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Entropy and the Uncertainty Principle

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A minimum principle is obtained for the sum of entropies of two distributions related as the absolute squares of a Fourier transform pair. The minimum is shown to be attained for a Gaussian pair. The joint entropy is calculated for two other Fourier pairs of interest. Applications to the uncertainty principle are made by defining a joint entropy for position and momentum. A generalized uncertainty principle, for any set of observables not simultaneously measurable, is conjectured.

LIST OF SYMBOLS

$\mathbf{x}$	Position vector
$\mathbf{p}$	Linear momentum vector
t	Time
$\psi(\mathbf{x}, t)$	Quantum mechanical wave function (position)
$\psi^*(\mathbf{x}, t)$	Conjugate of $\psi(x, t)$
$\phi(\mathbf{p}, t)$	Quantum mechanical wave function (linear momentum)
h	Planck's quantum of action
Δx_j	Uncertainty in position coordinate
Δp_j	Uncertainty in momentum coordinate
α, β, γ	Quantum mechanical observables
$\delta = \frac{2\pi i}{h}(\alpha\beta - \beta\alpha)$	Commutator of α, β
$\bar{\delta}$	Mean value of δ
$\mathrm{Var}(\alpha)$	Variance of α
$\psi(x)$	Position wave function
$\phi(p)$	Momentum wave function
H	Joint dimensionless entropy
L	Joint entropy
$\psi_{n,\lambda}(x)$	Approximation to $\psi(x)$
$c_\lambda(k)$	Fourier coefficient of $\psi_{n,\lambda}(x)$
$N_{n,\lambda}$	Normalization constant
$L_{n,\lambda}$	Approximate joint entropy
$M_{n,\lambda}$	Constrained approximate joint entropy

$\psi_1(x)$	Gaussian position wave function	$\psi_2(x)$	Position wave function (Cauchy case)
$\phi_1(p)$	Gaussian momentum wave function	$\phi_2(p)$	Momentum wave function (Cauchy case)
e	Natural base	$\psi_3(x)$	Position wave function (uniform case)
$\tilde{\Delta}x$	Entropic uncertainty in x	$\phi_3(p)$	Momentum wave function (uniform case)
$\tilde{\Delta}p$	Entropic uncertainty in p	γ	Euler's constant

1. THE QUANTUM MECHANICAL BACKGROUND

Let $\psi(\mathbf{x}, t)$ be the quantum mechanical wave function for a physical system in a given state. If $\int |\psi(\mathbf{x}, t)|^2 \, d\mathbf{x}$ is finite for one value of t, then it is a constant. If the integral is normalized to unity, then the probability that the position coordinates lie in the set A in configuration space at time t is given by $\int_A |\psi(\mathbf{x}, t)|^2 \, d\mathbf{x}$. If configuration space is $3n$-dimensional, then the probability that the momentum coordinates lie in the set B in momentum space is given by $\int_B |\phi(\mathbf{p}, t)|^2 \, d\mathbf{p}$, where

$$\phi(\mathbf{p}, t) = h^{-3n/2} \int \psi(\mathbf{x}, t) \exp\left[-\frac{2\pi i}{h}(\mathbf{x}\cdot\mathbf{p})\right] d\mathbf{x} \tag{1}$$

is the $3n$-dimensional Fourier transform of $\psi(\mathbf{x}, t)$. (See Kemble, 1937, Chapter III.)

According to the Heisenberg (1927) uncertainty principle, if Δx_j, Δp_j are the uncertainties, in some sense, in the simultaneous measurement of position coordinate x_j and momentum coordinate p_j, then $\Delta x_j \cdot \Delta p_j$ is of the order of h.

On the mathematical side, Weyl (1928) utilized a generalization of the Schwarz inequality to show that if $\mathrm{Var}(x_j)$, $\mathrm{Var}(p_j)$ are the statistical variances of x_j, p_j determined from the probability densities $|\psi(\mathbf{x}, t)|^2$ and $|\phi(\mathbf{p}, t)|^2$, then

$$\mathrm{Var}(x_j)\cdot\mathrm{Var}(p_j) \geqq (h/4\pi)^2. \tag{2}$$

H. P. Robertson (1929) derived, by the same method, the more general result that if $\alpha(\mathbf{p}, \mathbf{x})$ and $\beta(\mathbf{p}, \mathbf{x})$ are observables which are polynomials in the momentum coordinates, that

$$\mathrm{Var}\,\alpha\cdot\mathrm{Var}\,\beta \geqq \frac{h^2}{16\pi^2}|\bar{\delta}|^2, \tag{3}$$

where

$$\delta = \frac{2\pi i}{h} (\alpha\beta - \beta\alpha) \tag{4}$$

is the commutator of α and β, and

$$\bar{\delta} = \int \psi^*(\mathbf{x}, t)\delta\psi(\mathbf{x}, t)\, d\mathbf{x} \tag{5}$$

is the mean value of δ. He applied this to angular momentum with interesting results.

