_2} \int_\Omega |\partial_t g(u^\varepsilon)| \, dx \, dt &= \int_{t_1}^{t_2} \int_\Omega |g'(u^\varepsilon)| \, |u_t^\varepsilon| \, dx \, dt \\ &\leqslant \left(\int_{t_1}^{t_2} \int_\Omega |g'(u^\varepsilon)|^2 \, dx \, dt \right)^{1/2} \left(\int_{t_1}^{t_2} \int_\Omega (u_t^\varepsilon)^2 \, dx \, dt \right)^{1/2}, \end{aligned}$$

by Hölder's inequality. The first term is controlled by

$$\int_{t_1}^{t_2}\int_\Omega |g'(u^\varepsilon)|^2\,dx\,dt=\tfrac{1}{2}\int_{t_1}^{t_2}\int_\Omega ((u^\varepsilon)^2-1)^2\,dx\,dt$$
$$\leqslant 2\varepsilon M(t_2-t_1),$$

and the second term is controlled by (2.6). It follows that

$$\int_{t_1}^{t_2}\int_\Omega |\partial_t g(u^\varepsilon)|\,dx\,dt\leqslant [2\varepsilon M(t_2-t_1)]^{1/2}\cdot[\varepsilon^{-1}M]^{1/2},$$

which is the same as (2.9).

Since $g'(s)=(1/\sqrt{2})\,|s^2-1|$ is positive except at $s=\pm 1$, studying $g(u^\varepsilon)$ rather than u^ε amounts to making a harmless change of variables in the image space. The desired compactness follows rather easily from Proposition 2.2, since BV is compactly embedded in L^1.

THEOREM 2.3. *Let u^ε solve* (1.1) *with boundary conditions and initial data as discussed above. For any sequence of ε's tending to zero, there is a subsequence ε_j such that the limit*

$$\lim_{\varepsilon_j\to 0} u^{\varepsilon_j}(x,t)=v(x,t) \tag{2.10}$$

exists for a.e. $(x,t)\in\Omega\times(0,\infty)$. *The function $v(x,t)$ takes only the values* ± 1; *it satisfies*

$$\int_\Omega |v(x,t_1)-v(x,t_2)|\,dx\leqslant C\,|t_2-t_1|^{1/2} \tag{2.11}$$

and

$$\sup_{t\geqslant 0}\int_{\Omega\times\{t\}} |\nabla v|\leqslant C; \tag{2.12}$$

and its initial value is the limit of the initial data for u^ε,

$$\lim_{t\to 0} v(x,t)=v_0(x)\qquad a.e. \tag{2.13}$$

Proof. First let us fix $T<\infty$ and prove the existence of a subsequence which converges a.e. on $\Omega\times(0,T)$. Since $|g(s)|\sim C\,|s|^3$ when $|s|$ is large,

$$|g(u)|\leqslant C_1+C_2(u^2-1)^2$$

for a suitable choice of the constants C_1 and C_2; it follows using (2.5) that

$$\int_0^T \int_\Omega |g(u^\varepsilon)| \, dx \, dt \leqslant C \tag{2.14}$$

with C independent of ε. Now (2.8), (2.9), and (2.14) assert that $g(u^\varepsilon)$ remains in a bounded subset of $BV(\Omega \times (0, T))$, the space of bounded variation functions of space-time. By a standard compactness result, see, e.g., [24], there is a subsequence $g(u^{\varepsilon_j})$ which converges in L^1, say to g^*:

$$\| g(u^{\varepsilon_j}) - g^* \|_{L^1(\Omega \times (0, T))} \to 0. \tag{2.15}$$

Passing to a further subsequence if necessary we may also arrange that

$$g(u^{\varepsilon_j}) \to g^* \qquad \text{a.e. on} \quad \Omega \times (0, T).$$

Since g is monotone there is a unique function $v(x, t)$ such that $g^*(x, t) = g(v(x, t))$, and clearly $u^{\varepsilon_j} \to v$ a.e. By Fatou's lemma and (2.5) we have

$$\int_0^T \int_\Omega (v^2 - 1)^2 \, dx \, dt = 0,$$

so v takes the values ± 1.

To prove (2.11), we observe that $g(u^{\varepsilon_j}) \to g^*$ in $L^1(\Omega \times \{t\})$ for almost every $t \in (0, T)$, by (2.15); at time 0 we have

$$\int_\Omega |g(u^{\varepsilon_j}(x, 0)) - g(v_0(x))| \, dx \to 0 \tag{2.16}$$

by (2.2) and (2.8). Now, u^ε satisfies

$$|g(u^\varepsilon(x, t_1)) - g(u^\varepsilon(x, t_2))| \leqslant \int_{t_1}^{t_2} |\partial_t g(u^\varepsilon(x, t))| \, dt$$

for any $0 \leqslant t_1 < t_2$; integration in x yields

$$\int |g(u^\varepsilon(x, t_1)) - g(u^\varepsilon(x, t_2))| \, dx \leqslant \sqrt{2}M \, |t_2 - t_1|^{1/2}, \tag{2.17}$$

making use of (2.9). We pass to the limit $\varepsilon_j \to 0$ in (2.17) to conclude that

$$\int_\Omega |g^*(x, t_1) - g^*(x, t_2)| \, dx \leqslant \sqrt{2}M \, |t_2 - t_1|^{1/2} \tag{2.18}$$

for almost every $t_1, t_2 \in (0, T)$. Since v takes only the values ± 1,

$$|g^*(x, t_1) - g^*(x, t_2)| = |g(v(x, t_1)) - g(v(x, t_2))|$$
$$= \frac{c_0}{2} |v(x, t_1) - v(x, t_2)| \tag{2.19}$$

with $c_0 = g(1) - g(-1)$. Thus (2.18) yields (2.11) for a.e. t_1, t_2. We may redefine v at the exceptional times to make it continuous as a map from $[0, T]$ to $L^1(\Omega)$, and then (2.11) holds for every choice of t_1 and t_2.

