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Cohomology of abelian matched pairs and the Kac sequence ☆

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Abstract

The purpose of this paper is to introduce a cohomology theory for abelian matched pairs of Hopf algebras and to explore its relationship to Sweedler cohomology, to Singer cohomology and to extension theory. An exact sequence connecting these cohomology theories is obtained for a general abelian matched pair of Hopf algebras, generalizing those of Kac and Masuoka for matched pairs of finite groups and finite-dimensional Lie algebras. The morphisms in the low degree part of this sequence are given explicitly, enabling concrete computations.

0. Introduction

In this paper we discuss various cohomology theories for Hopf algebras and their relation to extension theory.

It is natural to think of building new algebraic objects from simpler structures, or to get information about the structure of complicated objects by decomposing them into simpler parts. Algebraic extension theories serve exactly that purpose, and the classification problem of such extensions is usually related to cohomology theories.

In the case of Hopf algebras, extension theories are proving to be invaluable tools for the construction of new examples of Hopf algebras, as well as in the efforts to classify finite-dimensional Hopf algebras [26].

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Hopf algebras, which occur for example as group algebras, as universal envelopes of Lie algebras, as algebras of representative functions on Lie groups, as coordinate algebras of algebraic groups and as Quantum groups, have many 'group like' properties. In particular, cocommutative Hopf algebras are group objects in the category of cocommutative coalgebras, and are very much related to ordinary groups and Lie algebras. In fact, over an algebraically closed field of characteristic zero, such a Hopf algebra is a semi-direct product of a group algebra by a universal envelope of a Lie algebra, hence just a group algebra if finite-dimensional (see [5,15,24] for the connected case, [9,10,30] for the general case).

In view of these facts it appears natural to try to relate the cohomology of Hopf algebras to that of groups and Lie algebras. The first work in this direction was done by M.E. Sweedler [29] and by G.I. Kac [13] in the late 1960s. Sweedler introduced a cohomology theory of algebras that are modules over a Hopf algebra (now called Sweedler cohomology). He compared it to group cohomology, to Lie algebra cohomology and to Amitsur cohomology. In that paper he also shows how the second cohomology group classifies cleft comodule algebra extensions. Kac considered Hopf algebra extensions of a group algebra kT by the dual of a group algebra k^N obtained from a matched pair of finite groups (N, T), and found an exact sequence connecting the cohomology of the groups involved and the group of Hopf algebra extensions $\mathrm{Opext}(kT, k^N)$

$$\begin{aligned}
0 &\to H^1(N \bowtie T, k^\bullet) \to H^1(T, k^\bullet) \oplus H^1(N, k^\bullet) \to \mathrm{Aut}\big(k^N \# kT\big) \\
&\to H^2(N \bowtie T, k^\bullet) \to H^2(T, k^\bullet) \oplus H^2(N, k^\bullet) \\
&\to \mathrm{Opext}\big(kT, k^N\big) \to H^3(N \bowtie T, k^\bullet) \to \cdots
\end{aligned}$$

which is now known as the Kac sequence. In the work of Kac all Hopf algebras are over the field of complex numbers and also carry the structure of a C^*-algebra. Such structures are now called Kac algebras. The generalization to arbitrary fields appears in recent work by A. Masuoka [19,20], where it is also used to show that certain groups of Hopf algebra extensions are trivial. Masuoka also obtained a version of the Kac sequence for matched pairs of Lie bialgebras [21], as well as a new exact sequence involving the group of quasi Hopf algebra extensions of a finite-dimensional abelian Singer pair [22].

In this paper we introduce a cohomology theory for general abelian matched pairs (T, N, μ, ν), consisting of two cocommutative Hopf algebras acting compatibly on each other with bismash product $H = N \bowtie T$, and obtain a general Kac sequence

$$\begin{aligned}
0 &\to H^1(H, A) \to H^1(T, A) \oplus H^1(N, A) \to \mathcal{H}^1(T, N, A) \to H^2(H, A) \\
&\to H^2(T, A) \oplus H^2(N, A) \to \mathcal{H}^2(T, N, A) \to H^3(H, A) \to \cdots
\end{aligned}$$

relating the cohomology $\mathcal{H}^*(T, N, A)$ of the matched pair with coefficients in a module algebra A to the Sweedler cohomologies of the Hopf algebras involved. For trivial coefficients the maps in the low degree part of the sequence are described explicitly. If T is finite-dimensional then abelian matched pairs (T, N, μ, ν) are in bijective correspondence with abelian Singer pairs (N, T^*), and we get a natural isomorphism

$\mathcal{H}^*(T,N,k) \cong H^*(N,T^*)$ between the cohomology of the abelian matched pair and that of the corresponding abelian Singer pair. In particular, together with results from [12] one obtains

$$\mathcal{H}^1(T,N,k) \cong H^1(N,T^*) \cong \mathrm{Aut}(T^* \# N)$$

and

$$\mathcal{H}^2(T,N,k) \cong H^2(N,T^*) \cong \mathrm{Opext}(N,T^*).$$

The sequence gives information about extensions of cocommutative Hopf algebras by commutative ones. It can also be used in certain cases to compute the (low degree) cohomology groups of Hopf algebras.

Such a sequence can of course not exist for non-abelian matched pairs, at least if the sequence is to consist of groups and not just pointed sets as in [27].

Together with the five term exact sequence for a smash product of Hopf algebras $H = N \rtimes T$ [17], generalizing that of K. Tahara [31] for a semi-direct product of groups,

$$\begin{aligned} 1 \to H^1_{\mathrm{meas}}\big(T, \mathrm{Hom}(N,A)\big) &\to \widetilde{H}^2(H,A) \to H^2(N,A)^T \\ &\to H^2_{\mathrm{meas}}\big(T, \mathrm{Hom}(N,A)\big) \to \widetilde{H}^3(H,A) \end{aligned}$$

it is possible in principle to give a procedure to compute the second cohomology group of any abelian matched pair of pointed Hopf algebras over a field of characteristic zero with a finite group of points and a reductive Lie algebra of primitives.

In Section 1 abelian Singer pairs of Hopf algebras are reviewed. In particular we talk about the cohomology of an abelian Singer pair, about Sweedler cohomology and Hopf algebra extensions [28,29].

In the second section abelian matched pairs of Hopf algebras are discussed. We introduce a cohomology theory for an abelian matched pair of Hopf algebras with coefficients in a commutative module algebra, and in Section 4 we see how it compares to the cohomology of an abelian Singer pair.

The generalized Kac sequence for an abelian matched pair of Hopf algebra is presented in Section 5. The homomorphisms in the low degree part of the sequence are given explicitly, so as to make it possible to use them in explicit calculations of groups of Hopf algebra extensions and low degree Sweedler cohomology groups.

Section 6 examines how the tools introduced combined with some additional observations can be used to describe explicitly the second cohomology group of some abelian matched pairs.

In the appendix some results from (co-)simplicial homological algebra used in the main body of the paper are presented.

Throughout the paper ${}_H\mathcal{V}$, ${}_H\mathcal{A}$ and ${}_H\mathcal{C}$ denote the categories of left H-modules, H-module algebras and H-module coalgebras, respectively, for the Hopf algebra H over the field k. Similarly, $\mathcal{V}^H$, $\mathcal{A}^H$ and $\mathcal{C}^H$ stand for the categories of right H-comodules, H-comodule algebras and H-comodule coalgebras, respectively.

We use the Sweedler sigma notation for comultiplication: $\Delta(c) = c_1 \otimes c_2$, $(1 \otimes \Delta)\Delta(c) = c_1 \otimes c_2 \otimes c_3$ etc. In the cocommutative setting the indices are clear from the context and we will omit them whenever convenient.

If V is a vector space, then V^n denotes its n-fold tensor power.

1. Cohomology of an abelian Singer pair

1.1. Abelian Singer pairs

Let (B, A) be a pair of Hopf algebras together with an action $\mu : B \otimes A \to A$ and a coaction $\rho : B \to B \otimes A$ so that A is a B-module algebra and B is an A-comodule coalgebra. Then $A \otimes B$ can be equipped with the cross product algebra structure as well as the cross product coalgebra structure. To ensure compatibility of these structures, i.e., to get a Hopf algebra, further conditions on (B, A, μ, ρ) are necessary. These are most easily expressed in term of the action of B on $A \otimes A$, twisted by the coaction of A on B,

$$\mu_2 = \big(\mu \otimes \mathrm{m}_A(1 \otimes \mu)\big)(14235)\big((\rho \otimes 1)\Delta_B \otimes 1 \otimes 1\big) : B \otimes A \otimes A \to A \otimes A,$$

i.e., $b(a \otimes a') = b_{1B}(a) \otimes b_{1A} \cdot b_2(a')$, and the coaction of A on $B \otimes B$, twisted by the action of B on A,

$$\rho_2 = \big(1 \otimes 1 \otimes \mathrm{m}_A(1 \otimes \mu)\big)(14235)\big((\rho \otimes 1)\Delta_B \otimes \rho\big) : B \otimes B \to B \otimes B \otimes A,$$

i.e., $\rho_2(b \otimes b') = b_{1B} \otimes b'_B \otimes b_{1A} \cdot b_2(b'_A)$.

Observe that for trivial coaction $\rho : B \to B \otimes A$ one gets the ordinary diagonal action of B on $A \otimes A$, and for trivial action $\mu : B \otimes A \to A$ the diagonal coaction of A on $B \otimes B$.

Definition 1.1. The pair (B, A, μ, ρ) is called an abelian Singer pair if A is commutative, B is cocommutative and the following are satisfied.

(1) (A, μ) is a B-module algebra (i.e., an object of ${}_B\mathcal{A}$),
(2) (B, ρ) is a A-comodule coalgebra (i.e., an object of $\mathcal{C}^A$),
(3) $\rho \mathrm{m}_B = (\mathrm{m}_B \otimes 1)\rho_2$, i.e., the diagram

$$\begin{tikzcd} B \otimes B \arrow[r, "\mathrm{m}_B"] \arrow[d, "\rho_2"'] & B \arrow[d, "\rho"] \\ B \otimes B \otimes A \arrow[r, "\mathrm{m}_B \otimes 1"] & B \otimes A \end{tikzcd}$$

commutes,

(4) $\Delta_A\mu = \mu_2(1\otimes\Delta_A)$, i.e., the diagram

$$\begin{tikzcd} B\otimes A \arrow[r, "\mu"] \arrow[d, "1\otimes\Delta_A"'] & A \arrow[d, "\Delta_A"] \\ B\otimes B\otimes A \arrow[r, "\mu_2"] & A\otimes A \end{tikzcd}$$

commutes.

The twisted action of B on A^n and the twisted coaction of A on B^n can now be defined inductively:

$$\mu_{n+1} = \big(\mu_n \otimes \mathrm{m}_A(1\otimes\mu)\big)(14235)\big((\rho\otimes 1)\Delta_B \otimes 1^n \otimes 1\big) : B\otimes A^n \otimes A \to A^n \otimes A$$

with $\mu_1 = \mu$ and

$$\rho_{n+1} = \big(1\otimes 1^n \otimes \mathrm{m}_A(1\otimes\mu)\big)(14235)\big((\rho\otimes 1)\Delta_B \otimes \rho_n\big) : B\otimes B^n \to B\otimes B^n \otimes A$$

with $\rho_1 = \rho$.

1.2. (Co-)modules over abelian Singer pairs

It is convenient to introduce the abelian category ${}_B\mathcal{V}^A$ of triples (V,ω,λ), where

(1) $\omega: B\otimes V \to V$ is a left B-module structure,
(2) $\lambda: V\to V\otimes A$ is a right A-comodule structure and
(3) the two equivalent diagrams

$$\begin{tikzcd} B\otimes V \arrow[r, "\omega"] \arrow[d, "1\otimes\lambda"'] & V \arrow[d, "\lambda"] \\ B\otimes V\otimes A \arrow[r, "\omega_{V\otimes A}"] & V\otimes A, \end{tikzcd} \qquad \begin{tikzcd} B\otimes V \arrow[r, "\omega"] \arrow[d, "\lambda_{B\otimes V}"'] & B \arrow[d, "\lambda"] \\ B\otimes V\otimes A \arrow[r, "\omega\otimes 1"] & V\otimes A \end{tikzcd}$$

commute, where the twisted action $\omega_{V\otimes A}: B\otimes V\otimes A \to V\otimes A$ of B on $V\otimes A$ is given by $\omega_{V\otimes A} = (\omega\otimes \mathrm{m}_A(1\otimes\mu))(14235)((\rho\otimes 1)\Delta_B\otimes 1\otimes 1)$ and the twisted coaction $\lambda_{B\otimes V}: B\otimes V\to B\otimes V\otimes A$ of A on $B\otimes V$ by $\lambda_{B\otimes V} = (1\otimes 1\otimes \mathrm{m}_A(1\otimes\mu))(14235)((\rho\otimes 1)\otimes\Delta_B\otimes\lambda)$.

It has been recently pointed out to us that these objects are a special case of entwined modules as introduced by Brzeziński and Majid in [3] (see also [2,4]).

The morphisms are B-linear and A-colinear maps. Observe that (B,m_B,ρ), (A,μ,Δ_A) and $(k,\varepsilon_B\otimes 1, 1\otimes\iota_A)$ are objects of ${}_B\mathcal{V}^A$. Moreover, $({}_B\mathcal{V}^A,\otimes,k)$ is a symmetric monoidal category, so that commutative algebras and cocommutative coalgebras are defined in $({}_B\mathcal{V}^A,\otimes,k)$ [11].

