A characterization of $[a, b]$-compact

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1. Introduction

Throughout this paper, $a$, $b$, $k$ and $r$ denote cardinal numbers with $a$ and $b$ infinite and $a \leq b$. The set of all cardinals $k$ such that $a \leq k \leq b$ is designated by $[a, b]$. The cardinality of a set $X$ is denoted by $|X|$.

The theory of $[a, b]$-compactness gives a unified approach to the important notions of compactness, the Lindelof property, countable compactness, and subsets having complete accumulation points. See, for example [1–4,7–9,11,12] and the references cited therein.

Before we proceed, we state the following definitions:

**Definition 1.1.** A space is $[a, b]$-compact if every open cover of $X$ of cardinality less than or equal $b$, has a subcover of cardinality less than $a$.

**Remark 1.1.** In Definition 1.1, if $a = b$ then $X$ is called $[a, a]$-compact. It is clear that $X$ is $[a, b]$-compact if and only if $X$ is $[k, k]$-compact for every $k$ with $a \leq k \leq b$.

**Definition 1.2.** A space is $[a, b]$ $r$-compact if it is $[k, k]$-compact for every regular cardinal $k$ with $a \leq k \leq b$.

(The readers may find the above definitions and some of their consequences in [4,6,13].)
There have been in the past some efforts to find sufficient conditions for an \([a, b]^r\)-compact space to be \([a, b]\)-compact. In 1929, Alexandroff and Urysohn [2], established the following theorem:

**Theorem 1.1.** ([2]) An \([a, b]^r\)-compact space is \([a, b]\)-compact if \(a = \omega_0\), where \(\omega_0 = |\mathbb{N}|\).

In 1962, Miscenko [10], established the following theorem:

**Theorem 1.2.** ([10]) If \(X\) is \([a, b]^r\)-compact, \(a\) is regular and for every open cover \(\mathcal{U}\) of \(X\), there is a closed cover \(\mathcal{F} = \{F_U \mid U \in \mathcal{U}\}\) such that \(F_U \subseteq U\) for all \(U \in \mathcal{U}\).

In 1970, Howes [5], established the following theorem:

**Theorem 1.3.** ([5]) If \(X\) is \([\omega_1, \infty]^r\)-compact, countably metacompact space, \(X\) is \([\omega_1, \infty]\)-compact, where \(\omega_1\) is the smallest uncountable cardinal and a space is countably metacompact if every countable open cover has a point-finite open refinement.

In 1974, Hodel and Vaughan [4], gave the following definition:

**Definition 1.3.** ([4]) A topological space \(X\) is said to satisfy \(I(\gamma)\) if for every increasing open cover \(\mathcal{U} = \{U_\gamma \mid 0 \leq \gamma \leq k\}\) of \(X\) with \(k < a\), there is a closed refinement \(\mathcal{F}\) of \(\mathcal{U}\) with \(|\mathcal{F}| \leq a\).

Taking into account the above definition, the same authors proved the following theorem:

**Theorem 1.4.** ([4]) Let \(a\) be regular. If \(X\) is \([a, b]^r\)-compact and satisfies \(I(a)\), then \(X\) is \([a, b]\)-compact.

In this paper, we find a necessary and sufficient condition for an \([a, b]^r\)-compact space to be \([a, b]\)-compact.

2. Main result

Before we proceed we will need the following definitions:

**Definition 2.1.** Let \(X\) be a set and \(\mathcal{U}\) be a cover of \(X\). A refinement \(\mathcal{V}\) of \(\mathcal{U}\) is a cover of \(X\) such that, for every \(V \in \mathcal{V}\) there exists \(U \in \mathcal{U}\) such that \(V \subseteq U\).

**Definition 2.2.** A family of sets \(\mathcal{V}\) is called star-\(k\) if and only if for every \(V \in \mathcal{V}\), \(|\mathcal{V}(V)| < k\), where \(\mathcal{V}(V) = \{V' \in \mathcal{V} \mid V' \cap V \neq \emptyset\}\).

**Definition 2.3.** A space \(X\) is \([a, b]\)-star-refinable if every open cover \(\mathcal{U}\) of \(X\) with \(|\mathcal{U}| = k \in [a, b]\), has a star-\(k\) open refinement.

**Notation.** Throughout this paper we will use the notation: Let \(\mathcal{U}\) be a cover of \(X\). Then for every \(x \in X\), \(St(x, \mathcal{U}) = \bigcup\{U \in \mathcal{U} \mid x \in U\}\).

The following two lemmas are needed for the proof of the main theorem:

**Lemma 2.1.** Let \(X\) be \([k, k]\)-compact, where \(k\) is a regular cardinal. Then:

(i) \(X\) is \([\lambda, \lambda]\)-compact for every \(\lambda > k\) with \(cf(\lambda) = k\).