2. ENTROPY AND THE UNCERTAINTY PRINCIPLE

In the Heisenberg-Weyl formulation, the measure of uncertainty in the joint distribution of observables α and β is

$$\Delta\alpha \cdot \Delta\beta = \sqrt{\text{Var } \alpha \text{ Var } \beta}.$$

Now there is another measure of uncertainty of a joint distribution, namely the entropy (Boltzmann, 1912; Szilard, 1929; Shannon, 1948).

Brillouin (1956) has shown that some relations exist between the concept of entropy, or information, and the uncertainty principle. His work has many points of contact with the present paper.

In Sections 3 and 4, we show that if $\psi(x)$ is square-integrable, and

$$\phi(p) = h^{-1/2} \int \psi(x) e^{-2\pi i p x/h}\, dx$$

is the Fourier transform of $\phi(x)$, then

$$-\int |\psi(x)|^2 \log |\psi(x)|^2\, dx - \int |\phi(p)|^2 \log |\phi(p)|^2\, dp \geqq \log\left(\frac{he}{2}\right)$$

The inequality becomes an equality when $\psi(x)$ and $\phi(p)$ are complex Gaussian pairs. This result motivates the definitions and assertions which follow.

Let $\{\alpha_1, \alpha_2, \cdots\}$ be a set of observables of a physical system and let γ be an observable (such as energy) of the system. Let $f(\alpha_1, \alpha_2, \cdots; t \mid c)$ be the joint distribution function of $\alpha_1, \alpha_2, \cdots$ at time t when the system is in an eigenstate c of γ. We define the joint entropy of $\alpha_1, \alpha_2, \cdots$ in the eigenstate c as

$$L(\alpha_1, \alpha_2, \cdots; t \mid c) = -\int \cdots \int f(a_1, a_2, \cdots; t \mid c) \cdot \log f(a_1, a_2, \cdots; t \mid c)\, da_1\, da_2 \cdots \tag{6}$$

provided a reasonable meaning can be given to the above integral. (In quantum mechanics the inequality $f(a_1, a_2, \cdots \cdot t \mid c) \geqq 0$ is not obvious.) L is not dimensionless, but if k is a constant with dimensions $[\alpha_1] \cdot [\alpha_2] \cdots$, then

$$H(\alpha_1, \alpha_2, \cdots; t \mid c) = L - \log k$$

is dimensionless. More generally, we can define $L(\alpha_1, \alpha_2, \cdots; t \mid \gamma)$ as

$$\textstyle\sum_c p(\gamma = c) L(\alpha_1, \alpha_2, \cdots; t \mid c), \qquad \text{etc.}$$

In case α_1 is linear position x, α_2 is linear momentum p, and the joint distribution function of x, p is independent, so that

$$f(x, p; t \mid c) = |\psi_c(x, t)|^2 \, |\phi_c(p, t)|^2,$$

for some eigenstate c of γ, then from Section 1, we see that the mathematical result mentioned above can be formulated physically as follows: *the joint dimensionless entropy of linear position and momentum has the minimum value* $\log(e/2) > 0$ (taking $k = h$, $H = L - \log h$). We conjecture that the above statement holds even when x and p are not statistically independent, in which case

$$f(x, p; t \mid c) = h^{-3n} \cdot \int \psi_c^*(x - r/2, t) \psi_c(x + r/2, t) \exp\left(-\frac{2\pi i p \cdot r}{h}\right) dr$$

(Wigner, 1932; see also Moyal, 1949). We further conjecture that *the joint entropy* $H(\alpha_1, \alpha_2, \cdots; t \mid \gamma)$ *has a positive minimum whenever* $\alpha_1, \alpha_2, \cdots$ *are not simultaneously observable*, provided the constants are suitable powers of h.