The free functor $F:\mathcal{V}^A\to {}_B\mathcal{V}^A$, defined by $F(X,\alpha)=(B\otimes X,\alpha_{B\otimes X})$ with twisted A-coaction $\alpha_{B\otimes X}=(1\otimes 1\otimes \mathrm{m}_A(1\otimes\mu))(14235)((\rho\otimes 1)\Delta_B\otimes\alpha)$ is left adjoint to the forgetful functor $U:{}_B\mathcal{V}^A\to\mathcal{V}^A$, with natural isomorphism $\theta:{}_B\mathcal{V}^A(FM,N)\to\mathcal{V}^A(M,UN)$ given by $\theta(f)(m)=f(1\otimes m)$ and $\theta^{-1}(g)(n\otimes m)=\mu_N(n\otimes g(m))$. The unit $\eta_M:M\to UF(M)$ and the counit $\varepsilon_N:FU(N)\to N$ of the adjunction are given by $\eta_M=\iota_B\otimes 1$ and $\varepsilon_N=\mu_N$, respectively, and give rise to a comonad $\mathbf{G}=(FU,\varepsilon,\delta=F\eta U)$.

Similarly, the cofree functor $L:{}_B\mathcal{V}\to {}_B\mathcal{V}^A$, defined by $L(Y,\beta)=(Y\otimes A,\beta_{Y\otimes A})$ with twisted B-action $\beta_{Y\otimes A}=(\beta\otimes \mathrm{m}_A(1\otimes\mu))(14235)((\rho\otimes 1)\Delta_B\otimes 1\otimes 1)$ is right adjoint to the forgetful functor $U:{}_B\mathcal{V}^A\to {}_B\mathcal{V}$, with natural isomorphism $\psi:{}_B\mathcal{V}(UM,N)\to {}_B\mathcal{V}^A(M,LN)$ given by $\psi(g)=(1\otimes g)\delta_M$ and $\psi^{-1}(f)=(1\otimes\varepsilon_A)f$. The unit $\eta_M:M\to LU(M)$ and the counit $\varepsilon_N:UL(N)\to N$ of the adjunction are given by $\eta_M=\delta_M$ and $\varepsilon_N=1\otimes\varepsilon_A$, respectively. They give rise to a monad (or triple) $\mathbf{T}=(LU,\eta,\mu=L\varepsilon U)$ on ${}_B\mathcal{V}^A$. The (non-commutative) square of functors

$$\begin{tikzcd} \mathcal{V} \arrow[r, "L"] \arrow[d, "F"'] & \mathcal{V}^A \arrow[d, "F"'] \\ {}_B\mathcal{V} \arrow[r, "L"] & {}_B\mathcal{V}^A \end{tikzcd}$$

together with the corresponding forgetful adjoint functors describes the situation. Observe that ${}_B\mathcal{V}^A(G(M),T(N))\cong\mathcal{V}(UM,UN)$. These adjunctions, monads and comonads restrict to coalgebras and algebras.

1.3. Cohomology of an abelian Singer pair

The comonad $\mathbf{G}=(FU,\varepsilon,\delta=F\eta U)$ defined on ${}_B\mathcal{V}^A$ can be used to construct B-free simplicial resolutions $\mathbf{X}_B(N)$ with $X_n(N)=G^{n+1}N=B^{n+1}\otimes N$, faces and degeneracies

$$\partial_i=G^i\varepsilon_{G^{n-i}(N)}:X_{n+1}\to X_n,\qquad s_i=G^i\delta_{G^{n-i}(N)}:X_n\to X_{n+1}$$

given by $\partial_i=1^i\otimes \mathrm{m}_B\otimes 1^{n+1-i}$ for $0\leqslant i\leqslant n$, $\partial_{n+1}=1^{n+1}\otimes\mu_N$, and $s_i=1^i\otimes\iota_B\otimes 1^{n+2-i}$ for $0\leqslant i\leqslant n$.

The monad $\mathbf{T}=(LU,\eta,\mu=L\varepsilon U)$ on ${}_B\mathcal{V}^A$ can be used to construct A-cofree cosimplicial resolutions $\mathbf{Y}_A(M)$ with $Y_A^n(M)=T^{n+1}M=M\otimes A^{n+1}$, cofaces and codegeneracies

$$\partial^i=T^{n+1-i}\eta_{T^i(M)}:Y^n\to Y^{n+1},\qquad s^i=T^{n_i}\mu_{T^i(M)}:Y^{n+1}\to Y^n$$

given by $\partial^0=\delta_M\otimes 1^{n+1}$, $\partial^i=1^{i-1}\otimes\Delta_A\otimes 1^{n+2-i}$ for $1\leqslant i\leqslant n+1$, and $s^i=1^{i+1}\otimes\varepsilon_A\otimes 1^{n+1-i}$ for $0\leqslant i\leqslant n$.

The total right derived functor [6] of

$${}_B\mathrm{Reg}^A=\mathcal{U}\,{}_B\mathrm{Hom}^A:\left({}_B\mathcal{C}^A\right)^{\mathrm{op}}\times {}_B\mathcal{A}^A\to \mathrm{Ab}$$

is now defined by means of the simplicial $\mathbf{G}$-resolutions $\mathbf{X}_B(M) = \mathbf{G}^{*+1}M$ and the cosimplicial $\mathbf{T}$-resolutions $\mathbf{Y}_A(N) = \mathbf{T}^{*+1}N$ as

$$R^*\big({}_B\mathrm{Reg}^A(M, N)\big) = H^*\big(\mathrm{Tot}\,{}_B\mathrm{Reg}^A\big(\mathbf{X}_B(M), \mathbf{Y}_A(N)\big)\big).$$

Definition 1.2. The cohomology of an abelian Singer pair (B, A, μ, ρ) is given by

$$H^*(B, A) = H^{*+1}(\mathrm{Tot}\,\mathbf{Z}_0),$$

where $\mathbf{Z}_0$ is the double cochain complex obtained from the double cochain complex $\mathbf{Z} = {}_B\mathrm{Reg}^A(\mathbf{X}(k), \mathbf{Y}(k))$ by deleting the 0th row and the 0th column.

1.4. The normalized standard complex

Use the natural isomorphism

$${}_B\mathcal{V}^A\big(FU(M), LU(N)\big) \cong \mathcal{V}(UM, UN)$$

to get the standard double complex

$$Z^{m,n} = \big({}_B\mathrm{Reg}^A\big(G^{m+1}(k)\big), T^{n+1}(k), \partial', \partial\big) \cong \big(\mathrm{Reg}\big(B^m, A^n\big), \partial', \partial\big).$$

For computational purposes it is useful to replace this complex by the normalized standard complex Z_+, where $Z_+^{m,n} = \mathrm{Reg}_+(B^m, A^n)$ is the intersection of the degeneracies, consisting of all convolution invertible maps $f : B^m \to A^n$ satisfying $f(1 \otimes \cdots \otimes \eta\varepsilon \otimes \cdots \otimes 1) = \eta\varepsilon$ and $(1 \otimes \cdots \otimes \eta\varepsilon \otimes \cdots \otimes 1)f = \eta\varepsilon$. In more detail, the normalized standard double complex is of the form

$$\begin{array}{ccccccc}
\mathrm{Reg}_+(k,k) & \xrightarrow{\partial^{0,0}} & \mathrm{Reg}_+(B,k) & \xrightarrow{\partial^{1,0}} & \mathrm{Reg}_+(B^2,k) & \xrightarrow{\partial^{2,0}} & \mathrm{Reg}_+(B^3,k)\cdots \\
{\scriptstyle\partial_{0,0}}\big\downarrow & & {\scriptstyle\partial_{1,0}}\big\downarrow & & {\scriptstyle\partial_{2,0}}\big\downarrow & & {\scriptstyle\partial_{3,0}}\big\downarrow \\
\mathrm{Reg}_+(k,A) & \xrightarrow{\partial^{0,1}} & \mathrm{Reg}_+(B,A) & \xrightarrow{\partial^{1,1}} & \mathrm{Reg}_+(B^2,A) & \xrightarrow{\partial^{2,1}} & \mathrm{Reg}_+(B^3,k)\cdots \\
{\scriptstyle\partial_{0,1}}\big\downarrow & & {\scriptstyle\partial_{1,1}}\big\downarrow & & {\scriptstyle\partial_{2,1}}\big\downarrow & & {\scriptstyle\partial_{3,1}}\big\downarrow \\
\mathrm{Reg}_+(k,A^2) & \xrightarrow{\partial^{0,2}} & \mathrm{Reg}_+(B,A^2) & \xrightarrow{\partial^{1,2}} & \mathrm{Reg}_+(B^2,A^2) & \xrightarrow{\partial^{2,2}} & \mathrm{Reg}_+(B^3,A^2)\cdots \\
{\scriptstyle\partial_{0,2}}\big\downarrow & & {\scriptstyle\partial_{1,2}}\big\downarrow & & {\scriptstyle\partial_{2,2}}\big\downarrow & & {\scriptstyle\partial_{3,2}}\big\downarrow \\
\mathrm{Reg}_+(k,A^3) & \xrightarrow{\partial^{0,3}} & \mathrm{Reg}_+(B,A^3) & \xrightarrow{\partial^{1,3}} & \mathrm{Reg}_+(B^2,A^3) & \xrightarrow{\partial^{2,3}} & \mathrm{Reg}_+(B^3,A^3)\cdots \\
{\scriptstyle\partial_{0,3}}\big\downarrow & & {\scriptstyle\partial_{1,3}}\big\downarrow & & {\scriptstyle\partial_{2,3}}\big\downarrow & & {\scriptstyle\partial_{3,3}}\big\downarrow \\
\vdots & & \vdots & & \vdots & & \vdots
\end{array}$$

The coboundary maps

$$d^i_{n,m} : \mathrm{Reg}_+\big(B^n, A^m\big) \to \mathrm{Reg}_+\big(B^{n+1}, A^m\big)$$

defined by

$$d^0_{n,m}\alpha = \mu_m(1_B \otimes \alpha), \qquad d^i_{n,m}\alpha = \alpha(1_{B^{i-1}} \otimes \mathrm{m}_B \otimes 1_{B^{n-i}}), \qquad d^{n+1}_{n,m}\alpha = \alpha \otimes \varepsilon,$$

for $1 \leqslant i \leqslant n$, are used to construct the horizontal differentials

$$\partial_{n,m} : \mathrm{Reg}_+\big(B^n, A^m\big) \to \mathrm{Reg}_+\big(B^{n+1}, A^m\big),$$

given by the 'alternating' convolution product

$$\partial_{n,m}\alpha = d^0_{n,m}\alpha * d^1_{n,m}\alpha^{-1} * d^2_{n,m}\alpha * \cdots * d^{n+1}_{n,m}\alpha^{(-1)^{n+1}}.$$

Dually the coboundaries

$$d'^i_{n,m} : \mathrm{Reg}_+\big(B^n, A^m\big) \to \mathrm{Reg}_+\big(B^n, A^{m+1}\big)$$

defined by

$$d'^0_{n,m}\beta = (\beta \otimes 1_A)\rho_n, \qquad d'^i_{n,m}\beta = (1_{A^{i-1}} \otimes \Delta_A \otimes 1_{A^{n-i}})\beta, \qquad d'^{n+1}_{n,m}\beta = \eta \otimes \beta,$$

for $1 \leqslant i \leqslant n$, determine the vertical differentials

$$\partial^{n,m} : \mathrm{Reg}_+\big(B^n, A^m\big) \to \mathrm{Reg}_+\big(B^n, A^{m+1}\big),$$

where

$$\partial^{n,m}\beta = d'^0_{n,m}\beta * d'^1_{n,m}\beta^{-1} * d'^2_{n,m}\beta * \cdots * d'^{n+1}_{n,m}\beta^{(-1)^{n+1}}.$$

The cohomology of the abelian Singer pair (B, A, μ, ρ) is by definition the cohomology of the total complex.

$$\begin{aligned} 0 &\to \mathrm{Reg}_+(B, A) \to \mathrm{Reg}_+\big(B^2, A\big) \oplus \mathrm{Reg}_+\big(B, A^2\big) \\ &\to \cdots \to \bigoplus_{i=1}^{n} \mathrm{Reg}_+\big(B^{n+1-i}, A^i\big) \to \cdots \end{aligned}$$

There are canonical isomorphisms $H^1(B, A) \simeq \mathrm{Aut}(A \# B)$, $H^2(B, A) \simeq \mathrm{Opext}(B, A)$ [12] (here $\mathrm{Opext}(B, A) = \mathrm{Opext}(B, A, \mu, \rho)$ denotes the abelian group of equivalence classes of those Hopf algebra extensions that give rise to the abelian Singer pair (B, A, μ, ρ)).

1.5. Special cases

In particular, for $A = k = M$ and N a commutative B-module algebra we get Sweedler cohomology of B with coefficients in N [29]

$$H^*(B, N) = H^*\big(\mathrm{Tot}\ {}_B\mathrm{Reg}\big(\mathbf{X}(k), N\big)\big) = H^*\big(\mathrm{Tot}\ {}_B\mathrm{Reg}\big(\mathbf{G}^{*+1}(k), N\big)\big).$$

In [29] it is also shown that if G is a group and $\mathbf{g}$ is a Lie algebra, then there are canonical isomorphisms $H^n(kG, A) \simeq H^n(G, \mathcal{U}(A))$ for $n \geqslant 1$ and $H^m(U\mathbf{g}, A) \simeq H^m(\mathbf{g}, A^+)$ for $m \geqslant 2$, where $\mathcal{U}(A)$ denotes the multiplicative group of units and A^+ denotes the underlying vector space.

For $B = k = N$ and M a cocommutative A-comodule coalgebra we get the dual version [7,29]

$$H^*(M, A) = H^*\big(\mathrm{Tot}\,\mathrm{Reg}^A\big(M, \mathbf{Y}(k)\big)\big) = H^*\big(\mathrm{Tot}\,\mathrm{Reg}^A\big(M, \mathbf{T}^{*+1}(k)\big)\big).$$

2. Cohomology of an abelian matched pair

2.1. Abelian matched pairs

Here we consider pairs of cocommutative Hopf algebras (T, N) together with a left action $\mu : T \otimes N \to N$, $\mu(t \otimes n) = t(n)$, and a right action $\nu : T \otimes N \to T$, $\nu(t \otimes n) = t^n$. Then we have the twisted switch

$$\tilde{\sigma} = (\mu \otimes \nu)\Delta_{T\otimes N} : T \otimes N \to N \otimes T$$

or, in shorthand $\tilde{\sigma}(t \otimes n) = t_1(n_1) \otimes t_2^{n_2}$, which in case of trivial actions reduces to the ordinary switch $\sigma : T \otimes N \to N \otimes T$.