(ii) Furthermore, if \(X\) is \([a, b]^r\)-compact, then \(X\) is \([k, k]\)-compact for every singular cardinal \(k\) such that \(a \leq k \leq b\) and \(cf(k) \geq a\).

**Proof.** Let \(\mathcal{U} = \{U_\gamma \mid \gamma < \lambda\}\) be an open cover of \(X\), choose cardinals \(\lambda, \beta < k\) with \(sup(\lambda, \beta) = \lambda\). For every \(\beta < k\), let \(V_\beta = \bigcup\{V_\beta \mid \gamma < \lambda_\beta\}\) and \(V = \{V_\beta \mid \beta < k\}\). Then \(V\) is an open cover of \(X\), with \(|V| = k\), and has a subcover \(V'\) of cardinality \(\mu < k\), since \(X\) is \([k, k]\)-compact. Let \(V' = \{V_\beta \mid \beta < \mu\}\), then \(V_{\mu+1} = X\), but \(V_{\mu+1} = \bigcup\{U_\gamma \mid \gamma < \lambda_{\mu+1}\}\). Put \(\mathcal{U}' = \{U_\gamma \mid \gamma < \lambda_{\mu+1}\}\), we have \(|\mathcal{U}'| = \lambda_{\mu+1} < \lambda\) and since \(\mathcal{U}'\) is a subcover of \(\mathcal{U}\), \(X\) is \([\lambda, \lambda]\)-compact. The proof of Part (i) is complete.

Part (ii) is direct from Part (i).

The proof of the lemma is complete.

**Lemma 2.2.** If \(\mathcal{W}\) is an open cover of a topological space \(X\) and \(\xi\) is a regular cardinal such that \(|\mathcal{W}| \geq \xi\), and \(\mathcal{W}\) has no subcover of cardinality smaller than \(\xi\), and for every \(W \in \mathcal{W}\), \(|\mathcal{W}(W)| < \xi\), then every infinite cardinal \(\lambda \leq \xi\) there exists an open cover of \(X\) of cardinality \(\lambda\) with no subcover of smaller cardinality.
Proof. Pick any point \( x_0 \in X \) and \( W_0 \in \mathcal{W} \) with \( x_0 \in W_0 \). By recursion pick for all \( \alpha < \xi \) points, \( x_0 \notin \bigcup_{\beta < \alpha} \text{St}(x_\beta, \mathcal{W}) = Y \), and sets \( W_\beta \in \mathcal{W} \) with \( x_\beta \in W_\beta \). This is possible because at step \( \alpha \), \( \bigcup_{\beta < \alpha} \text{St}(x_\beta, \mathcal{W}) \neq X \). For otherwise, every one of the \( \xi \) or more elements of \( \mathcal{W} \) would intersect \( Y \), but for all \( \beta \), \( \text{St}(x_\beta, \mathcal{W}) \) is a union of less than \( \xi \) elements of \( \mathcal{W} \), hence one of the less than \( \xi \) many elements would have to intersect at least \( \xi \) many elements of \( \mathcal{W} \) (since \( \xi \) is regular), which contradicts the hypothesis.

The recursion gives us a set \( S = \{ x_\alpha : \alpha < \xi \} \) and a family \( \{ W_\alpha : \alpha < \xi \} \) of open sets with \( x_\alpha \in W_\alpha \) for all \( \alpha < \xi \). We also have

\[ (\forall \mathcal{W} \in \mathcal{W}) \quad |\mathcal{W} \cap S| \leq 1, \quad \text{i.e., no \( \mathcal{W} \) contains two elements of \( S \).} \]

Now define \( B = \bigcup \{ \text{St}(x, \mathcal{W}) : x \notin \bigcup_{\alpha < \xi} \text{St}(x_\alpha, \mathcal{W}) \} \). Then

\[ B = \{ B \} \cup \{ \text{St}(x_\alpha, \mathcal{W}) : \alpha < \xi \} \]

is an open cover of \( X \), and \( \text{St}(x_\alpha, \mathcal{W}) \) is the only member of \( B \) that contains \( x_\alpha \). Thus \( B \) is a cover of \( X \) of cardinality \( \xi \) with no subcover of smaller cardinality. If \( \lambda < \xi \) then put \( C_\lambda = \bigcup \{ \text{St}(x_\alpha, \mathcal{W}) : \lambda < \beta < \xi \} \). Then \( \{ B, C_\lambda \} \cup \{ \text{St}(x_\alpha, \mathcal{W}) : \tau < \lambda \} \) is an open cover of \( X \) of cardinality \( \lambda \) with no subcover of smaller cardinality. The proof is complete. \( \square \)

Theorem 2.1. A space \( X \) is \( [a, b] \)-compact (\( a \) is a regular cardinal), if and only if \( X \) is \( [a, b]^\prime \)-compact and \( [a, b] \)-star-refinable.

