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On the Continuous Dependence of Solutions of Boundary Value Problems for Ordinary Differential Equations

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In the literature there are various theorems on the continuous dependence of solutions of boundary value problems for ordinary differential equations, cf. [1–11]. All these results present two peculiar features that restrict considerably their range of applicability: they apply only to special boundary value problems, and they require the hypothesis of "unrestricted" uniqueness. Unrestricted uniqueness can be roughly described in modern language by saying that the nonlinearity is below the first eigenvalue of the linear part. This assumption looks unnatural when compared with the initial value problem where uniqueness is required only for the limit problem.

It is the purpose of this paper to prove a general theorem about continuous dependence of solutions to boundary value problems that avoids these inconveniences and is the analogue of the general theorem on continuous dependence for the Cauchy problem (Theorem 3.2 at p. 14 of Hartman [12]). This theorem provides a positive answer to the question raised on p. 123 of Conti [13]. It includes all the above mentioned theorems as special cases, as far as assumptions on boundary conditions are concerned, while for the hypotheses on Cauchy problems a comment is needed: unlike some of the earlier results, we require local uniqueness on Cauchy problems but we avoid completely the assumption of global existence. The proof is quite different from the arguments in papers [1–11] since it is a simple application of the continuity of the Brouwer topological degree (the proof written below is long due to the technicality involved with the fact that global existence for the Cauchy problems has not been assumed).

As an application, it is possible to generalize to nonlinear functional boundary value problems many of the results known in the linear cases.

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Since this program can be carried out along the line of the nice paper Opial [18], we prefer to omit it and to present two applications which do not seem obtainable in a traditional way:

(i) an existence theorem for the perturbation of functional boundary value problems admitting unique solvability; and

(ii) the results announced in Piccinini, Stampacchia, and Vidossich [14, Chap. IV, Ref. 118] related to the use of the spectral theory of the two-point boundary value problem to get existence theorems for the n-point boundary value problem. This has been one starting point of this research, since the continuous dependence of solutions to boundary value problems was needed, and no sufficiently general result was available.

1. The General Theorem

In this section we prove a general theorem on the continuous dependence of solutions of functional boundary value problems. By *functional boundary value problem* we mean a problem of the type

$$x' = f(t, x), \qquad L(x) = r,$$

where $f\colon [a, b] \times \mathbb{R}^N \to \mathbb{R}^N$ and L is a mapping from the space of continuous functions $C([a, b], \mathbb{R}^N)$ into $\mathbb{R}^N$. This type of problem has been introduced by Whyburn in the forties for linear L and has been intensively studied in the fifties and sixties by Conti, Lasota, and Opial; the first treatment for nonlinear operators L appeared in the seventies by McCandless. For a comprehensive bibliography on the subject, cf. Conti [15], Mawhin [16], and Piccinini, Stampacchia and Vidossich [14].

In Theorem 1 special attention has been paid in order to avoid completely the assumption of global existence for the Cauchy problems. This would be a "bad" hypothesis as the works on $x'' = a(t)\, x^n$ show. Note that in the approximating boundary value problems no assumptions have been made on existence or uniqueness. Theorem 1 shows that existence is "spread around" for sufficiently near problems, a fact that could be useful in numerical analysis.

Theorem 1. *For each* $n \geqslant 0$, *let* $f_n\colon [a, b] \times \mathbb{R}^N \to \mathbb{R}^N$ *satisfy the generalized Caratheodory conditions, let* $L_n\colon C([a, b], \mathbb{R}^N) \to \mathbb{R}^N$ *be continuous, and let* $r_n \in \mathbb{R}^N$. *Assume that*

(a) $\lim_n r_n = r_0$;

(b) $\lim_n f_n = f_0$ *and* $\lim_n L_n = L_0$ *uniformly on compact sets of* $[a, b] \times \mathbb{R}^N$ *and* $C([a, b], \mathbb{R}^N)$, *respectively*;

(c) *each initial value problem*

$$x' = f_n(t, x), \qquad x(a) = u$$

has at most one local solution for $u \in \mathbb{R}^N$;

(d) *the functional boundary value problem*

$$x' = f_0(t, x), \qquad L_0(x) = r$$

has at most one solution for each $r \in \mathbb{R}^N$.