3. THE MINIMIZING EQUATIONS

The mathematical problem here considered is to find the minimum value of

$$L(\psi) = -\int |\psi(x)|^2 \log |\psi(x)|^2 \, dx - \int |\phi(p)|^2 \log |\phi(p)|^2 \, dp \tag{7}$$

given that

$$\int |\psi(x)|^2 \, dx = \int |\phi(p)|^2 \, dp = 1. \tag{8}$$

The technique adopted is to approximate $\psi(x)$ by finite Fourier series

$$\psi_{n,\lambda}(x) = \frac{1}{\lambda h^{\frac{1}{2}}} \sum_{k=-n}^{n} c_\lambda(k) \exp\left(\frac{2\pi i k x}{h\lambda}\right) \tag{9}$$

of increasing order and period, minimize with respect to the Fourier coefficients $c_\lambda(k)$ of $\psi_{n,\lambda}(x)$, under the given constraints, and pass to the limit after the minimizing equations are set up. The limiting equation is then solved, and the minimum value of $L(\psi)$ computed. (This heuristic limiting procedure can be justified by standard arguments, which will appear in a more extended treatment of the mathematical problem to be published elsewhere). After submission of this paper, the author was apprised of the important paper by Hirschman (1957), who proved by an elegant method that $L(\psi) \geqq \log h$ (in our notation), and conjectured the stronger result shown here.

As expected, the minimizing wave functions are Gaussian, as in Weyl's version of the Heisenberg principle. If

$$\psi_{n,\lambda}(x) = \frac{1}{\lambda h^{1|2}} \sum_{k=-n}^{n} c_\lambda(k) \exp\left(\frac{2\pi i k x}{h\lambda}\right),$$

then (10)

$$c_\lambda(k) = \frac{1}{h^{1|2}} \int_{-h\lambda/2}^{h\lambda/2} \psi_{n,\lambda}(x) \exp\left(-\frac{2\pi i k x}{h\lambda}\right) dx$$

By the Parseval equality,

$$\begin{aligned} N_{n,\lambda}^2 &= \int_{-\lambda h/2}^{\lambda h/2} |\psi_{n,\lambda}(x)|^2 \, dx \\ &= \frac{1}{\lambda^2} h \int_{-\lambda h/2}^{\lambda h/2} \sum_{k,j} c_\lambda(k) c_\lambda(j) \exp\left(\frac{2\pi i (k-j) x}{h\lambda}\right) dx \\ &= \frac{1}{\lambda} \sum_{k=-n}^{n} |c_\lambda(k)|^2. \end{aligned} \tag{11}$$

If

$$\int_{-\lambda h/2}^{\lambda h/2} |\psi_{n,\lambda}(x)|^2 \, dx = N_{n,\lambda}^2 > 0,$$

then

$$\frac{|\psi_{n,\lambda}(x)|^2}{N^2_{n,\lambda}}$$

is a probability density on $[-\lambda h/2, \lambda h/2]$, and

$$\frac{|c_\lambda(k)|^2}{\lambda N^2_{n,\lambda}}$$

is a discrete probability distribution over the $2n + 1$ integers

$$\{-n, -n+1, \cdots, n\}.$$

The corresponding entropy is

$$\begin{aligned} L_{n,\lambda} &= -\int_{-\lambda h/2}^{\lambda h/2} \frac{|\psi_{n,\lambda}(x)|^2}{N^2_{n,\lambda}} \log \frac{|\psi_{n,\lambda}(x)|^2}{N^2_{n,\lambda}}\, dx \\ &\qquad - \sum_{k=-n}^{n} \frac{|c_\lambda(k)|^2}{\lambda N^2_{n,\lambda}} \log \frac{|c_\lambda(k)|^2}{\lambda N^2_{n,\lambda}} \\ &= \frac{-1}{N^2_{n,\lambda}} \left[\int_{-\lambda h/2}^{\gamma h/2} |\psi_{n,\lambda}(x)|^2 \log |\psi_{n,\lambda}(x)|^2\, dx - \log N^2_{n,\lambda} \right] \\ &\qquad \frac{-1}{\lambda N^2_{n,\lambda}} \left[\sum_{k=-n}^{n} |c_\lambda(k)|^2 \log |c_\lambda(k)|^2 - \log (\lambda N^2_{n,\lambda}) \right]. \end{aligned} \tag{12}$$

Clearly we must minimize $L_{n,\lambda}$ with respect to the $c_\lambda(k)$ for fixed λ, n, and

$$\int_{-\lambda h/2}^{\lambda h/2} |\psi_{n,\lambda}(x)|^2\, dx = \frac{1}{\lambda} \sum_{k=-n}^{n} |c_\lambda(k)|^2.$$

In accordance with Lagrange's procedure, we look for the stationary values of

$$M_{n,\lambda} = L_{n,\lambda} + \mu \int_{-\lambda h/2}^{\lambda h/2} |\psi_{n,\lambda}(x)|^2\, dx. \tag{13}$$

Following Hausdorff's method (Titchmarsh, 1948), we let

$$c_\lambda(k) = u_k + iv_k, \qquad |k| \leqq n, \tag{14}$$

set

$$\frac{\partial M_{n,\lambda}}{\partial u_k} + i \frac{\partial M_{n,\lambda}}{\partial v_k} = 0,$$

and determine μ implicitly by multiplication by $c_\lambda{}^*(k)$ and summation over k.

