HISTORIA MATHEMATICA 22 (1995), 262–303

The Axiomatization of Linear Algebra: 1875–1940

GREGORY H. MOORE

Department of Mathematics, McMaster University, Hamilton, Ontario, Canada L8S 4K1

Modern linear algebra is based on vector spaces, or more generally, on modules. The abstract notion of vector space was first isolated by Peano (1888) in geometry. It was not influential then, nor when Weyl rediscovered it in 1918. Around 1920 it was rediscovered again by three analysts—Banach, Hahn, and Wiener—and an algebraist, Noether. Then the notion developed quickly, but in two distinct areas: functional analysis, emphasizing infinite-dimensional normed vector spaces, and ring theory, emphasizing finitely generated modules which were often not vector spaces. Even before Peano, a more limited notion of vector space over the reals was axiomatized by Darboux (1875). © 1995 Academic Press, Inc.

L'algèbre linéaire moderne a pour concept fondamental l'espace vectoriel ou, plus généralement, le concept de module. Peano (1888) fut le premier à identifier la notion abstraite d'espace vectoriel dans le domaine de la géométrie, alors qu'avant lui Darboux avait déjà axiomatisé une notion plus étroite. La notion telle que définie par Peano eut d'abord peu de répercussion, même quand Weyl la redécouvrit en 1918. C'est vers 1920 que les travaux d'analyse de Banach, Hahn, et Wiener ainsi que les recherches algébriques d'Emmy Noether mirent vraiment l'espace vectoriel à l'honneur. A partir de là, la notion se développa rapidement, mais dans deux domaines distincts: celui de l'analyse fonctionnelle où l'on utilisait surtout les espaces vectoriels de dimension infinie, et celui de la théorie des anneaux où les plus importants modules etaient ceux qui sont générés par un nombre fini d'éléments et qui, pour la plupart, ne sont pas des espaces vectoriels. © 1995 Academic Press, Inc.

Grundlage der modernen linearen Algebra sind Vektorräume oder, allgemeiner, Moduln. Der abstrakte Begriff des Vektorraumes wurde zuerst von Peano (1888) in der Geometrie herausgearbeitet. Er gewann damals kaum Einfluß, und auch dann noch nicht, als er 1918 von Weyl wiederentdeckt wurde. Um 1920 wurde er von drei Analytikern—Banach, Hahn, Wiener—und einer Algebraikerin, Noether, nochmals wiederentdeckt. Danach wurde der Begriff rasch weiterentwickelt, allerdings auf zwei unterschiedlichen Gebieten: in der Funktionalanalysis unter Betonung endlich-dimensionaler normierter Vektorraüme und in der Ringtheorie unter Betonung endlich erzeugter Moduln, die oft keine Vektorräume sind. Noch vor Peano wurde ein eingeschränkterer Begriff des Vektorraumes über den reellen Zahlen von Darboux (1875) axiomatisiert. © 1995 Academic Press, Inc.

MSC 1991 Subject Classifications: 01A55, 01A60, 03B30, 03E25, 13-03, 13C05, 15-03, 15A03.

KEY WORDS: axiomatization, vector space, module over a ring, Hamel basis, Banach space, Hilbert space, algebra over a field.

1. INTRODUCTION

Modern linear algebra is based on the theory of vector spaces over a field or, more generally, on the theory of modules over a ring.[1] Around 1930 this formulation of linear algebra unified the subject and made it a part of abstract, or modern,

[1] This view of linear algebra is standard today. See, e.g., the *Encyclopedic Dictionary of Mathematics* [70, 1:31].

0315-0860/95 $12.00

algebra. Yet linear algebra has its historical roots in various branches of mathematics, not just one. In algebra, its roots include linear equations, bilinear forms, and matrices, as well as quaternions and hypercomplex number systems; in geometry, directed line segments, affine geometry, and projective geometry; and in analysis, linear differential equations and infinite-dimensional spaces of various sorts. Finally, linear algebra has its roots in the 19th-century physics of Maxwell, Gibbs, and Heaviside.

Given the complexity and variety of these mathematical phenomena, it is not surprising that the general notion of vector space (as opposed to the older notion of a vector as a directed line segment or as an n-tuple) was isolated and adopted relatively late. Although this general notion was first formulated by Peano in a geometric context in 1888, it remained almost unknown at the time. Three decades later it was rediscovered independently in two quite distinct branches of mathematics: functional analysis and ring theory. In ring theory, its roots were in Dedekind's work on algebraic number theory.

The present article discusses the question: How did the fundamental notions of vector space and module come to be isolated and then axiomatized? A related question has been raised at least twice before. In 1980 Gray wished

> to draw attention to an interesting problem in the history of mathematics ... : to describe the origins of the concept of a vector space and its recognition as a central topic in mathematics. Moves towards the explicit treatment of topics, and their axiomatization, are always of interest; in addition, it seems that vector spaces have had a curious and convoluted development. [44, 65]

And in 1981 Mac Lane raised a similar question:

> When did mathematicians first generally accept the definition of a vector space as a set of elements subject to suitably axiomatized operations of addition and multiplication by scalars, and not just as a set of n-tuples of scalars closed under these two operations? This abstract description did not come into early use, but has the evident advantages of conceptual simplicity and geometric invariance. [68, 15]

The relevant period for axiomatization and acceptance extends from 1875 to about 1940. In this context we are also concerned with the interactions between linear algebra and set theory, since infinite-dimensional vector spaces turned out to involve fundamental set-theoretic matters, particularly when a basis was uncountable.

The period in question can be considered to end with the publication in 1941 of *A Survey of Modern Algebra* by the young algebraists Garrett Birkhoff and Saunders Mac Lane, who were then at Harvard. In that textbook, which was extremely influential in the United States, the notion of vector space over a field was defined in an abstract way. A vector space V over a field F was said to be a set V with an operation + that is an Abelian group and satisfies the following four axioms: There is an operation, called the scalar product, which for any a in F and any x in V gives ax and is such that

(V1) $a(x + y) = ax + ay$ (V2) $(a + b)x = ax + bx$
(V3) $(ab)x = a(bx)$ (V4) $1x = x$.

They defined a basis for a vector space and allowed the basis to be infinite, but showed, for a given space, that it always has the same cardinality (a result later proved equivalent to the Axiom of Choice). Finally, they defined the notion of inner product axiomatically, as a real-valued product (x, y) of two vectors such that for any vectors x, y, z, and any scalar a,

(P1) $(x + y, z) = (x, y) + (x, z)$ (P2) $(ax, y) = a(x, y)$
(P3) $(x, y) = (y, x)$ (P4) $(x, x) > 0$ unless $x = 0$

[13, 167, 181, 183]. As we shall see below, Birkhoff and Mac Lane were not at all enthusiastic about modules at the time.

What role did axiomatization play in the development of linear algebra? First, it should be made clear what role it did not play. It was not merely a question of rigor, or of "tidying up" concepts. Rather, in linear algebra the process of axiomatization helped to create linear algebra as a distinct subject. Certainly the name "linear algebra" emerged in this way, as well as the view that it is a unified subject embracing linear phenomena in many different parts of mathematics, from differential equations to ring theory.

In this context, the abstract notions of vector space and module played crucial roles. As Banach remarked, it was much more efficient to prove a result once about all vector spaces than to prove it repeatedly for 10 different function spaces. But Banach's work also illustrates a new kind of theorem that emerged from considering abstract vector spaces, be they finite-dimensional or infinite-dimensional: the Hahn–Banach theorem. This theorem on extending linear functionals defined on vector spaces would prove to be extremely fruitful in analysis.

During the 1880s axiomatization was going on in other parts of mathematics as well. It resulted in distinctly new concepts, such as that of order-type, due to Cantor and published in 1887 [23]. It was part of a fundamental shift that occurred in mathematics during the period from about 1880 to 1940—the consideration of a wide variety of mathematical "structures," defined axiomatically and studied both individually and as the classes of structures—groups, fields, lattices, etc.—satisfying those axioms. (The general notion of an algebraic structure was first clearly formulated by Birkhoff in 1933 [12].) This approach is so common now that it is almost superfluous to mention it explicitly, but it represented a major conceptual shift in answering the question: What is mathematics?

The importance of this shift can be illustrated by two quotations, one from the end of our period and one from the beginning. In 1951 Weyl, who felt considerable ambivalence about axiomatization, expressed very well the new role that it was playing: "Whereas the axiomatic method was formerly used merely for the purpose of elucidating the foundations on which we build, it has now become a tool for concrete mathematical research" [131, 523]. And in 1888, at the beginning of our period, Dedekind expressed best of all how axiomatization played a fruitful role:

> The greatest and most fruitful advances in mathematics and the other sciences have been achieved, above all, through the creation and introduction of new concepts, after the frequent

> recurrence of complex phenomena, which were only laboriously mastered by the old concepts, has forced them upon us. [28, preface]

2. THE PREHISTORY OF AXIOMS FOR VECTOR SPACES

The name "vector" stems from Hamilton (1845), who used it for what he called the "vector part," as opposed to the "scalar part," of a quaternion. But the idea of a vector as a directed line segment was decades older. It formed part of the barycentric calculus of Möbius (1827), the calculus of equipollences of Bellavitis (1835), and the calculus of extension, or "Ausdehnungslehre," of Grassmann (1844). The explicit use of vectors occurs even earlier in the work of Wessel (1797) and Argand (1806). The older term "radius vector" ("rayon vecteur") is found in French mathematical physics, such as Ampère's *Théorie mathématique des phénomènes électrodynamiques* (1826). This term appears already in 1776 in the celebrated *Encyclopédie*, edited by Diderot, in the article "Rayon vecteur" by the astronomer J.-J. de la Lande. He wrote that a radius vector is the "ligne droite qui va ... du soleil au centre de la planète; on l'appelle *vecteur*, parce qu'on le conçoit comme portant la planète à une de ses extrêmités ...."[2] What de la Lande had in mind becomes clearer when we recall that "vecteur," or "vector," comes from the past participle "vectus" of the Latin verb meaning "to carry or transport.")

For some decades, the systems of Grassmann and Hamilton competed for influence. From the 1840s to the 1870s, the Hamiltonian system was much better known in most of Europe than the Grassmannian. From the 1870s to the 1890s, publications on the Grassmannian system increased substantially [25, 113], and, as we shall see below, Peano was a part of that trend. Both Grassmann's and Hamilton's systems included much more than vectors. During the 19th century, a vector was *not* identified with a point, as is often done today. (A detailed historical analysis of the idea of vector, prior to its abstract and axiomatic treatment, can be found in Crowe's book [25].)

In the late 19th century and the early 20th, the notion of vector was generally treated as a directed line segment AB, or as the difference B − A of two points A and B. Physicists treated a vector as a quantity (e.g., momentum or force) possessing both direction and magnitude. As directed line segments, two vectors were considered to be equal when they had the same length and the same direction. Usually vectors were considered to have at most three dimensions. At times, particularly by those in the Grassmannian tradition, vectors were allowed to have any finite number of dimensions. A more general concept of vector—abstract vector spaces—first arose under a different name, that of "linear system," in the work of Peano.

3. PEANO AND HIS LINEAR SYSTEMS

Giuseppe Peano treated vector-like notions in three different ways at different periods in his career. The first way, beginning in 1887, was as n-tuples, with addition and scalar multiplication defined by the corresponding operation on each coordi-

[2] Vol. 4 of the supplement, p. 580. We owe to Thomas Archibald the information about Ampère and de la Lande.

nate. He did not identify these n-tuples with vectors, but we would so regard them now. The second way, beginning in 1888, was as the "difference" B − A of two points A and B (i.e., as a directed line segment). Here he took a Grassmannian approach to what he called the geometric calculus. The third way, which also began in 1888, was what he called linear systems (and which we would now call vector spaces). The first way was not axiomatic, and the second way was axiomatized by Peano only a decade later, in 1898. At that time, he no longer used his third approach via linear systems.

From today's perspective, it was Peano's third approach that was the most important, since his axiomatization of linear system was essentially the modern concept of vector space over the real numbers. Dieudonné [35, 72] and Mac Lane [69, 187] find it odd that Peano's axioms for linear systems were not adopted soon after he presented them in 1888. But it was not odd at all. Peano only discussed those axioms in detail on one occasion, and then at the end of an elementary book that would appeal only to followers of Grassmann. Such followers would be unlikely to adopt Peano's axiomatization, which was precisely where he deviated from Grassmann. Moreover, Peano himself later took a quite different axiomatic approach to vectors. Given these circumstances, we would only expect someone under the personal influence of Peano to be likely to adopt his axioms for linear systems. This was in fact what occurred. The first person to adopt Peano's axioms for linear systems, in 1896, was Cesare Burali-Forti, his colleague at the military academy in Turin; somewhat later, Peano's axioms were adopted by a second Italian, Salvatore Pincherle. We now turn to the details.