Let x_0 *be the solution to* $x' = f_0(t, x)$, $L_0(x) = r_0$. *Then for each* $\varepsilon > 0$ *there exists* n_ε *such that the functional boundary value problem*

$$x' = f_n(t, x), \qquad L_n(x) = r_n$$

has a solution x_n *for* $n \geqslant n_\varepsilon$ *satisfying the condition*

$$\|x_0 - x_n\|_\infty < \varepsilon.$$

We have denoted by $\|\cdot\|_\infty$ the sup norm, while by the statement $f(t, x)$ *satisfies the generalized Caratheodory conditions* it is meant that f is measurable in t, continuous in x a.e. in t and for each $M > 0$ there is a summable function h such that

$$\|f(t, x)\| \leqslant h(t) \qquad (|x| \leqslant M \text{ a.e. in } t).$$

Direct consequences of Theorem 1 are the following:

COROLLARY 1. *Under the hypothesis of Theorem* 1, *assume further that the boundary value problem*

$$x' = f_n(t, x), \qquad L_n(x) = r_n$$

has a unique solution x_n. *Then we have* $\lim_n x_n = x_0$ *uniformly on* $[a, b]$.

COROLLARY 2. *If the Cauchy problems for* $x' = f(t, x)$ *enjoy local uniqueness and if the boundary value problems*

$$x' = f(t, x), \qquad L(x) = r \tag{1}$$

have at most one solution for each $r \in \mathbb{R}^N$, *then the set* U *of all* $r \in \mathbb{R}^N$ *for which* (1) *has a solution is an open subset of* $\mathbb{R}^N$.

Proof of Theorem 1. Fix $\varepsilon > 0$. We claim that

(*) *there exists a pair* (U, m) *with* U *a neighbourhood of* $x_0(a)$ *in* $\mathbb{R}^N$ *and* m *an integer, such that for each pair* $(u, n) \in U \times \{n \in N \mid n \geqslant m\}$ *the Cauchy problem*

$$x' = f_n(t, x), \qquad x(a) = u$$

has a unique solution u_n *defined on* $[a, b]$ *and satisfying* $\|u_n - x_0\|_\infty < \varepsilon$.

In fact, if (*) fails, then for every pair (U, m) there exists a pair (u, n) such that $u \in U$ and $n \geqslant m$ for which either the solution u_n to

$$x' = f_n(t, x), \qquad x(a) = u$$

does not exist on $[a, b]$ or u_n exists on $[a, b]$ but we have $\|u_n - x_0\|_\infty \geqslant \varepsilon$. For each $m \geqslant 1$ we take U equal to the ball $B(x_0(a), 1/m)$ of centre $x_0(a)$ and radius $1/m$. We find $u_m \in B(x_0(a), 1/m)$ and $n_m \geqslant m$ such that the solution $\bar{u}_m$ of

$$x' = f_{n_m}(t, x), \qquad x(a) = u_m$$

either does not exist on $[a, b]$ or, alternatively, $\|\bar{u}_m - x_0\|_\infty \geqslant \varepsilon$. By virtue of Theorem 3.2 on p. 14 of Hartman [12] (whose proof works also for f_n's satisfying the generalized Caratheodory conditions), there is a subsequence $(\bar{u}_{m_k})_k$ of $(\bar{u}_m)_m$ such that each $\bar{u}_{m_k}$ is defined on $[a, b]$ and $\lim_k \bar{u}_{m_k} = x_0$ uniformly on $[a, b]$. This contradicts the definition of $\bar{u}_{m_k}$ and we conclude that (*) must hold.