Definition 2.1. Such a configuration (T, N, μ, ν) is called an abelian matched pair if

(1) N is a left T-module coalgebra, i.e., $\mu : T \otimes N \to N$ is a coalgebra map,
(2) T is a right N-module coalgebra, i.e., $\nu : T \otimes N \to T$ is a coalgebra map,
(3) N is a left T-module algebra with respect to the twisted left action $\tilde{\mu} = (1 \otimes \mu)(\tilde{\sigma} \otimes 1) : T \otimes N \otimes N \to N$, in the sense that the diagrams

$$\begin{tikzcd} T \otimes N \otimes N \arrow[r, "1\otimes \mathrm{m}_N"] \arrow[d, "\tilde{\mu}"'] & T \otimes N \arrow[d, "\mu"] \\ N \otimes N \arrow[r, "m_N"] & N, \end{tikzcd} \qquad \begin{tikzcd} T \otimes k \arrow[r, "1\otimes \iota_N"] \arrow[d, "\varepsilon_T\otimes 1"'] & T \otimes N \arrow[d, "\mu"] \\ k \arrow[r, "\iota_N"] & N \end{tikzcd}$$

commute, i.e., $\mu(t \otimes nm) = \sum \mu(t_1 \otimes n_1)\mu(\nu(t_2 \otimes n_2) \otimes m)$ and $\mu(t \otimes 1) = \varepsilon(t)1_N$, or in shorthand $t(nm) = t_1(n_1)t_2^{n_2}(m)$ and $t(1_N) = \varepsilon(t)1_N$,

(4) T is a right N-module algebra with respect to the twisted right action $\tilde{\nu} = (\nu \otimes 1)(1 \otimes \tilde{\sigma}) : T \otimes T \otimes N \to T \otimes T$, in the sense that the diagrams

$$\begin{tikzcd} T \otimes T \otimes N \arrow[r, "m_T \otimes 1"] \arrow[d, "\tilde{\nu}"'] & T \otimes N \arrow[d, "\nu"'] \\ T \otimes T \arrow[r, "m_T"] & T, \end{tikzcd} \qquad \begin{tikzcd} k \otimes N \arrow[r, "\iota_T \otimes 1"] \arrow[d, "1 \otimes \varepsilon_N"'] & T \otimes N \arrow[d, "\nu"'] \\ k \arrow[r, "\iota_T"] & T \end{tikzcd}$$

commute, i.e., $\nu(ts \otimes n) = \sum \nu(t \otimes \mu(s_1 \otimes n_1))\nu(s_2 \otimes n_2)$ and $\nu(1_T \otimes n) = \varepsilon(n)1_T$, or in shorthand $(ts)^n = t^{s_1(n_1)} s_2^{n_2}$ and $1_T^n = \varepsilon(n)1_T$.

The bismash product Hopf algebra $(N \bowtie T, \mathrm{m}, \Delta, \iota, \varepsilon, S)$ is the tensor product coalgebra $N \otimes T$ with unit $\iota_{N \otimes T} : k \to N \otimes T$, twisted multiplication

$$m = (m \otimes \mathrm{m})(1 \otimes \tilde{\sigma} \otimes 1) : N \otimes T \otimes N \otimes T \to N \otimes T,$$

in short $\tilde{\sigma}(t \otimes n) = t_1(n_1) \otimes t_2^{n_2}$, $(n \otimes t)(m \otimes s) = nt_1(m_1) \otimes t_2^{m_2} s$, and antipode

$$S = \tilde{\sigma}(S \otimes S)\sigma : N \otimes T \to N \otimes T,$$

i.e., $S(n \otimes t) = S(t_2)(S(n_2)) \otimes S(t_1)^{S(n_1)}$. For a proof that this is a Hopf algebra see [14]. To avoid ambiguity we will often write $n \bowtie t$ for $n \otimes t$ in $N \bowtie T$. We also identify N and T with the Hopf subalgebras $N \bowtie k$ and $k \bowtie T$, respectively, i.e., $n \equiv n \bowtie 1$ and $t \equiv 1 \bowtie t$. In this sense we write $n \bowtie t = nt$ and $tn = t_1(n_1)t_2^{n_2}$.

If the action $\nu : T \otimes N \to T$ is trivial, then the bismash product $N \bowtie T$ becomes the smash product (or semi-direct product) $N \rtimes T$. An action $\mu : T \otimes N \to N$ is compatible with the trivial action $1 \otimes \varepsilon : T \otimes N \to T$, i.e., $(T, N, \mu, 1 \otimes \varepsilon)$ is a matched pair, if and only if N is a T-module bialgebra and $\mu(t_1 \otimes n) \otimes t_2 = \mu(t_2 \otimes n) \otimes t_1$. Note that the last condition is trivially satisfied if T is cocommutative.

To make calculations more transparent we start to use the abbreviated Sweedler sigma notation for the cocommutative setting whenever convenient.

Lemma 2.2 [21, Proposition 2.3]. *Let (T, N, μ, ν) be an abelian matched pair.*

(1) *A left T-module, left N-module (V, α, β) is a left $N \bowtie T$-module if and only if $t(nv) = t(n)(t^n(v))$, i.e., if and only if with the twisted action $\tilde{\alpha} = (1 \otimes \alpha)(\tilde{\sigma} \otimes 1) : T \otimes N \otimes V \to N \otimes V$ the square*

$$\begin{tikzcd} T \otimes N \otimes V \arrow[r, "1 \otimes \beta"] \arrow[d, "\tilde{\alpha}"'] & T \otimes V \arrow[d, "\alpha"'] \\ N \otimes V \arrow[r, "\beta"] & V \end{tikzcd}$$

commutes.

(2) *A right T-module, right N-module (W, α, β) is a right $N \bowtie T$-module if and only if $(v^t)^n = (v^{t(n)})^{t^n}$, i.e., if and only if with the twisted action $\tilde{\beta} = (\beta \otimes 1)(1 \otimes \tilde{\sigma}) : W \otimes T \otimes N \to W \otimes T$ the square*

$$\begin{array}{ccc} W \otimes T \otimes N & \xrightarrow{\alpha \otimes 1} & W \otimes N \\ {\scriptstyle \tilde{\beta}} \downarrow & & \downarrow {\scriptstyle \beta} \\ W \otimes T & \xrightarrow{\alpha} & W \end{array}$$

commutes.

(3) *Let (V, α) be a left T-module and (W, β) a right N-module. Then*

(i) *$N \otimes V$ is a left $N \bowtie T$-module with N-action on the first factor and T-action given by*

$$\tilde{\alpha} = (1 \otimes \alpha)\tilde{\sigma} : T \otimes N \otimes V \to N \otimes V,$$

that is $t(n \otimes v) = t_1(n_1) \otimes t_2^{n_2}(v)$.

(ii) *$W \otimes T$ is a right $N \bowtie T$-module with T-action on the right factor and N-action given by*

$$\tilde{\beta} = (\beta \otimes 1)(1 \otimes \tilde{\sigma}) : W \otimes T \otimes N \to W \otimes T,$$

that is $(w \otimes t)^n = w^{t_2(n_2)} \otimes t_1^{n_1}$. Moreover, $W \otimes T$ is a left $N \bowtie T$-module by twisting the action via the antipode of $N \bowtie T$.

(iii) *The map $\psi : (N \bowtie T) \otimes V \otimes W \to (W \otimes T) \otimes (N \otimes V)$ defined by $\psi((n \bowtie t) \otimes v \otimes w) = w^{S(t)(S(n))} \otimes S(t)^{S(n)} \otimes n \otimes tv$, is a $N \bowtie T$-homomorphism, when $N \bowtie T$ is acting on the first factor of $(N \bowtie T) \otimes V \otimes W$ and diagonally on $(W \otimes T) \otimes (N \otimes V)$ by $(nt)(w \otimes s \otimes \mathrm{m} \otimes v) = w^{(sS(t))(S(n))} \otimes (sS(t))^{S(n)} \otimes nt(m) \otimes t^m(v)$.*
In particular, $(W \otimes T) \otimes (N \otimes V)$ is a free left $N \bowtie T$-module in which any basis of the vector space $(W \otimes k) \otimes (k \otimes V)$ is a $N \bowtie T$-free basis.

Observe that the inverse of $\psi : (N \bowtie T) \otimes V \otimes W \to (W \otimes T) \otimes (N \otimes V)$ is given by

$$\psi^{-1}\big((w \otimes t) \otimes (n \otimes v)\big) = \big(n \bowtie S(t^n)\big) \otimes \big(w^{t(n)} \otimes t^n(v)\big).$$

The twisted actions can now be extended by induction to higher tensor powers

$$\mu_{p+1} = (1 \otimes \mu_p)\big(\tilde{\sigma} \otimes 1^p\big) : T \otimes N^{p+1} \to N^{p+1}$$

so that $\mu_{p+1}(t \otimes n \otimes \mathbf{m}) = \mu(t \otimes n) \otimes \mu_p(\nu(t \otimes n) \otimes \mathbf{m})$, $t(n \otimes \mathbf{m}) = t(n) \otimes t^n(\mathbf{m})$ and

$$\nu_{q+1} = (\nu_q \otimes 1)\big(1^q \otimes \tilde{\sigma}\big) : T^{q+1} \otimes N \to T^{q+1}$$

so that $\nu_{q+1}(\mathbf{t}\otimes s\otimes n)=\nu_q(\mathbf{t}\otimes\mu(s\otimes n))\otimes\nu(s\otimes n)$, $(\mathbf{t}\otimes s)^n=\mathbf{t}^{s(n)}\otimes s^n$. Observe that the squares

$$\begin{array}{ccc} T\otimes N^{p+1} & \xrightarrow{\mu_{p+1}} & N^{p+1} \\ {\scriptstyle 1\otimes f}\downarrow & & \downarrow{\scriptstyle f} \\ T\otimes N^p & \xrightarrow{\mu_p} & N^p, \end{array} \qquad \begin{array}{ccc} T^{q+1}\otimes N & \xrightarrow{\nu_{q+1}} & T^{q+1} \\ {\scriptstyle g\otimes 1}\downarrow & & \downarrow{\scriptstyle g} \\ T^q\otimes N & \xrightarrow{\nu_q} & T^q \end{array}$$

commute when $f=1^{i-1}\otimes \mathrm{m}_N\otimes 1^{p-i}$ for $1\leqslant i\leqslant p$ and $g=1^{j-1}\otimes \mathrm{m}_T\otimes 1^{q-j}$ for $1\leqslant j\leqslant q$, respectively.

By part 3 (iii) of the lemma above $T^{i+1}\otimes N^{j+1}$ can be equipped with the $N\bowtie T$-module structure defined by $(nt)(\mathbf{r}\otimes s\otimes \mathrm{m}\otimes\mathbf{k})=\mathbf{r}^{(sS(t))(S(n))}\otimes(sS(t))^{S(n)}\otimes nt(m)\otimes t^m(\mathbf{k})$.

Corollary 2.3. *The map $\psi:(N\bowtie T)\otimes T^i\otimes N^j\to T^{i+1}\otimes N^{j+1}$, defined by $\psi((nt)\otimes(\mathbf{r}\otimes\mathbf{k}))=\mathbf{r}^{S(t)S(n)}\otimes S(t)^{S(n)}\otimes n\otimes t(\mathbf{k})$, is an isomorphism of $N\bowtie T$-modules.*

The content of the Lemma 2.2 can be summarized in the square of 'free' functors between monoidal categories

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{F_T} & {}_T\mathcal{V} \\ {\scriptstyle F_N}\downarrow & & \downarrow{\scriptstyle \widetilde{F}_N} \\ {}_N\mathcal{V} & \xrightarrow{\widetilde{F}_T} & {}_{N\bowtie T}\mathcal{V} \end{array}$$

each with a corresponding tensor preserving right adjoint forgetful functor.

2.2. The distributive law of a matched pair

The two comonads on ${}_{N\bowtie T}\mathcal{V}$ given by

$$\widetilde{\mathbf{G}}_T=(\widetilde{G}_T,\delta_T,\varepsilon_T),\qquad \widetilde{\mathbf{G}}_N=(\widetilde{G}_N,\delta_N,\varepsilon_N)$$

with $\widetilde{G}_T=\widetilde{F}_T\widetilde{U}_T$, $\delta_T(t\otimes x)=t\otimes 1\otimes x$, $\varepsilon_T(t\otimes x)=tx$, and with $\widetilde{G}_N=\widetilde{F}_N\widetilde{U}_N$, $\delta_N(n\otimes x)=n\otimes 1\otimes x$, $\varepsilon_N(n\otimes x)=nx$, satisfy a distributive law [1]

$$\tilde{\sigma}:\widetilde{G}_T\widetilde{\mathbf{G}}_N\to\widetilde{\mathbf{G}}_N\widetilde{G}_T$$

given by $\tilde{\sigma}(t\otimes n\otimes -)=\tilde{\sigma}(t\otimes n)\otimes -=t_1(n_1)\otimes t_2^{n_2}\otimes -$. The equations for a distributive law

$$\widetilde{G}_N\delta_T\cdot\tilde{\sigma}=\tilde{\sigma}\widetilde{G}_T\cdot\widetilde{G}_T\tilde{\sigma}\cdot\delta_T\widetilde{G}_N,\qquad \delta_N\widetilde{G}_T\cdot\tilde{\sigma}=\widetilde{G}_N\tilde{\sigma}\cdot\tilde{\sigma}\widetilde{G}_N\cdot\widetilde{G}_T\delta_N$$

and

$$\varepsilon_N \widetilde{G}_T \cdot \tilde{\sigma} = \widetilde{G}_T \varepsilon_N, \qquad \widetilde{G}_N \varepsilon_T \cdot \tilde{\sigma} = \varepsilon_T \widetilde{G}_N$$

are easily verified.