Peano wrote about vectors (i.e., using that name) only in the context of geometry, but he did so many times. However, his earliest discussion of what we would now regard as vectors occurred in analysis rather than geometry. In a paper of 1887 he gave a proof of the existence of a solution to n first-order homogeneous linear differential equations in n variables, and, in a preliminary discussion, he introduced "number complexes of order n." Such complexes were n-tuples of real numbers, and he defined both their addition and their product by a number in what is now the usual coordinatewise way: If $\mathbf{a} = [a_1, \ldots, a_n]$, $\mathbf{b} = [b_1, \ldots, b_n]$, and k is a real number, then $\mathbf{a} + \mathbf{b} = [a_1 + b_1, \ldots, a_n + b_n]$ and $k\mathbf{a} = [ka_1, \ldots, ka_n]$. He noted that this addition was commutative and associative, while the product was distributive with respect to both factors. After introducing the standard basis and the norm (though without using these terms), he turned to vector functions and their integrals. Then he discussed linear transformations on complexes, representing such a transformation by a matrix and considering the product of two transformations [80, Sects. 2–3]. The following year this paper was translated into French and reprinted, with some modifications, in *Mathematische Annalen* [82].[3] He took a similar ap-

[3] Gray [44, 66] has stated that Peano in his paper [82] "gave an explicit axiomatic definition of an n-dimensional vector space over the reals." In fact, Peano did not do so. In [82] he gave a definition of n-tuples with an addition and scalar multiplication defined by coordinates, as he had done the previous year. It was in the context of geometry, rather than differential equations, that Peano stated in his book [81] of 1888 an explicit axiomatic definition of the concept of vector space over the reals.

proach to such complexes, and once more mentioned the same properties, when he extended his previous result to the theorem that n first-order differential equations, continuous in the neighborhood of a point but not necessarily linear, have a solution there [84, 186–187]. Nowhere in either paper did he use the term "vector," and nowhere did he give any hint of an axiomatic treatment.

Peano took a similar approach via n-tuples in 1895 when he published the first volume of his well known *Formulaire de mathématiques*, which expressed much of mathematics (but not geometry) in his logical symbolism. Number complexes occurred in Part V, "Classes de nombres." While he had not introduced an inner product in 1887 or 1890, he did so in the *Formulaire*, calling it $\mathbf{a} \mid \mathbf{b}$ and giving it the usual definition in terms of coordinates. He noted its properties $\mathbf{a} \mid \mathbf{b} = \mathbf{b} \mid \mathbf{a}$, $\mathbf{a} \mid (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \mid \mathbf{b}) + (\mathbf{a} \mid \mathbf{c})$, and $k(\mathbf{a} \mid \mathbf{b}) = (k\mathbf{a}) \mid \mathbf{b}$. Once again, he made no reference to vectors and did not use an axiomatic approach [87, 58–59].

By contrast, Peano did refer explicitly to vectors and did take an axiomatic approach to generalizing them a few years earlier in his book *Calcolo geometrico secondo l'Ausdehnungslehre di H. Grassmann, preceduto dalle operazioni della logica deduttiva* [81]. This book of 1888 treated the "geometric calculus," which, according to Peano, had originated with Leibniz and was developed especially by Möbius, Bellavitis, Hamilton, and Grassmann. Peano's book was intended to introduce students to Grassmann's approach, which largely included the others, in a way that was clearer and more accessible than Grassmann had done. Only in the last two chapters of the book, so Peano wrote in the introduction, did he introduce new ideas of his own.

Vectors appeared in the book when Peano discussed geometrical formations, a notion adopted from Grassmann. A formation of the first species was a finite expression of the form $m\mathrm{A} + n\mathrm{B} + p\mathrm{C} + \ldots$, where A, B, C, ... were points and $m, n, p, \ldots$ were real numbers. A vector was defined to be a formation of first species that can be put in the form $\mathrm{B} - \mathrm{A}$ [81, 37]. Thus Peano conceived of vectors in quite a traditional way.

It was Peano's final chapter, entitled "Transformations of Linear Systems," that is of most interest. He began it with a definition of linear system (that is, of vector space over the real numbers):

> There exist systems of objects for which the following definitions are given:
>
> (1) There is defined an *equivalence* between two objects of the system, i.e., a proposition, denoted by $\mathbf{a} = \mathbf{b} \ldots$
>
> (2) There is defined a *sum* of two objects $\mathbf{a}$ and $\mathbf{b}$. That is, there is defined an object, denoted by $\mathbf{a} + \mathbf{b}$, which also belongs to the given system and satisfies the conditions:[4]
>
> $$(\mathbf{a} = \mathbf{b}) < (\mathbf{a} + \mathbf{c} = \mathbf{b} + \mathbf{c}),\ \mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a},\ \mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}.$$
>
> (3) Letting $\mathbf{a}$ be an object of the system and m be a positive integer, we mean by $m\mathbf{a}$ the sum of m objects equal to $\mathbf{a}$. It is easy to see that if $\mathbf{a}, \mathbf{b}, \ldots$ are objects of the system and $m, n, \ldots$ are positive integers, then
>
> $$(\mathbf{a} = \mathbf{b}) < (m\mathbf{a} = m\mathbf{b});\ m(\mathbf{a} + \mathbf{b}) = m\mathbf{a} + m\mathbf{b};\ (m + n)\mathbf{a} = m\mathbf{a} + n\mathbf{a};\ m(n\mathbf{a}) = (mn)\mathbf{a};\ 1\mathbf{a} = \mathbf{a}.$$

[4] [Here < means "implies."]

We assume that a meaning is assigned to $m\mathbf{a}$ for any real number m in such a way that the previous equations are still satisfied. The object $m\mathbf{a}$ is said to be the *product* of the (real) number m by the object $\mathbf{a}$.

(4) Finally, we assume that there exists an object of the system, which we ... denote by 0, such that, for any object $\mathbf{a}$, the product of the number 0 by the object $\mathbf{a}$ is always the object 0, i.e.,

$$0\boldsymbol{a} = 0.$$

If we let $\mathbf{a} - \mathbf{b}$ mean $\mathbf{a} + (-1)\mathbf{b}$, then it follows that:

$$\mathbf{a} - \mathbf{a} = 0, \quad \mathbf{a} + 0 = \mathbf{a}.$$

DEF. Systems of objects for which definitions (1)–(4) are introduced in such a way as to satisfy the given conditions are called *linear systems*. [81, 141–142]

As examples of linear systems, he mentioned the real numbers, the complex numbers, formations of first species, vectors in a plane or in space, and formations of higher species. (He considered points in space not to be a linear system, since their sum was a formation of first species rather than a point.)

Peano's most innovative example of a linear system was that of the polynomial functions of a real variable. He noted that if the polynomial functions were restricted to those of degree at most n, then they would form a linear system of dimension $n + 1$. But if one considered all such polynomial functions, he added, then the linear system would have an infinite dimension [81, 142, 143, 154].

This tantalizing reference to an infinite-dimensional linear space was not pursued further by Peano. But he did give another example of a linear system, in which the objects were functions, that was quite surprising for the time: the set of all linear transformations from a linear system A to a linear system B (i.e., Hom(A,B) in modern notation). But Peano was content merely to mention this example, without exploring it further [81, 147].

What interested him most was to extend the infinitesimal calculus by considering functions from the reals to an arbitrary linear system and then defining the continuity, derivative, and integral of such functions. Going further in this vein, he discussed Hom(A,A) and defined a Taylor series for the exponential function $e^{\mathbf{R}}$, where $\mathbf{R}$ was any linear transformation on the linear system A [81, 150].

Peano's book *Calcolo geometrico* was not unread, but his chapter on linear systems exerted no influence outside of Italy. In particular, when Alfred Lotze discussed this book in detail in his article on systems of geometric analysis for the *Enzyklopädie der mathematischen Wissenschaften* [66, 1543–1546], he made no mention of linear systems. Likewise, although Peano's book had a long and positive review in the *Jahrbuch über die Fortschritte der Mathematik*, where his linear systems were described favorably as a generalization of vectors in space, the review does not seem to have stimulated anyone to pursue axiomatic linear systems any further.

What is surprising is that Peano himself hardly mentioned his linear systems in later works. Almost his only reference is found in a brief note of 1894, which gave some corrections to a long article on linear systems by Emmanuel Carvallo (1891). In that article Carvallo used the term "système linéaire" in a much more restricted way than Peano had, namely, as a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ represented by a 3×3 matrix, without mentioning Peano. Carvallo credited the "calcul des systèmes linéaires" to Edmond Laguerre [24, 177–179]. In the note replying to Carvallo, Peano restricted his linear systems to n-tuples [86, 136]. This suggests that Peano was influenced in formulating his axiomatic notion of linear system (1888) by his work on n-tuples, in the context of linear differential equations, the previous year.

To avoid confusion, we must say what Laguerre meant by linear system. This was actually the idea of an $n \times n$ matrix, which Laguerre formulated independently of Cayley.[5] In 1867 Laguerre saw his linear systems or matrices as a way of giving a simple representation of complex numbers, quaternions, Cauchy's "clefs algébriques" (a kind of vector), and Galois imaginaries [62, 230].

Peano published works on his Grassmannian geometric calculus several times (e.g., [85; 89]) without taking an axiomatic approach to vectors or even mentioning his linear systems. The only other work that discussed such systems was his paper of 1895 on linear transformations of vectors in a plane. In it he defined a linear system as follows: "A system of entities is said to be *linear*, if the entities can be added, and multiplied by real numbers, and if the sum and this multiplication preserve the usual properties" [88, Sect. 1]. Although he did not state what these "usual properties" were, he referred to his *Calcolo geometrico* [81]. When he turned to vectors, however, he considered only those in a plane and did not define them axiomatically [88, Sect. 2].

4. THE LIMITED INFLUENCE OF PEANO'S LINEAR SYSTEMS: BURALI-FORTI AND PINCHERLE

Given how little attention Peano paid to his linear systems, it is not surprising that others ignored them as well. Moreover, Peano's axioms for such systems had an abstract form that was highly unusual in 1888.[6] There is every reason to regard Peano's linear systems as premature. It appears that only three mathematicians adopted his axioms.

The first to do so was Burali-Forti, who was influenced by Peano in many ways, such as recognizing the importance of symbolic logic, and who came to share Peano's enthusiasm for Grassmannian methods. In 1896 Burali-Forti published the first of two articles on the use of such methods in projective geometry, adopting Peano's approach rather than Grassmann's. Burali-Forti followed Peano [88] in calling a system "linear" when, for all its elements, there was defined a "sum," as well as a product by a real number, "and such operations enjoy the properties of the corresponding operations on numbers" [16, 183]. He was mainly concerned with n-dimensional linear systems of geometric forms, for n at most four, in the context

[5] As Hawkins has noted [52, 108], matrix algebra was formulated independently of Cayley not only by Laguerre but also by Eisenstein, Frobenius, and Sylvester.

[6] Only Dedekind gave an equally abstract form to his postulates for the positive integers in the same year, as did Peano independently a year later.

of projective transformations. When he used Grassmannian methods the following year in differential geometry [17], he did not mention Peano's linear systems.

Burali-Forti did not use such systems again until 1910, in his book *Eléments de calcul vectoriel*, written jointly with Roberto Marcolongo, who was professor of rational mechanics at Naples. Even then, linear systems were not defined axiomatically, but Grassmannian geometric forms (of dimension at most 3) were shown to have the properties that would make them a linear system [18, 18 and 182].

By contrast, Burali-Forti and Marcolongo explicitly stated axioms for linear systems in their book *Transformations linéaires* (1912), where such systems formed the basis of their work. But the emphasis, as in their earlier book, was on applications in mechanics. They began their book as follows: "We briefly set forth the foundations of the general theory of *linear systems* and *linear operators*. Generally, these matters are familiar in large part" [19, 1]. They then stated axioms for a linear system more or less as Peano had done in his 1888 book.[7] They pointed out, as Peano had, the important fact that the linear operators between two linear systems themselves form a linear system [19, 6]. In the second edition of their book, they gave a similar treatment [20, 16–18]. Despite their claim, the axiomatic notion of linear system was certainly not familiar to most mathematicians in 1912.

The third person to adopt Peano's axioms for a linear system was the Italian analyst, Salvatore Pincherle. During the 1890s, Pincherle emphasized linear spaces of functions and linear operators on them. His interest grew from Vito Volterra's work on the functional calculus and from Peano's approach to linear systems. In 1897 Pincherle wrote in *Mathematische Annalen*:

> It remains for us to cite certain works which also concern the functional calculus, but which treat it from a new viewpoint, that can make certain generalities in this calculus very clear and almost intuitive. This is the viewpoint of vectors or the geometric calculus .... On this subject Peano has written some very interesting pages [81, 141] where ... he gives the simplest properties of distributive [i.e., linear] operations applied to elements determined by *n* coordinates .... The author notes ... that one could also consider linear systems with an infinite number of dimensions. [95, 330]

Although he did not state Peano's axioms for linear systems, he indicated where Peano had given them and drew attention to infinite-dimensional vector spaces. Pincherle's own interests were strongly focused on linear operators on infinite-dimensional spaces.

Four years later, Pincherle published a book about such operators on what he called a linear set or linear space ("insieme o spazio lineare") and gave a variant of Peano's axioms [96, 1–4]. Perhaps Pincherle's most intriguing example of a linear space was the set of analytic functions, which he regarded as a space with countably infinite dimension [96, 73 and 465].