Clearly

$$\frac{\partial M_{n,\lambda}}{\partial u_k} = \frac{-1}{N^2_{n,\lambda}}\left[\int_{-\lambda h/2}^{\lambda h/2} (1 + \log|\psi_{n,\lambda}(x)|^2)\left(\psi^*_{n,\lambda}(x)\exp\left(\frac{2\pi ikx}{h\lambda}\right)\right.\right.$$
$$\left.\left.+\ \psi_{n,\lambda}(x)\exp\left(-\frac{2\pi ikx}{h\lambda}\right)\right)dx\,\frac{1}{\lambda h^{1|2}} + \frac{1}{\lambda}\cdot 2u_k(1 + \log|c_\lambda(k)|^2)\right] \quad (15)$$
$$+\ \mu\int_{-\lambda h/2}^{\lambda h/2}\left(\psi^*_{n,\lambda}(x)\exp\left(\frac{2\pi ikx}{h\lambda}\right) + \psi_{n,\lambda}(x)\exp\left(-\frac{2\pi ikx}{h\lambda}\right)\right)dx$$

$$\frac{\partial M_{n,\lambda}}{\partial v_k} = -\left[\int_{-\lambda h/2}^{\lambda h/2}(1 + \log|\psi_{n,\lambda}(x)|^2)\left(i\psi^*_{n,\lambda}(x)\exp\left(\frac{2\pi ikx}{h\lambda}\right)\right.\right.$$
$$\left.\left.-\ i\psi_{n,\lambda}(x)\exp\left(-\frac{2\pi ikx}{h\lambda}\right)\right)dx\cdot\frac{1}{\lambda h^{1|2}} + \frac{1}{\lambda}2v_k(1 + \log|c_\lambda(k)|^2)\right]\frac{1}{N^2_{n,\lambda}} \quad (16)$$
$$+\ i\mu\int_{-\lambda h/2}^{\lambda h/2}\left(\psi^*_{n,\lambda}(x)\exp\left(\frac{2\pi ikx}{h\lambda}\right) - \psi_{n,\lambda}(x)\exp\left(-\frac{2\pi ikx}{h\lambda}\right)\right)dx.$$

Hence

$$\frac{\partial M_{n,\lambda}}{\partial u_k} + i\,\frac{\partial M_{n,\lambda}}{\partial v_k} = -\left[\frac{2}{\lambda h^{1|2}}\int_{-\lambda h/2}^{\lambda h/2}\psi_{n,\lambda}(x)(1 + \log|\psi_{n,\lambda}(x)|^2)\right.$$
$$\left.\exp\left(-\frac{2\pi ikx}{h\lambda}\right)dx + \frac{2}{\lambda}\,c_\lambda(k)(1 + \log|c_\lambda(k)|^2)\right]\times\frac{1}{N^2_{n,\lambda}} \quad (17)$$
$$+\ 2\mu\int_{-\lambda h/2}^{\lambda h/2}\psi_{n,\lambda}(x)\exp\left(-\frac{2\pi ikx}{h\lambda}\right)dx.$$

Thus

$$\frac{\partial M_{n,\lambda}}{\partial u_k} + i\cdot\frac{\partial M_{n,\lambda}}{\partial v_k} = 0$$

yields

$$\frac{1}{h^{1|2}}\int_{-\lambda h/2}^{\lambda h/2}\psi_{n,\lambda}(x)\log|\psi_{n,\lambda}(x)|^2\exp\left(-\frac{2\pi ikx}{h\lambda}\right)dx$$
$$= c_\lambda(k)[\lambda\mu h^{1|2}N^2_{n,\lambda} - 2 - \log|c_\lambda(k)|^2] \quad (18)$$

Multiplication by $c_\lambda{}^*(k)$ and summation leads to

$$\frac{1}{N_{n,\lambda}^2}\left[\int_{-\lambda h/2}^{\lambda h/2} |\psi_{n,\lambda}(x)|^2 \log |\psi_{n,\lambda}(x)|^2 \, dx \right. \\ \left. + \frac{1}{\lambda}\sum_{j=-n}^{n} |c_\lambda(j)|^2 \log |c_\lambda(j)|^2\right] = \lambda\mu h^{1/2}\cdot N_{n,\lambda}^2 - 2. \tag{19}$$