To prove (2.13), we take $t_1 = 0$ in (2.17). Passage to the limit $\varepsilon_j \to 0$ gives

$$\int_\Omega |g(v_0(x)) - g(v(x, t_2))| \, dx \leqslant \sqrt{2}\, M(t_2)^{1/2}$$

for $t_2 > 0$, making use of (2.16). By (2.19) this gives

$$\int_\Omega |v_0(x) - v(x, t_2)| \, dx \leqslant \frac{2}{c_0} \sqrt{2}\, M(t_2)^{1/2};$$

we deduce (2.13) by passing to the limit $t_2 \to 0$.

It remains to prove (2.12). Here we understand $\int_\Omega |\nabla v|$ as the total variation of the vector-valued measure ∇v, or equivalently as twice the perimeter of the interface separating the sets $\{v = +1\}$ and $\{v = -1\}$, see, e.g., [24]. From (2.15), (2.8), and the lower semicontinuity of the total variation under L^1 convergence, we have

$$\operatorname*{ess\,sup}_{0 < t < T} \int_{\Omega \times \{t\}} |\nabla g^*| \, dx \leqslant M. \tag{2.20}$$

Now, $g^* = g(v)$; and since v takes only the values $\pm 1, \nabla g^* = (c_0/2) \nabla v$. Thus (2.20) gives

$$\operatorname*{ess\,sup}_{0 < t < T} \int_{\Omega \times \{t\}} |\nabla v| \leqslant \frac{2}{c_0} M. \tag{2.21}$$

Since v is continuous in time with values in L^1, (2.21) remains valid with "ess sup" replaced by "sup", and this yields (2.12) for $0 \leqslant t \leqslant T$.

The assertions of the theorem have at this stage been established on an arbitrary finite time interval $(0, T)$. Applying this for a sequence of times $T_i \to \infty$, and taking a diagonal subsequence of $\{u^\varepsilon\}$ in the usual manner, we easily deduce the assertions for the infinite interval $(0, \infty)$.

Remark 2.4. These results can naturally be reformulated as statements about w^ε, the solution of (1.2). The fact that w^ε evolves on a time scale of order ε^{-2} emerges quite clearly from (2.17): setting $t_1 = 0$ and $u^\varepsilon(x, t) = w^\varepsilon(x, \varepsilon^{-2}t)$, that relation gives

$$\int_\Omega |g(w^\varepsilon(x, 0)) - g(w^\varepsilon(x, s))| \, dx \leqslant \sqrt{2} \, M\varepsilon \, |s|^{1/2}$$

for any $s > 0$. Thus nothing happens until $s \sim \varepsilon^{-2}$.

Remark 2.5. In work on the stationary problem it is customary to prove compactness in L^1; the corresponding assertion in the present context is that $u^{\varepsilon_j} \to v$ in $L^1_{\text{loc}}(\Omega \times (0, \infty))$. If the initial data are uniformly bounded independent of ε then $u^\varepsilon(x, t)$ remains bounded by an application of the maximum principle, and L^1_{loc} convergence follows from (2.10). If the initial data are unbounded one can still deduce L^1_{loc} convergence by arguing as in [32, 38].

3. The Radial Case

This section proves that the formal picture is asymptotically correct in the radial case, for certain boundary and initial data. Our attention is henceforth restricted to radially symmetric solutions; rescaling if necessary, there is no loss of generality in assuming that

$$\Omega = \{x \in \mathbb{R}^n : |x| \leqslant 1\}.$$

The evolution of $u = u^\varepsilon(r, t)$ is governed by

$$u_t - u_{rr} - \frac{n-1}{r} u_r + \varepsilon^{-2}(u^3 - u) = 0, \tag{3.1}$$

which is (1.1) in radial coordinates. We consider only the case of a Dirichlet condition at $\partial\Omega$,

$$u(1, t) = 1, \qquad t \geqslant 0; \tag{3.2}$$

of course at $r = 0$, u must satisfy $u_r(0, t) = 0$. Our analysis requires that the initial data "have a single transition sphere," and that they "make the transition from -1 to $+1$ rather efficiently." More precisely, we require that

$$\begin{aligned} u^\varepsilon(r, 0) &< 0 \qquad \text{for} \quad r < r_0 \\ u^\varepsilon(r, 0) &> 0 \qquad \text{for} \quad r > r_0 \end{aligned} \tag{3.3}$$

for some r_0, $0 < r_0 < 1$, independent of ε; evidently, r_0 is the "radius of the initial transition sphere." We also require that at time $t = 0$

$$\int_0^1 \psi_0(r) \left[\frac{\varepsilon}{2} (u_r^\varepsilon)^2 + \frac{\varepsilon^{-1}}{4} ((u^\varepsilon)^2 - 1)^2 \right] r^{n-1}\, dr \leqslant c_0 + C\varepsilon^{1/2} \tag{3.4}$$

with

$$\psi_0(r) = r_0^{-(n-1)} e^{-(n-1)[r/r_0 - 1]}, \qquad c_0 = 2\sqrt{2}/3. \tag{3.5}$$

Finally, we assume that the initial data are uniformly bounded, independent of ε,

$$|u^\varepsilon(r, 0)| \leqslant C. \tag{3.6}$$

Data meeting these requirements are easily constructed by the methods of [38]. (We remark that (3.6) is almost redundant: (3.4) implies a uniform bound for $u^\varepsilon(r, 0)$ except near $r = 0$.)