Pincherle viewed himself as the successor of Laguerre, Carvallo, and Peano in treating linear systems. But while Pincherle saw his three predecessors as concerned

[7] One of the factors that likely encouraged them to use his axioms was their emphasis on using vectors intrinsically, i.e., without coordinates [19, viii].

with finite-dimensional spaces, he emphasized the infinite-dimensional [96, 465]. What is interesting is that he saw a common subject here. For Laguerre, who introduced the term "système linéaire," was working with matrices [62, 216], and Carvallo (1891) did likewise, closely following Laguerre's approach. Peano, on the other hand, did not refer to matrices at all, but only to linear systems (i.e., vector spaces) and their linear transformations [81, 141–170]. Yet Peano did see Carvallo's linear systems (i.e., matrices) as instances of his own linear systems, without explicitly stating that square matrices of fixed order were an example of a vector space [86].

In 1912 Pincherle referred again to Laguerre, Carvallo, and Peano in his article on functionals for the French encyclopedia of mathematics [97, 21–22] and even wrote of "linear space." But the definition he gave was not axiomatic, merely the linear combinations of n linearly independent elements for some n, where the scalars were real. Even at this late date the axiomatic approach did not come altogether naturally to Pincherle.

5. PEANO'S SECOND AXIOMATIZATION OF VECTOR SPACE

One of the reasons that Peano's 1888 axiomatization of linear system had such a limited impact was that a major shift in his treatment of vectors occurred in 1898. During the previous decade he had handled vectors by presupposing geometry, so that a vector was defined in the traditional way as a directed line segment. Now he reversed the process, and decided to axiomatize geometry by using vectors. This was not his first axiomatization of geometry. A decade earlier, his axioms for geometry used as primitive ideas "point," "segment," and a ternary relation, "c is an interior point of the segment ab" [83, Sect. 1]. Now his approach and motivation had changed:

> In this work I propose to consider which ideas are encountered in the theory of vectors and to classify them into primitive ideas, which are extracted from the observation of physical space, and into derived ideas, whose definition is given; and to consider which propositions must be assumed to be primitive and which are deduced as consequences of them, by purely logical processes, without recourse to intuition.
>
> Thus the theory of vectors appears to be developed without presupposing any previous geometric study. And since, by means of this theory, all of geometry can be treated, there results thereby the theoretical possibility of substituting the theory of vectors for elementary geometry itself. [90, 513]

He took two ideas as primitive, that of point and that of a four-place relation of "equidifference" between points, which he wrote $a - b = c - d$. He gave various interpretations to establish the independence of his axioms.

Peano stated 11 axioms. The first three required the equidifference relation to be an equivalence relation. The next permitted the interchange of middle terms:

4. If $a - b = c - d$, then $a - c = b - d$.

Then the set of vectors was defined to be all those x such that $x = b - a$ for some points a and b. In particular, the zero vector 0 was defined to be the unique x such that, for any point a, $x = a - a$. The first axiom specifically concerning vectors was the following:

5. If a is a point and u is a vector, then there is a point b with $b - a = u$.

A notion of addition was defined such that if a is a point and u is a vector, then $a + u$ is a point. From this, an addition was defined between vectors by the condition that, for any vectors u and v, $u + v$ is the unique x such that for any point a, $x = ((a + u) + v) - a$. Then he proved that vector addition satisfies the commutative and associative laws. The inverse $-u$ of a vector $u = a - b$ was defined to be $b - a$, and it was shown that $u + (-u) = 0$.

Peano next turned to the product of a vector by a number. At first he introduced this operation only for integers, and then gave two further axioms:

6. If a is a positive integer and u is a vector, then $au = 0$ yields $u = 0$.
7. If a is a positive integer and u is a vector, then there is a vector v such that $av = u$.

Then he noted: "The definition of the product of an irrational number by a vector presents grave difficulties at this stage. We define it symbolically only after introducing the ideas of distance and limit" [90, 525]. It is unclear why Peano laboriously extended the scalar product from the case where the scalars were positive integers to that of rational numbers and finally to the case where the scalars were real numbers, since he took the real numbers as given in his next axiom (below). But a somewhat similar process was at work in Weyl's later axiomatization of vectors (see Section 7 below).

Before introducing distance and limit, Peano gave four axioms for the inner product of vectors u and v, which he denoted by $u \mid v$:

$u \mid v$ is a real number; $u \mid v = v \mid u$;
$(u + v) \mid w = u \mid w + v \mid w$; if $u \neq 0$, then $u \mid u$ is a positive real number.

Finally, the length of u was defined as the square root of $u \mid u$ [90, 530]. In contrast to his other axioms for vectors, these axioms for inner product later became standard. (Gray has raised the question "when did the concept of a vector space with additional structure, such as an inner product, first emerge" [44, 68]. The answer is in 1898, in this work of Peano.)

He ended his article with four more axioms. The first of them was the fulfillment of his promise to extend the scalar product to irrational numbers. The others stated in effect that the space was three-dimensional [90, 533].

In 1899 Peano devoted a section to vectors in the second edition of his *Formulaire de mathématiques*. There he changed what he had done in the first edition (where he had used n-tuples) and instead adopted the approach of his 1898 paper, but with a few modifications [91]. He explicitly added two new axioms, stating that all the points formed a class and that this class was nonempty. Next he added an axiom 4′ which in 1898 he had thought to follow from his other axioms but which he now showed to be independent: If $a - c = b - c$, then $a = b$. Peano continued to use this revised axiom system for vectors in the later editions of the *Formulaire*, e.g., [92, 192–209; 93, 165–179].

Peano's 1898 axiomatization of vectors had little influence, except on Bertrand Russell. In 1903 Russell discussed the matter while giving a vector definition of

Euclidean space in the *Principles of Mathematics*. He observed that this definition "is inappropriate when Euclidean space is considered as the limit of certain non-Euclidean spaces, but is very appropriate to quaternions and the vector Calculus" [100, 432]. This definition consisted, more or less, in a translation of Peano's 1898 axioms into Russell's logic of relations. "This definition is," he concluded, "by no means the only one which can be given of Euclidean space, but it is, I think, the simplest" [100, 434].

6. A CONCRETE APPROACH TO AXIOMATIZING VECTORS: DARBOUX, SCHIMMACK, AND HAMEL

A different approach to axiomatizing vectors emerged from the work of Gaston Darboux. In 1875 he published an article analyzing various proofs of the composition of forces in statics (i.e., the parallelogram law), beginning with one due to Daniel Bernoulli in 1726. Darboux set himself the task of treating this matter in pure geometry and then determining which assumptions are necessary. He found four [26, 283–288]: Given n directed segments, all beginning at the same point O, the law of composition is such that:

1. The total resultant is unique and is unchanged by permuting the order of the partial resultants.
2. The total resultant is unchanged by a rotation of the segments about O.
3. The law of composition reduces to algebraic addition for segments having the same direction.
4. The direction and magnitude of the resultant are continuous functions of the segments.

In 1903 Darboux's four axioms for vectors were taken up by two Germans, Rudolf Schimmack and Georg Hamel.

From 1903 to 1907, Schimmack was an assistant at the Mathematical Institute in Göttingen, and he received his doctorate from Göttingen in 1908. His dissertation [104] was devoted to axiomatizing vector addition. In fact, it was an elaboration of his 1903 paper [103] revising Darboux's axioms. That paper was presented by Hilbert to the Göttingen Academy of Sciences.

While Darboux had proposed four axioms, Schimmack offered seven. Following Darboux and the traditional approach, Schimmack defined a vector as a directed line segment in Euclidean space. Then he split Darboux's first axiom into three, giving the uniqueness of vector addition, its commutativity, and its associativity. Darboux's third axiom was replaced by two, the first stating that there is a zero vector and the second that if a vector $\mathfrak{B}$ has length r, then for any positive scalar a the length of the vector $\mathfrak{B} + a\mathfrak{B}$ is $(1 + a)r$. Schimmack accepted Darboux's second and fourth axioms without change [103, 318].

Much of Schimmack's paper was devoted, in a Hilbertian style, to questions of the consistency and independence of the seven axioms. In particular, the independence of Darboux's fourth axiom was equivalent to the existence of a discontinuous real function f satisfying the functional equation $f(x + y) = f(x) + f(y)$ for all real x and y. Since no one had shown the existence of such a discontinuous solution, Schimmack left the independence of this axiom as an open question [103, 321].

This functional equation was to be involved repeatedly with vectors. Long before, Cauchy had shown that its only continuous solutions are $f(x) = kx$, where k is any constant. Darboux himself considered the possible solutions in 1875 and again five years later in a paper on projective geometry [27]. As we shall see, it was Hamel who, in the context of axioms for vectors, found a discontinuous solution.

In 1903, apparently unaware of Schimmack's paper, Hamel gave his own analysis of Darboux's axioms. This analysis agreed in essentials with that of Schimmack. Hamel too noted that in order to show the independence of Darboux's fourth axiom it would suffice to find a discontinuous solution f to the equation $f(x + y) = f(x) + f(y)$. Much of Hamel's paper [47] was devoted to the independence of Darboux's second axiom.

Like Schimmack, Hamel had strong ties to Göttingen. In 1901 Hamel had written his doctoral dissertation in geometry there under Hilbert. Hamel's paper giving a discontinuous solution to $f(x + y) = f(x) + f(y)$ was published by Hilbert in the *Mathematische Annalen* in 1905 as part of a series of papers debating the merits of Ernst Zermelo's theorem that every set can be well-ordered. This theorem had only been published in October 1904, but on 30 November, Hamel sent Hilbert a paper for the *Annalen* responding to Zermelo. In his covering letter Hamel wrote:

> I would like to submit to you a short note in which I prove that (1) there is a basis for all numbers and that (2) there exist discontinuous solutions for the functional equation $f(x + y) = f(x) + f(y)$. And in fact I give all the solutions to it. The proof relies on the proposition that the continuum is equivalent to a well-ordered set. And since, thanks to the proof of Herr Zermelo, this proposition has been made as secure as can be expected in the present condition of set theory, the same is true, I hope, for the two propositions given above. If, honored Professor, you should consider my note to be suitable for the *Annalen*, it would please me very much.
>
> I remain your grateful and devoted student,
>
> G. Hamel

Hamel concluded his paper by pointing out that the axioms for vector addition require a continuity axiom, i.e., Darboux's fourth axiom needs to be adopted [48, 460]. It is interesting that Hamel, like Darboux before him, would not take the real numbers as given in treating vectors axiomatically, but insisted on including continuity as one of the axioms.

It is worth examining more closely what Hamel called "a basis for all numbers." In modern terms, this was a basis for the vector space of the real numbers over the field of the rational numbers. But at this time there was no general concept of the "basis" of a vector space. In fact, there were at least two different ideas waiting to be discovered: first, that of maximal linearly independent set, and, second, that of maximal orthogonal set. For finite-dimensional vector spaces, these two notions were equivalent. For infinite-dimensional vector spaces with an inner product, they disagreed completely. It was the first of them that Hamel used in a new way in his paper. But this concept of the "basis" of a vector space over the real numbers (in terms of linear independence) was only given a general formulation by Banach [7, 231] and by Hausdorff [50, 174; 51, 295] more than two decades later.

And so, in the interim, the notion of "Hamel basis" was not treated generally but in the particular way that Hamel had done, namely, as a set H of real numbers such that every real number was a unique linear combination of some finite number of members of H with rational coefficients. In 1908 Zermelo cited the existence of such a Hamel basis and of a discontinuous solution to the functional equation $f(x + y) = f(x) + f(y)$ as evidence for the truth of his Axiom of Choice [138, 114–115].

Until about 1930, Hamel bases were investigated primarily in the context of analysis and of descriptive set theory. Thus in 1920 the Polish set theorist Wacław Sierpiński showed, using the Axiom of Choice, that while some Hamel bases are Lebesgue measurable, others are not. Any Hamel basis that was Lebesgue measurable had measure zero. On the other hand, no Hamel basis was a Borel set or even an analytic set. Without the Axiom of Choice, the existence of a Hamel basis implied the existence of a set of reals that was not Lebesgue measurable [108].

Sierpiński had been influenced by earlier work on Hamel bases done by the Austrian mathematician, Celestyn Burstin. In 1916 Burstin gave a Hamel basis intersecting every perfect set of real numbers when he partitioned the real line into continuum many non-Lebesgue-measurable sets. Burstin also proved that in the space of real functions on the interval [0, 1] there are continuum many, all non-measurable, that are orthogonal [21, 217]. By contrast, Erhard Schmidt had shown previously for the same space that there are only countably many continuous functions that are orthogonal, while Frigyes Riesz had extended Schmidt's result by replacing "continuous" with "measurable" [98, 740].

Burstin used a Hamel basis again in 1929 when discussing ordered vector spaces over the reals (i.e., spaces with an ordering that preserves vector addition). There he showed that any vector space over the reals can be ordered so as to satisfy the Archimedean Axiom [22]. What had changed in the interim was that abstract vector spaces had been accepted.

7. WEYL AND FINITE-DIMENSIONAL SPACES

At the end of the First World War, the situation with regard to vector spaces was the following. The general notion of a vector space over the real numbers was barely alive in Italy. A much more limited notion of vector space, axiomatized by Darboux in France and studied by Schimmack and Hamel in Germany, was not developed after 1905. But it led to a specific "Hamel basis" that continued to be a subject of research in analysis. Then, in 1918, another mathematician axiomatized the notion of vector space over the reals. This was Hermann Weyl, who gave no indication of being aware of Peano's earlier axiomatization but who might have seen it in Pincherle's works on functional analysis.