Substitution in the preceding yields

$$\frac{1}{h^{1/2}}\left[\int_{-\lambda h/2}^{\lambda h/2} \psi_{n,\lambda}(x) \log |\psi_{n,\lambda}(x)|^2 \exp\left(-\frac{2\pi ikx}{h\lambda}\right) dx \right. \\ = \frac{c_\lambda(k)}{N_{n,\lambda}^2}\left[\int_{-\lambda h/2}^{\lambda h/2} |\psi_{n,\lambda}(x)|^2 \log |\psi_{n,\lambda}(x)|^2 \, dx \right. \\ \left. + \frac{1}{\lambda}\sum_{j=-n}^{n} |c_\lambda(j)|^2 \log |c_\lambda(j)|^2\right] - c_\lambda(k) \log |c_\lambda(k)|^2. \tag{20}$$

If we now let $n \to \infty$, $\lambda \to \infty$ in such a way that $\psi_{n,\lambda}(x) \to \psi(x)$, $N_{n,\lambda}^2 \to 1$, $k/\lambda \to p$, $1/\lambda \to dp$, we formally obtain

$$c_\lambda(k) \to \phi(p) = \frac{1}{h^{1/2}}\int_{-\infty}^{\infty} \psi(x) \exp\left(-\frac{2\pi ipx}{h}\right) dx, \\ \psi(x) = \frac{1}{h^{1/2}}\int_{-\infty}^{\infty} \phi(p) \exp\left(\frac{2\pi ipx}{h}\right) dp \tag{21}$$

and as the stationary condition

$$\frac{1}{h^{1/2}}\int_{-\infty}^{\infty} \psi(x) \log |\psi(x)|^2 \exp\left(-\frac{2\pi ipx}{h}\right) dx \\ = \phi(p)\left[\int_{-\infty}^{\infty} |\psi(x)|^2 \log |\psi(x)|^2 \, dx \right. \\ \left. + \int_{-\infty}^{\infty} |\phi(q)|^2 \log |\phi(q)|^2 \, dq\right] - \phi(p) \log |\phi(p)|^2 \\ = \phi(p)[-L(\psi) - \log |\phi(p)|^2]. \tag{22}$$

This condition can also be derived more rapidly by a formal application of the calculus of variations to the original expression (7) under the constraint (8), but the present state of the theory (Caratheodory, 1945) is not sufficient to justify this application.

4. SOLUTION OF THE MINIMIZING EQUATION

The nonlinear minimizing equation (22) seems to be extremely difficult, However, it is plausible that a solution would also furnish a minimum to the joint uncertainty $\Delta x \cdot \Delta p$. As Weyl (1928) proved, the $\psi_1(x)$ which minimizes $\Delta x \cdot \Delta p$ is a complex Gaussian wave packet, with

$$\psi_1(x) = (2\pi\sigma^2)^{-1/4} \exp\left(-\frac{(x-\bar{x})^2}{4\sigma^2} + \frac{2\pi i x \bar{p}}{h}\right) \tag{23}$$

$$|\psi_1(x)|^2 = (2\pi\sigma^2)^{-1/2} \exp\left(-\frac{(x-\bar{x})^2}{2\sigma^2}\right) \tag{24}$$

A familiar calculation yields

$$\phi_1(p) = \left(\frac{2}{h}\right)^{1/2} (2\pi\sigma^2)^{1/4} \exp\left(-\frac{4\pi^2\sigma^2(p-\bar{p})^2}{h^2} - \frac{2\pi i \bar{x}(p-\bar{p})}{h}\right) \tag{25}$$

and thus

$$|\phi_1(p)|^2 = \left(\frac{8\pi\sigma^2}{h^2}\right)^{1/2} \exp\left(-\frac{8\pi^2\sigma^2}{h^2}(p-\bar{p})^2\right) \tag{26}$$

The mean and variance of (26) are, respectively, $\bar{p}$ and $h^2/16\pi^2\sigma^2$. Note that

$$\text{Var } x \cdot \text{Var } p = \sigma^2 \cdot \frac{h^2}{16\pi^2\sigma^2} = \left(\frac{h}{4\pi}\right)^2,$$

$$\Delta x \cdot \Delta p = \frac{h}{4\pi},$$

so the Weyl minimum is indeed attained.