The formal picture asserts that the transition sphere "flows by mean curvature," i.e., with normal velocity equal to the sum of its principal curvatures. This flow takes it to a sphere of radius $\rho(t)$ at time t, where

$$\dot{\rho} = \frac{-(n-1)}{\rho(t)}, \qquad \rho(0) = r_0. \tag{3.7}$$

It is easy to see that $\rho(t) = (r_0^2 - 2(n-1)t)^{1/2}$; in particular, the sphere shrinks to a point at time

$$T_{\max} = \frac{r_0^2}{2(n-1)}. \tag{3.8}$$

Thus $u^\varepsilon(r, t)$ is expected to resemble

$$f(r, t) = \begin{cases} -1, & r < \rho(t) \\ +1, & r > \rho(t) \end{cases} \tag{3.9}$$

for $0 < t < T_{\max}$. We will prove that this picture is asymptotically correct in the following sense:

THEOREM 3.1. *If Ω and u^ε are as above then for any $T < T_{\max}$,*

$$\lim_{\varepsilon \to 0} \int_0^T \int_\Omega |u^\varepsilon(r, t) - f(r, t)|\, r^{n-1}\, dr\, dt = 0, \tag{3.10}$$

where f is defined by (3.9).

The rest of this section is devoted to the proof of Theorem 3.1. It is convenient to work in a moving coordinate system with respect to which u^ε should be asymptotically stationary. The distance to the moving sphere is $R = r - \rho(t)$, so the appropriate change of variables is

$$v(R, \tau) = u(R + \rho(\tau), \tau). \tag{3.11}$$

Note that the "spatial" domain of v changes in time: v is defined for

$$-\rho(\tau) < R < 1 - \rho(\tau), \qquad 0 < \tau < T_{\max}.$$

If u satisfies (3.1) then one easily computes that

$$v_\tau - v_{RR} + \frac{(n-1)R}{\rho(\tau)(R + \rho(\tau))} v_R + \varepsilon^{-2}(v^3 - v) = 0; \tag{3.12}$$

the boundary conditions for u yield

$$v_R(-\rho(\tau), \tau) = 0, \qquad v(1 - \rho(\tau), \tau) = 1. \tag{3.13}$$

Equation (3.12) can be written as

$$v_\tau - \frac{1}{\phi}(\phi v_R)_R + \varepsilon^{-2}(v^3 - v) = 0,$$

if the integrating factor $\phi = \phi(R, \tau)$ satisfies

$$\phi_R = \frac{-(n-1)R}{\rho(\tau)(R + \rho(\tau))}\phi. \tag{3.15}$$

We choose

$$\phi(R, \tau) = e^{-(n-1)R/\rho(\tau)}\left(1 + \frac{R}{\rho(\tau)}\right)^{(n-1)}; \tag{3.16}$$

one readily verifies that this ϕ satisfies (3.15) and also

$$0 \leqslant \phi(R, \tau) \leqslant 1 \tag{3.17}$$

$$\phi(-\rho(\tau), \tau) = 0, \qquad \phi(0, \tau) = 1 \tag{3.18}$$

$$\phi_\tau \leqslant 0. \tag{3.19}$$

Using these properties we will show that

$$E_\phi[v^\varepsilon](\tau) = \int_{-\rho(\tau)}^{1-\rho(\tau)} \phi(R, \tau)\left[\frac{\varepsilon}{2}(v_R^\varepsilon)^2 + \frac{\varepsilon^{-1}}{4}((v^\varepsilon)^2 - 1)^2\right] dR \tag{3.20}$$

is a decreasing function of τ, with

$$\frac{d}{dt} E_\phi[v^\varepsilon](\tau) \leqslant -\varepsilon \int_{-\rho(\tau)}^{1-\rho(\tau)} \phi(R, \tau)\, |v_\tau^\varepsilon|^2\, dR. \tag{3.21}$$

(See Proposition 3.2.) Our condition (3.4) on the initial data is designed to bound the initial value of $E_\phi[v^\varepsilon]$: when transformed to a statement about v^ε, (3.4) becomes

$$E_\phi[v^\varepsilon](0) \leqslant c_0 + C\varepsilon^{1/2}. \tag{3.22}$$

Note that the weighted energy $\tilde{E}_\varepsilon[u](t)$ discussed in the introduction (see (1.7)) is simply $E_\phi[v^\varepsilon](\tau)$, expressed in the original variables. The initial data of v^ε converge in L^1 as $\varepsilon \to 0$ to a Heaviside function,

$$\lim_{\varepsilon \to 0} v^\varepsilon(R, 0) = \begin{cases} +1, & R > 0 \\ -1, & R < 0; \end{cases} \tag{3.23}$$

this is an easy consequence of (3.3), (3.4), and the compactness results in [32, 38]. Our goal, Theorem 3.1, asserts in essence that $v^\varepsilon(R, \tau)$ is asymptotically independent of τ.

The proof of Theorem 3.1 is somewhat involved, so we pause at this point to explain the strategy. Let us assume for the moment that $R \mapsto v^\varepsilon(R, \tau)$ has "transition layer structure" at every time, and let $R = z^\varepsilon(\tau)$ be the "location of the transition" at time τ. (This discussion is strictly heuristic, so we do not propose to define these concepts precisely.) The structure of the initial data gives

$$z^\varepsilon(0) \approx 0.$$

In Section 2 we used the energy estimate (2.3) to show that $t \mapsto v^\varepsilon(x, t)$ is Hölder continuous in t with exponent $\frac{1}{2}$; a similar argument using the weighted energy estimate (3.21) will give

$$|z^\varepsilon(\tau_1) - z^\varepsilon(\tau_2)| \lesssim C\, |\tau_2 - \tau_1|^{1/2} \left(E_\phi[v^\varepsilon](\tau_1) - E_\phi[v^\varepsilon](\tau_2)\right)^{1/2} \tag{3.24}$$

for $0 \leqslant \tau_1 < \tau_2$. Now, it turns out that there is a certain minimum energy associated to the presence of a transition layer. This is a consequence of the inequality

$$\frac{\varepsilon}{2} v_R^2 + \frac{\varepsilon^{-1}}{4} (v^2 - 1)^2 \geqslant \frac{1}{\sqrt{2}} |v^2 - 1|\, |v_R| = |g(v)_R| \tag{3.25}$$

with g as in (2.7). We expect a sharp transition, i.e., we expect $R \mapsto v(R, \tau)$

to pass from $v \approx -1$ to $v \approx +1$ over a narrow range of R near $z^\varepsilon(\tau)$; so (3.25) suggests that