Let (U, m) be the pair defined by (*). By repeating the argument used to prove (*), we see that we can take U so small that each initial value problem

$$x' = f_0(t, x), \qquad x(a) = u$$

with $u \in U$ has a (unique) solution u_0 defined on $[a, b]$ and $\|x_0 - u_0\|_\infty < \varepsilon$. Select such a U. For each $n \geqslant m$ or $n = 0$ and each $u \in U$, let u_n be the unique solution to

$$x' = f_n(t, x), \qquad x(a) = u$$

defined on $[a, b]$, existing by (*) and the above remark. Now we fix a ball B of centre $x_0(a)$ in $\mathbb{R}^N$ such that $\bar{B} \subset U$, and define for $n = 0$ and $n \geqslant m$ a function $F_n \colon \bar{B} \to \mathbb{R}^N$ as

$$F_n(u) = L_n(u_n),$$

where, as said above, u_n is the unique solution to $x' = f_n(t, x)$, $x(a) = u$. By the uniqueness of Cauchy problems, the mapping $u \rightsquigarrow u_n$ is continuous.

Therefore, F_n is continuous. We claim that

$$\lim_n F_n = F_0 \tag{2}$$

uniformly on $\bar{B}$. If not, there exist $\delta > 0$, $n_k \uparrow \infty$, and $y_k \in \bar{B}$ such that

$$\|F_{n_k}(y_k) - F_0(y_k)\| \geqslant \delta \qquad (k \geqslant 1). \tag{3}$$

Passing to a subsequence if necessary, we assume $\lim_k y_k = y_0$ for a suitable $y_0 \in \bar{B}$. Let $\bar{y}_k$ be the unique solution to the Cauchy problem

$$x' = f_{n_k}(t, x), \qquad x(a) = y_k.$$

Using again Theorem 3.2 on p. 14 of Hartman [12], we see there exists a subsequence $(\bar{y}_{n_{k_i}})_i$ of $(\bar{y}_{n_k})_k$ such that

$$\lim_i \bar{y}_{k_i} = \bar{y}_0$$

uniformly on $[a, b]$. Since $\lim_n L_n = L_0$ uniformly on compact sets and since $\{\bar{y}_{k_i} \mid i \geqslant 1\} \cup \{\bar{y}_0\}$ is a compact subset of $C([a, b], \mathbb{R}^N)$, we have

$$\lim_n L_n(\bar{y}_{k_i}) = L_0(\bar{y}_{k_i})$$

uniformly with respect to i. This contradicts (3) since $L_{n_{k_i}}(\bar{y}_{k_i}) = F_{n_{k_i}}(y_{k_i})$ and $L_0(\bar{y}_{k_i}) = F_0(y_{k_i})$. Therefore (2) must hold. Since F_0 is injective by (d), we have $r_0 \notin F_0(\partial B)$. It follows that

$$c = \operatorname{dist}(F_0(\partial B), r_0) > 0$$

since $F_0(\partial B)$ is closed. From this, from (2), and from (a), we get the existence of n_0 such that

$$\operatorname{dist}(F_n(\partial B), r_n) \geqslant \frac{c}{3} \qquad (n \geqslant n_0).$$

This implies that the Brouwer topological degree $\deg(F_n, B, r_n)$ is well defined for $n \geqslant n_0$. By the continuous dependence of the topological degree there is $n_1 \geqslant n_0$ such that

$$\deg(F_n, B, r_n) = \deg(F_0, B, r_0) \qquad (n \geqslant n_1).$$

Since F_0 is injective by (d), we have

$$\deg(F_0, B, r_0) = \pm 1$$

by virtue of Theorem 3.3.3 of Lloyd [17]. Therefore we have

$$\deg(F_n, B, r_n) = \pm 1 \qquad (n \geqslant n_1)$$

and the equation

$$F_n(x) = r_n$$

has at least one solution x_n in B for $n \geqslant n_1$. From (*) it follows that the solution $\bar{x}_n$ to the Cauchy problem

$$x' = f_n(t, x), \qquad x(a) = x_n$$

satisfies the condition $\|\bar{x}_n - x_0\|_\infty < \varepsilon$. Thus the theorem is proven by taking $n_\varepsilon = n_1$. Q.E.D.