Proof. Assume that \( X \) is \( [a, b] \)-compact (\( a \) is a regular cardinal). Then, in view of the definition of \( [a, b] \)-compact, it is obvious that \( X \) is \( [a, b]^\prime \)-compact, and \( [a, b] \)-star-refinable.

Let \( X \) be \( [a, b] \)-compact, and \( [a, b] \)-star-refinable.

Assume, for the sake of contradiction, that \( X \) is not \( [a, b] \)-compact.

Let \( k \) be the smallest singular cardinal with \( k > a \) such that \( X \) is not \( [k, k] \)-compact, then by Lemma 2.1, \( cf(k) < a \). Let \( U \) be an open cover of \( X \) with \( |U| = k \), and \( U \) has no subcover of smaller cardinality. By the hypothesis \( U \) has an open refinement \( V \), where \( V \) is star-\( k \). Then \( V \) has no subcover of cardinality less than \( k \).

Step 1: We may assume that \( V \) has a subcover of cardinality \( k \), for if it did not, then setting \( \xi = k^+ \) and \( \mathcal{W} = V \), the hypothesis of Lemma 2.2 is satisfied. Taking \( \lambda = a \) the lemma says that there is an open cover of \( X \) of cardinality \( a \) with no subcover of cardinality less than \( a \). Since \( a \) is regular, this contradicts that \( X \) is \( [a, b]^\prime \)-compact. By passing to this subcover of cardinality \( k \) we may assume that \( |V| = k \).

Step 2: Since \( k \) is singular, let \( \mu = cf(k) \) and let \( \{ \lambda_\alpha : \alpha < \mu \} \) be an increasing sequence of cardinals such that \( k = \sum_{\alpha < \mu} \lambda_\alpha \). Since \( a < k \) we may as well assume that \( \lambda_0 = a \). In view of Lemma 2.1, we have \( \mu < a \). Write

\[ V = \bigcup_{\alpha < \mu} V_\alpha \]

where \( |V_\alpha| \leq \lambda_\alpha \).

Step 3: For \( \beta < \mu \) let \( A_\beta = \{ V \in V : |V(V)| \leq \lambda_\beta \} \). Then

\[ V = \bigcup_{\alpha < \mu} A_\alpha. \]

Note further that \( V = \bigcup \{ V_\alpha \cap A_\alpha : \alpha < \mu \} \) because both \( V_\alpha \)'s and \( A_\alpha \)'s are increasing.

Define

\[ O_\alpha = \bigcup \{ V \in V : V \cap W = \emptyset \text{ for all } W \in V_\alpha \cap A_\alpha \}. \]

Step 4: For \( \alpha < \mu \) the family

\[ \{ O_\alpha \} \cup \{ V \in V : V \cap W \neq \emptyset \text{ for some } W \in V_\alpha \cap A_\alpha \} \]

satisfies two properties: (i) it covers \( X \), and (ii) it has cardinality at most \( \lambda_\alpha < \mu \) since there are at most \( \lambda_\alpha \) many \( W \in V_\alpha \cap A_\alpha \) and each such \( W \) covers at most \( \lambda_\alpha \) sets in \( V \).

Step 5: By \( [a, b]^\prime \)-compactness and definition of \( k \) as “smallest”, the open cover in Step 4 has a subcover of cardinality less than \( a \); call it \( V_\alpha \). Put \( W_\alpha \setminus V_\alpha = W_\alpha \setminus O_\alpha \). Then \( W_\alpha \setminus V_\alpha \) covers \( \bigcup \{ V_\alpha \cap A_\alpha \} \).

Step 6: \( \bigcup_{\alpha < \mu} W_\alpha \setminus V_\alpha \) covers \( X \).

Step 7: \( |\bigcup_{\alpha < \mu} W_\alpha \setminus V_\alpha| < a \). This follows because \( a \) is regular, \( |W_\alpha \setminus V_\alpha| < a \) for all \( \alpha < a \) and \( \mu < a \).

Step 8: Since \( \bigcup_{\alpha < \mu} W_\alpha \setminus V_\alpha \subset V \), we see that \( V \) has a subcover of cardinality less than \( a \) and since \( V \) is a refinement of \( U \), so does \( U \). This contradicts the original assumption about \( U \), and that completes the proof. Thus \( X \) is \( [a, b] \)-compact.

The proof of the theorem is complete. \( \square \)
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References