$$E_\phi[v^\varepsilon](\tau) \gtrsim \phi(z^\varepsilon(\tau), \tau) \cdot c_0 \tag{3.26}$$

with $c_0 = g(1) - g(-1) = 2\sqrt{2}/3$. The right hand side of (3.26) can be estimated using the Taylor expansion of $R \mapsto \phi(R, \tau)$ at $R = 0$:

$$\phi(R, \tau) \approx 1 - \frac{n-1}{2\rho^2(\tau)} R^2.$$

Finally, we recall from (3.22) that the initial weighted energy is controlled:

$$E_\phi[v^\varepsilon](0) \lesssim c_0.$$

Taking $\tau_1 = 0$, we use these estimates to control the location of the transition at time $\tau_2 = \tau > 0$:

$$\begin{aligned} |z^\varepsilon(\tau)| &\approx |z^\varepsilon(\tau) - z^\varepsilon(0)| \\ &\lesssim C\tau^{1/2}(E_\phi[v^\varepsilon](0) - E_\phi[v^\varepsilon](\tau))^{1/2} \\ &\lesssim C\tau^{1/2}\left(c_0 - c_0\left[1 - \frac{n-1}{2\rho^2(\tau)}|z^\varepsilon(\tau)|^2\right]\right)^{1/2} \\ &\lesssim C'\tau^{1/2}\rho^{-1}(\tau)\,|z^\varepsilon(\tau)|. \end{aligned} \tag{3.27}$$

If τ is chosen small enough so that

$$C'\tau^{1/2}\rho^{-1}(\tau) < 1, \tag{3.28}$$

then (3.27) forces $|z^\varepsilon(\tau)| \approx 0$. In other words, on the time interval determined by (3.28)—which is short, but independent of ε— the transition $z^\varepsilon(\tau)$ moves a distance that tends to zero with ε. Repetition of this argument finitely many times gives $|z^\varepsilon(\tau)| \approx 0$ for τ strictly smaller than $T_{\max}$ (at which time $\rho(\tau) \to 0$ and $v^\varepsilon(R, \tau)$ ceases to be defined).

The preceding outline can be made fully rigorous; this naturally involves proving that $v^\varepsilon(\cdot, \tau)$ has the anticipated "transition layer structure," see [6] or [7, Sect. 4]. Here, however, we take a slightly different approach, which avoids discussing the transition layer explicitly. Rather than estimate $|z^\varepsilon(\tau_1) - z^\varepsilon(\tau_2)|$, we shall control the L^1 difference between $g(v^\varepsilon)(\cdot, \tau_1)$ and $g(v^\varepsilon)(\cdot, \tau_2)$, where g is given by (2.7). The argument sketched above can be rephrased in terms of this L^1 difference, because

$$\int |g(v^\varepsilon)(R, \tau_1) - g(v^\varepsilon)(R, \tau_2)|\, dR \approx c_0\, |z^\varepsilon(\tau_1) - z^\varepsilon(\tau_2)| \tag{3.29}$$

with $c_0 = g(1) - g(-1)$, if v^ε has the expected transition layer form.

We turn now to the task of executing these ideas.

Proposition 3.2. *The function* $v^\varepsilon(R, \tau)$, *defined by* (3.11), *satisfies the weighted energy relation* (3.21).

Proof. Using the definition of $E_\phi[v^\varepsilon]$, (3.20), we compute that

$$\frac{d}{d\tau} E_\phi[v^\varepsilon](\tau) = \int_{-\rho(\tau)}^{1-\rho(\tau)} \phi(\varepsilon v_R v_{R\tau} + \varepsilon^{-1}(v^3 - v)\, v_\tau)\, dR$$
$$+ \int_{-\rho(\tau)}^{1-\rho(\tau)} \phi_\tau \left(\frac{\varepsilon}{2} v_R^2 + \frac{\varepsilon^{-1}}{4}(v^2-1)^2\right) dR$$
$$+ \phi\left(\frac{\varepsilon}{2} v_R^2 + \frac{\varepsilon^{-1}}{4}(v^2-1)^2\right)(-\dot\rho)\Big|_{-\rho(\tau)}^{1-\rho(\tau)}.$$

The second term on the right is negative, by (3.19). We integrate by parts in the first term and use (3.14) to obtain

$$\frac{d}{d\tau} E_\phi[v^\varepsilon](\tau) \leqslant -\varepsilon \int_{-\rho(\tau)}^{1-\rho(\tau)} \phi v_\tau^2\, dR + B(\tau), \tag{3.30}$$

where $B(\tau)$ consists of boundary terms

$$B(\tau) = \varepsilon\phi v_R v_\tau \Big|_{-\rho(\tau)}^{1-\rho(\tau)} - \phi\left(\frac{\varepsilon}{2} v_R^2 + \frac{\varepsilon^{-1}}{4}(v^2-1)^2\right)(\dot\rho)\Big|_{-\rho(\tau)}^{1-\rho(\tau)}.$$

At $R = -\rho(\tau)$ the weight ϕ vanishes, by (3.18), and at $R = 1-\rho(\tau)$ we have $v = 1$, by (3.13); therefore

$$B = \left(\varepsilon\phi v_R v_\tau - \frac{\varepsilon}{2}\phi v_R^2 \dot\rho\right)\Big|_{R=1-\rho(\tau)}$$

Now, differentiation of (3.11) gives

$$v_\tau(R, \tau) = \dot\rho(\tau)\, u_r(R + \rho(\tau), \tau) + u_t(R + \rho(\tau), \tau),$$
$$v_R(R, \tau) = u_r(R + \rho(\tau), \tau).$$

Since u satisfies the Dirichlet condition (3.2) at $r = 1$, we have $u_t(1, \tau) = 0$, and so

$$v_\tau(1-\rho(\tau), \tau) = \dot\rho(\tau)\, v_R(1-\rho(\tau), \tau).$$

Therefore

$$B = \dot\rho(\tau) \cdot \frac{\varepsilon}{2} \phi v_R^2 \big|_{R=1-\rho(\tau)} \leqslant 0. \tag{3.31}$$

The desired energy relation is an immediate consequence of (3.30) and (3.31).