Proposition 2.4 [1, Theorem 2.2]. *The composite*

$$\mathbf{G} = \mathbf{G}_N \circ_{\tilde{\sigma}} \mathbf{G}_T$$

with $G = (G_N G_T,\ \delta = G_N \tilde{\sigma} G_T \cdot \delta_N \delta_T$ *and* $\varepsilon = \varepsilon_N \varepsilon_T)$ *is again a comonad on* ${}_{N \bowtie T}\mathcal{V}$*. Moreover,* $\mathbf{G} = \mathbf{G}_{N \bowtie T}$.

The antipode can be used to define a left action

$$\nu_S = S\nu(S \otimes S)\sigma : N \otimes T \to T$$

by $n(t) = \nu_S(n \otimes t) = S\nu(S \otimes S)\sigma(n \otimes t) = S(S(t)^{S(n)})$ and a right action

$$\mu_S = S\mu(S \otimes S)\sigma : N \otimes T \to N$$

by $n^t = \mu_S(n \otimes t) = S\mu(S \otimes S)\sigma(n \otimes t) = S(S(t)(S(n))$. The inverse of the twisted switch is then

$$\tilde{\sigma}^{-1} = (\nu_S \otimes \mu_S)\Delta_{N \otimes T} : N \otimes T \to T \otimes N$$

given by $\tilde{\sigma}^{-1}(n \otimes t) = n_1(t_1) \otimes n_2^{t_2}$, and induces the inverse distributive law

$$\tilde{\sigma}^{-1} : G_N G_T \to G_T G_N.$$

2.3. Matched pair cohomology

For every cocommutative Hopf algebra H the category of H-modules ${}_H\mathcal{V}$ is symmetric monoidal. The tensor product of two H-modules V and W has underlying vector space the ordinary vector space tensor product $V \otimes W$ and diagonal H-action. Algebras and coalgebras in ${}_H\mathcal{V}$ are known as H-module algebras and H-module coalgebras, respectively. The adjoint functors and comonads of the last section therefore restrict to the situations where $\mathcal{V}$ is replaced by $\mathcal{C}$ or $\mathcal{A}$. In particular, if (T, N, μ, ν) is an abelian matched pair, $H = N \bowtie T$ and C is a H-module coalgebra then $\mathbf{X}_H(C)$ is a canonical simplicial free H-module coalgebra resolution of C and by the Corollary 2.3 the composite $\mathbf{X}_N(\mathbf{X}_T(C))$ is a simplicial double complex of free H-module coalgebras.

Definition 2.5. The cohomology of an abelian matched pair (T, N, μ, ν) with coefficients in the commutative $N \bowtie T$-module algebra is defined by

$$\mathcal{H}^*(T, N, A) = H^{*+1}\big(\mathrm{Tot}(\mathbf{B}_0)\big),$$

where $\mathbf{B}_0$ is the double cochain complex obtained from the double cochain complex $\mathbf{B} = C({}_{N\bowtie T}\operatorname{Reg}(\mathbf{X}_N(\mathbf{X}_T(k), A)))$ by deleting the 0th row and the 0th column.

2.4. The normalized standard complex

Let $H = N \bowtie T$ be a bismash product of an abelian matched pair of Hopf algebras and let the algebra A be a left N and a right T-module such that it is a left H-module via $nt(a) = n(a^{S(t)})$, i.e., $(n(a))^{S(t)} = (t(n))(a^{S(t^n)})$.

Note that $\operatorname{Hom}(T^p, A)$ becomes a left N-module via $n(f)(\mathbf{t}) = n(f(\nu_p(\mathbf{t}, n)))$ and $\operatorname{Hom}(N^q, A)$ becomes a right T-module via $f^t(\mathbf{n}) = (f(\mu_q(t, \mathbf{n})))^t = S(t)(f(\mu_q(t, \mathbf{n})))$.

The simplicial double complex $G_T^p G_N^q(k) = (T^p \otimes N^q)_{p,q}$, $p, q \geqslant 1$, of free H-modules has horizontal face operators $1 \otimes d_N^* : T^p \otimes N^{q+1} \to T^p \otimes N^q$, vertical face operators $d_T^* \otimes 1 : T^{p+1} \otimes N^q \to T^p \otimes N^q$, horizontal degeneracies $1 \otimes s_N^* : T^p \otimes N^q \to T^p \otimes N^{q+1}$ and vertical degeneracies $s_T^* \otimes 1 : T^p \otimes N^q \to T^{p+1} \otimes N^q$, where

$$d_N^i = 1^i \otimes \mathrm{m} \otimes 1^{q-i-1}, \qquad d_N^q = 1^q \otimes \varepsilon, \qquad s_N^i = 1^i \otimes \eta \otimes 1^{q-i}$$

for $0 \leqslant i \leqslant q-1$, and

$$d_T^j = 1^{p-j-1} \otimes \mathrm{m} \otimes 1^j, \qquad d_T^p = \varepsilon \otimes 1^p, \qquad s_T^j = 1^{p-j} \otimes \eta \otimes 1^j$$

for $0 \leqslant j \leqslant p-1$.

These maps preserve the H-module structure on $T^p \otimes N^q$. Apply the functor ${}_{\mathrm{H}}\operatorname{Reg}(\_, A) : {}_H\mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ to get a cosimplicial double complex of abelian groups

$$\mathbf{B} = {}_{\mathrm{H}}\operatorname{Reg}\big(\mathbf{X}_N\big(\mathbf{X}_T(k), A\big)\big)$$

with $B^{p,q} = {}_{\mathrm{H}}\operatorname{Reg}(T^{p+1} \otimes N^{q+1}, A)$, coface operators ${}_{\mathrm{H}}\operatorname{Reg}(d_{N*}, A)$, ${}_{\mathrm{H}}\operatorname{Reg}(d_{T*}, A)$ and codegeneracies are ${}_{\mathrm{H}}\operatorname{Reg}(s_{N*}, A)$, ${}_{\mathrm{H}}\operatorname{Reg}(s_{T*}, A)$.

The isomorphism described in Corollary 2.3 induces an isomorphism of double complexes $\mathbf{B}(T, N, A) \cong \mathbf{C}(T, N, A)$ given by

$${}_{\mathrm{H}}\operatorname{Reg}\big(T^{p+1} \otimes N^{q+1}, A\big) \xrightarrow{{}_{\mathrm{H}}\operatorname{Reg}(\psi, A)} {}_{\mathrm{H}}\operatorname{Reg}\big(H \otimes T^p \otimes N^q, A\big) \xrightarrow{\theta} \operatorname{Reg}\big(T^p \otimes N^q, A\big)$$

for $p, q \geqslant 0$, where $C^{p,q} = \operatorname{Reg}(T^p \otimes N^q, A)$ is the abelian group of convolution invertible linear maps $f : N^p \otimes T^q \to A$.

The vertical differentials $\delta_N : C^{p,q} \to C^{p+1,q}$ and the horizontal differentials

$$\delta_T : C^{p,q} \to C^{p,q+1}$$

are transported from $\mathbf{B}$ and turn out to be the twisted Sweedler differentials on the N and T parts, respectively. The coface operators are

$$\delta_{Ni} f(\mathbf{t} \otimes \mathbf{n}) = \begin{cases} sf(\mathbf{t} \otimes n_1 \otimes \cdots \otimes n_i n_{i+1} \otimes \cdots \otimes n_{q+1}), & \text{for } i = 1, \ldots, q, \\ n_1(f(\nu_q(\mathbf{t} \otimes n_1) \otimes n_2 \otimes \cdots \otimes n_{p+1})), & \text{for } i = 0, \\ f(\mathbf{t} \otimes n_1 \otimes \cdots \otimes n_q)\varepsilon(n_{q+1}), & \text{for } i = q+1, \end{cases}$$

where $\mathbf{t} \in T^p$ and $\mathbf{n} = n_1 \otimes \cdots \otimes n_{q+1} \in N^{q+1}$, and similarly

$$\delta_{Tj} f(\mathbf{t} \otimes \mathbf{n}) = \begin{cases} sf(t_{p+1} \otimes \cdots \otimes t_{j+1}t_j \otimes \cdots \otimes t_1 \otimes \mathbf{n}), & \text{for } j = 1, \ldots, p, \\ (f(t_{p+1} \otimes \cdots \otimes t_2 \otimes \mu_p(t_1 \otimes \mathbf{n})))^{t_1}, & \text{for } j = 0, \\ \varepsilon(t_{p+1}) f(t_p \otimes \cdots \otimes t_1 \otimes \mathbf{n}), & \text{for } j = p+1, \end{cases}$$

where $\mathbf{t} = t_1 \otimes \cdots \otimes t_{q+1} \in T^{q+1}$ and $\mathbf{n} \in N^q$. The differentials in the associated double cochain complex are the alternating convolution products

$$\delta_N f = \delta_{N0} f * \delta_{N1} f^{-1} * \cdots * \delta_{Nq+1} f^{\pm 1}$$

and

$$\delta_T f = \delta_{T0} f * \delta_{T1} f^{-1} * \cdots * \delta_{Tp+1} f^{\pm 1}.$$

In the associated normalized double complex $\mathbf{C}_+$, the (p,q) term $C_+^{p,q} = \mathrm{Reg}_+(T^p \otimes N^q, A)$ is the intersection of the degeneracy operators, that is, the abelian group of convolution invertible maps $f : T^p \otimes N^q \to A$ with $f(t_p \otimes \cdots \otimes t_1 \otimes n_1 \otimes \cdots \otimes n_q) = \varepsilon(t_p) \ldots \varepsilon(n_q)$, whenever one of t_i or one of n_j is in k. Then $\mathcal{H}^*(N, T, A) \cong H^{*+1}(\mathrm{Tot}\,\mathbf{C}_0)$, where $\mathbf{C}_0$ is the double complex obtained from $\mathbf{C}_+$ by replacing the edges by zero.

The groups of cocycles $\mathcal{Z}^i(T, N, A)$ and the groups coboundaries $\mathcal{B}^i(T, N, A)$ consist of i-tuples of maps $(f_j)_{1 \leqslant j \leqslant i}$, $f_j : T^j \otimes N^{i+1-j} \to A$ that satisfy certain conditions.

We introduce the subgroups $\mathcal{Z}_p^i(T, N, A) \leqslant \mathcal{Z}^i(T, N, A)$, that are spanned by i-tuples in which the f_j's are trivial for $j \neq p$ and subgroups $\mathcal{B}_p^i = \mathcal{Z}_p^i \cap \mathcal{B}^i \subset \mathcal{B}_i$. These give rise to subgroups of cohomology groups $\mathcal{H}_p^i = \mathcal{Z}_p^i / \mathcal{B}_p^i \simeq (\mathcal{Z}_p^i + \mathcal{B}^i)/\mathcal{B}^i \subseteq \mathcal{H}^i$ which have a nice interpretation when $i = 2$ and $p = 1, 2$; see Section 4.3.

3. The homomorphism $\pi : \mathcal{H}^2(T, N, A) \to H^{1,2}(T, N, A)$

If T is a finite group and N is a finite T-group, then we have the following exact sequence [16]

$$H^2(N, k^\bullet) \xrightarrow{\delta_T} \mathrm{Opext}\big(kT, k^N\big) \xrightarrow{\pi} H^1\big(T, H^2(N, k^\bullet)\big).$$

Here we define a version of homomorphism π for arbitrary smash products of cocommutative Hopf algebras.

We start by introducing the Hopf algebra analogue of $H^i(T, H^j(N, k^\bullet))$. For positive i, j and an abelian matched pair of Hopf algebras (T, N), with the action of N on T trivial, we define

$$\begin{aligned} Z^{i,j}(T, N, A) = \big\{\alpha \in \mathrm{Reg}_+\big(T^i \otimes N^j, A\big) \mid \delta_N \alpha = \varepsilon, \text{ and } \\ \exists \beta \in \mathrm{Reg}_+\big(T^{i+1} \otimes N^{j-1}, A\big)\colon\ \delta_T \alpha = \delta_N \beta\big\}, \\ B^{i,j}(T, N, A) = \big\{\alpha \in \mathrm{Reg}_+\big(T^i \otimes N^j, A\big) \mid \exists \gamma \in \mathrm{Reg}_+\big(T^i \otimes N^{j-1}, A\big), \\ \exists \gamma' \in \mathrm{Reg}_+\big(T^{i-1} \otimes N^j, A\big)\colon\ \alpha = \delta_N \gamma * \delta_T \gamma'\big\}, \end{aligned}$$

$$H^{i,j}(T,N,A) = Z^{i,j}(T,N,A)/B^{i,j}(T,N,A).$$

Remark. If $j = 1$, then

$$H^{i,1}(T,N,A) \simeq \mathcal{H}_i^i(T,N,A) \simeq H^i_{\mathrm{meas}}\bigl(T, \mathrm{Hom}(N,A)\bigr),$$

where the H^i_{meas} denotes the measuring cohomology [17].

Proposition 3.1. *If $T = kG$ is a group algebra, then there is an isomorphism*

$$H^i\bigl(G, H^j(N,A)\bigr) \simeq H^{i,j}(kG,N,A).$$

Remark. Here the *right* action of G on $H^j(N,A)$ is given by precomposition. We can obtain symmetric results in case we start with a *right* action of $T = kG$ on N, hence a *left* action of G on $H^j(N,A)$.

Proof of Proposition 3.1. By inspection we have

$$Z^i\bigl(G, H^j(N,A)\bigr) = Z^{i,j}(kG,N,A)/\bigl\{\alpha : G \to B^j(N,A)\bigr\},$$
$$B^i\bigl(G, H^j(N,A)\bigr) = B^{i,j}(kG,N,A)/\bigl\{\alpha : G \to B^j(N,A)\bigr\}.$$

Here we identify regular maps from $(kG)^i \otimes N^j$ to A with set maps from $G^{\times i}$ to $\mathrm{Reg}(N^j, A)$ in the obvious way. □

The following is a straightforward generalization of Theorem 7.1 in [17].