Weyl axiomatized real vector spaces in his book, *Raum, Zeit, Materie* (1918), based on lectures on general relativity that he had given the previous year in Zurich. In the foreword he described his book as an intermingling of philosophical, mathematical, and physical thought. His aim was to build the theory of relativity systematically from its foundations by emphasizing the underlying principles. Perhaps it was this emphasis on principles that encouraged him to give axioms for vectors rather than treating them only in traditional fashion. Perhaps too, his acquaintance with Hilbert's axiomatic method played a role.

In Weyl's book, vectors were part of the foundations of geometry. Here his concern was to restrict the general notion of congruence-preserving transformation to that of translation in order to treat affine geometry axiomatically. He conceived of a vector intuitively as a displacement in space.

Weyl's axioms for affine geometry were in two parts. The first part, which axiomatized vectors, was essentially Peano's first (1888) set of axioms. But one axiom of this part separated him quite strongly from Peano's approach. Whereas Peano explicitly allowed linear systems to be infinite-dimensional, Weyl ruled out that possibility. The last of Weyl's axioms was his Axiom of Dimension: "There are n linearly independent vectors, but every $n + 1$ vectors are linearly dependent" [129, Chap. I, Sect. 2].

What Weyl had axiomatized was the notion of a finite-dimensional vector space over the reals. The matter is slightly more complicated, since he was ready to abandon his axioms (β), which concerned scalar multiplication (i.e., axioms V1–V4 in Sect. 1 above). The reason for abandoning them was that, for rational scalars, his axioms (β) followed from his remaining axioms, if scalar multiplication was defined in the appropriate way. Moreover, the continuity of the real numbers allowed that definition to be extended so as to dispense with the axioms (β) altogether. But Weyl's discomfort with the principle of the continuity of the real numbers induced him to keep his axioms (β) and, instead, "to banish continuity, which is so difficult to grasp, from the logical structure of geometry" [129, Chap. I, Sect. 2].

Then Weyl completed his axiomatization of affine geometry with a second set of axioms which connected the concepts of point and vector, axioms that were reminiscent of Peano's second (1898) axiomatization of vectors:

> 1. Any two points determine a vector $\mathfrak{a}$; in symbols, $\overrightarrow{AB} = \mathfrak{a}$. If A is any point and $\mathfrak{a}$ is any vector, then there is one and only one point B for which $\overrightarrow{AB} = \mathfrak{a}$.
> 2. If $\overrightarrow{AB} = \mathfrak{a}$ and $\overrightarrow{BC} = \mathfrak{b}$, then $\overrightarrow{AC} = \mathfrak{a} + \mathfrak{b}$. [129, Chap. 1, Sect. 2]

This concern to link vectors with points brought Weyl in line with the traditional geometric treatment of vectors as directed line segments. Moreover, he immediately turned to treating vectors in terms of coordinates by assigning basis vectors as appropriate n-tuples [129, Chap. I, Sect. 3]. Vectors themselves were then superseded by tensors: "From now on we shall no longer use the term 'vector' as being synonymous with 'displacement' but to signify a 'tensor of the first order'" [129, Chap. I, Sect. 5].

Weyl began his section on metric geometry by treating the notion of inner product intuitively in 3-dimensional Euclidean space. His definition of inner product was traditional and presupposed the concepts of length and projection: the inner product of $\mathfrak{a}$ and $\mathfrak{b}$ is the length of $\mathfrak{a}$ multiplied by the directed length of $\mathfrak{b}$ projected on $\mathfrak{a}$. But then he introduced his "Metric Axiom," stating that an inner product $\mathfrak{a} \cdot \mathfrak{b}$ is a symmetric bilinear form which is positive definite and that there is a vector $\mathfrak{e}$ with $\mathfrak{e} \cdot \mathfrak{e} = 1$. What Weyl had actually done was to axiomatize the notion of a finite-dimensional inner-product space, much as Peano had done [129, Chap. 1, Sect. 4]. Weyl's axioms for the inner product were the modern ones (P1–P4 in Sect. 1 above).

How influential were Weyl's axioms for a finite-dimensional vector space? Perhaps even less influential than those of Peano.[8] Certainly Weyl himself made no priority claim for these axioms in a later historical article where he discussed vector spaces, praised Hilbert spaces, and belittled Banach spaces [131, 536, 541, 549]. In 1928, when Weyl's book *Gruppentheorie und Quantenmechanik* appeared, he again gave his axioms for a finite-dimensional vector space, but he now omitted the axioms connecting points and vectors. What is surprising is that he insisted on keeping the axiom that made all his vector spaces finite-dimensional, since later in the book he discussed complex Hilbert space and, at that point, temporarily dropped the axiom of finite dimensionality. He did so with applications to quantum mechanics explicitly in mind [130, Chap. I, Sects. 1–7].

In that book Weyl also showed a confusion that was fairly common in the early history of infinite-dimensional vector spaces; he regarded Hilbert space as having a countably infinite dimension. Moreover, he insisted that there was no distinction between a space having countably infinite dimension and one having as its dimension the power of the continuum [130, 29–30]. The confusion was between the number of the space's coordinates, which was countably infinite, and the maximal number of linearly independent vectors (i.e., the cardinality of a Hamel basis), which was uncountable. For finite-dimensional spaces these two concepts agreed, and Weyl did not distinguish between them [130, 5]. But he still did not distinguish between them for Hilbert space, and that was an error. Likewise he described the space of continuous functions as having continuum many dimensions [130, 29].

To sum up, neither Peano nor Weyl played a decisive role in the diffusion of axiomatic vector spaces. The notion had to be discovered a third time by three mathematicians working independently in three different countries: Stefan Banach in Poland, Hans Hahn in Austria, and Norbert Wiener in the United States.

8. THE DEFINITIVE AXIOMATIZATION OF VECTOR SPACE: THE ROLE OF A NORM

Banach, Hahn, and Wiener all discovered the notion of a normed vector space through their researches in analysis. All were interested in a notion that generalized both algebraic and topological properties of various spaces. All of them (with the possible exception of Banach) knew Fréchet's notion of metric space and saw its relevance to their concerns. We first discuss the work of Hahn and Wiener, leaving Banach's contribution until later since it was the most important.

In 1922 the Viennese analyst Hans Hahn formulated the notion of a normed vector space, which he called a linear space, in work that was on the border of classical analysis and functional analysis [45]. Hahn was moved to do so by the desire to unify his treatment of singular integrals (i.e., limits of definite integrals)

[8] Up to 1933, the only known reference to those axioms is in Stone's book on abstract Hilbert spaces, a book primarily influenced by von Neumann [111, 17].

and Issai Schur's results on linear transformations of infinite series. After introducing his definition of linear space, Hahn used the norm to make it a metric space. He then defined metric completeness for it, giving a complete normed vector space. By contrast, Hahn showed no interest in vector spaces without a norm, and did not even formulate such a notion.

Hahn's axioms for a norm came directly from a paper published in the same journal the previous year by another Viennese mathematician, Eduard Helly. Whereas Hahn formulated those axioms in the general setting of a vector space, Helly had done so for an n-dimensional Euclidean space and for one infinite-dimensional space of sequences [53, 61 and 67]. But Helly had not given any axioms for a vector space or even hinted at such a concept. His aims were more concrete.

Most of Hahn's paper was devoted to discussing 21 different normed vector spaces, which were all function spaces, and to linear transformations on these spaces. He was especially concerned with conditions that were equivalent to convergence of sequences of points in these spaces. More generally, he was interested in bounded or convergent sequences of operators on such spaces [45, 5–10].

The lack of general results in Hahn's 1922 paper contrasts with his second article (1927) on complete normed vector spaces, where he mentioned Banach's independent discovery of these spaces. Hahn's second article was concerned with linear systems of equations in such spaces. It was motivated by problems in integral equations, but formulated its results in terms of linear subspaces of normed vector spaces. He used transfinite induction to define a chain of such linear subspaces well-ordered by inclusion. The high point of his paper was his version of the Hahn–Banach Theorem:

> If R is a complete normed vector space having a complete subspace R_0, and if $f_0(x)$ is a linear functional defined on R_0 and having norm M, then there is a linear functional $f(x)$ defined on all of R that agrees with $f_0(x)$ on R_0 and also has norm M.[9] [46, 217]

Besides this extension theorem, Hahn began to study the dual or adjoint space of a complete normed vector space V, i.e., the space of all bounded continuous real-valued functionals on V.[10]

9. WIENER AND NORMED VECTOR SPACES

While Weyl's concerns were with affine geometry and mathematical physics, Wiener's were with functional analysis and involved a good deal of topology. While Hahn's work was rooted in real analysis, Wiener's was much more oriented to abstract spaces in the spirit of Fréchet.

The young Wiener first stated axioms for a normed vector space in a paper that he gave in 1920 to the International Congress of Mathematicians at Strasburg. His aim was to determine those general "spaces" in which the concepts of sequential limit, neighborhood, and homeomorphism agree. As an initial step, he took two

[9] For a normed vector space V, the norm of a bounded linear functional $f(x)$ on V is the least number M such that $\| f(x) \| \leq M \| x \|$ for all x.

[10] On Hahn's work, see [9, 70–72, 78–84].

undefined notions: a set K and a set Σ of bijections on K. The concepts of derived set and closed set were defined in terms of invariants of these bijections. Fréchet's metric spaces were emphasized.

In this context, Wiener introduced what he called a "vector system," or system (Ve), as a set K of points and a set σ of vectors with operations $\oplus$ (vector addition), $\odot$ (scalar multiplication), and $\| \ \|$ (norm) satisfying 14 axioms. In effect, his axioms defined something more or less like a normed vector space, but with no mention of completeness. Any two points A and B in K determined a vector AB in σ. One axiom stated that, given any A in K and α in σ, there was some B in K such that AB $= \alpha$. Other axioms relating points and vectors were reminiscent of Peano: If A, B, C, D are in K, then AC $=$ AB $\oplus$ BC, and AB $=$ CD implies BA $=$ DC [132, 312–313].

Wiener's axiom system (Ve) had certain failings. It did not state that $\oplus$ was commutative, associative, or had an identity, as one might expect. Moreover, his operation $n \odot \alpha$ was defined only for non-negative real numbers n—in contrast to the approach of Weyl, Banach, and Hahn, who defined scalar multiplication for all real numbers. Consequently, Wiener did not consider the inverse of his vector addition.

Wiener realized that his norm $\|AB\|$ between points A and B made his vector space into a metric space. Among his examples of a normed vector space were n-dimensional Euclidean space, Hilbert space, the space of continuous real functions on a closed interval, and the space of bounded real sequences. While limits were to be taken uniformly in the last two spaces, he was not sure whether the space of real sequences was a normed vector space if limits were taken pointwise rather than uniformly [132, 313].

In 1922 Wiener published two more articles on normed vector spaces [133; 134]. The latter article proposed "to develop a categorical theory of the structure of the [real] line in terms of bicontinuous, biunivocal transformations, or, in other words, to give a complete postulational characterization of the analysis situs group of the line" [134, 329]. Here too normed vector spaces played a role, but a minor one. The only new idea was what he called a system (Vr), or "a restricted vector system," which was a system (Ve) in which, among other things, the commutative law held for vector addition [134, 333].

Three decades later, in his autobiography, Wiener discussed his work of this period and its connection with Fréchet:

> Fréchet's generalized theory of limits ... applies to many sorts of spaces, including vector spaces, but is not ... confined to those spaces in which the elements may be regarded as steps [vectors]. On the other hand, this geometry of steps constitutes a very important part of Fréchet's general theory and was worth solidifying with an appropriate set of postulates. Fréchet had not done this, nor did he consider these particular vector systems as peculiarly important....
>
> I gave a full set of axioms for vector spaces. Fréchet liked it, but did not seem particularly struck with the result. But then, a few weeks later, he became quite excited when he saw an article published by Stefan Banach ... which contained results practically identical with those I had given, neither more nor less general .... [136, 59–60]

Wiener added that his work and Banach's "came for a time to be known as the theory of Banach–Wiener spaces" but that, after publishing a little more, he gradually left the field. The principal factor, he remarked, which

> led me to abandon the theory of Banach spaces ... was that my work on the Brownian motion was now coming to a head. Differential space, the space of the Brownian motion, is itself in fact a sort of vector space, very close to the Banach spaces, and it presented itself as a successful rival for my attentions because it had a physical character most gratifying to me. In addition, it was wholly mine ... whereas I was only a junior partner in the theory of Banach spaces. [136, 64]

10. BANACH AND THE RISE OF BANACH SPACES

Banach's research on vector spaces was much more influential than that of the mathematicians discussed earlier. Out of his research, which combined in a judicious fashion a concern with an abstract axiomatic framework and with applications in analysis, there came a vigorous tradition of investigating normed vector spaces, especially complete normed vector spaces, i.e., Banach spaces.