Proof of Corollary 1. Obvious. Q.E.D.

Proof of Corollary 2. If $r_0 \in U$ is not an interior point of U, then there exists a sequence $(r_n)_n$ in $\mathbb{R}^N$ such that $\lim_n r_n = r_0$ and

$$x' = f(t, x), \qquad L(x) = r_n$$

has no solution. But this contradicts the conclusion of Theorem 1 when we apply it by taking $f_n = f$ and $L_n = L$. Q.E.D.

2. Application to the Existence of Solutions to Functional Boundary Value Problems

We provide only one of the applications of Theorem 1 to functional boundary value problems because, as mentioned in the introduction of the paper, all the others considered by the author turned out, on second thought, to be more or less a natural extension of the argument in Opial [18] via the use of Theorem 1.

THEOREM 2. *Let $f: [a, b] \times \mathbb{R}^N \to \mathbb{R}^N$ be a C^1-function and let $L: C([a, b], \mathbb{R}^N) \to \mathbb{R}^N$ be a bounded linear operator. Assume that*

(a) *$(\partial/\partial x) f$ is bounded;*

(b) *the functional boundary value problem*

$$x' = f(t, x), \qquad L(x) = r$$

has a unique solution x_r for every $r \in \mathbb{R}^N$.

Then for every bounded continuous function $g: [a, b] \times \mathbb{R}^N \to \mathbb{R}^N$ and every $r \in \mathbb{R}^N$, the functional boundary value problem

$$x' = f(t, x) + g(t, x), \qquad L(x) = r$$

has at least one solution.

Proof. For each $x_0 \in \mathbb{R}^N$, denote by $x(t, t_0, x_0)$ the unique solution to the Cauchy problem

$$x' = f(t, x), \qquad x(t_0) = x_0$$

existing in $[a, b]$ since $(\partial/\partial x) f$ is bounded. It is well known that the function

$$U(t, t_0, x_0) = \frac{\partial}{\partial x_0} x(t, t_0, x_0)$$

satisfies the following ordinary differential equation in the space of $N \times N$ matrices

$$\frac{\partial}{\partial t} U(t, t_0, x_0) = f_x(t, x(t, t_0, x_0)) \cdot U(t, t_0, x_0), \; U(t_0, t_0, x_0) = \mathrm{id}, \tag{4}$$

where $f_x = (\partial/\partial x) f$. From this and the Gronwall lemma it follows

$$\| U(t, t_0, x_0) \| \leqslant e^{M(b-a)}, \tag{5}$$

where M is an upper bound for the norm of f_x, existing by (a). For each $n \geqslant 1$ set $\delta_n = (b-a)/n$ and define $g_n \colon [a, b] \times C([a, b], \mathbb{R}^N) \to \mathbb{R}^N$ by

$$g_n(t, x) = \begin{cases} 0 & \text{if } a \leqslant t < a + \delta_n \\ g(t - \delta_n, x(t - \delta_n)) & \text{if } t \geqslant a + \delta_n. \end{cases}$$

Clearly g_n satisfies the generalized Caratheodory conditions. Fix n. We claim that the "functional" Cauchy problem

$$y' = f(t, y(t)) + g_n(t, y), \qquad y(a) = x_0$$

has a unique absolutely continuous solution $y_n(t, x_0)$ defined on $[a, b]$. In fact, $y_n(\cdot, x_0)$ can be defined as follows. On $[a, a + \delta_n]$ we set $y_n(t, x_0) = x(t, a, x_0)$. The Cauchy problem

$$z' = f(t, z) + g_n(t, x(\cdot, a, x_0)), \qquad z(a + \delta_n) = y_n(a + \delta_n, x_0)$$