COROLLARY 3.3. *If* $0 \leqslant \tau_1 < \tau_2 < T_{\max}$ *then*

$$\varepsilon \int_{\tau_1}^{\tau_2} \int_{-\rho(\tau)}^{1-\rho(\tau)} \phi v_\tau^2 \, dR \, d\tau \leqslant E_\phi[v^\varepsilon](\tau_1) - E_\phi[v^\varepsilon](\tau_2). \tag{3.32}$$

Proof. This follows from (3.21) by integration with respect to τ.

Remark 3.4. We are unable to prove the analogue of (3.32) when u has a Neumann boundary condition $u_r = 0$ at $r = 1$, because in that case the boundary term B comes out positive.

Our next goal is a rigorous version of (3.24). We wish to work on a space-time cylinder $(-a, a) \times (0, T)$, whereas $v^\varepsilon(R, \tau)$ is defined for $0 < \tau < T_{\max}$ and $-\rho(\tau) < R < 1 - \rho(\tau)$. Since $\rho(\tau) \to 0$ as $\tau \to T_{\max}$, we must first choose $T < T_{\max}$; then there exists $a > 0$, depending on T, such that

$$[-a, a] \subset (-\rho(\tau), 1 - \rho(\tau)), \qquad 0 < \tau < T. \tag{3.33}$$

Since the weight ϕ vanishes only at $R = -\rho(\tau)$, we may also arrange that

$$\phi(R, \tau) \geqslant \phi_{\min} \qquad \text{for} \quad -a < R < a, 0 < \tau < T \tag{3.34}$$

with $\phi_{\min} > 0$, depending only on T. For $0 \leqslant \tau_1 < \tau_2 \leqslant T$, we set

$$d^\varepsilon(\tau_1, \tau_2) = \int_{-a}^{+a} |g(v^\varepsilon)(R, \tau_1) - g(v^\varepsilon)(R, \tau_2)| \, dR. \tag{3.35}$$

PROPOSITION 3.5. *Let* $T < T_{\max}$ *and* $a > 0$ *be as above. Then there is a constant* C, *depending on* T *but not on* ε, *such that*

$$d^\varepsilon(\tau_1, \tau_2) \leqslant C(\tau_2 - \tau_1)^{1/2} \left(E_\phi[v^\varepsilon](\tau_1) - E_\phi[v^\varepsilon](\tau_2)\right)^{1/2} \tag{3.36}$$

whenever $0 \leqslant \tau_1 < \tau_2 \leqslant T$.

Proof. By (3.34) and the Fundamental Theorem of Calculus we have

$$\begin{aligned} d^\varepsilon(\tau_1, \tau_2) &\leqslant \int_{-a}^{+a} \int_{\tau_1}^{\tau_2} |\partial_\tau g(v^\varepsilon)| \, d\tau \, dR \\ &\leqslant (\phi_{\min})^{-1} \int_{-a}^{+a} \int_{\tau_1}^{\tau_2} \phi \, |\partial_\tau g(v^\varepsilon)| \, d\tau \, dR \\ &\leqslant (\phi_{\min})^{-1} \int_{\tau_1}^{\tau_2} \int_{-\rho(\tau)}^{1-\rho(\tau)} \phi \, |\partial_\tau g(v^\varepsilon)| \, dR \, d\tau, \end{aligned} \tag{3.37}$$

using Fubini's Theorem and the positivity of ϕ in the last step. We estimate (3.37) by the same argument that was used to prove (2.9). Hölder's inequality yields

$$\iint \phi\,|\partial_\tau g(v^\varepsilon)| \leqslant \left(\iint \phi\,|g'(v^\varepsilon)|^2\right)^{1/2}\left(\iint \phi v_\tau^2\right)^{1/2}. \tag{3.38}$$

Since $|g'(v^\varepsilon)|^2 = \frac{1}{2}((v^\varepsilon)^2-1)^2$, we have

$$\begin{aligned}\int_{\tau_1}^{\tau_2}\int_{-\rho(\tau)}^{1-\rho(\tau)} \phi\,|g'(v^\varepsilon)|^2\,dR\,d\tau &\leqslant 2\varepsilon\int_{\tau_1}^{\tau_2} E_\phi[v^\varepsilon](\tau)\,d\tau\\ &\leqslant \varepsilon C(\tau_2-\tau_1),\end{aligned}$$

using the fact that $E_\phi[v^\varepsilon](\tau)$ is decreasing, and the hypothesis (3.22), which controls it at the initial time. The second term on the right hand side of (3.38) is controlled by (3.32). Combining these estimates leads easily to (3.36).

Now we shall derive a lower bound on $E_\phi[v^\varepsilon](\tau)$, in essence a rigorous version of (3.26). Recall from our informal discussion that $|z^\varepsilon(\tau)| \approx \text{const}\cdot d^\varepsilon(0,\tau)$ (see especially (3.29)); therefore the lower bound we prove involves not $\phi(z^\varepsilon(\tau),\tau)$ but rather the value of ϕ at $R\approx \text{const}\cdot d^\varepsilon(0,\tau)$. We continue to hold $T<T_{\max}$ fixed, and to work in the cylinder $|R|\leqslant a$, $0\leqslant\tau\leqslant T$, with a satisfying (3.33)–(3.34). We note that

$$\phi(R,\tau)\geqslant\phi(-R,\tau) \qquad \text{for} \quad 0\leqslant R\leqslant a,\ 0\leqslant\tau\leqslant T, \tag{3.39}$$

as a consequence of the definition (3.16) and the fact that $e^{-x}(1+x)\geqslant e^{x}(1-x)$ for all positive x. For convenience of notation we write $d^\varepsilon(\tau)$ instead of $d^\varepsilon(0,\tau)$.