The information in a Gaussian distribution of variance σ^2 is well known to be $\frac{1}{2} \log (2\pi\sigma^2 e)$. Thus

$$\begin{aligned} L(\psi_1) &= \frac{1}{2}\log (2\pi\sigma^2 e + \frac{1}{2}\log\left(\frac{2\pi \cdot h^2 e}{16\pi^2\sigma^2}\right) \\ &= \log (he/2) = \log h + \log (e/2), \end{aligned} \tag{27}$$

and

$$H(\psi_1) = L(\psi_1) - \log h = \log(e/2) = 0.30685 \cdots$$

It remains to calculate

$$\chi_1(p) = \frac{1}{h^{1/2}} \int \psi_1(x) \log |\psi_1(x)|^2 \exp\left(-\frac{2\pi i x p}{h}\right) dx$$

and to check whether ψ_1 yields a solution of the minimizing equation. Differentiation of (25) with respect to σ^2 and a brief manipulation yields

$$\chi_1(p) = \frac{(2\pi\sigma^2)^{-(1/4)}}{h^{1/2}} \cdot \exp\left(-\frac{4\pi^2\sigma^2(p-\bar{p})^2}{h^2} - \frac{2\pi i\bar{x}(p-\bar{p})}{h}\right) \cdot \left[-\frac{1}{2}(4\pi\sigma^2)^{1/2}\log(2\pi\sigma^2) - 2\sigma\pi^{1/2} + \frac{16\sigma^3\pi^{5/2}(p-\bar{p})^2}{h^2}\right] \tag{28}$$

On the other hand

$$\phi_1(p)(-L(\psi_1) - \log|\phi_1(p)|^2) = \left(\frac{2}{h}\right)^{1/2}(2\pi\sigma^2)^{1/4}\exp\left(-\frac{4\pi^2\sigma^2(p-\bar{p})^2}{h^2} - \frac{2\pi i\bar{x}(p-\bar{p})}{h}\right) \cdot \left(\log\left(\frac{2}{eh}\right) - \frac{1}{2}\log\left(\frac{8\pi\sigma^2}{h^2}\right) + \frac{8\pi^2\sigma^2(p-\bar{p})^2}{h^2}\right)$$

Term-by-term comparison shows that

$$\chi_1(p) = \phi_1(p)\,(-L(\psi_1) - \log|\phi_1(p)|^2)$$

for all p, $\bar{p}$, $\bar{x}$, σ^2. Hence $\psi_1(x)$ is indeed a solution of the minimizing equation (22), and we have shown that

$$L(\psi) \geqq L(\psi_1) = \log(he/2), \qquad H(\psi) \geqq H(\psi_1) = \log(e/2) > 0. \tag{29}$$

Note that as $h \to 0$, the inequality degenerates to $L(\psi) \geqq -\infty$, just as $\Delta x \cdot \Delta p \geqq h/4\pi$ degenerates to $\Delta x \cdot \Delta p \geqq 0$.

It is interesting that if we define "entropic uncertainties" $\tilde{\Delta}x$ in position and $\tilde{\Delta}p$ in momentum by

$$\tilde{\Delta}x = \exp\left(-\int |\psi(x)|^2 \log|\psi(x)|^2\,dx\right)$$

and

$$\tilde{\Delta}p = \exp\left(-\int |\phi(p)|^2 \log|\phi(p)|^2\,dp\right),$$

then it follows from the preceding that

$$\tilde{\Delta}x \cdot \tilde{\Delta}p = \exp(L(\psi)) \geqq \exp L(\psi_1) = he/2. \tag{30}$$

This closely resembles the Weyl formulation of the Heisenberg principle.

5. JOINT INFORMATION FOR NONMINIMIZING WAVE PACKETS

In order to see how sharp the minimum is, the calculation of $L(\psi)$ for some simple nonminimizing wave packets may be of interest. We consider two examples, one for reasons of mathematical convenience, the other for its physical significance.

Let

$$\psi_2(x) = \frac{1}{\sqrt{2\sigma}} \exp\left(-\frac{|x - \bar{x}|}{2\pi} + \frac{2\pi i \bar{p} x}{h}\right),$$

so that

$$|\psi_2(x)|^2 = \frac{1}{2\sigma} \exp\left(-\frac{|x - \bar{x}|}{\sigma}\right).$$

Now

$$\int |\psi_2(x)|^2 \log |\psi_2(x)|^2 \, dx = \frac{1}{2\sigma} \log \frac{1}{2\sigma} \int_{-\infty}^{\infty} \exp\left(-\left|\frac{x - \bar{x}}{\sigma}\right|\right) dx$$
$$- \frac{1}{2\sigma^2} \int_{-\infty}^{\infty} |x - \bar{x}| \exp\left(-\frac{|x - \bar{x}|}{\sigma}\right) dx = \log \frac{1}{2\sigma e}.$$