has a unique solution x_2 on $[a, b]$ since f_x is bounded. We set $y_n(t, x_0) = x_2(t)$ for $a + \delta_n \leqslant t \leqslant a + 2\delta_n$. The Cauchy problem

$$x' = f(t, x) + g_n(t, x_2), \qquad x(a + 2\delta_n) = y_n(a + 2\delta_n, x_0)$$

has a unique solution x_3 on $[a, b]$ since f_x is bounded. We set $y_n(t, x_0) = x_3(t)$ for $a + 2\delta_n \leqslant t \leqslant a + 3\delta_n$. Proceeding in this way, we get the existence and uniqueness of $y_n(\cdot, x_0)$ on $[a, b]$. Now differentiate

$v(s) = x(t, s, y_n(s, x_0))$ and integrate the result from a to t. From this and the relationship between $(\partial/\partial t_0)\, x(t, t_0, x_0)$ and $(\partial/\partial x_0)\, x(t, t_0, x_0)$, it follows that $y_n(\cdot, x_0)$ satisfies the "Volterra" equation

$$y(t) = x(t, a, x_0) + \int_a^t U(t, s, y_n(s, x_0))\, g_n(s, y)\, ds. \qquad (6)$$

Assume that u and v are two solutions to (6). Then $u = v = x(\cdot, a, x_0)$ on $[a, a + \delta_n]$ by virtue of the definition of g_n. It follows that $g_n(t, u) = g_n(t, v)$ on $[a, a + 2\delta_n]$ and therefore u and v are equal on $[a + \delta_n, a + 2\delta_n]$ since the right-hand side of (6) is equal there. Proceeding in this way, we obtain $u = v$ on $[a, b]$. Thus (6) has a unique solution. Now consider the mapping $r \rightsquigarrow x_r$, x_r being defined in (b). By Theorem 1, $r \rightsquigarrow x_r$ is continuous. By this and by the continuous dependence of solutions to Cauchy problems admitting uniqueness, it follows that the mapping

$$r \rightsquigarrow x(\cdot, a, x_r(a))$$

is a continuous function from $\mathbb{R}^N$ into $C([a, b], \mathbb{R}^N)$. Now define an operator $G_n: C([a, b], \mathbb{R}^N) \to C([a, b], \mathbb{R}^N)$ by

$$G_n(u)(t) = \int_a^t U(t, s, y_n(s, u(a)))\, g_n(s, u)\, ds \qquad (n \geqslant 0)$$

with $g_0 = g$. Since the solutions to (4) as well as $y_n(\cdot, x_0)$ depend continuously on the parameter x_0, G_n is continuous. From (5) and the Ascoli theorem it follows that G_n is completely continuous, i.e., G_n maps bounded subsets of $C([a, b], \mathbb{R}^N)$ into compact subsets. By continuity it follows that the mapping $u \rightsquigarrow x(\cdot, a, x_{r - LG_n(u)}(a))$ transforms bounded subsets of $C([a, b], \mathbb{R}^N)$ into compact subsets. Therefore the operator defined by

$$F_n(u) = x(\cdot, a, x_{r - LG_n(u)}(a)) + G_n(u)$$

is a completely continuous operator $C([a, b], \mathbb{R}^N) \to C([a, b], \mathbb{R}^N)$ taking values into a bounded set. Then from the Schauder fixed point theorem we get the existence of a function u_n such that $u_n = F_n(u_n)$, i.e.,

$$u_n(t) = x(t, a, x_{r - LG_n(u_n)}(a)) + \int_a^t U(t, s, y_n(s, u_n(a)))\, g_n(s, u_n)\, ds. \qquad (7)$$

Taking $t = a$ we get

$$u_n(a) = x_{r - LG_n(u_n)}(a)$$

and so we have

$$u_n(t) = x(t, a, u_n(a)) + \int_a^t U(t, s, y_n(s, u_n(a)))\, g_n(s, u_n)\, ds. \tag{8}$$