Banach's paper of 1922, which was his doctoral dissertation of 1920, introduced the notion of Banach space:

> The aim of the present work is to establish certain theorems valid in different functional domains, which I will specify in what follows. Nevertheless, in order not to have to prove them for each particular domain, which would be painful, I have chosen to take a different route ...: I consider sets of elements about which I postulate certain properties; I deduce from them certain theorems, and I then prove for each particular functional domain that the postulates adopted are true for it. [3, 134]

Here Banach used the axiomatic method, but in a way quite different from Hilbert. Banach's aim was not to characterize a certain mathematical domain by axioms, as Hilbert had done for Euclidean geometry, but to establish theorems true for a *class* of domains by giving axioms for that class, and then to show that those theorems were true for a particular domain by showing merely that it satisfied the axioms. This version of the axiomatic method, so common in mathematics now, was relatively new in 1920 and not common in analysis, having been used rarely except in certain branches of algebra (primarily group theory and field theory).

Banach considered his 1922 paper to be in the tradition of Volterra's functions of lines (i.e., real-valued functions whose arguments were curves). But Banach's axiomatic approach, based on set theory, was quite different from earlier work in that tradition. He began his paper with a first group of 13 axioms, which defined the notion of vector space (over the reals). Here he cited as examples Grassmannian forms, quaternions, hypercomplex number systems, and vectors of the traditional sort. His second group of axioms was that for a norm, and his third was that every Cauchy sequence converged. (Here it would have been natural to express matters in terms of a metric space, but he did not do so in this paper).

The first part of Banach's paper was largely devoted to the connection between boundedness and continuity for operators on Banach space. (His notion of continuity was that of sequential continuity.) The second part was concerned with additive operators on such spaces. The third and final part gave three further axioms. These were formulated for spaces of measurable real functions on an interval and stated in terms of asymptotic convergence, a notion due to Hardy and Landau [3, 163]. Banach then deduced various theorems about asymptotic convergence, and ended by showing that all of his axioms, those for a Banach space and those for asymptotic convergence, held for 10 different kinds of function spaces.

The first reaction in print to Banach's paper was by Wiener, who submitted a note to the journal where it had appeared. There he acknowledged Banach's axioms for a (real) Banach space and extended them to complex Banach spaces. Wiener also pointed out there that "postulates not unlike those of M. Banach have been given by me on several occasions ... I have here employed M. Banach's postulates rather than my own because they are in a form more immediately adopted to the treatment of the problem in hand" [135, 143].

11. FRÉCHET'S RESPONSE TO BANACH SPACES

Apparently the first mathematician to respond to the axioms for normed vector spaces was the analyst Maurice Fréchet. In 1925 he published two papers on such spaces, and then returned to the subject in his book *Les espaces abstraits* (1928).

In his first paper [40] he compared the two systems of axioms proposed by Banach and Wiener. Fréchet was unaware that normed vector spaces had been axiomatized by Hahn as well. But Fréchet realized—unlike Banach, Hahn, and Wiener—that the notion of vector space had been axiomatized previously, and cited the book *Le operazione distributive* by Pincherle (1901) as containing such an axiomatization. According to Fréchet, Pincherle had credited these axioms to Laguerre (1867) and Peano (1888). In fact, Laguerre's work contained no such axiomatization and dealt only with matrices, whereas Peano's did axiomatize the notion of vector space over the reals. Fréchet held that "the merit of MM. Banach and Wiener consists in having recognized that these postulates [for vector spaces] are verified by spaces much more general than n-dimensional spaces and in having given some examples" [40, 52]. But, we must stress, Peano and Pincherle had already recognized the same thing some years earlier. The importance of Banach's and Wiener's work was, first, in formulating the axiomatic notion of norm for a vector space—a notion which, from Fréchet's remarks, one might expect to find in Peano or Pincherle, but which is not there—and, second, in causing the notion of normed vector space to become widely accepted.

Fréchet adopted a hybrid of Banach's and Wiener's systems and called it an "espace (D) vectoriel," i.e., a vector space whose norm yields a metric space. Wiener had failed to include certain postulates needed for vector addition, postulates that Banach had stated and that Fréchet now adopted. On the other hand, Fréchet was happier with Wiener's approach concerning the relation between points and vectors. That is, Wiener had introduced axioms relating points and vectors, so that a vector was determined by a pair of points in the traditional way. Banach had only introduced vectors and not a separate category of points. Fréchet insisted on having both points and vectors in his metric vector spaces and in defining them by axioms like those of Wiener.

Fréchet's interest in vector spaces was in the context of what, following E. H. Moore, he called general analysis. This was, for Fréchet, a topological generalization of function spaces to allow for abstract elements. Thus it is not surprising that, when Fréchet returned to the subject of vector spaces later in 1925, his concerns were primarily topological. His aim was to generalize normed vector spaces, which were metric spaces, so as to include certain interesting function spaces that were not metric spaces.

His main generalization was that of "topological affine space." Its axioms were the same as for a normed vector space, except that he replaced the triangle inequality by three weaker conditions:

> (a) Every accumulation point of a subset of a line lies on that line.
> (b) A point A on a line is an accumulation point of a set M on that line if and only if the norm of $A - B$ has zero as least upper bound for any B in M.
> (c) Every translation is continuous, and so is its inverse. [41, 42]

The notions of accumulation point and continuity were defined in terms of sequential limit, based on his L-spaces, which were more general than metric spaces [39, 5–6]. In contrast to a normed vector space, a topological affine space need not be a metric space. In the case where it was, he called it a metric affine space. Here the metric might differ from the length of a vector as given by the norm [41, 27]. When he gave a systematic treatment of such matters in his book *Les espaces abstraits* [42], he did not go substantially beyond his papers of 1925.

These topological affine spaces were the first glimmering of the important notion of topological vector space. In 1925 Fréchet was the first to consider axiomatized vector spaces that had a topological but not a metric structure, and his topological affine spaces were studied two years later by the young André Weil, who considered them to be the abstract foundation for linear analysis [128]. The concept of topological vector space, i.e., a vector space with a topology in which vector addition and scalar multiplication are continuous, was formulated a few years later by Kolmogorov [60], who was influenced not by Fréchet but by Banach's book of 1932 on linear operators.

12. BANACH RETURNS TO VECTOR SPACES

After Banach published his dissertation on Banach spaces in 1922, he turned to other subjects for research. He came back to vector spaces seven years later in two papers on extensions of linear functionals [4; 5]. The second paper contained his independent discovery of the Hahn–Banach theorem. These papers appeared in the first volume of a new journal, *Studia Mathematica*, which he had founded at Lwów with Hugo Steinhaus. Devoted to functional analysis, its first two volumes contained several papers, by mathematicians at Lwów, which used normed vector spaces: Orlicz [77] on orthogonality for spaces of measurable functions, Mazur [72] on the zeroes of linear operators, and Schauder [102] on the inverse of a continuous linear operator. Thus Lwów was well on its way to becoming a world center of research on normed vector spaces.

At that time Banach's interests turned to generalizing Banach spaces by means of topological groups, i.e., groups $(G, *, ^{-1})$ that have a topology which makes the group operations $*$ and $^{-1}$ continuous. In particular, he invented the notion of a G-space, i.e., a group that is a complete metric space and whose two group operations are sequentially continuous. A typical theorem was that every subgroup H of a G-space such that H has the Baire property either is of first category or is both open and closed [6, 104]. Every Banach space was a G-space when considered as a group with respect to its vector addition.

Influenced by Fréchet's paper [41], Banach also defined the notion of F-space, i.e., a vector space that is also a complete metric space, sequentially continuous in each argument of its scalar multiplication, such that for its metric $d(x,y) = d(x - y, 0)$. This condition on the metric ensured that translations preserve distance [6, 113]. Banach's F-spaces—now called Fréchet spaces—were inspired by Fréchet's metric affine spaces but were not identical with them.

Banach's seminal book (1932) on the theory of linear functionals unified and simplified the earlier work on Banach spaces. But the book took a much more general view. After some preliminaries, it began with his G-spaces, on which he considered additive operators U (i.e., with $U(x + y) = U(x) + U(y)$.) In a G-space, an additive operator continuous at one point was continuous everywhere—a theorem that generalized Hamel's result (1905) for the case when U was a real function.

The second chapter of Banach's book discussed vector spaces. While in his 1922 paper introducing normed vector spaces he had only considered properties that depend on the norm, now he considered arbitrary (real) vector spaces. He defined the general notion of a (Hamel) basis H for a vector space E as a set of vectors such that each vector in E is a unique linear combination of finitely many vectors in H. Then he pointed out that every vector space has a basis and that any two bases will have the same cardinality [7, 231]. This appears to be the first time that vector spaces over the real numbers were treated as a distinct topic, in complete axiomatic generality and without consideration of a norm or restriction to a finite dimension.

However, the main topic of that chapter on vector spaces was the Hahn–Banach Theorem. In 1929 Banach had formulated this theorem in the context of normed vector spaces, but had stated one version that did not use a norm: if a functional $p(x)$ on a vector space E is subadditive (i.e., $p(x + y) \leq p(x) + p(y)$) and homogeneous (i.e., $p(\alpha x) = \alpha p(x)$ for all nonnegative real α), then there is a linear functional $f(x)$ on E such that $-p(-x) \leq f(x) \leq p(x)$ [4, 226]. In his book he revised this theorem, turning it into an extension theorem for linear functionals:

> If a functional $p(x)$ on a vector space E is subadditive and homogeneous, and if a functional $f(x)$ is additive and homogeneous on a subspace K of E, and if $f(x) \leq p(x)$ on K, then $f(x)$ can be extended to a functional $F(x)$ that is additive and homogeneous on E and such that $F(x) \leq p(x)$ on E. [7, 28]

This was the definitive form of the Hahn–Banach Theorem, which was to prove enormously fruitful.

Banach next devoted a chapter of his book to his F-spaces. Every Banach space was an F-space, and every F-space was a G-space. But in contrast to G-spaces, every F-space was connected. Moreover, every linear operator on an F-space was homogeneous (and hence was linear in the traditional sense). As an application of his methods on F-spaces, he deduced a classical result in analysis—the existence of a continuous function not differentiable on a set of positive measure [7, 43]. Thus Banach had a hierarchy of generalizations of Euclidean space and proved a series of highly abstract theorems relating them.

13. NORMED VECTOR SPACES AND TOPOLOGICAL ALGEBRA

One of the most fruitful ways of viewing the development of normed vector spaces during the 1920s is as a line of development within topological algebra—the study of algebraic structures having a topology in which the algebraic operations are continuous. The first explicit example of topological algebra was the notion of topological group, which arose in 1925, soon after normed vector spaces. This notion was formulated independently by Otto Schreier in Hamburg [107] and Franciszek Leja in Warsaw [64]. Both used Fréchet's L-spaces rather than Hausdorff's topological spaces. Two years later, Leja proposed a version of topological group in a Hausdorff space, the version that was to be influential [65]. Banach was led to propose his G-spaces by specializing Leja's notion of topological group to a metric space [6; 7].

Meanwhile, other notions of topological algebra had been formulated. In 1933 van Dantzig summed up the situation as follows: "The general program of topological algebra would now run: ... to systematically investigate and classify all topological groups, topological rings, and topological fields (and, if one wishes, topological modules, which can be defined in a completely analogous way)" [113, 589]. This was the beginning of the notion of a topological module, but the more restricted notion of topological vector space was first formulated and developed the following year by Kolmogorov in Moscow.

Kolmogorov took his notion of vector space from Banach's book, requiring further that the vector space have a topology on the vectors such that addition and scalar multiplication are continuous. Already in Kolmogorov's paper the importance of convexity was apparent. When he gave conditions under which a topological vector space is also a normed vector space, they were in terms of convexity [60]. Topological vector spaces provided a natural framework in which to generalize normed vector spaces from the perspective of topological groups.

A year later, von Neumann independently invented the notion of topological vector space while generalizing "completeness" from metric spaces to topological spaces [122]. Von Neumann's aim was to extend this concept of completeness to normed vector spaces that did not satisfy Hausdorff's first countability axiom, and here too the notion of convexity was important.

By 1937, topological vector spaces had become an area of active research interest, especially at the California Institute of Technology, where doctoral dissertations were soon completed by D. H. Hyers [57], E. W. Paxson [94], and J. P. LaSalle [63]. In 1938 John Wehausen of Brown contributed an article [127], and by 1943 George Mackey was publishing on the subject as well [71]. Over the next two decades, the subject became a central one in functional analysis.

Meanwhile, in 1937 L. V. Kantorovich of Leningrad formulated a notion related to topological algebra, namely, that of partially ordered vector space [59]. This notion was developed not long afterward by Birkhoff in his book *Lattice Theory*, under the name of "vector lattice." The work of Kantorovich and Birkhoff on this subject quickly found its way into E. T. Bell's *Development of Mathematics* [8, 261–262].

14. THE ROLE OF HILBERT SPACE

Until 1927, Hilbert space played almost no role in the developments discussed above. The idea of Hilbert space originated in Hilbert's work, beginning in 1904, on linear integral equations. But Hilbert space, i.e., the set of square-summable real sequences, was not treated geometrically there. Indeed, in 1909 Hilbert made it clear that his aim was a "methodologically unified restructuring of algebra and analysis" in the context of integral equations [56, 60]. At that time vectors were seen as geometric objects rather than as algebraic ones, and so it is not surprising that they played no role in his approach.