We also find

$$\phi_2(p) = (2\sigma h)^{-1/2} \cdot \frac{\exp(-2\pi i \bar{x}(p - \bar{p})h^{-1})}{(4\sigma)^{-1} + 4\pi^2(p - \bar{p})^2 \sigma h^{-2}}$$

$$|\phi_2(p)|^2 = \frac{(h/\sigma)^3}{32\pi^4} \cdot \frac{1}{((p - \bar{p})^2 + (h/4\pi\sigma)^2)^2}$$

and

$$\int |\phi_2(p)|^2 \log |\phi_2(p)|^2 \, dp = \frac{(h/\sigma)^3}{32\pi^4}\left[\log \frac{(h/\sigma)^3}{32\pi^4} \int \frac{dp}{((p - \bar{p})^2 + (h/4\pi\sigma)^2)^2} - 2 \int \frac{\log((p - \bar{p})^2 + (h/4\pi\sigma)^2)\, dp}{((p - \bar{p})^2 + (h/4\pi\sigma)^2)^2}\right]$$

From the result

$$\int_0^{\infty} \frac{\log(r^2 + x^2)}{q^2 + x^2} \, dx = \frac{\pi}{q} \log(r + q)$$

and from differentiating the above with respect to q, we find after simplification

$$\int |\phi_2(p)|^2 \log |\phi_2(p)|^2 \, dp = \log\left(\frac{e^2 \sigma}{2h}\right)$$

Hence

$L(\psi_2) = \log(4h/e)$,

$H(\psi_2) = \log(4/e) = 0.38629 \cdots > H(\psi_1) = \log(e/2) = 0.30685 \cdots$

The wave packet $\psi_3(x)$ defined by

$$\psi_3(x) = \begin{cases} \dfrac{1}{\sqrt{2A}} \exp\left(\dfrac{2\pi i \bar{p} x}{h}\right), & |x - \bar{x}| \leqq A \\ 0 \quad , & |x - \bar{x}| > A \end{cases}$$

has a physical interpretation to be discussed in a later communication. Obviously, it gives the uniform density

$$|\psi_3(x)|^2 = \begin{cases} \dfrac{1}{2A}, & |x - \bar{x}| \leqq A \\ 0, & |x - \bar{x}| > A \end{cases},$$

and so

$$\int |\psi_3(x)|^2 \log |\psi_3(x)|^2 \, dx = \log \frac{1}{2A}.$$

Now

$$\phi_3(p) = (2Ah)^{-1/2} \exp\left(-\frac{2\pi i \bar{x}(p - \bar{p})}{h}\right) \cdot \frac{\sin(2\pi A(p - \bar{p})h^{-1})}{(\pi/h)(p - \bar{p})}$$

and

$$|\phi_3(p)|^2 = \frac{h}{2A}\left(\frac{\sin(2\pi A(p - p)h^{-1})}{\pi(p - p)}\right)^2.$$

It follows that

$$\int |\phi_3(p)|^2 \log |\phi_3(p)|^2 \, dp = \frac{1}{\pi}\left[\log \frac{2A}{h} \int_{-\infty}^{\infty} \frac{\sin^2 u}{u^2} \, du + \int_{-\infty}^{\infty} \frac{\sin^2 u}{u^2} \log\left(\frac{\sin^2 u}{u^2}\right) du\right]$$

and, since $\int_{-\infty}^{\infty} u^{-2} \sin^2 u \, du = \pi$,

$$H(\psi_3) = L(\psi_3) - \log h = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin^2 u}{u^2} \log\left(\frac{\sin^2 u}{u^2}\right) du.$$

Unfortunately, this definite integral does not seem to have been calculated previously. The derivation of its value may be of interest, inasmuch as similar integrals arise frequently in "continuous" applications of information theory.

The function $\log | 2 \sin u |$ can be expanded into the Fourier series

$$-\sum_{k=1}^{\infty} \frac{\cos 2ku}{k},$$

convergent for $u \neq n\pi$, $n = 0, \pm 1, \cdots$. Since $(\sin^2 u/u^2)$ is absolutely integrable on $(-\infty, \infty)$ the series

$$\frac{\sin^2 u}{u^2} \log | 2 \sin u | = -\sum_{k=1}^{\infty} \frac{\sin^2 u}{u^2} \frac{\cos 2ku}{k}$$

is term-wise integrable, and thus

$$Q = \int \frac{\sin^2 u}{u^2} \log \left| \frac{4 \sin^2 u}{4u^2} \right| du = -\pi \log 4 - 4 \int_{-\infty}^{\infty} \frac{\sin^2 u}{u^2} \log | u | \, du$$
$$- 2 \sum_{k=1}^{\infty} \frac{1}{k} \int \frac{\sin^2 u \cos 2ku}{u^2} du.$$