By virtue of the above, the function $y_n(\cdot, u_n(a))$ must satisfy Eq. (6). Since (6) can have only one solution as shown above, we have $u_n = y_n(\cdot, u_n(a))$ and so u_n, $n \geqslant 1$, is a solution to the perturbed equation $y' = f(t, y) + g_n(t, y)$. Now we apply L on both sides of (7) and we get

$$\begin{aligned} Lu_n &= r - LG_n(u_n) + LG_n(u_n) \\ &= r. \end{aligned} \tag{9}$$

Since $(G_n)_n$ is uniformly bounded, $(u_n)_n$ and hence $(u_n')_n$ are uniformly bounded. Then by the Ascoli theorem there is a subsequence $(u_{n_k})_k$ converging uniformly to a function u_∞. Taking limits in (8) and (9) we get $u_\infty' = f(t, u_\infty) + g(t, u_\infty)$, $L(u_\infty) = r$ and so u_∞ is the desired solution.

Q.E.D.

3. Applications to Boundary Value Problems for n^{th}-Order Equations Using Eigenvalues of Second Order Equations

In this section we shall prove an existence theorem for boundary value problems of n^{th}-order equations by using the properties of the spectral theory for the two-point boundary value problem.

In place of the Picard problem we can use any Sturm–Liouville problem for second order equations. Moreover, our argument can be used to handle those non-symmetric problems that admit a nice symmetric lower part.

THEOREM 3. *Let* $f\colon [a, b] \times \mathbb{R}^{n-1} \to \mathbb{R}$ *be continuous together with* $(\partial/\partial x_{n-1}) f(t, x, ..., x_{n-1})$. *Let* λ_m *be the* m^{th} *eigenvalue of the Picard problem*

$$y'' + \lambda y = 0, \qquad y(a) = y(b) = 0. \tag{10}$$

Let $L\colon C([a, b], \mathbb{R}^{n-2}) \to \mathbb{R}^{n-2}$ *be a continuous function such that the functional boundary value problem*

$$x^{(n-2)} = u(t), \qquad L(x, ..., x^{(n-3)}) = r$$

has a unique solution u_r *for every* $u \in C([a, b])$ *and* $r \in \mathbb{R}^{n-2}$. If $f(\cdot, \cdot, ..., \cdot, 0)$ *is bounded and if one of the following conditions holds*:

(i) *there exists* $p \in L^\infty$ *such that*

$$\frac{\partial}{\partial x_{n-1}} f(t, x_1, ..., x_{n-1}) \leqslant p(t) \qquad (\text{a.e. } t; \quad \text{all} \quad x_i)$$

and $p \leqslant \lambda_1$ *a.e. with strict inequality in a set of positive measure; or alternatively*

(ii) *there exist* $p, q \in L^\infty$ *such that*

$$p(t) \leqslant \frac{\partial}{\partial x_{n-1}} f(t, x_1, ..., x_{n-1}) \leqslant q(t) \qquad (\text{a.e. } t; \quad \text{all} \quad x_i)$$

and for a suitable m we have

$$\lambda_m \leqslant p(t), \qquad q(t) \leqslant \lambda_{m+1}$$

a.e. with strict inequality in a set of positive measure;

then the n^{th}*-order boundary value problem*

$$x^{(n)} + f(t, x, ..., x^{(n-2)}) = 0,$$
$$x^{(n-2)}(a) = A, \quad x^{(n-2)}(b) = B, \quad L(x, ..., x^{(n-3)}) = r$$

has at least one solution for any $A, B \in \mathbb{R}$ *and* $r \in \mathbb{R}^{n-2}$.

If L is linear, then it follows from the Fredholm alternative that the uniqueness assumption on

$$x^{(n-2)} = u(t), \qquad L(x, ..., x^{(n-3)}) = r$$

is satisfied whenever $x \equiv 0$ is the only solution to $x^{(n-2)} = 0$, $L(x, ..., x^{(n-3)}) = 0$.