The beginnings of a geometric approach to Hilbert space first appeared in a 1908 paper by Erhard Schmidt, in a chapter entitled "Geometry in a Function Space." In that space the functions $A(x)$ were all the square-summable sequences. Schmidt cited Fréchet's thesis (1906), where a metric space whose points were all real sequences was described as having infinitely many dimensions, though without any mention of linear structure [39, 39]. But Schmidt added: "For the geometric significance of the concepts and theorems developed in this chapter I am grateful to [Gerhard] Kowalewski. It stands out even more clearly if $A(x)$ is defined, not as a function, but as a vector in a space of infinitely many dimensions" [105, 56]. Schmidt did not make any use of the addition or scalar multiplication of vectors, though he did use inner products for both real and complex Hilbert space.

The name "Hilbert space" took some time to emerge. Apparently it was first used by Frigyes Riesz in his book (1913) on systems of linear equations with infinitely many variables [99, 78]. He followed Schmidt in interpreting the square-summable sequences

> as vectors in an infinite-dimensional space. Given a finite or infinite number of vectors, the vectors orthogonal to each of them constitute a certain linear variety [subspace], and the set of vectors orthogonal to this latter variety will be the smallest linear variety containing the given vectors. [99, 73]

Despite this interpretation, Riesz did not take an axiomatic approach to vectors or to Hilbert space.

Hausdorff also used the term "Hilbert space" in his influential book on set theory

(1914), where he treated Hilbert space as a kind of metric space. He followed Fréchet in also considering a metric defined on all real sequences, not just the square-summable ones. But Hausdorff went further in a way that illustrated his set-theoretic concerns. Rather than restricting himself to the usual infinite sequences (i.e., function on ω), he introduced a metric space, modeled on Hilbert space, whose points were ω_1-sequences with all but countably many coordinates equal to zero. This example he called a "Euclidean space with $\aleph_1$ dimensions" [49, 289]. However, he did not explicitly refer to the notion of vector or linear space.

One other person must be mentioned in this context in order to rectify an erroneous historical account. Bourbaki claimed that Otto Toeplitz, in a 1909 paper [112] on solving infinite systems of linear equations, "introduced (but by means of coordinates) the most general vector space on the reals ...; and he also points out that linear algebra as thus conceived [i.e., without determinants] is certainly applicable to any commutative field" [15, 89]. In fact, Toeplitz did nothing of the sort. He did not introduce any vector space, much less the most general one, and he did not refer to a commutative field at all.

The most general vector space over the reals, expressed in terms of coordinates, was not formulated by Toeplitz but by Hausdorff, who did so in the 1927 edition of his book on set theory. There, in the chapter on point sets, he referred to what he called "linear spaces." He considered the set of all real-valued functions on an arbitrary set M. For him, a linear space was the set of all such functions with addition and scalar mutliplication defined coordinatewise [50, 95]. He introduced a norm, defined by the usual axioms, on his linear spaces, but did not mention the axioms for a vector space.

In fact, Hilbert space was not treated axiomatically until the work of von Neumann, beginning in 1927—some five years after the papers of Banach, Hahn, and Wiener on normed vector spaces. Von Neumann's work on Hilbert space was explicitly motivated by the desire to give a mathematical foundation to the quantum mechanics of Heisenberg and Schrödinger. After noting that the complex Hilbert space of square-summable sequences has the same formal properties as a certain space of twice-differentiable functions, von Neumann sought "to characterize these spaces through their common properties and to describe a space possessing all those given characteristic properties as abstract Hilbert space" [119, 14].

At this point, von Neumann's aim was to characterize complex Hilbert space up to isomorphism. That is, there was to be only one such space. This contrasts with the later approach, in which there are many nonisomorphic Hilbert spaces.

Von Neumann gave five axioms for his complex Hilbert space. It was defined to be a "linear space" (i.e., a complex vector space) with Hermitian bilinear form (i.e., a complex inner product); this made it a metric space, which was assumed to be separable, complete, and infinite-dimensional [119, 15–17]. For his abstract Hilbert space, he showed that every orthonormal system is countable. He then developed a calculus of linear operators on his abstract Hilbert space and applied this calculus to eigenvalue problems.

Abstract Hilbert space did not have to wait long for others to develop it further.

Marshall Stone, then at Harvard, was stimulated by von Neumann's paper of 1927 to work on linear transformations in abstract Hilbert space. In 1929 Stone published a paper on what he called "the fundamental problem of the theory of linear transformations," namely, to prove the existence of linear subspaces of abstract Hilbert space that were "invariant under a given transformation T and to determine these subspaces" [110, 199]. Three years later, Stone published a lengthy book on those transformations [111].

Meanwhile von Neumann had returned to the subject. In 1930 he published a paper on Hermitian operators in Hilbert space, and for a second time stated his five axioms characterizing abstract Hilbert space [120].

This 1930 paper raises the question of where von Neumann got his axioms for a vector space. In his first statement of 1927 he did not mention any source, but in 1930 he made a statement about his first two axioms A and B, i.e., the axioms for a complex vector space with an inner product making the space into a metric space: "Moreover, the conditions A and B were stated by H. Weyl (*Raum, Zeit, Materie* ... Sects. 2–4) in connection with an axiom system for finite-dimensional vector spaces" [120, 65]. The question at once arises whether von Neumann took his axioms A and B from Weyl, or whether he was merely noting the work of another author along similar lines. We believe that it was the latter, both from the way in which von Neumann made his statement and from the fact that he gave no credit to Weyl in 1927.

Moreover, von Neumann credited two other people with closely related concepts. In another paper of 1930, he mentioned Hausdorff (1914) as having stated the notion of a vector space that was also a metric space [121, 371]. And in 1935 he and the physicist Pascual Jordan credited the idea of a normed vector space to Banach and Hahn [58, 719]. On both occasions von Neumann did not mention Weyl at all. Thus there is little reason to think that von Neumann obtained his axioms for vector spaces with inner product from Weyl.[11] It is more likely that he invented them himself, independent of their earlier formulation by others.

In 1935 the modern notion of Hilbert space finally emerged. That year Fréchet published a paper investigating conditions under which a normed vector space E is isometric to abstract Hilbert space, and found that such a condition is that every linear subspace of E having dimension at most three be isometric to a Euclidean space [122]. That same year von Neumann and Jordan showed that the condition can be weakened to dimension at most two, and gave an equivalent condition on the norm for vectors x and y:

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

This latter condition was a kind of generalized parallelogram law. Lastly, they introduced their result in terms of what they called "generalized Hilbert space," i.e., a complete complex vector space with an inner product, precisely the modern

[11] Consequently, we must disagree with Monna [73, 80], who implies than von Neumann did get those axioms from Weyl.

definition of Hilbert space. Thus they dropped the condition that a Hilbert space be infinite-dimensional, since this only added the finite-dimensional Euclidean spaces. More importantly, they omitted the condition that the space be separable, and noted that then "essentially new hyper-Hilbert spaces arise, but they are nevertheless similar to Hilbert space under most aspects" [122, 719].

This emergence of the modern notion of Hilbert space continued a trend that had been under way for several years, namely, to try to prove theorems about such spaces with the minimum of assumptions. Thus in 1932 Hausdorff raised the question as to which results known for Banach spaces actually hold for normed vector spaces in general [51, 294]. And in 1934 Heinrich Löwig in Prague proved that several theorems, which had previously been shown to hold in Hilbert space, were actually true for any Euclidean space, i.e., any vector space with an inner product; here the main concern was to omit the assumption that the metric is separable [67, 1]. Nonseparable Hilbert spaces would be the subject of much future research.

15. THE ORIGINS OF MODULES

Thus far we have discussed the geometric and analytic origins of abstract vector spaces. Now we turn to the algebraic roots of the modern concept of module. The term "module," but not the modern concept, is found in Dedekind's work on algebraic number theory. In his well-known 10th supplement of 1871 to Dirichlet's lectures on number theory, Dedekind introduced the notion of an "ideal" over the set $\mathfrak{o}$ of algebraic integers in an algebraic number field Ω. A subset A of $\mathfrak{o}$ was defined to be an ideal if it was closed under addition, subtraction, and under multiplication by a member of $\mathfrak{o}$. From a modern perspective, it is clear that A is a module over the ring $\mathfrak{o}$. But Dedekind introduced the term "module" with a more restricted meaning, namely, any subset M of the complex numbers closed under addition and subtraction. He used the notation $a \equiv b \pmod{M}$ to mean that $a - b \in M$. Dedekind's notion of module, which was quite general for the time and which included his ideals as a special case, was modeled on Gauss's use of $a \equiv b \pmod{m}$, where m was an integer rather than a set, in his *Disquisitiones arithmeticae* [36, Sects. 161–163].

Dedekind recognized the connection between linear forms and his work on number theory. He restricted attention to those fields Ω of algebraic numbers with what he called a "basis," i.e., such that Ω consisted of the linear combinations of n linearly independent elements with rational coefficients [36, Sect. 159]. From a modern perspective, Ω is an n-dimensional vector space over the rationals, but no such general concept existed at the time. Such a finite-dimensional field Ω of complex algebraic numbers has come to be called simply an "algebraic number field," and we shall follow this usage.

Likewise, Dedekind introduced the notion of a "finite module" M, i.e., the linear combinations of n algebraic numbers with integer coefficients, and these n numbers were what he called a "basis" for the module [36, Sect. 161]. From a modern perspective, M was a finitely generated module over the integers. But Dedekind's use of the word "basis" later caused some confusion since, in modern usage, a basis for a vector space must be linearly independent. Dedekind explicitly allowed his basis for a module (but not for a field) to be linearly dependent. To avoid ambiguity, we will speak of a basis—whether for a vector space or a module—only when it is linearly independent.

In his last edition of Dirichlet's lectures (1894), Dedekind introduced a restricted notion of ring, which he called an "Ordnung," as a module containing 1 and closed under multiplication. Such an "Ordnung" was, Dedekind remarked, closed under addition, subtraction, and multiplication [37, 170].

Thus far, Dedekind had introduced his various notions in the context of algebraic number theory, where his object of study was any field of complex algebraic numbers that was finite-dimensional over the rationals. In 1882 he published a joint paper with Weber that extended these same tools to algebraic function fields in order to give a rigorous foundation to Riemann's theory of algebraic functions of a complex variable. Dedekind and Weber defined a field of algebraic functions as a system of algebraic functions (of a complex variable z) closed under addition, subtraction, multiplication, and division. But at once, in analogy with Dedekind's earlier number theory, they restricted themselves to considering those fields Ω that were n-dimensional over the field of rational functions. They then spoke of a basis for Ω [29, 186].

Dedekind and Weber also extended the earlier notion of module. They defined a function module as a subsystem of Ω that was closed under addition and subtraction, as well as under multiplication by any polynomial in z. In modern terms, they defined a module over the complex polynomial ring $\mathbb{C}[z]$. But at once they restricted discussion to function modules that were "finite," i.e., finitely generated over the field of rational functions. They showed that any such module has what they called a rationally irreducible basis, i.e., a finite basis over the field of rational functions [29, 195 and 199].

A decade later, in the context of Galois theory, Weber unified the various notions of field previously considered (algebraic number field, algebraic function field, finite field) by introducing the modern abstract concept of field. In contrast to the fields in Dedekind's work, fields as axiomatized by Weber did not always have characteristic zero [123].

The 1882 paper of Dedekind and Weber strongly influenced a book that Kurt Hensel and Georg Landsberg published in 1902 on the theory of algebraic functions of a complex variable. Hensel and Landsberg argued that the most appropriate way to approach the subject was an "arithmetic" one (i.e., one motivated by algebraic number theory). In particular, they borrowed Dedekind's and Weber's notion of function module with its (finite) basis [54, 159]. Although Hensel and Landsberg used various function fields, they did not employ the general notion of field or vector space.[12]

In 1910 Weber's abstract fields were analyzed in depth by Ernst Steinitz, who,

[12] Gray has stated that Hensel and Landsberg's book "took up the concepts of fields and vector spaces ... in function theory" [44, 67]. However, this is somewhat misleading, since they did not use these general concepts.

in the course of doing so, used the idea of linear independence. In fact, Steinitz began by calling an element x transcendental with respect to a given field K if 1, x, x^2, x^3, ... are all linearly independent over K; otherwise, x was called algebraic over K. He was particularly interested in the structure of extension fields L of K. It could happen that L was what he called a "finite extension" of K, i.e., an n-dimensional vector space over K (although he did not speak of vector spaces). He did refer to L as having a "basis" over K [109, 199]. Despite the work of Hamel (1905), nowhere did Steinitz allow a basis to be infinite, and so he could not use the idea that, e.g., the field of all algebraic numbers has a basis over the rational numbers.

Meanwhile, Dedekind's notion of module was pursued by Hilbert in his 1897 report on algebraic number theory for the Deutsche Mathematiker-Vereinigung. There Hilbert used the term "Zahlring" or "Ring" for what Dedekind had called an "Ordnung." Since everything considered was an algebraic number, Hilbert could emphasize the parallelism created by closure: an algebraic number field K was closed under addition, subtraction, multiplication, and division; a ring was a set of algebraic integers of K closed under addition, subtraction, and multiplication; and a module was such a set closed under addition and subtraction. An ideal was intermediate, being such a set closed under addition and subtraction, as well as closed under multiplication by the algebraic integers of K. Just as for Dedekind, the concern was with structures that were finitely generated. In particular, this applied to Hilbert's rings [55, Sects. 1, 2, 31, 35].