PROPOSITION 3.6. *There are positive constants c_1 and c_2, independent of ε, such that*

$$E_\phi[v^\varepsilon](\tau)\geqslant\phi(-c_1\,d^\varepsilon(\tau)-\varepsilon^{1/2},\tau)\cdot(c_0-c_2\varepsilon^{1/2}) \tag{3.40}$$

whenever ε is sufficiently small and

$$c_1\,d^\varepsilon(\tau)+\varepsilon^{1/2}\leqslant a. \tag{3.41}$$

Here $c_0=g(1)-g(-1)=2\sqrt{2}/3$, as usual. The value of c_2 depends on T; a suitable choice for c_1 is $c_1=|g(0)-g(\frac{1}{2})|^{-1}$.

Proof. This is really a property of the functional E_ϕ; τ is fixed, and the dynamics of v plays no role. The proof has two parts: first we show that

v^ε takes values near *both* $+1$ *and* -1 in a certain neighborhood of 0; then integration of (3.25) yields the desired estimate.

Let $A = (-c_1\, d^\varepsilon(\tau) - \varepsilon^{1/2}, c_1\, d^\varepsilon(\tau) + \varepsilon^{1/2})$. The precise goal of the first step is to locate points R_1, $R_2 \in A$ such that

$$v^\varepsilon(R_1, \tau) \leqslant -1 + c\varepsilon^{1/4}, \qquad v^\varepsilon(R_2, \tau) \geqslant 1 - c\varepsilon^{1/4}. \tag{3.42}$$

To obtain R_1, let

$$I^+ = (-c_1\, d^\varepsilon(\tau) - \varepsilon^{1/2}, 0) \cap \{R: v^\varepsilon(R, \tau) \geqslant \tfrac{1}{2}\}$$
$$I^- = (-c_1\, d^\varepsilon(\tau) - \varepsilon^{1/2}, 0) \cap \{R: v^\varepsilon(R, \tau) < \tfrac{1}{2}\}.$$

Using (3.41), we have

$$\begin{aligned} d^\varepsilon(\tau) &= \int_{-a}^{+a} |g(v^\varepsilon)(R, 0) - g(v^\varepsilon)(R, \tau)|\, dR \\ &\geqslant \int_{I^+} |g(v^\varepsilon)(R, 0) - g(v^\varepsilon)(R, \tau)|\, dR \\ &\geqslant |g(0) - g(\tfrac{1}{2})|\, m(I^+), \end{aligned}$$

since $v^\varepsilon(R, 0) < 0$ and $v^\varepsilon(R, \tau) \geqslant \frac{1}{2}$ for $R \in I^+$. Substitution of

$$m(I^+) = c_1\, d^\varepsilon(\tau) + \varepsilon^{1/2} - m(I^-)$$

yields

$$d^\varepsilon(\tau) \cdot \left(\frac{1}{|g(0) - g(\frac{1}{2})|} - c_1 \right) \geqslant \varepsilon^{1/2} - m(I^-).$$

Taking $c_1 \geqslant |g(0) - g(\frac{1}{2})|^{-1}$ we conclude that

$$m(I^-) \geqslant \varepsilon^{1/2}. \tag{3.43}$$

Now recall that

$$E_\phi[v^\varepsilon](\tau) \leqslant C$$

since $E_\phi[v^\varepsilon]$ decreases as a function of τ, and the initial value is controlled by (3.22). Since $\phi \geqslant \phi_{\min}$ on A, we have

$$\begin{aligned} \frac{\varepsilon^{-1}}{4} \phi_{\min} \int_{I^-} ((v^\varepsilon)^2 - 1)^2\, dR &\leqslant \int_A \phi \cdot \frac{\varepsilon^{-1}}{4} ((v^\varepsilon)^2 - 1)^2\, dR \\ &\leqslant E_\phi[v^\varepsilon](\tau), \end{aligned}$$

and so

$$\int_{I^-} ((v^\varepsilon)^2 - 1)^2 \, dR \leqslant C\phi_{\min}^{-1} \varepsilon. \tag{3.44}$$

It follows from (3.43) and (3.44) that there exists $R_1 \in I^-$ satisfying

$$((v^\varepsilon)^2 (R_1, \tau) - 1)^2 \leqslant C\phi_{\min}^{-1} \varepsilon^{1/2}. \tag{3.45}$$

Since $v^\varepsilon(R_1, \tau) < \frac{1}{2}$ from the definition of I^-, (3.45) yields

$$v^\varepsilon(R_1, \tau) \leqslant -1 + c\varepsilon^{1/4}$$

with c depending on T but not on ε, provided that ε is sufficiently small. This establishes the first part of (3.42). The argument for the existence of R_2 satisfying the second part of (3.42) is essentially the same.

We now proceed to the second step, which obtains (3.40) as a consequence of (3.42). The main point is the following property of the function g:

$$\text{if } \xi_1 < -1 + \alpha \text{ and } \xi_2 > 1 - \alpha \text{ with } 0 < \alpha < 1,$$
$$\text{then } |g(\xi_1) - g(\xi_2)| \geqslant c_0 - c\alpha^2.$$

This is an easy consequence of the definition (2.7). Taking $\xi_1 = v^\varepsilon(R_1, \tau)$ and $\xi_2 = v^\varepsilon(R_2, \tau)$, we conclude using (3.42) that

$$|g(v^\varepsilon)(R_1, \tau) - g(v^\varepsilon)(R_2, \tau)| \geqslant c_0 - c_2 \varepsilon^{1/2}$$

for a suitable choice of the constant c_2. Therefore

$$\begin{aligned} E_\phi[v^\varepsilon](\tau) &\geqslant \int_A \phi \left(\frac{\varepsilon}{2} (v_R^\varepsilon)^2 + \frac{\varepsilon^{-1}}{4} ((v^\varepsilon)^2 - 1)^2 \right) dR \\ &\geqslant \int_A \phi \cdot |\partial_R g(v^\varepsilon)| \, dR \\ &\geqslant (\min_A \phi) \cdot |g(v^\varepsilon)(R_1, \tau) - g(v^\varepsilon)(R_2, \tau)| \\ &\geqslant (\min_A \phi) \cdot (c_0 - c_2 \varepsilon^{1/2}), \end{aligned} \tag{3.46}$$

using (3.25) in the second step. It follows from (3.39) (using the property $R\phi_R \geqslant 0$) that

$$\min_A \phi = \phi(-c_1 \, d^\varepsilon(\tau) - \varepsilon^{1/2}, \tau),$$

so (3.46) is the same as (3.40), and the proof is complete.