Theorem 3.2. *The homomorphism $\pi : \mathcal{H}^2(T,N,A) \to H^{1,2}(T,N,A)$, induced by $(\alpha,\beta) \mapsto \alpha$, makes the following sequence*

$$H^2(N,A) \oplus \mathcal{H}^2_2(T,N,A) \xrightarrow{\delta_T + \iota} \mathcal{H}^2(T,N,A) \xrightarrow{\pi} H^{1,2}(T,N,A)$$

exact.

Proof. It is clear that $\pi\delta_T = 0$ and obviously also $\pi(\mathcal{H}^2_2) = 0$.

Suppose a cocycle pair $(\alpha,\beta) \in \mathcal{Z}^2(T,N,A)$ is such that $\alpha \in B^{1,2}(T,N,A)$. Then for some $\gamma : T \otimes N \to A$ and some $\gamma' : N \otimes N \to A$ we have $\alpha = \delta_N\gamma * \delta_T\gamma'$, and hence $(\alpha,\beta) = (\delta_N\gamma, \beta) * (\delta_T\gamma', \varepsilon) \sim (\delta_N\gamma^{-1}, \delta_T\gamma) * (\delta_N\gamma,\beta) * (\delta_T\gamma', \varepsilon) = (\varepsilon, \delta_T\gamma * \beta) * (\delta_T\gamma', \varepsilon) \in \mathcal{Z}^2_2(T,N,A) * \delta_T(Z^2(N,A))$. □

4. Comparison of abelian Singer pairs and abelian matched pairs

4.1. Abelian Singer pairs vs. abelian matched pairs

In this section we sketch a correspondence from abelian matched pairs to abelian Singer pairs. For more details we refer to [21].

Definition 4.1. We say that an action $\mu : A \otimes M \to M$ is locally finite, if every orbit $A(m) = \{a(m) \mid a \in A\}$ is finite-dimensional.

Lemma 4.2 [25, Lemma 1.6.4]. *Let A be an algebra and C a coalgebra.*

(1) *If M is a right C-comodule via $\rho : M \to M \otimes C$, $\rho(m) = m_0 \otimes \mathrm{m}_1$, then M is a left C^*-module via $\mu : C^* \otimes M \to M$, $\mu(f \otimes \mathrm{m}) = f(m_1)m_0$.*
(2) *Let M be a left A-module via $\mu : A \otimes M \to M$. Then there is (a unique) comodule structure $\rho : M \to M \otimes A^\circ$, such that $(1 \otimes \mathrm{ev})\rho = \mu$ if and only if the action μ is locally finite. The coaction is then given by $\rho(m) = \sum f_i \otimes \mathrm{m}_i$, where $\{m_i\}$ is a basis for $A(m)$ and $f_i \in A^\circ \subseteq A^*$ are coordinate functions of $a(m)$, i.e., $a(m) = \sum f_i(a)m_i$.*

Let (T, N, μ, ν) be an abelian matched pair and suppose $\mu : T \otimes N \to N$ is locally finite. Then the lemma above gives a coaction $\rho : N \to N \otimes T^\circ$, $\rho(n) = n_N \otimes n_{T^\circ}$, such that $t(n) = \sum n_N \cdot n_{T^\circ}(t)$.

There is a left action $\nu' : N \otimes T^* \to T^*$ given by pre-composition, i.e., $\nu'(n \otimes f)(t) = f(t^n)$. If μ is locally finite, it is easy to see that ν' restricts to $T^\circ \subseteq T^*$.

Lemma 4.3 [21, Lemma 4.1]. *If (T, N, μ, ν) is an abelian matched pair with μ locally finite then the quadruple (N, T°, ν', ρ) forms an abelian Singer pair.*

Remark. There is also a correspondence in the opposite direction [18].

4.2. Comparison of abelian Singer and abelian matched pair cohomologies

Let (T, N, μ, ν) be an abelian matched pair of Hopf algebras, with μ locally finite and (N, T°, ν', ρ) the abelian Singer pair associated to it as above.

The embedding $\mathrm{Hom}(N^i, (T^\circ)^j) \subseteq \mathrm{Hom}(N^i, (T^j)^*) \simeq \mathrm{Hom}(T^j \otimes N^i, k)$ induced by the inclusion $T^{\circ j} = (T^j)^\circ \subseteq (T^j)^*$ restricts to the embedding $\mathrm{Reg}_+(N^i, (T^\circ)^j) \subseteq \mathrm{Reg}_+(T^j \otimes N^i, k)$. A routine calculation shows that it preserves the differentials, i.e., that it gives an embedding of double complexes, which is an isomorphism in case T is finite-dimensional.

There is no apparent reason for the embedding of complexes to induce an isomorphism of cohomology groups in general. It is our conjecture that this is not always the case.

In some cases we can compare the multiplication part of $H^2(N, T^\circ)$ (see the following section) and $\mathcal{H}^2_2(N, T, k)$. We use the following lemma for this purpose.

Lemma 4.4. *Let (T, N, μ, ν) be an abelian matched pair with the action μ locally finite. If $f : T \otimes N^i \to k$ is a convolution invertible map, such that $\delta_T f = \varepsilon$, then for each $\mathbf{n} \in N^i$, the map $f_{\mathbf{n}} = f(\_, \mathbf{n}) : T \to k$ lies in the finite dual $T^\circ \subseteq T^*$.*

Proof. It suffices to show that the orbit of $f_{\mathbf{n}}$ under the action of T (given by $(s(f_{\mathbf{n}})(t) = f_{\mathbf{n}}(ts))$ is finite-dimensional (see [8,25,30] for the description of finite duals). Using the fact that $\delta_T f = \varepsilon$ we get $s(f_{\mathbf{n}})(t) = f_{\mathbf{n}}(ts) = \sum f_{\mathbf{n}_1}(s_1) f_{\mu_i(s_2 \otimes \mathbf{n}_2)}(t)$.

Let $\Delta(\mathbf{n}) = \sum_j \mathbf{n}'_j \otimes \mathbf{n}''_j$. The action $\mu_i : T \otimes N^i \to N^i$ is locally finite, since $\mu : T \otimes N \to N$ is, and hence we can choose a finite basis $\{\mathbf{m}_p\}$ for $\mathrm{Span}\{\mu_i(s \otimes \mathbf{n}''_j) \mid s \in T\}$. Now note that $\{f_{\mathbf{m}_p}\}$ is a finite set which spans $T(f_{\mathbf{n}})$. $\square$

Corollary 4.5. *If (T, N, μ, ν) is an abelian matched pair, with μ locally finite and $(N, T^\circ, \omega, \rho)$ is the corresponding abelian Singer pair, then $\mathcal{H}^1(T, N, k) = H^1(N, T^\circ)$.*

4.3. The multiplication and comultiplication parts of the second cohomology group of an abelian Singer pair

Here we discuss in more detail the Hopf algebra extensions that have an "unperturbed" multiplication and those that have an "unperturbed" comultiplication, more precisely we look at two subgroups $H^2_m(B, A)$ and $H^2_c(B, A)$ of $H^2(B, A) \simeq \mathrm{Opext}(B, A)$, one generated by the cocycles with a trivial multiplication part and the other generated by the cocycles with a trivial comultiplication part [16]. Let

$$Z^2_c(B, A) = \big\{\beta \in \mathrm{Reg}_+(B, A \otimes A) \mid (\eta\varepsilon, \beta) \in Z^2(B, A)\big\}.$$

We shall identify $Z^2_c(B, A)$ with a subgroup of $Z^2(B, A)$ via the injection $\beta \mapsto (\eta\varepsilon, \beta)$. Similarly let

$$Z^2_m(B, A) = \big\{\alpha \in \mathrm{Reg}_+(B \otimes B, A) \mid (\alpha, \eta\varepsilon) \in Z^2(B, A)\big\}.$$

If

$$B^2_c(B, A) = B^2(B, A) \cap Z^2_c(B, A) \text{ and } B^2_m(B, A) = B^2(B, A) \cap Z^2_m(B, A)$$

then we define

$$H^2_c(B, A) = Z^2_c(B, A)/B^2_c(B, A) \text{ and } H^2_m(B, A) = Z^2_m(B, A)/B^2_m(B, A).$$

The identification of $H^2_c(B, A)$ with a subgroup of $H^2(B, A)$ is given by

$$H^2_c(B, A) \xrightarrow{\sim} \big(Z^2_c(B, A) + B^2(B, A)\big)/B^2(B, A) \leqslant H^2(B, A),$$

and similarly for $H^2_m \leqslant H^2$.

Note that in case T is finite-dimensional $H^2_c(N, T^*) \simeq \mathcal{H}^2_2(T, N, k)$ and $H^2_m(N, T^*) \simeq \mathcal{H}^2_1(T, N, k)$ with $\mathcal{H}^i_p(T, N, k)$ as defined in Section 2.4.

Proposition 4.6. *Let (T, N, μ, ν) be an abelian matched pair, with μ locally finite and let $(N, T^\circ, \omega, \rho)$ be the corresponding abelian Singer pair. Then*

$$H_m^2(N, T^\circ) \simeq \mathcal{H}_1^2(T, N, k).$$

Proof. Observe that we have an inclusion $Z_m^2(N, T^\circ) = \{\alpha : N \otimes N \to T^\circ \mid \partial\alpha = \varepsilon, \partial'\alpha = \varepsilon\} \subseteq \{\alpha : T \otimes N \otimes N \to k \mid \delta_T\alpha = \varepsilon, \delta_N\alpha = \varepsilon\} = \mathcal{Z}_1^2(T, N, k)$. The inclusion is in fact an equality by Lemma 4.4. Similarly the inclusion $B_m^2(N, T^\circ) \subseteq \mathcal{B}_1^2(T, N, k)$ is an equality as well. □

5. The generalized Kac sequence

5.1. The Kac sequence of an abelian matched pair

We now start by sketching a conceptual way to obtain a generalized version of the Kac sequence for an arbitrary abelian matched pair of Hopf algebras relating the cohomology of the matched pair to Sweedler cohomology. Since it is difficult to describe the homomorphisms involved in this manner, we then proceed in the next section to give an explicit description of the low degree part of this sequence.

Theorem 5.1. *Let $H = N \bowtie T$, where (T, N, μ, ν) be an abelian matched pair of Hopf algebras, and let A be a commutative left H-module algebra. Then there is a long exact sequence of abelian groups*

$$\begin{aligned} 0 &\to H^1(H, A) \to H^1(T, A) \oplus H^1(N, A) \to \mathcal{H}^1(T, N, A) \to H^2(H, A) \\ &\to H^2(T, A) \oplus H^2(N, A) \to \mathcal{H}^2(T, N, A) \to H^3(H, A) \to \cdots. \end{aligned}$$

Moreover, if T is finite-dimensional then (N, T^*) is an abelian Singer pair, $H^*(T, k) \cong H^*(k, T^*)$ and $\mathcal{H}^*(T, N, k) \cong H^*(N, T^*)$.

Proof. The short exact sequence of double cochain complexes

$$0 \to \mathbf{B}_0 \to \mathbf{B} \to \mathbf{B}_1 \to 0,$$

where $\mathbf{B}_1$ is the edge double cochain complex of $\mathbf{B} = {}_H\mathrm{Reg}(\mathbf{X}_T\mathbf{X}_N(k), A)$ as in Section 2.3, induces a long exact sequence in cohomology

$$\begin{aligned} 0 &\to H^1\big(\mathrm{Tot}(\mathbf{B})\big) \to H^1\big(\mathrm{Tot}(\mathbf{B}_1)\big) \to H^2\big(\mathrm{Tot}(\mathbf{B}_0)\big) \to H^2\big(\mathrm{Tot}(\mathbf{B})\big) \\ &\to H^2\big(\mathrm{Tot}(\mathbf{B}_1)\big) \to H^3\big(\mathrm{Tot}(\mathbf{B}_0)\big) \to H^3\big(\mathrm{Tot}(\mathbf{B})\big) \to H^3\big(\mathrm{Tot}(\mathbf{B}_1)\big) \to \cdots \end{aligned}$$

where $H^0(\mathrm{Tot}(\mathbf{B}_0)) = 0 = H^1(\mathrm{Tot}(\mathbf{B}_0))$ and $H^0(\mathrm{Tot}(\mathbf{B})) = H^0(\mathrm{Tot}(\mathbf{B}_1))$ have already been taken into account. By Definition 2.5 $H^{*+1}(\mathrm{Tot}(\mathbf{B}_0)) = \mathcal{H}^*(T, N, A)$ is the cohomology of the matched pair (T, N, μ, ν) with coefficients in A. Moreover, $H^*(\mathrm{Tot}(\mathbf{B}_1)) \cong H^*(T, A) \oplus H^*(N, A)$ is a direct sum of Sweedler cohomologies.