In 1914 Abraham Fraenkel was stimulated by Hilbert's "Zahlring," as well as by hypercomplex number systems and rings of matrices, to define an abstract concept of ring. Fraenkel stressed how rings played an important role not only in number theory but also in other branches of mathematics. He was primarily concerned with the decomposition of rings, in the spirit of Steinitz's decompositions of fields, but did not use the modern notions of ideal and module. Moreover, Fraenkel's notion of ring differed from the modern concept in having special conditions designed to deal with zero divisors [38].

The modern concepts of ring, of ideal, and of module over a ring all appeared for the first time in Emmy Noether's ground-breaking paper "Idealtheorie in Ringbereichen" (1921). She extended the notion of ideal from algebraic number fields and polynomial rings to all commutative rings satisfying what she called the "finiteness condition," i.e., having the property that every ideal was finitely generated [74, 30]. Her chief concern was to prove that every ideal can be decomposed, in that general setting, into a product of ideals of a given kind (e.g., prime ideals, irreducible ideals).

Noether's original definition of module was relative to a double domain (Σ, T), where Σ was a ring and T was defined as what we would now call a module over Σ (i.e., take the conditions for a vector space over a field, require only a ring rather than a field, and drop the axiom requiring $1x = x$ for scalar multiplication). Noether then defined a module in (Σ, T) to be any subset M of T closed under subtraction as well as under left multiplication by an element of Σ [74, 54–55]. This definition was strongly influenced by Dedekind's notion of ideal. In particular, a module in (Σ, T) was an ideal, as defined by Dedekind's closure conditions but in an arbitrary ring, if Σ was a commutative ring equal to T. She considered the simplest example of a module to be where Σ was the ring of integers and T a set of linear forms, i.e., in the context of algebraic number theory. Beyond that, she did not mention any connection with vectors.

Noether's concern with algebraic number theory *à la* Dedekind made it natural for her to formulate the notion of a module over a ring. It was equally natural for her immediately to restrict attention to modules that are "finite," i.e., finitely generated. She followed Dedekind in calling a set A a "basis" for a module if A is finite and generates the module, without requiring that the elements of A be linearly independent [74, 55].

In her work later in the 1920s, Noether tended to emphasize the role of ideals more than that of modules. But both concepts were central in her important paper of 1927, which characterized abstractly those rings whose ideal theory agrees with that of the ring of integers of a (finitely generated) algebraic number field [75, 26]. And late in the 1920s she reformulated the definition of an algebra over a field to incorporate the concepts of ring and module. This reformulation is the subject of the next section.

16. ALGEBRAS OVER A FIELD

The modern notion of an algebra over a field evolved from the 19th-century idea of a linear associative algebra or hypercomplex number system—a concept going back to Hamilton (1853) and Benjamin Peirce (1870/1881). The modern notion differs from the earlier one in several ways. During the earlier period, such algebras were considered only over the real or complex numbers rather than over an arbitrary field and, moreover, were always finite-dimensional. The modern notion is stated in terms of vector spaces and rings. More specifically, A is an algebra over a field F if A is a vector space over F and is also a ring with unit, where the vector addition and scalar multiplication coincide with the corresponding operations on the ring.

In 1903 the concept of a (finite-dimensional) algebra over an *arbitrary* field F, rather than over the real or complex numbers, was first proposed by Leonard Dickson of the University of Chicago [32]. He presented two different definitions for such an algebra. The first definition was given in the traditional 19th-century way as a multiplication table for n elements linearly independent with respect to F. The second definition was in terms of n-tuples of elements of F, where addition and subtraction were defined in a coordinate-wise fashion. Multiplication was not defined in this way but was taken as a primitive operation required to be associative, have a right identity element, and satisfy a certain condition on left zero-divisors. (The distributive law was then deduced.) Dickson devoted most of his article to showing the independence of his postulates. It is not accidental that his article immediately followed another in which he gave an independent set of postulates for abstract fields [31].

Wedderburn, who had come to Chicago in 1904, followed Dickson's lead in considering algebras over an arbitrary field. Indeed, the aim of Wedderburn's well known paper of 1907 was to extend Elie Cartan's results on algebras over the real numbers or the complex numbers so as to apply to algebras over any field (see [78]). But, in contrast to Dickson and others at the time, he took seriously the possibility of infinite-dimensional algebras, and hoped to come back to this "interesting class of algebras" on another occasion [124, 78–79].

In 1924 Wedderburn finally published a paper on infinite-dimensional algebras. In it he gave two definitions of an algebra. The first of them, which he described as a modification of Hamilton's, was as a certain set of functions. More specifically, he began with an arbitrary set G and defined addition of functions in a coordinatewise fashion. A product of two functions was not defined explicitly but was required to be associative and to satisfy left and right distributive laws. (This definition generalized Dickson's second definition of 1903.) Wedderburn's own second definition was in terms of postulates and stated that A is an algebra if A is a ring and if some field F is associated with A in a certain rather complicated way. He summed up these postulates by stating that they are "broadly equivalent to saying that the elements of A correspond to an affine geometry in which these elements are the points of the geometry" [125, 400].

There was no mention of vector spaces in Wedderburn's paper of 1924, nor of rings or modules. (We have used a ring to state his definition more easily.) Likewise, when he came back to algebras a decade later in his book *Lectures on Matrices*, there was not a word about vector spaces or modules, and vectors appeared there only as n-tuples [126, 1]. In that book he defined an algebra as an ordered ring (though, again, he did not use that term and did not refer to the research that had been done on rings during the previous decade), but he soon restricted his algebras to those having a finite basis over some ordered ring [126, 147–148]. What is striking is that in 1934, more than a decade after vector spaces and modules had been introduced, Wedderburn apparently did not find them relevant to matrices or to algebras over a field.[13]

Meanwhile, in 1923, Dickson published an influential book on algebras over a field, giving a new definition of such algebras that differed from those accepted earlier and from his own definitions of 1903. An algebra A over a field F was defined to be a system consisting of two operations on A, addition and multiplication, and a third operation between F and A, i.e., scalar multiplication. Multiplication, addition, and scalar multiplication were assumed to be associative, and the last two to be commutative. Scalar multiplication was taken to be distributive with respect to both the addition on A and the field addition. The final assumption was that the

[13] Parshall [79, 535] has pointed out that Wedderburn's health and creativity were never fully restored after 1931. And in a private communication of May 1994, she mentioned his nervous breakdowns in the late 1920s and early 1930s, adding that Nathan Jacobson had informed her that throughout the 1930s Wedderburn did not embrace the new concepts of abstract algebra, such as ring annd vector space. Thus she is inclined to think that it was more a matter of his health, and its restrictions on his work, than his not finding the new ideas relevant.

algebra had a finite basis [33, 10]. There was no mention of rings, modules, or vectors, although matrices played an important role.

Under these circumstances, it is ironic that in 1935 Max Deuring, in his book *Algebren*, credited Dickson's book with the modern definition of an algebra over a field as a ring and also a module over the field [30, 1]. But Deuring quickly turned to finite-dimensional algebras, where Dickson's definition was equivalent to the modern one.

The modern definition is due to Emmy Noether, who published it in her 1929 paper on finite-dimensional algebras [76, 654], although Emil Artin had referred to her definition already in 1927 when he recognized the need to broaden the older definition of algebras [2, 251]. Her aim was to reunite the theory of algebras with the theory of group representations, which had been unified in the work of Frobenius but had since pursued different paths. This was to be done by treating both theories as special cases of the theory of noncommutative rings which satisfy certain finiteness conditions and by investigating certain classes of modules [76, 642]. With these tools she developed an ideal theory for noncommutative rings. Along the way, she showed that a finitely generated module over a division ring has a basis [76, 654]. And she now gave the modern definition of module over a ring [76, 646], rather than her more complicated (but equivalent) definition of 1921.

The heart of Noether's approach to algebras was the notion of an Abelian group with operators, i.e., a group of homomorphisms on an Abelian group into itself, a notion that she credited to Wolfgang Krull [61] and Otto Schmidt [106]. In 1929 she pointed out that every module over a ring is an Abelian group with operators. Thus group theory began to be the framework within which the theory of modules and the theory of algebras over a field were treated.

One striking feature of Noether's 1929 paper occurred in a footnote: "As B. L. van der Waerden has communicated to me, one can obtain an invariant connection, independent of the specific choice of basis, by separating the concepts *linear transformation* and *matrix.* A linear transformation is a homomorphism of two modules of linear forms; a matrix is an expression (the representation) of this homomorphism by a definite choice of basis" [76, 670]. Here we have the essential modern connection between the notions of linear transformation, matrix, and module (or vector space). Two years later, van der Waerden's insight about the proper way of viewing that connection was presented to a much wider audience in his textbook.

17. MODERN ALGEBRA

When van der Waerden came to Göttingen in 1924, he was an algebraic geometer, concerned with giving that subject a rigorous foundation [118, 32]. But he quickly learned from Emmy Noether that to do so required the tools of modern algebra. In 1927 he began to treat algebraic geometry using Noether's tools in his formulation of the theory of zeroes of polynomial ideals [114, 183]. His highly influential two-volume textbook of 1930–1931, *Moderne Algebra*, was largely based on lectures by Noether and Artin.

In the first volume van der Waerden defined a module as an additive Abelian group with a domain of operators (i.e., homomorphisms) that forms a ring and satisfies axioms V2 and V3 (see Sect. 1 above) [115, 133]. The second volume of 1931 devoted an entire chapter to modules and related notions [116, Chap. XV]. For a ring with unit 1, modules satisfying $1x = x$ for all x were called unitary. Finitely generated unitary modules which have a basis were given particular prominence and were called "modules of linear forms" over a ring K. He noted that a module M of linear forms is characterized by the (finite) cardinality n of its basis and by its ring K. Thus the elements of M could be taken to be n-tuples, which he called "vectors."

In his reminiscences [118, 36] van der Waerden tells us that this chapter contained material largely known in 1924. Entitled "Lineare Algebra" (perhaps the first time this term was used in the modern sense, although the term can be found earlier in Weyl [129, 22]), it gives a clear statement of linear algebra as the study of modules over a ring and their homomorphisms, i.e., linear transformations. Matrices are treated as a way of writing such homomorphisms when the module has a finite basis. Thus, at the very beginning of the 1930s, this unified and modern approach to linear algebra was presented to an international audience.

On the other hand, van der Waerden still treated linear independence for field extensions as Steinitz had done two decades earlier, rather than treating an extension field L of a field K as a vector space over K. Thus vector spaces had not yet been explicitly applied to the theory of fields [116, 95].

While linear algebra only appeared in the second volume of the first edition, in the first volume of the second edition (1937) van der Waerden introduced a new section (Sect. 14) entitled "Vector Spaces and Hypercomplex Systems" within the chapter on rings and fields. The notion of a finite-dimensional module of linear forms over a ring was treated prominently (although it was called a "vector space"). Then he considered the notion of an algebra over a ring, noting that the ring was usually (though not necessarily) a field. He concluded by introducing the concept of an infinite-dimensional algebra, offering polynomial rings as the simplest example [117, 46–49].

Some years later, van der Waerden expanded that section on modules to an entire chapter on vector spaces over a division ring. Thus linear algebra came to occupy an increasingly central position in his book.

18. VECTOR SPACES VS. MODULES

It did not take long for van der Waerden's approach to reach the United States. In 1937 A. A. Albert published his textbook, *Modern Higher Algebra*, in which he too began with groups and rings. He introduced what he called a "linear set," i.e., the notion of a module over a ring, early in the book [1, 16].

But the most influential American book to treat modern algebra during this period was *A Survey of Modern Algebra* by Birkhoff and Mac Lane (1941). Mac Lane had encountered what he later called "the Chicago view of vectors as n-tuples (or denumerable tuples) of numbers" while a graduate student there in 1930. At Göttingen shortly afterward, in a seminar by Weyl, Mac Lane "made the shocking discovery that a vector is better considered as an element in an axiomatically defined vector space" [14, 27]. Mac Lane added that he could have made that discovery earlier by reading Weyl's *Raum, Zeit, Materie*. The fact that Mac Lane did not learn this from Weyl's book, but from Weyl's lectures, is further evidence of the limited influence of that book on the emergence of abstract vector spaces.

During 1937–1938, Birkhoff taught an undergraduate course at Harvard on algebra, and its second semester began with an axiomatic treatment of vector spaces over a field. Matrices "were introduced as linear operators on finite-dimensional vector spaces" [14, 29]. In 1939–1940, Mac Lane was teaching the same course, where he too treated vector spaces and linear transformations conceptually. The course notes of Birkhoff and Mac Lane formed the basis for their joint book, *A Survey of Modern Algebra.*

In *Survey* vector spaces over a field were treated prominently [13, Chap. VII] and were allowed to be infinite-dimensional. The polynomials over a field were given as an example of such a space [13, 169]. Matrices were treated as linear transformations of vector spaces, while modules were confined to a footnote near the end of the book.