A simple application of Theorem 3 is furnished by the following n-point boundary value problem

$$x^{(n)} = f(t, x, ..., x^{(n-2)}),$$

$$x^{(n-2)}(a) = A, \quad x^{(n-2)}(b) = B, \quad x^{(m_i)}(t_i) = r_i \quad \left(i = 0, ..., k; \sum_{i=1}^{k} m_i = n-3\right)$$

with $t_i \in [a, b]$. In this case, $L(z) = (z_{m_1}(t_1), ..., z_{m_k}(t_k))$.

To prove Theorem 3 we need the following

LEMMA *Let* λ_1 *be the first eigenvalue of* (10) *and let* $p \in L^\infty$ *be such that* $p \leqslant \lambda_1$ *with strict inequality in a set of positive measure. Then for every continuous g and every* $A, B \geqslant 0$, *the Picard problem*

$$x'' + p(t)x + g(t) = 0, \qquad x(a) = A, \quad x(b) = B$$

has a unique solution x_g with absolutely continuous first derivative. If $p > 0$ and $g \geqslant 0$ a.e. and if $A, B \geqslant 0$ then we have $x_g \geqslant 0$.

Proof. Let μ_1 be the first positive eigenvalue of

$$v'' + \mu p(t) v = 0, \qquad v(a) = v(b) = 0.$$

Suppose $\mu_1 \leqslant 1$. Let us compare the eigenvalue problems

$$u'' + \lambda' \lambda_1 u = 0, \qquad u(a) = u(b) = 0 \tag{11}$$

$$z'' + \lambda'' \mu_1 p(t) z = 0, \qquad z(a) = z(b) = 0 \tag{12}$$

and call λ_1', λ_1'' their first positive eigenvalues. Since $\mu_1 p \leqslant p$, we have $\operatorname{meas}\{\mu_1 p < \lambda_1\} > 0$. Therefore from Proposition 1.12A of de Figueiredo [19] it follows

$$\lambda_1' < \lambda_1''.$$

Obviously $\lambda_1'' = 1$. Thus $\lambda_1' < 1$. But this goes against the fact that for $\lambda' = 1$ problem (11) has a positive solution (namely, any eigenfunction corresponding to λ_1) and therefore $\lambda' = 1$ must be the first eigenvalue. This contradiction shows that the case $\mu_1 \leqslant 1$ cannot occur, i.e.,

$$\mu_1 > 1. \tag{13}$$

Now let G be the Green function corresponding to the problem

$$-x'' = h(t), \qquad x(a) = x(b) = 0.$$

Let I be the identity of $C([a, b])$ and let $L: C([a, b]) \to C([a, b])$ be the linear operator defined by

$$Lx(t) = \int_a^b G(t, s)\, p(s)\, x(s)\, ds.$$

From (13) it follows that $\operatorname{Ker}(I - L)$ is reduced to the origin. Then by the Fredholm alternative for any $g \in C([a, b])$ and any $A, B \in \mathbb{R}$, there is at least one solution $x_{A,B}$ of

$$x(t) - Lx(t) = \int_a^b G(t, s)\, g(s)\, ds + A + (B - A)\frac{(t-a)}{b-a}. \tag{14}$$

This function $x_{A,B}$ is a solution to the given problem. The uniqueness of solutions follows from (13). It remains to show that $x_{A,B} \geqslant 0$ whenever the additional assumptions hold. Let x_0 be the solution to (14) corresponding to $A = B = 0$ and a fixed $g \geqslant 0$. Since $p > 0$ a.e., it follows from

Theorem 1.14 of de Figueriredo [19] that $x_0 \geqslant 0$. Then it is enough to show that $w = x_{A,B} - x_0$ is not negative. If $w < 0$ in at least one point, then there exist $a \leqslant \alpha < \beta \leqslant b$ such that $w < 0$ in $]\alpha, \beta[$ and $w(\alpha) = w(\beta) = 0$. Moreover, w is a solution to

$$w'' + p(t)w = 0.$$

Therefore it follows from the uniqueness portion of the lemma (the part already established), applied on $[\alpha, \beta]$, that we must have $w \equiv 0$ on $[\alpha, \beta]$, a contradiction. Q.E.D.