We remark for later use that we have actually proved a little more than (3.40). In fact, we have proved the "local" lower bound

$$\int_A \phi\left[\frac{\varepsilon}{2}(v_R^\varepsilon)^2+\frac{\varepsilon^{-1}}{4}((v^\varepsilon)^2-1)^2\right]dR$$
$$\geqslant \phi(-c_1\, d^\varepsilon(\tau)-\varepsilon^{1/2},\tau)(c_0-c_2\varepsilon^{1/2}) \tag{3.47}$$

with $A=(-c_1\, d^\varepsilon(\tau)-\varepsilon^{1/2}, c_1\, d^\varepsilon(\tau)+\varepsilon^{1/2})$.

Our next goal is a rigorous version of (3.27). We continue to work on the cylinder $(-a, a)\times(0, T)$, with a satisfying (3.33) and (3.34). We place one more smallness condition on a,

$$\phi(R,\tau)\geqslant 1-\frac{(n-1)^2}{\rho(T)^2}R^2 \qquad \text{for} \quad |R|\leqslant a, 0\leqslant\tau\leqslant T. \tag{3.48}$$

This holds when $a/\rho(T)$ is sufficiently small, as a consequence of the definition (3.16) and the inequality $e^{-(n-1)x}(1+x)^{n-1}\geqslant 1-(n-1)^2x^2$, which is valid for x in a neighborhood of 0 and $n\geqslant 2$.

PROPOSITION 3.7. *There is a positive constant τ_0, depending on T but not on ε, such that*

$$d^\varepsilon(\tau)\leqslant\varepsilon^{1/4} \qquad \textit{for} \quad 0\leqslant\tau\leqslant\tau_0 \tag{3.49}$$

whenever ε is sufficiently small.

Proof. By Proposition 3.5, $d^\varepsilon(\tau)$ is Hölder continuous in τ, uniformly in ε; therefore by choosing τ_0 appropriately we may be sure that (3.41) holds for small values of ε. Combining (3.22), (3.36), and (3.40), we have

$$(d^\varepsilon(\tau))^2\leqslant C\tau(E_\phi[v^\varepsilon](0)-E_\phi[v^\varepsilon](\tau))$$
$$\leqslant C\tau\{c_0[1-\phi(-c_1\, d^\varepsilon(\tau)-\varepsilon^{1/2},\tau)]+C\varepsilon^{1/2}\}.$$

It follows using (3.48) that

$$(d^\varepsilon(\tau))^2\leqslant C'\tau\{(d^\varepsilon(\tau))^2+\varepsilon^{1/2}\} \tag{3.50}$$

for a suitable choice of the constant C', depending on T but not on ε. Choosing τ_0 so that $C'\tau_0\leqslant\frac{1}{2}$, we obtain (3.49) as an easy consequence of (3.50).

The preceding proposition controls $d^\varepsilon(\tau)$ for τ close to 0. The next one uses an inductive argument to control it for all τ, $0\leqslant\tau\leqslant T$.

Proposition 3.8. *There is a constant C, depending only on T, such that*

$$d^\varepsilon(\tau) \leqslant C\varepsilon^{1/4} \qquad \text{for} \quad 0 \leqslant \tau \leqslant T \tag{3.51}$$

provided that ε is sufficiently small.

Proof. We shall prove inductively that

$$d^\varepsilon(\tau) \leqslant 2^N \varepsilon^{1/4} \qquad \text{for} \quad 0 \leqslant \tau \leqslant \min(N\tau_1, T) \tag{3.52}$$

for a suitable choice of the constant $\tau_1 > 0$, provided that ε is sufficiently small. Here N is a positive integer. The smallness condition on ε will depend on N, but the value of τ_1 will be independent of N and ε. The desired assertion (3.51) clearly follows, by taking $N = [T/\tau_1] + 1$.

The initial step of the induction is provided by Proposition 3.7: Relation (3.52) with $N=1$ is a consequence of (3.49), provided that $\tau_1 \leqslant \tau_0$. Assuming that (3.52) holds with $N = 1, 2, \dots, k$, we seek to prove it with $N = k+1$. Of course, we may assume that $k\tau_1 \leqslant T$, since otherwise the desired assertion is trivial. By the inductive hypothesis and the Hölder continuity of $d^\varepsilon(\tau)$ (Proposition 3.5), we can arrange that (3.41) hold for $k\tau_1 \leqslant \tau \leqslant \min((k+1)\tau_1, T)$ by choosing τ_1 and ε sufficiently small. (The smallness condition on ε depends on N, but that for τ_1 does not.)

Let τ satisfy $k\tau_1 \leqslant \tau \leqslant \min((k+1)\tau_1, T)$. From (3.36) we have

$$\begin{aligned}[d^\varepsilon(k\tau_1, \tau)^2] &\leqslant C(\tau - k\tau_1)(E_\phi[v^\varepsilon](k\tau_1) - E_\phi[v^\varepsilon](\tau)) \\ &\leqslant C\tau_1(E_\phi[v^\varepsilon](0) - E_\phi[v^\varepsilon](\tau)).\end{aligned} \tag{3.53}$$