From $\sin^2 u = \tfrac{1}{2}(1 - \cos 2u)$, the multiplication formula for trigonometric functions, and the integral

$$\int_{-\infty}^{\infty} \frac{1 - \cos ax}{2x^2} dx = a\pi,$$

we find that

$$\int \frac{\sin^2 u \cos 2ku}{u^2} du = 0 \quad \text{for} \quad k = 1, 2, 3, \cdots,$$

and thus

$$Q = -\pi \log 4 - 4 \int_{-\infty}^{\infty} \frac{\sin^2 u}{u^2} \log | u | du.$$

Let

$$A_q = \int_0^{\infty} \frac{\sin^2 u}{u^q} du$$

and recall that

$$\int_0^{\infty} e^{-uz} z^{q-1} dz = \frac{\Gamma(q)}{u^q}.$$

Thus

$$A_q = \int_0^\infty \frac{\sin^2 u}{\Gamma(q)} \left(\int_0^\infty e^{-uz} z^{q-1} dz \right) du = \int_0^\infty \int_0^\infty \frac{\sin^2 u}{\Gamma(q)} e^{-uz} z^{q-1} dz\, du$$

$$= \int_0^\infty \left(\frac{1}{\Gamma(q)} \int_0^\infty \sin^2 u e^{-uz} du \right) z^{q-1} dz.$$

Now

$$\int_0^\infty (\sin^2 u) e^{-uz} du = \int_0^\infty \frac{1}{2} (1 - \cos 2u) e^{-uz} du$$

$$= \frac{1}{2} \left[\frac{e^{-uz}}{z} - e^{-uz} \frac{(-z \cos 2u + 2 \sin 2u)}{z^2 + 4} \right]_0^\infty$$

$$= \frac{1}{2} \left(\frac{1}{z} - \frac{z}{z^2 + 4} \right)$$

$$= \frac{2}{z(z^2 + 4)}$$

hence

$$A_q = \int_0^\infty \frac{1}{\Gamma(q)} \frac{2}{z(z^2 + 4)} z^{q-1} dz = \frac{2}{\Gamma(q)} \int_0^\infty \frac{z^{q-2}}{z^2 + 4} dz.$$

The right member converges at 0 when $q > 1$ and at ∞ when $q < 3$. Thus

$$A_q = \frac{2^{q-3}}{\Gamma(q)} \int_0^\infty \frac{y^{q-2}}{y^2 + 1} dy$$

$$= \frac{2^{q-2}}{\Gamma(q)} \cdot \frac{\pi}{2} \sec \frac{(q - 2)\pi}{2}$$

$$= - \frac{2^{q-3}\pi}{\Gamma(q) \cos (q\pi/2)}$$

for $1 < q < 3$.

Differentiating A_q with respect to q, we find

$$\frac{dA_q}{dq} = - \frac{\pi}{8} \left[\frac{2^q \log 2}{\Gamma(q) \cos (q\pi/2)} - \frac{2^q \Gamma'(q)}{(\Gamma(q))^2 \cos (q\pi/2)} + \frac{2^q}{\Gamma(q)} \cdot \frac{\pi}{2} \sec \frac{q\pi}{2} \tan \frac{q\pi}{2} \right]$$

At $q = 2$, we have

$$\left.\frac{dA_q}{dq}\right|_{q=2} = -\frac{\pi}{8}\left[\frac{4 \log 2}{-1} - \frac{4}{-1}\,\Gamma'(2)\right] = \pi\left[\frac{\log 2}{2} - \frac{\Gamma'(2)}{2}\right]$$

Now

$$\frac{\Gamma'(q)}{\Gamma(q)} = \int_0^1 \frac{x^{q-1} - 1}{x - 1}\,dx - \gamma,$$

where γ is Euler's constant. Hence

$$\Gamma'(2) = \int_0^1 dx - \gamma = 1 - \gamma,$$

and

$$\left.\frac{dA_q}{dq}\right|_{q=2} = \frac{\pi}{2}\,[\log 2 + \gamma - 1].$$

On the other hand,

$$\frac{dA_q}{dq} = -\int_0^\infty \frac{\sin^2 u}{u^q} \log|u|\,du,$$

so

$$\int_0^\infty \frac{\sin^2 u}{u^2} \log|u|\,du = -\left.\frac{dA_q}{dq}\right|_{q=2}$$

Hence

$$Q = -\pi \log 4 - 4 \cdot \frac{\pi}{2}(1 - \gamma - \log 2) = -2\pi(1 - \gamma).$$

Finally, then $H(\psi_3) = -(Q/\pi) = 2(1 - \gamma) = 0.84557 \cdots$.

Thus the joint entropy of a uniform position distribution is nearly three times that of a Gaussian wave packet.

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After completion of this work, I was informed by the editors of the papers by Bourret (1958) and Stam (1959) in this journal bearing on this problem.[1]

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[1] Because of editorial problems, the paper by Stam will appear in the next issue.

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