Proof of Theorem 3. Assume (i). By changing p with $\max\{p(t), \lambda_1/2\}$ if necessary, we assume $p > 0$. Let

$$M > \sup_{t,x} |f(t, x_1, ..., x_{n-2}, 0)|.$$

By the above lemma there is a unique positive solution β to the Picard problem

$$v'' + p(t)v + M = 0, \qquad v(a) = |A|, \quad v(b) = |B|.$$

Setting

$$g(t, x_1, ..., x_{n-1}) = \int_0^1 \frac{\partial}{\partial x_{n-1}} f(t, x_1, ..., x_{n-2}, \xi x_{n-1})\, d\xi,$$

the given equation can be rewritten in the form

$$x^{(n)} + g(t, x, ..., x^{(n-2)})x^{(n-2)} + f(t, x, ..., x^{(n-3)}, 0) = 0 \tag{15}$$

with $g(t, \cdot) \leqslant p(t)$ a.e.. Fix $r \in \mathbb{R}^{n-2}$. For every $u \in C([a, b])$, let $\bar{u}$ be the unique solution to the functional boundary value problem

$$z^{(n-2)} = u(t), \qquad L(z, ..., z^{(n-3)}) = r.$$

Clearly β and $-\beta$ are an upper and a lower solution to the Picard problem

$$\begin{gathered} y'' + q(t, \bar{u}(t), ..., \bar{u}^{(n-3)}(t), u(t))y + f(t, \bar{u}(t), ..., \bar{u}^{(n-3)}(t), 0) = 0, \\ y(a) = A, \qquad y(b) = B. \end{gathered} \tag{16}$$

Therefore a well-known theorem ensures that (16) has a solution y_u such that $-\beta \leqslant y_u \leqslant \beta$. The solution to (16) is unique by virtue of the above lemma. Consider the map $T: C([a, b]) \to C([a, b])$ defined by

$$T(u) = y_u.$$

By Theorem 1, $u \rightsquigarrow \bar{u}$ and T are continuous. Since $\|y_u\|_\infty \leqslant \|\beta\|_\infty$, it is easily seen from the Ascoli theorem that T maps bounded sets into relatively compact sets. Then we apply the Schauder fixed point theorem to the restriction of T on the ball $\overline{B(0, \|\beta\|_\infty)}$ in $C([a, b])$ and we get the existence of a function u_0 such that $u_0 = T(u_0)$. It is easily seen that $\bar{u}_0$ is a solution to Eq. (15), hence to the given boundary value problem.

Assume (ii). Let g and $\bar{u}$ be as above. The Picard problem (16) has a unique solution y_u by Mawhin–Ward [20]. We claim that the set $\{y_u \mid u \in C^0\}$ is bounded. In fact, if there is $\|y_{u_n}\|_\infty \to \infty$, then we set $z_n = y_{u_n}/\|y_{u_n}\|_\infty$ and by using the weak compactness of the coefficients of

$$z_n'' + g(t, \bar{u}_n, ..., \bar{u}_n^{(n-3)}, u_n) z_n + \frac{1}{\|y_{u_n}\|_\infty} f(t, \bar{u}_n, ..., \bar{u}^{(n-3)}, 0) = 0$$

we get a contradiction against the fact that $z'' = \bar{g}(t) z$, $z(a) = 0 = z(b)$ can have only $z \equiv 0$ as a solution whenever $p \leqslant \bar{g} \leqslant q$. Now we can conclude as in case (i). Q.E.D.

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