From (3.40) and (3.48) we have

$$\begin{aligned}E_\phi[v^\varepsilon](\tau) &\geqslant (c_0 - c_2\varepsilon^{1/2})\left(1 - \frac{(n-1)^2}{\rho(T)^2}(c_1\, d^\varepsilon(\tau) + \varepsilon^{1/2})^2\right) \\ &\geqslant c_0 - C(\varepsilon^{1/2} + [d^\varepsilon(\tau)]^2);\end{aligned}$$

therefore, using (3.22),

$$E_\phi[v^\varepsilon](0) - E_\phi[v^\varepsilon](\tau) \leqslant C([d^\varepsilon(\tau)]^2 + \varepsilon^{1/2}). \tag{3.54}$$

The triangle inequality and the inductive hypothesis yield

$$\begin{aligned}[d^\varepsilon(\tau)]^2 &\leqslant 2[d^\varepsilon(k\tau_1)]^2 + 2[d^\varepsilon(k\tau_1, \tau)]^2 \\ &\leqslant 2 \cdot 2^{2k}\varepsilon^{1/2} + 2[d^\varepsilon(k\tau_1, \tau)]^2.\end{aligned} \tag{3.55}$$

Taken together, (3.52)–(3.55) yield

$$[d^\varepsilon(\tau)]^2 \leqslant 2 \cdot 2^{2k}\varepsilon^{1/2} + C'\tau_1([d^\varepsilon(\tau)]^2 + \varepsilon^{1/2}), \tag{3.56}$$

where C' is a constant depending on T but not τ_1. We choose τ_1 so that $C'\tau_1 \leqslant \frac{1}{4}$. Then (3.56) yields

$$[d^\varepsilon(\tau)]^2 \leqslant \tfrac{4}{3} \cdot 2^{2k+1} \varepsilon^{1/2} + \tfrac{1}{3} \varepsilon^{1/2} \leqslant 2^{2k+2} \varepsilon^{1/2}.$$

This proves (3.52) for $N = k+1$, completing the induction.

The preceding result is the main ingredient in the proof of Theorem 3.1. Indeed, the theorem asserts in essence that $v^\varepsilon(R, \tau)$ is asymptotically independent of τ, and Proposition 3.8 says this is so for $R \in (-a, a)$. We also have to rule out the appearance of a new "transition" outside $(-a, a)$. This will be done using (3.47): it dictates that almost all the "energy" is consumed by the transition at $R = 0$, so there is not enough left for a transition to develop elsewhere.

Proof of Theorem 3.1. Fixing $T < T_{\max}$, our goal is to show that (3.10) holds. If not, then there is a sequence $\varepsilon_j \to 0$ and a constant $\delta > 0$ such that

$$\int_0^T \int_\Omega |u^{\varepsilon_j}(r, t) - f(r, t)| \, r^{n-1} \, dr \, dt \geqslant \delta. \tag{3.56'}$$

We apply the compactness results of Section 2; note that the crucial hypothesis $E[u^\varepsilon](0) \leqslant M$ follows from (3.4), since $\psi_0(r)$ (defined by (3.5)) is bounded below. By Theorem 2.3 there is a subsequence (still denoted u^{ε_j}) which converges a.e.,

$$u^{\varepsilon_j} \to u^* \qquad \text{a.e.}, \tag{3.57}$$

with u^* taking only the values ± 1. We have assumed that the initial data are uniformly bounded (see (3.6)); by an application of the maximum principle, $\{u^\varepsilon(r, t)\}$ remains uniformly bounded for all time. We may thus pass to the limit $\varepsilon_j \to 0$ in (3.56)$'$, concluding that

$$\int_0^T \int_\Omega |u^*(r, t) - f(r, t)| \, r^{n-1} \, dr \, dt \geqslant \delta. \tag{3.58}$$

We shall derive a contradiction by showing that in fact $u^* \equiv f$.

Consider the functions v^{ε_j} and v^* corresponding to u^{ε_j} and u^* through the change of variables (3.11). From (3.23) we have

$$\lim_{\varepsilon \to 0} v^{\varepsilon_j}(R, 0) = \begin{cases} +1, & R > 0 \\ -1, & R < 0. \end{cases}$$

Applying Proposition 3.8 and the dominated convergence theorem, we conclude that

$$v^*(R,\tau)=\begin{cases}+1, & R>0\\ -1, & R<0\end{cases} \quad \text{for} \quad -a<R<a, 0<\tau<T. \tag{3.59}$$

To handle values of R outside $(-a, a)$, we apply Proposition 3.8 together with (3.47) and (3.48) to see that

$$\int_{-c\varepsilon^{1/4}}^{c\varepsilon^{1/4}} \phi\left[\frac{\varepsilon}{2}(v_R^\varepsilon)^2+\frac{\varepsilon^{-1}}{4}((v^\varepsilon)^2-1)^2\right]dR \geqslant c_0 - C\varepsilon^{1/2}$$

for each τ, $0\leqslant\tau\leqslant T$. Since $E_\phi[v^\varepsilon](\tau)\leqslant c_0+C\varepsilon^{1/2}$, we conclude that

$$\int_{-\rho(\tau)}^{-c\varepsilon^{1/4}} \phi\left[\frac{\varepsilon}{2}(v_R^\varepsilon)^2+\frac{\varepsilon^{-1}}{4}((v^\varepsilon)^2-1)^2\right]dR \leqslant C\varepsilon^{1/2}.$$

It follows using (3.25) that

$$\int_{-\rho(\tau)}^{-c\varepsilon^{1/4}} \phi\,|g(v^\varepsilon)_R|\,dR \leqslant C\varepsilon^{1/2}. \tag{3.60}$$

Since ϕ is strictly positive except at the endpoint $-\rho(\tau)$, we may pass to the limit $\varepsilon=\varepsilon_j\to 0$ in (3.60) to see that $g(v^*)$ is constant on $(-\rho(\tau), 0)$. By (3.59), its value must be $g(-1)$. A similar argument shows that $g(v^*)=g(+1)$ for $R\in(0, 1-\rho(\tau))$. Using the monotonicity of g and returning to the original variables, we have shown that

$$u^*(r,t)=\begin{cases}+1, & r>\rho(t)\\ -1, & r<\rho(t),\end{cases}$$

or in other words $u^*=f$. This contradicts (3.58), completing the proof.

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