From the cosimplicial version of the Eilenberg–Zilber theorem (see Appendix A) it follows that $H^*(\mathrm{Tot}(\mathbf{B})) \cong H^*(\mathrm{Diag}(\mathbf{B}))$. On the other hand, Barr's theorem [1, Theorem 3.4] together with Corollary 2.3 says that $\mathrm{Diag}\,\mathbf{X}_T(\mathbf{X}_N(k)) \simeq \mathbf{X}_H(k)$, and gives an equivalence

$$_H\mathrm{Reg}\big(\mathrm{Diag}\,\mathbf{X}_T\big(\mathbf{X}_N(k)\big), A\big) \simeq \mathrm{Diag}\big(_H\mathrm{Reg}\big(\mathbf{X}_T\big(\mathbf{X}_N(k)\big), A\big) = \mathrm{Diag}(\mathbf{B})\big).$$

Thus, we get

$$H^*(H, A) = H^*\big(_H\mathrm{Reg}\big(\mathbf{X}_H(k), A\big)\big) \cong H^*\big(\mathrm{Diag}(\mathbf{B})\big) \cong H^*\big(\mathrm{Tot}(\mathbf{B})\big),$$

and the proof is complete. □

5.2. Explicit description of the low degree part

The aim of this section is to define explicitly homomorphisms that make the following sequence

$$\begin{aligned} 0 \;\to\; & H^1(H, A) \xrightarrow{\mathrm{res}_2} H^1(T, A) \oplus H^1(N, A) \xrightarrow{\delta_N * \delta_T} \mathcal{H}^1(T, N, A) \xrightarrow{\phi} H^2(H, A) \\ & \xrightarrow{\mathrm{res}_2} H^2(T, A) \oplus H^2(N, A) \xrightarrow{\delta_N * \delta_T^{-1}} \mathcal{H}^2(T, N, A) \xrightarrow{\psi} H^3(H, A). \end{aligned}$$

exact. This is the low degree part of the generalized Kac sequence. Here $H = N \bowtie T$ is the bismash product Hopf algebra arising from a matched pair $\mu : T \otimes N \to N$, $\nu : T \otimes N \to T$. Recall that we abbreviate $\mu(t, n) = t(n)$, $\nu(t, n) = t^n$. We shall also assume that A is a trivial H-module.

We define $\mathrm{res}_2 = \mathrm{res}_2^i : H^i(H, A) \to H^i(T, A) \oplus H^i(N, A)$ to be the map $(\mathrm{res}_T, \mathrm{res}_N)\Delta$, more precisely if $f : H^i \to A$ is a cocycle, then it gets sent to a pair of cocycles (f_T, f_N), where $f_T = f|_{T^i}$ and $f_N = f|_{N^i}$.

By $\delta_N * \delta_T^{(-1)^{i+1}}$, we denote the composite

$$\begin{aligned} H^i(T, A) \oplus H^i(N, A) & \xrightarrow{\delta_N \oplus \delta_T^{\pm 1}} \mathcal{H}_i^i(T, N, A) \oplus \mathcal{H}_1^i(T, N, A) \\ & \xrightarrow{\iota \oplus \iota} \mathcal{H}^i(T, N, A) \oplus \mathcal{H}^i(T, N, A) \xrightarrow{*} \mathcal{H}^i(T, N, A). \end{aligned}$$

When $i = 1$, the map just defined, sends a pair of cocycles $a \in Z^1(T, A)$, $b \in Z^1(N, A)$ to a map $\delta_N a * \delta_T b : T \otimes N \to A$ and if $i = 2$ a pair of cocycles $a \in Z^2(T, A)$, $b \in Z^2(N, A)$ becomes a cocycle pair $(\delta_N a, \varepsilon) * (\varepsilon, \delta_T b^{-1}) = (\delta_N a, \delta_T b^{-1}) : (T \otimes T \otimes N) \oplus (T \otimes N \otimes N) \to A$. Here δ_N and δ_T are the differentials for computing the cohomology of a matched pair described in Section 2.4.

The map $\phi : \mathcal{H}^1(T, N, A) \to H^2(H, A)$ assigns to a cocycle $\gamma : T \otimes N \to A$, a map $\phi(\gamma) : H \otimes H \to A$, which is characterized by $\phi(\gamma)(nt, n't') = \gamma(t, n')$.

The homomorphism $\psi : \mathcal{H}^2(T, N, A) \to H^3(H, A)$ is induced by a map that sends a cocycle pair $(\alpha, \beta) \in \mathcal{Z}^2(T, N, A)$ to the cocycle $f = \psi(\alpha, \beta) : H \otimes H \otimes H \to A$ given by

$$f\bigl(nt, n't', n''t''\bigr) = \varepsilon(n)\varepsilon\bigl(t''\bigr)\alpha\bigl(t^{n'}, t', n''\bigr)\beta\bigl(t, n', t'\bigl(n''\bigr)\bigr).$$

A direct, but lengthy computation shows that the maps just defined induce homomorphisms that make the sequence above exact [18]. The most important tool in computations is the following lemma about the structure of the second cohomology group $\mathcal{H}^2(H, A)$ [18].

Lemma 5.2. *Let $f : H \otimes H \to A$ be a cocycle. Define maps $g_f : H \to A$, $h : H \otimes H \to A$ and $f_c : T \otimes N \to A$ by $g_f(nt) = f(n \otimes t)$, $h = f * \delta g_f$ and $f_c(t \otimes n) = f(t \otimes n) f^{-1}(t(n) \otimes t^n)$. Then*

(1) $h(nt, n't') = f_T(t^{n'}, t') f_N(n, t'(n')) f_c(t, n')$,
(2) $h_T = f_T$, $h_N = f_N$, $h|_{N\otimes T} = \varepsilon$, $h|_{T\otimes N} = h_c = f_c$, $g_h = \varepsilon$,
(3) *the maps f_T and f_N are cocycles and $\delta_N f_T = \delta_T f_c^{-1}$, $\delta_T f_N = \delta_N f_c^{-1}$,*
(4) *if $a : T \otimes T \to A$, $b : N \otimes N \to A$ are cocycles and $\gamma : T \otimes N \to A$ is a convolution invertible map, such that $\delta_N a = \delta_T \gamma$ and $\delta_T b = \delta_N \gamma$, then the map $f = f_{a,b,\gamma} : H \otimes H \to A$, defined by*

$$f\bigl(nt, n't'\bigr) = a\bigl(t^{n'}, t'\bigr) b\bigl(n, t\bigl(n'\bigr)\bigr)\gamma^{-1}\bigl(t, n'\bigr)$$

is a cocycle and $f_T = a$, $f_N = b$, $f_c = f|_{T\otimes N} = \gamma^{-1}$ and $f|_{N\otimes T} = \varepsilon$.

5.3. The locally finite case

Suppose that the action $\mu : T \otimes N \to N$ is locally finite and let $(N, T^\circ, \omega, \rho)$ be the abelian Singer pair corresponding to the matched pair (T, N, μ, ν) as in Section 4.1.

By Corollary 4.5 we have $\mathcal{H}^1(T, N, k) = H^1(N, T^\circ)$.

From the explicit description of the generalized Kac sequence, we see that $(\delta_N * \delta_T^{-1})|_{H^2(T,A)} = \delta_N : H^2(T, A) \to \mathcal{H}_2^2(N, T, A)$ and similarly that $(\delta_N * \delta_T^{-1})|_{H^2(N,A)} = \delta_T^{-1} : H^2(N, A) \to \mathcal{H}_1^2(N, T, A)$. By Proposition 4.6 we have the equality $\mathcal{H}_1^2(T, N, k) = H_m^2(N, T^\circ)$. Recall that $H_m^2(N, T^\circ) \subseteq H^2(N, T^\circ) \simeq \operatorname{Opext}(N, T^\circ)$.

If the action ν is locally finite as well, then there is also a (right) Singer pair $(T, N^\circ, \omega', \rho')$. By 'right' we mean that we have a right action $\omega' : N^\circ \otimes T \to N^\circ$ and a right coaction $\rho' : T \to N^\circ \otimes T$. In this case we get that $\mathcal{H}_2^2(T, N, k) \simeq H_m^{2\,\prime}(T, N^\circ) \subseteq \operatorname{Opext}'(T, N^\circ)$. The dash refers to the fact that we have a right Singer pair.

Define $H_{mc}^2 = H_m^2 \cap H_c^2$ and $H_{mc}^{2\,\prime} = H_m^{2\,\prime} \cap H_c^{2\,\prime}$ and note $H_{mc}(N, T^\circ) \simeq \mathcal{H}_2^2(N, T, k) \cap \mathcal{H}_1^2(N, T, k) \simeq H_{mc}'^2(T, N^\circ)$. Hence

$$\begin{aligned}\operatorname{im}\bigl(\delta_N * \delta_T^{-1}\bigr) &\subseteq \mathcal{H}_1^2(T, N, k) + \mathcal{H}_2^2(T, N, k) \simeq \frac{\mathcal{H}_1^2(T, N, k) \oplus \mathcal{H}_2^2(T, N, k)}{\mathcal{H}_1^2(T, N, k) \cap \mathcal{H}_2^2(T, N, k)}\\ &= \frac{H_m^2(N, T^\circ) \oplus H_m^{2\,\prime}(T, N^\circ)}{\langle H_{mc}^2(N, T^\circ) \equiv H_{mc}^{2\,\prime}(T, N^\circ)\rangle}.\end{aligned}$$

In other words, $\mathrm{im}(\delta_N * \delta_T^{-1})$ is contained in a subgroup of $\mathcal{H}^2(T, N, k)$, that is isomorphic to the pushout

$$\begin{array}{ccc} H^{2\,\prime}_{mc}(T, N^\circ) \simeq H^2_{mc}(N, T^\circ) & \longrightarrow & H^2_m(N, T^\circ) \\ \downarrow & & \downarrow \\ H^{2\,\prime}_m(T, N^\circ) & \longrightarrow & X. \end{array}$$

Hence if both actions μ and ν of the abelian matched pair (T, N, μ, ν) are locally finite then we get the following version of the low degree part of the Kac sequence:

$$\begin{aligned} 0 \longrightarrow H^1(H,k) &\xrightarrow{\mathrm{res}_2} H^1(T,k) \oplus H^1(N,k) \xrightarrow{\delta_N * \delta_T} H^1(N, T^\circ) \xrightarrow{\phi} H^2(H,k) \\ &\xrightarrow{\mathrm{res}_2} H^2(T,k) \oplus H^2(N,k) \xrightarrow{\delta_N * \delta_T^{-1}} X \xrightarrow{\psi|_X} H^3(H,k). \end{aligned}$$

5.4. The Kac sequence of an abelian Singer pair

Here is a generalization of the Kac sequence relating Sweedler and Doi cohomology to Singer cohomology.

Theorem 5.3. *For any abelian Singer pair (B, A, μ, ρ) there is a long exact sequence*

$$\begin{aligned} 0 \to H^1(\mathrm{Tot}\, Z) &\to H^1(B,k) \oplus H^1(k, A) \to H^1(B, A) \to H^2(\mathrm{Tot}\, Z) \\ &\to H^2(B,k) \oplus H^2(k, A) \to H^2(B, A) \to H^3(\mathrm{Tot}\, Z) \to \cdots, \end{aligned}$$

where Z is the double complex from Definition 1.2. *Moreover, we always have $H^1(B, A) \cong \mathrm{Aut}(A \,\#\, B)$, $H^2(B, A) \cong \mathrm{Opext}(B, A)$ and $H^*(\mathrm{Tot}\, Z) \cong H^*(\mathrm{Diag}\, Z)$. If A is finite-dimensional then $H^*(\mathrm{Tot}\, Z) = H^*(A^* \bowtie B, k)$.*

Proof. The short exact sequence of double cochain complexes

$$0 \to Z_0 \to Z \to Z_1 \to 0,$$

where Z_1 is the edge subcomplex of $Z = {}_B\mathrm{Reg}^A(\mathbf{X}_B(k), \mathbf{Y}_A(k))$, induces a long exact sequence

$$\begin{aligned} 0 \to H^1(\mathrm{Tot}\, Z) &\to H^1(\mathrm{Tot}\, Z_1) \to H^2(\mathrm{Tot}\, Z_0) \to H^2(\mathrm{Tot}\, Z) \\ &\to H^2(\mathrm{Tot}\, Z_1) \to H^3(\mathrm{Tot}\, Z_0) \to H^3(\mathrm{Tot}\, Z) \to H^3(\mathrm{Tot}\, Z_0) \to \cdots \end{aligned}$$

where $H^0(\mathrm{Tot}\, Z_0) = 0 = H^1(\mathrm{Tot}\, Z_0)$ and $H^0(\mathrm{Tot}\, Z) = H^0(\mathrm{Tot}\, Z_1)$ have already been taken into account. By definition $H^*(\mathrm{Tot}\, Z_0) = H^*(B, A)$ is the cohomology of the abelian Singer pair (B, A, μ, ρ), and by [12] we have $H^1(B, A) \cong \mathrm{Aut}(A \,\#\, B)$ and $H^2(B, A) \cong \mathrm{Opext}(B, A)$. Moreover, we clearly have $H^*(\mathrm{Tot}\, Z_1) \cong H^*(B, k) \oplus H^*(k, A)$, where the summands are Sweedler and Doi cohomologies. By the cosimplicial Eilenberg–Zilber theorem (see Appendix A) there is a natural isomorphism $H^*(\mathrm{Tot}(\mathbf{Z})) \cong H^*(\mathrm{Diag}(\mathbf{Z}))$. Finally, if A is finite-dimensional then $\mathbf{Z} = {}_B\mathrm{Reg}^A(\mathbf{X}(k), \mathbf{Y}(k)) \cong {}_{A^* \bowtie B}\,\mathrm{Reg}(\mathbf{B}(k), k)$, where $\mathbf{B}(k) = \mathbf{X}_{A^*}(\mathbf{X}_B(k))$. □

6. On the matched pair cohomology of pointed cocommutative Hopf algebras over fields of zero characteristic

In this section we describe a method which gives information about the second cohomology group $\mathcal{H}^2(T, N, A)$ of an abelian matched pair.

6.1. The method

Let (T, N) be an abelian matched pair of pointed Hopf algebras, and A a trivial $N \bowtie T$-module algebra.

(1) Since $\mathrm{char}\, k = 0$ and T and N are pointed we have $T \simeq UP(T) \rtimes kG(T)$ and $N \simeq UP(N) \rtimes kG(N)$ and $N \bowtie T \simeq U(P(T) \bowtie P(N)) \rtimes k(G(T) \bowtie G(N))$ [9,10]. If H is a Hopf algebra then $G(H)$ denotes the group of points and $P(H)$ denotes the Lie algebra of primitives.

(2) We can use the generalized Tahara sequence [17] (see introduction) to compute $H^2(T)$, $H^2(N)$, $H^2(N \bowtie T)$. In particular if $G(T)$ is finite then the cohomology group $H^2_{\mathrm{meas}}(kG(T), \mathrm{Hom}(UP(T), A)) = H^{2,1}(kG(T), UP(T), A) = \mathcal{H}^2_2(kG(T), UP(T), A)$ is trivial and there is a direct sum decomposition $H^2(T) = H^2(P(T))^{G(T)} \oplus H^2(G(T))$; we get a similar decomposition for $H^2(N)$ if $G(N)$ is finite and for $H^2(N \bowtie T)$ in the case $G(T)$ and $G(N)$ are both finite.

(3) Since the Lie algebra cohomology groups $H^i(\mathbf{g})$ admit a vector space structure, the cohomology groups $H^{1,2}(G, \mathbf{g}, A) \simeq H^1(G, H^2(\mathbf{g}, A))$ are trivial if G is finite (any additive group of a vector space over a field of zero characteristic is uniquely divisible).

(4) The exactness of the sequence from Theorem 3.2 implies that the maps $\delta_T : H^2(G(\_)) \to \mathcal{H}^2(kG(\_), UP(\_), A)$ are surjective if $G(\_)$ is finite, hence by the generalized Kac sequence the kernels of the maps $\mathrm{res}^3_2 : H^3(\_) \to H^3(P(\_)) \oplus H^3(G(\_))$ are trivial. This then gives information about the kernel of the map $\mathrm{res}^3_2 : H^3(N \bowtie T) \to H^3(T) \oplus H^3(N)$.

(5) Now use the exactness of the generalized Kac sequence

$$H^2(N \bowtie T) \xrightarrow{\mathrm{res}^2_2} H^2(T) \oplus H^2(N) \xrightarrow{\delta_T + \delta_N^{-1}} \mathcal{H}^2(T, N, A)$$
$$\longrightarrow H^3(N \bowtie T) \xrightarrow{\mathrm{res}^3_2} H^3(T) \oplus H^3(N)$$

to get information about $\mathcal{H}^2(T, N, A)$.

6.2. Examples

Here we describe how the above procedure works on concrete examples.

In the first three examples we restrict ourselves to a case in which one of the Hopf algebras involved is a group algebra.

Let $T = UP(T) \rtimes kG(T)$ and $N = kG(N)$ and suppose that the matched pair of T and N arises from actions $G(T) \times G(N) \to G(N)$ and $(G(N) \rtimes G(T)) \times P(T) \to P(T)$. If the groups $G(T)$ and $G(N)$ are finite and their orders are relatively prime, then the generalized Kac sequence shows that there is an injective homomorphism

$$\Phi : \frac{H^2(P(T))^{G(T)}}{H^2(P(T))^{G(N)\rtimes G(T)}} \oplus \frac{H^2(G(N))}{H^2(G(N))^{G(T)}} \to \mathcal{H}^2(T, N, A).$$

Theorem 3.2 guarantees that the map $H^3(N \bowtie T) = H^3(U(P(T)) \rtimes k(G(N) \rtimes G(T))) \to H^3(P(T)) \oplus H^3(G(N) \rtimes G(T))$ is injective. Since the orders of $G(T)$ and $G(N)$ are assumed to be relatively prime the map $H^3(G(N) \rtimes G(T)) \to H^3(G(N)) \oplus H^3(G(T))$ is also injective. Hence the map

$$\mathrm{res}_2^3 : H^3(N \bowtie T) \to H^3(N) \oplus H^3(T)$$

must be injective as well, since the composite $H^3(N \bowtie T) \to H^3(N) \oplus H^3(T) \to H^3(G(N)) \oplus H^3(P(T)) \oplus H^3(G(T))$ is injective. Hence by the exactness of the generalized Kac sequence Φ is an isomorphism.

Example 6.1. Let $\mathbf{g} = k \times k$ be the abelian Lie algebra of dimension 2 and let $G = C_2 = \langle a \rangle$ be the cyclic group of order two. Furthermore assume that G acts on $\mathbf{g}$ by switching the factors, i.e., $a(x, y) = (y, x)$. Recall that $U\mathbf{g} = k[x, y]$ and that $H^i_{\mathrm{Sweedler}}(U\mathbf{g}, A) = H^i_{\mathrm{Hochschild}}(U\mathbf{g}, A)$ for $i \geqslant 2$ and that $H^i_{\mathrm{Hochschild}}(k[x, y], k) = k^{\oplus\binom{i}{2}}$. A computation shows that G acts on $k \simeq H^2(k[x, y], k)$ by $a(t) = -t$ and hence $H^2(k[x, y], k)^G = 0$. Thus the homomorphism π (Theorem 3.2) is the zero map and the homomorphism $k \simeq H^2(k[x, y], k) \xrightarrow{\delta_T} \mathcal{H}^2(kC_2, k[x, y], k)$ is an isomorphism.

Example 6.2 (*symmetries of a triangle*). Here we describe an example arising from the action of the dihedral group D_3 on the abelian Lie algebra of dimension 3 (basis consists of vertices of a triangle). More precisely let $\mathbf{g} = k \times k \times k$, $G = C_2 = \langle a \rangle$, $H = C_3 = \langle b \rangle$, the actions $G \times \mathbf{g} \to \mathbf{g}$, $H \times \mathbf{g} \to \mathbf{g}$ and $H \times G \to H$ are given by $a(x, y, z) = (z, y, x)$, $b(x, y, z) = (z, x, y)$ and $b^a = b^{-1}$ respectively. A routine computation reveals the following

- C_2 acts on $k \times k \times k \simeq H^2(k[x, y, z], k)$ by $a(u, v, w) = (-w, -v, -u)$, hence the G stable part is

$$H^2\big(k[x, y, z], k\big)^G = \big\{(u, 0, -u)\big\} \simeq k.$$

- $H = C_3$ acts on $k \times k \times k$ by $b(u, v, w) = (w, u, v)$ and the H stable part is $H^2(k[x, y, z], k)^H = \{(u, u, u)\} \simeq k$.
- The $D_3 = C_2 \rtimes C_3$ stable part $H^2(k[x, y, z], k)^{D_3}$ is trivial.

Thus we have an isomorphism $k \times k^\bullet/(k^\bullet)^3 \simeq \mathcal{H}^2(k[x, y, z] \rtimes kC_2, kC_3, k)$.

Remark. The above also shows that there is an isomorphism

$$k \times k \times k \simeq \mathcal{H}^2\bigl(k[x, y, z], kD_3, k\bigr).$$

Example 6.3. Let $\mathbf{g} = sl_n$, $G = C_2 = \langle a\rangle$, $H = C_n = \langle b\rangle$, where a is a matrix that has 1's on the skew diagonal and zeroes elsewhere and b is the standard permutation matrix of order n. Let H and G act on sl_n by conjugation in $\mathcal{M}_n$ and let G act on H by conjugation inside GL_n. Furthermore assume that A is a finite-dimensional trivial $U\mathbf{g} \rtimes k(H \rtimes G)$-module algebra. By Whitehead's second lemma $H^2(\mathbf{g}, A) = 0$ and hence we get an isomorphism $\mathcal{U}A/(\mathcal{U}A)^n \simeq \mathcal{H}^2(Usl_n \rtimes kC_2, kC_n, A)$ if n is odd.

Example 6.4. Let $H = U\mathbf{g} \rtimes kG$, where $\mathbf{g}$ is an abelian Lie algebra and G is a finite abelian group and assume the action of H on itself is given by conjugation, i.e., $h(k) = h_1 k S(h_2)$. In this case it is easy to see that $H^2(H, A)^H = H^2(H, A)$ for any trivial H-module algebra A and hence the homomorphism in the generalized Kac sequence $\delta_{H,1} \oplus \delta_{H,2} : H^2(H, A) \oplus H^2(H, A) \to \mathcal{H}^2(H, H, A)$ is trivial. Hence $\mathcal{H}^2(H, H, A) \simeq \ker(H^3(H \rtimes H, A) \to H^3(H, A) \oplus H^3(H, A))$.

Appendix A. Simplicial homological algebra

This is a collection of notions and results from simplicial homological algebra used in the main text. The emphasis is on the cohomology of cosimplicial objects, but the considerations are similar to those in the simplicial case [32].

A.1. Simplicial and cosimplicial objects

Let $\boldsymbol{\Delta}$ denote the simplicial category [23]. If $\mathcal{A}$ is a category then the functor category $\mathcal{A}^{\boldsymbol{\Delta}^{\mathrm{op}}}$ is the category of simplicial objects while $\mathcal{A}^{\boldsymbol{\Delta}}$ is the category of cosimplicial objects in $\mathcal{A}$. Thus a simplicial object in $\mathcal{A}$ is given by a sequence of objects $\{X_n\}$ together with, for each $n \geqslant 0$, face maps $\partial_i : X_{n+1} \to X_n$ for $0 \leqslant i \leqslant n+1$ and degeneracies $\sigma_j : X_n \to X_{n+1}$ for $0 \leqslant j \leqslant n$ such that

$$\partial_i\partial_j = \partial_{j-1}\partial_i \quad \text{for } i < j, \qquad \sigma_i\sigma_j = \sigma_{j+1}\sigma_i \quad \text{for } i \leqslant j,$$
$$\partial_i\sigma_j = \begin{cases} \sigma_{j-1}\partial_i, & \text{if } i < j,\\ 1, & \text{if } i = j, j+1,\\ \sigma_j\partial_{i-1}, & \text{if } i > j+1.\end{cases}$$

A cosimplicial object in $\mathcal{A}$ is a sequence of objects $\{X^n\}$ together with, for each $n \geqslant 0$, coface maps $\partial^i : X^n \to X^{n+1}$ for $0 \leqslant i \leqslant n+1$ and codegeneracies $\sigma^j : X^{n+1} \to X^n$ such that

$$\partial^j \partial^i = \partial^i \partial^{j-1} \quad \text{for } i < j, \qquad \sigma^j \sigma^i = \sigma^i \sigma^{j+1} \quad \text{for } i \leqslant j,$$
$$\sigma^j \partial^i = \begin{cases} \partial^i \sigma^{j-1}, & \text{if } i < j, \\ 1, & \text{if } i = j, j+1, \\ \partial^{i-1} \sigma^j, & \text{if } i > j+1. \end{cases}$$

Two cosimplicial maps $f, g : X \to Y$ are homotopic if for each $n \geqslant 0$ there is a family of maps $\{h^i : X^{n+1} \to Y^n \mid 0 \leqslant i \leqslant n\}$ in $\mathcal{A}$ such that

$$h^0 \partial^0 = f, \qquad h^n \partial^{n+1} = g,$$
$$h^j \partial^i = \begin{cases} \partial^i h^{j-1}, & \text{if } i < j, \\ h^{i-1} \partial^i, & \text{if } i = j \neq 0, \\ \partial^{i-1} h^j, & \text{if } i > j+1, \end{cases}$$
$$h^j \sigma^i = \begin{cases} \sigma^i h^{j+1}, & \text{if } i \leqslant j, \\ \sigma^{i-1} h^j, & \text{if } i > j. \end{cases}$$

Clearly, homotopy of cosimplicial maps is an equivalence relation.

If X is a cosimplicial object in an abelian category $\mathcal{A}$, then $C(X)$ denotes the associated cochain complex in $\mathcal{A}$, i.e., an object of the category of cochain complexes $\mathrm{Coch}(\mathcal{A})$.

Lemma A.1. *For a cosimplicial object X in the abelian category $\mathcal{A}$ let*

$$N^n(X) = \bigcap_{i=0}^{n-1} \ker \sigma^i \quad \textit{and} \quad D^n(X) = \sum_{j=0}^{n-1} \operatorname{im} \partial^j.$$

Then $C(X) \cong N(X) \oplus D(X)$. Moreover, $C(X)/D(X) \cong N(X)$ is a cochain complex with differentials given by $\partial^n : X^n/D^n \to X^{n+1}/D^{n+1}$, and $\pi^*(X) = H^*(N^*(X))$ is the sequence of cohomotopy objects of X.

Theorem A.2 (Cosimplicial Dold–Kan correspondence [32, 8.4.3]). *If $\mathcal{A}$ is an abelian category then*:

(1) $N : \mathcal{A}^{\boldsymbol{\Delta}} \to \mathrm{Coch}(\mathcal{A})$ *is an equivalence and $N(X)$ is a summand of $C(X)$.*
(2) $\pi^*(X) = H^*(N(X)) \cong H^*(C(X))$.
(3) *If $\mathcal{A}$ has enough injectives, then $\pi^* = H^*N : \mathcal{A}^{\boldsymbol{\Delta}} \to$* $\mathrm{Coch}(\mathcal{A})$ *and $H^*C : A^{\boldsymbol{\Delta}} \to$* $\mathrm{Coch}(\mathcal{A})$ *are the sequences of right derived functors of $\pi^0 = H^0 N : \mathcal{A}^{\boldsymbol{\Delta}} \to \mathcal{A}$ and $H^0 C : \mathcal{A}^{\boldsymbol{\Delta}} \to \mathcal{A}$, respectively.*

Proof. (1) If $y \in N^n(X) \cap D^n(X)$ then $y = \sum_{i=0}^{n-1} \partial^i(x_i)$, where each $x_i \in X^{n-1}$. Suppose that $y = \partial^0(x)$ and $y \in N^n(X)$, then $0 = \sigma^0(y) = \sigma^0 \partial^0(x) = x$ and hence $y = \partial^0(x) = 0$.

Now proceed by induction on the largest j such that $\partial^j(x_j) \neq 0$. So let $y = \sum_{i=0}^{j} \partial^i(x_i)$ such that $\partial^j(x_j) \neq 0$, i.e., $y \notin \sum_{i<j} \operatorname{im} \partial^i$, and $y \in N^n(X)$. Then

$$0 = \sigma^j(y) = \sum_{i \leqslant j} \sigma^j \partial^i(x_i) = x_j + \sum_{i<j} \sigma^j \partial^i(x_i) = x_j + \sum_{i<j} \partial^i \sigma^{j-1}(x_i).$$

This implies that $x_j = -\sum_{i<j} \partial^i \sigma^{j-1}(x_i)$ and hence

$$\partial^j(x_j) = -\sum_{i<j} \partial^j \partial^i \sigma^{j-1}(x_i) = -\sum_{i<j} \partial^i \partial^{j-1} \sigma^{j-1}(x_i) \in \sum_{i<j} \operatorname{im} \partial^i,$$

a contradiction. Thus, $N^n(X) \cap D^n(X) = 0$.

Now let us show that $D^n(X) + N^n(X) = C^n(X)$. Suppose that $y = \partial^0(x)$ for some $x \in X_{n-1}$ and $y \in N^n(x) = \bigcap_{i=0}^{n-1} \ker \sigma^i$. Then $0 = \sigma^0(y) = \sigma^0 \partial^0(x) = x$, so that $\sigma^i(y) \neq 0$. If $y' = y - \partial^i \sigma^i(y)$ then $y - y' \in D^n(X)$. For $i < j$ we get $\sigma^j(y') = \sigma^j(y) - \sigma^j \partial^i \sigma^i(y) = \sigma^j(y) - \partial^i \sigma^{j-1} \sigma^i(y) = \sigma^j(y) - \partial^i \sigma^i \sigma^j(y) = 0$. Moreover, $\sigma^i(y') = \sigma^i(y) - \sigma^i \partial^i \sigma^i(y) = \sigma^i(y) - \sigma^i(y) = 0$, so that $i-1$ is the largest index for which $\sigma^{i-1} y' \neq 0$. By induction, there is a $z \in D^n(X)$ such that $y - z \in N^n(X)$, and hence $y \in D^n(X) + N^n(X)$.

It now follows that

$$\bigcap_{i=0}^{n-1} \ker \sigma^i = N^n(X) \cong X^n / D^n(X) = X^n \Big/ \sum_{i=0}^{n-1} \operatorname{im} \partial^i.$$

The differential $\partial^n : N^n(X) \to N^{n+1}(X)$ is given by $\partial^n(x + D^n(X)) = \partial^n(x) + D^{n+1}(X)$.

(2) By definition, see [32, 8.4.3].

(3) The functors $N : \mathcal{A}^{\boldsymbol{\Delta}} \to \operatorname{Coch}(\mathcal{A})$ and $C : \mathcal{A}^{\boldsymbol{\Delta}} \to \operatorname{Coch}(\mathcal{A})$ are exact. $\square$

The inverse equivalence $K : \operatorname{Coch}(\mathcal{A}) \to \mathcal{A}^{\boldsymbol{\Delta}}$ has a description, similar to that for the simplicial case [32, 8.4.4].

A.2. *Cosimplicial bicomplexes*

The category of cosimplicial bicomplexes in the abelian category $\mathcal{A}$ is the functor category $\mathcal{A}^{\boldsymbol{\Delta} \times \boldsymbol{\Delta}} = (\mathcal{A}^{\boldsymbol{\Delta}})^{\boldsymbol{\Delta}}$. In particular, in a cosimplicial bicomplex $X = \{X^{p,q}\}$ in $\mathcal{A}$

(1) horizontal and vertical cosimplicial identities are satisfied,
(2) horizontal and vertical cosimplicial operators commute.

The associated (unnormalized) cochain bicomplex $C(X)$ with $C(X)^{p,q} = X^{p,q}$ has horizontal and vertical differentials

$$d_h = \sum_{i=0}^{p+1} (-1)^i \partial_h^i : X_{p,q} \to X^{p+1,q}, \qquad d_v = \sum_{j=0}^{q+1} (-1)^{p+j} \partial_v^j : X^{p,q} \to X^{p,q+1}$$

so that $d_h d_v = d_v d_h$. The normalized cochain bicomplex $N(X)$ is obtained from X by taking the normalized cochain complex of each row and each column. It is a summand of CX. The cosimplicial Dold–Kan theorem then says that $H^{**}(CX) \cong H^{**}(NX)$ for every cosimplicial bicomplex.

The diagonal $\mathrm{diag}: \Delta \to \Delta \times \Delta$ induces the diagonalization functor

$$\mathrm{Diag} = \mathcal{A}^{\mathrm{diag}}: \mathcal{A}^{\Delta\times\Delta} \to \mathcal{A}^{\Delta},$$

where $\mathrm{Diag}^p(X) = X^{p,p}$ with coface maps $\partial^i = \partial_h^i \partial_v^i : X^{p,p} \to X^{p+1,p+1}$ and codegeneracies $\sigma^j = \sigma_h^j \sigma_v^j : X^{p+1,p+1} \to X^{p,p}$ for $0 \leqslant i \leqslant p+1$ and $0 \leqslant j \leqslant p$, respectively.

Theorem A.3 (The cosimplicial Eilenberg–Zilber theorem). *Let $\mathcal{A}$ be an abelian category with enough injectives. There is a natural isomorphism*

$$\pi^*(\mathrm{Diag}\, X) = H^*\big(C\,\mathrm{Diag}(X)\big) \cong H^*\big(\mathrm{Tot}(X)\big),$$

where $\mathrm{Tot}(X)$ denotes the total complex associated to the double cochain complex CX. Moreover, there is a convergent first quadrant cohomological spectral sequence

$$E_1^{p,q} = \pi_v^q\big(X^{p,*}\big), \qquad E_2^{p,q} = \pi_h^p \pi_v^q(X) \Rightarrow \pi^{p+q}(\mathrm{Diag}\, X).$$

Proof. It suffices to show that $\pi^0\,\mathrm{Diag} \cong H^0(\mathrm{Tot}\, X)$, and that

$$\pi^*\,\mathrm{Diag},\ H^*\,\mathrm{Tot}: \mathcal{A}^{\Delta\times\Delta} \to \mathcal{A}^{\mathbf{N}}$$

are sequences of right derived functors.

First observe that $\pi^0(\mathrm{Diag}\, X) = \mathrm{eq}(\partial_h^0\partial_v^0, \partial_h^0\partial_v^0 : X^{0,0} \to X^{1,1})$, while $H^0(\mathrm{Tot}(X)) = \ker((\partial_h^0 - \partial_h^1, \partial_v^0 - \partial_v^1): X^{0,0} \to X^{10} \oplus X^{01})$. But $\partial_h^0\partial_v^0 x = \partial_h^1\partial_v^1 x$ implies that $\partial_v^0 x = \sigma_h^0\partial_h^0\partial_v^0 x = \sigma_h^0\partial_h^1\partial_v^1 x = \partial_v^1 x$, since $\sigma_h^0\partial_h^0 = 1 = \sigma_h^0\partial_h^1$, and similarly $\partial_h^0 x = \sigma_v^0\partial_h^0\partial_v^0 x = \sigma_v^0\partial_h^1\partial_v^1 x = \partial_h^1 x$, since $\sigma_v^0\partial_v^0 = 1 = \sigma_v^0\partial_v^1$, so that $\pi^0(\mathrm{Diag}\, X) \subseteq H^0(\mathrm{Tot}(X))$.

Conversely, if $\partial_h^0 x = \partial_h^1 x$ and $\partial_v^0 x = \partial_v^1 x$ then $\partial_h^0\partial_v^0 x = \partial_h^0\partial_v^1 x = \partial_v^1\partial_h^0 x = \partial_v^1\partial_h^1 x = \partial_h^1\partial_v^1 x$, and hence $H^0(\mathrm{Tot}(X)) \subseteq \pi^0(\mathrm{diag}\, X)$.

The additive functors $\mathrm{Diag}: \mathcal{A}^{\Delta\times\Delta} \to \mathcal{A}^{\Delta}$ and $\mathrm{Tot}: \mathcal{A}^{\Delta\times\Delta} \to \mathrm{Coch}(\mathcal{A})$ are obviously exact, while π^*, H^* are cohomological δ-functors, so that both $\pi^*\,\mathrm{Diag}$, $H^*\,\mathrm{Tot}: \mathcal{A}^{\Delta\times\Delta} \to \mathrm{Coch}(\mathcal{A})$ are cohomological δ-functors.

The claim is that these cohomological δ-functors are universal, i.e., the right derived functors of $\pi^0\,\mathrm{Diag}$, $H^0\,\mathrm{Tot}\,C: \mathcal{A}^{\Delta\times\Delta} \to \mathcal{A}$, respectively. Since $\mathcal{A}$ has enough injectives, so does $\mathrm{Coch}(\mathcal{A})$ by [32, Example 2.3.4], and hence by the Dold–Kan equivalence $\mathcal{A}^{\Delta}$ and $\mathcal{A}^{\Delta\times\Delta}$ have enough injectives. Moreover, by the next lemma, both Diag and Tot preserve injectives. It therefore follows that

$$\pi^*\,\mathrm{Diag} = \big(R^*\pi^0\big)\,\mathrm{Diag} = R^*\big(\pi^0\,\mathrm{Diag}\big),$$
$$H^*\,\mathrm{Tot} = \big(R^*H^0\big)\,\mathrm{Tot} = R^*\big(H^0\,\mathrm{Tot}\big).$$

The canonical cohomological first quadrant spectral sequence associated with the cochain bicomplex $C(X)$ has

$$E_1^{p,q} = H_v^q\big(C^{p,*}(X)\big) = \pi_v^q\big(X^{p,*}\big), \qquad E_2^{p,q} = H_h^p\big(C\big(\pi_v^q(X)\big)\big) = \pi_h^p \pi_v^q(X)$$

and converges finitely to $H^{p+q}(\mathrm{Tot}(X)) \cong \pi^{p+q}(\mathrm{diag}\, X)$. □

Lemma A.4. *The functors* $\mathrm{Diag}: \mathcal{A}^{\Delta\times\Delta} \to \mathcal{A}^{\Delta}$ *and* $\mathrm{Tot}: \mathcal{A}^{\Delta\times\Delta} \to \mathrm{Coch}\,\mathcal{A}$ *preserve injectives.*

Proof. A cosimplicial bicomplex J is an injective object in $\mathcal{A}^{\Delta\times\Delta}$ if and only if

(1) each $J^{p,q}$ is an injective object of $\mathcal{A}$,
(2) each row and each column is cosimplicially null-homotopic, i.e., the identity map is cosimplicially homotopic to the zero map,
(3) the vertical homotopies $h_v^j : J^{*,q} \to J^{*,q-1}$ for $0 \leqslant j \leqslant q-1$ are cosimplicial maps.

It then follows that $\mathrm{Diag}(J)$ is an injective object in $\mathcal{A}^{\Delta}$, since $J^{p,p}$ is injective in $\mathcal{A}$ for every $p \geqslant 0$ and the maps $h^i = h_h^i h_v^i : J^{p,p} \to J^{p-1,p-1}$, $0 \leqslant i \leqslant p-1$ and $p > 0$, form a contracting cosimplicial homotopy, i.e., the identity map of $\mathrm{Diag}\, J$ is cosimplicially null-homotopic.

On the other hand $\mathrm{Tot}(J)$ is a non-negative cochain complex of injective objects in $\mathcal{A}$, so it is injective in $\mathrm{Coch}(\mathcal{A})$ if and only if it is split-exact, that is if and only if it is exact. But every column of the associated cochain bicomplex $C(J)$ is acyclic, since $H_v^*(J^{p,*}) = \pi^*(J^{p,*}) = 0$. The exactness of $\mathrm{Tot}(J)$ now follows from the convergent spectral sequence with $E_1^{p,q} = H^q(C^{p,*}(J)) = 0$ and $E_2^{p,q} = H_h^p(H_v^q(C(J))) \Rightarrow H^{p+q}(\mathrm{Tot}(J))$. □

A.3. The cosimplicial Alexander–Whitney map

The cosimplicial Alexander–Whitney map gives an explicit formula for the isomorphism in the Eilenberg–Zilber theorem. For $p+q=n$ let

$$g_{p,q} = d_h^n d_h^{n-1} \cdots d_h^{p+1} d_v^0 \cdots d_v^0 : X^{p,q} \to X^{n,n}$$

and $g^n = (g^{p,q}) : \mathrm{Tot}^n(X) \to X^{n,n}$. This defines a natural cochain map $g : \mathrm{Tot}(X) \to C(\mathrm{Diag}\, X)$, which induces a morphism of universal δ-functors

$$g^* : H^*\big(\mathrm{Tot}(X)\big) \to H^*\big(C(\mathrm{Diag}\, X)\big) = \pi^*(\mathrm{Diag}\, X).$$

Moreover, $g^0 : \mathrm{Tot}^0(X) = X^0 = C^0(\mathrm{Diag}\, X)$, and hence

$$g^0 : H^0\big(\mathrm{Tot}(X)\big) \to H^0\big(C(\mathrm{Diag}\, X)\big) = \pi^0(\mathrm{Diag}\, X).$$

The cosimplicial Alexander–Whitney map is therefore (up to equivalence) the unique cochain map inducing the isomorphism in the Eilenberg–Zilber theorem. The inverse map $f : C(\operatorname{Diag} X) \to \operatorname{Tot}(X)$ is given by the shuffle coproduct formula

$$f^{p,q} = \sum_{(p,q)\text{-shuffles}} (-1)^{\mu} \sigma_h^{\mu(n)} \cdots \sigma_h^{\mu(p+1)} \sigma_v^{\mu(p)} \cdots \sigma_v^{\mu(1)} : X^{n.n} \to X^{p,q},$$

and is a natural cochain map. It induces a natural isomorphism

$$\pi^0(\operatorname{Diag} X) = H^0\big(C(\operatorname{Diag} X)\big) \cong H^0\big(\operatorname{Tot}(X)\big),$$

and thus

$$f^* : \pi^*(\operatorname{Diag} X) = H^*\big(C(\operatorname{Diag} X)\big) \cong H^*\big(\operatorname{Tot}(X)\big)$$

is the unique isomorphism of universal δ-functors given in the cosimplicial Eilenberg–Zilber theorem. In particular, f^* is the inverse of g^*.

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