By contrast, Mac Lane took a very positive view of modules in 1981 when discussing the history of modern algebra:

> Today the notion of a left module over a ring appears as an absolutely fundamental notion ... Moreover, from the viewpoint of abstract algebra this general notion is the natural one to emphasize and use first. The notion in this sense was a long time in developing ... Although a 1929 paper of Emmy Noether indicates a clear understanding of ... modules, even by 1940 there was little general tendency to make use of the notion of a module. I can recollect that I did not understand its importance at that time, and I was not alone. The first real recognition of the central role of a module is the Bourbaki volume on linear algebra (1947). [68, 22]

But while Mac Lane later recognized the importance of modules, Birkhoff was more ambivalent. In his 1976 recollections of how *Survey* came to be written, Birkhoff stated that the book showed

> the power of the axiomatic approach for treating 'vector spaces' over arbitrary fields, including an elegant deductive treatment of linear independence and dimension due in part to Hassler Whitney ... I have reviewed the design of Birkhoff–Mac Lane ... partly to emphasize how completely it deviated from van der Waerden in spirit and content. [10, 68–69]

Birkhoff also distanced himself from Bourbaki's approach to modules and vectors spaces:

> Whereas Mac Lane and I had tried to temper the purism of van der Waerden's 'modern' algebra in our book, Bourbaki was ultramodern. For instance, his book on *Linear Algebra* discusses vector spaces after modules ... And his first theorem about vector spaces states 'Every vector space (over a field K) is a free K-module.' Matrices come much later. [10, 78]

Birkhoff notwithstanding, the recent history of linear algebra has continued to confirm the fundamental importance of modules over a ring, with vector spaces over a field as a vital but subsidiary notion. This was how the subject was treated in Mac Lane and Birkhoff's *Algebra* (1967), where modules precede vector spaces—presumably under the influence of Mac Lane.

19. CONCLUSION

Dieudonné [35, 72] and Mac Lane [69, 187] found it odd that Peano's 1888 axioms for an abstract vector space (over the reals) were not accepted by mathematicians at the time. But, as we have seen, it was not odd at all. Peano's concept of vector space—given in the context of geometry—was premature, in the same sense that Wussing attributes to Cayley's 1854 axiomatization of finite groups [137, 232]. For three decades after Peano wrote, mathematicians felt little need for a general notion of vector space. By contrast, linear independence was increasingly used in various contexts (e.g., algebraic number theory). Darboux's 1875 attempt to characterize 3-dimensional vectors did not lead to abstract vector spaces or a general concept of basis, but led instead to the (particular) Hamel basis of the reals over the rationals (1905).

It is worth noting that the so-called American Postulate Theorists (Dickson, Huntington, E. H. Moore, Veblen, etc. [101]), who soon after 1900 gave axioms for groups, fields, algebras over a field, and Boolean algebras, did not attempt to axiomatize vector spaces or modules.

By 1920 the situation had changed so much that axioms for a vector space were invented four times—by Weyl in 1918 for finite-dimensional spaces and by Banach, Hahn, and Wiener for normed infinite-dimensional spaces two years later. The context had changed from geometry to analysis, now strongly tied to metric spaces. The interaction between algebraic and topological notions, an important one for normed vector spaces, was soon treated in a more general context, that of topological groups. Hilbert space (i.e., square-summable sequences) was only studied as a particular object until 1934–1935, when the modern notion of Hilbert space as a class of structures emerged in the work of Fréchet and von Neumann at the same time as topological vector spaces (a generalization of normed vector spaces).

Meanwhile, stimulated by Dedekind's work on algebraic number theory, Noether formulated the general concept of a module over a ring in 1920. (Dedekind's 1871 concept of module agreed with Noether's for algebraic number fields.) Dedekind had also used the idea of a "basis" for an algebraic field extension. Noether shifted the context by introducing the general notion of ring and developing a general theory of (finitely generated) ideals over rings. For Noether, linear forms provided the simplest example of a module.

Algebras over a field were axiomatized by Dickson in 1903 (but not in a coordinate-free way), well before he was aware of vector spaces. The modern definition of an algebra was a late arrival, only appearing in Noether's work of 1927–1929. Like her predecessors, she restricted attention to finite-dimensional algebras.

Infinite-dimensional algebras were first studied with insight by combining the two traditions that for two decades had proceeded independently: Banach spaces in functional analysis and rings/ideals/modules in algebra. This was done by Izrail Gelfand of Moscow in his influential paper of 1941 on what he called "normed rings," now known as Banach algebras [43]. In particular, maximal ideals played a major role, as did considerations of topological algebra.

What had changed between 1888 and 1920 to account for acceptance of Banach's, but not Peano's, vector spaces? In 1888 the concrete notion of vector in the plane and space was still struggling to be accepted. By 1920 the need for a more abstract approach had emerged in analysis and independently in algebra. In analysis it first took root with the work of Fréchet in 1906 on metric spaces and with that of Hausdorff in 1914 on topological spaces. In 1920 Banach quite consciously took an axiomatic approach influenced by set theory.[14] Mac Lane's view of what had changed sheds light on an important aspect [68, 20]:

> The formal definition of a vector space by axioms and not by n-tuples ... could have been introduced and understood by Grassmann in 1842, it was introduced by Peano in 1888, but it was not introduced and effectively advertised before Weyl (1918) and Banach (1922).... In the conceptual parts of mathematics, it is not the discovery but the courage and conviction of importance that plays a central role.

By the late 1930s, many interrelated notions were being researched vigorously: Hilbert spaces, topological vector spaces, topological groups, vector lattices. From the side of analysis, the key element was the abstract study of interactions between algebraic and topological aspects. From the algebraic side, the study was almost always of finitely generated modules (for which there was in general no basis) and so proceeded more or less independently on the side of analysis, where the use of abstractions was intended to study infinite-dimensional Hilbert spaces and Banach spaces. Even in 1941, the algebraic and analytic investigations of vector spaces and modules were proceeding more or less independently. Nevertheless, van der Waerden's book, in its various editions, played a substantial role in making the notion of vector space a central one and in placing matrices in the context of vector spaces. Linear algebra had finally come of age.

ACKNOWLEDGMENTS

We thank Liliane Beaulieu, Israel Kleiner, Karen Parshall, Ian Tweddle, and Jeffrey Zucker for their helpful comments on this paper and the Niedersächsische Staats- und Universitätsbibliothek Göttingen for permission to publish Hamel's letter.

REFERENCES

1. A. Adrian Albert, *Modern Higher Algebra*, Chicago: University of Chicago Press, 1937.
2. Emil Artin, Zur Theorie der hyperkomplexen Zahlen, *Abhandlungen aus dem mathematischen Seminar der Hamburgischen Universität* **5** (1927), 215–260.
3. Stefan Banach, Sur les opérations dans les ensembles abstraites et leur application aux équations intégrales, *Fundamenta Mathematicae* **3** (1922), 133–181.
4. Stefan Banach, Sur les fontionnelles linéaires, *Studia Mathematica* **1** (1929), 211–216.
5. Stefan Banach, Sur les fontionnelles linéaires, II, *Studia Mathematica* **1** (1929), 223–239.
6. Stefan Banach, Über metrische Gruppen, *Studia Mathematica* **3** (1931), 101–113.
7. Stefan Banach, *Théorie des opérations linéaires*, Warsaw: Garasiński, 1932.
8. Eric Temple Bell, *The Development of Mathematics*, New York: McGraw–Hill, 1945.

[14] Birkhoff's claim that functional analysis adopted the abstract approach from van der Waerden's *Moderne Algebra* is incorrect [11, 773].

9. Michael Bernkopf, The Development of Function Spaces, *Archive for History of Exact Sciences* **3** (1966), 1–96.

10. Garrett Birkhoff, The Rise of Modern Algebra to 1936, in *Men and Institutions in American Mathematics*, Graduate Studies of Texas Tech University, 1976, pp. 41–85.

11. Garrett Birkhoff, Current Trends in Algebra, *American Mathematical Monthly* **80** (1973), 760–782.

12. Garrett Birkhoff, On the Combination of Subalgebras, *Proceedings of the Cambridge Philosophical Society* **80** (1933), 441–464.

13. Garrett Birkhoff and Saunders Mac Lane, *A Survey of Modern Algebra*, New York: Macmillan Co., 1941.

14. Garrett Birkhoff and Saunders Mac Lane, *A Survey of Modern Algebra*: The Fiftieth Anniversary of Its Publication, *Mathematical Intelligencer* 14, No. 1 (1991), 26–31.

15. Nicolas Bourbaki, *Eléments d'histoire des mathématiques*, Paris: Masson, 1984.

16. Cesare Burali-Forti, Il metodo del Grassmann nella geometria proiettiva, *Rendiconti del Circolo matematico di Palermo* **10** (1896), 177–195.

17. Cesare Burali-Forti, *Introduction à la géométrie différentielle suivant la méthode de H. Grassmann*, Paris: Gauthier–Villars, 1897.

18. Cesare Burali-Forti and Roberto Marcolongo, *Eléments de calcul vectoriel avec de nombreuses applications à la géométrie, à la mécanique et à la physique mathématique*, Paris: Hermann, 1910.

19. Cesare Burali-Forti and Roberto Marcolongo, *Transformations linéaires*, Pavia: Mattei, 1912; Vol. I of *Analyse vectorielle générale*.

20. Cesare Burali-Forti and Roberto Marcolongo, *Trasformazioni lineari*, Bologna: Zanichelli, 1929; Vol. I of *Analisi vettoriale generale e applicazioni*.

21. Celestyn Burstin, Die Spaltung des Kontinuums in c in Lebesgueschem Sinne nicht-messbare Mengen, *Sitzungsberichte der Akademie der Wissenschaften zu Wien, Mathematisch-Naturwissenschaftliche Klasse, Abteilung IIa* **125** (1916), 209–217.

22. Celestyn Burstin, Ein Beitrag zur Theorie der Ordnung der linearen Systeme, *Tôhoku Mathematical Journal* **31** (1929), 296–299.

23. Georg Cantor, Mitteilungen zur Lehre vom Transfiniten, *Zeitschrift für Philosophie und philosophische Kritik* **91** (1887), 81–125.

24. Emmanuel Carvallo, Sur les systèmes linéaires, le calcul des symboles différentiels et leur application à la physique mathématique, *Monatshefte für Mathematik und Physik* **2** (1891), 177–216, 225–266, 311–330.

25. Michael J. Crowe, *A History of Vector Analysis: The Evolution of the Idea of a Vectorial System*, Notre Dame, IN: University of Notre Dame, 1967.

26. Gaston Darboux, Sur la composition des forces en statiques, *Bulletin des sciences mathématiques* **9** (1875), 281–288.

27. Gaston Darboux, Sur le théorème fondamental de la géométrie projective, *Mathematische Annalen* **17** (1880), 55–61.

28. Richard Dedekind, *Was sind und was sollen die Zahlen*? Braunschweig: Vieweg, 1888.

29. Richard Dedekind and Heinrich Weber, Theorie der algebraischen Functionen einer Veränderlichen, *Journal für die reine und angewandte Mathematik* **92** (1881), 181–290.

30. Max Deuring, *Algebren*, Berlin: Springer-Verlag, 1935.

31. Leonard E. Dickson, Definition of a Field by Independent Postulates, *Transactions of the American Mathematical Society* **4** (1903), 13–20.

32. Leonard E. Dickson, Definitions of a Linear Associative Algebra by Independent Postulates, *Transactions of the American Mathematical Society* **4** (1903), 21–26.

33. Leonard E. Dickson, *Algebras and Their Arithmetics*, Chicago: University of Chicago, 1923.

34. Jean Dieudonné, Le développement historique de la notion de groupe, *Bulletin de la Société mathématique de Belgique* **28** (1976), 267–296.

35. Jean Dieudonné, *History of Functional Analysis*, Amsterdam: North-Holland, 1981.

36. P. G. Lejeune Dirichlet, *Vorlesungen über Zahlentheorie*, 2d ed., ed. Richard Dedekind, Braunschweig: Vieweg, 1871.

37. P. G. Lejeune Dirichlet, *Vorlesungen über Zahlentheorie*, 4th ed., ed. Richard Dedekind, Braunschweig: Vieweg, 1894.

38. Abraham A. Fraenkel, Über die Teiler der Null und die Zerlegungen von Ringen, *Journal für die reine und angewandte Mathematik* **145** (1914), 139–176.

39. Maurice Fréchet, Sur quelques points du calcul fonctionnel, *Rendiconti del Circolo matematico di Palermo* **22** (1906), 1–74.

40. Maurice Fréchet, Les espaces vectoriels abstraits, *Bulletin of the Calcutta Mathematical Society* **15** (1925), 51–62.

41. Maurice Fréchet, Les espaces abstraits topologiquement affines, *Acta Mathematica* **47** (1925), 25–52.

42. Maurice Fréchet, *Les espaces abstraits et leur théorie considérée comme introduction à l'analyse générale*, Paris: Gauthier–Villars, 1928.

43. Izrail Gelfand, Normierte Ringe, *Matematicheskii Sbornik* **9** (1941), 3–24.

44. Jeremy Gray, The History of the Concept of a Finite-Dimensional Vector Space, *Historia Mathematica* **7** (1980), 65–70.

45. Hans Hahn, Über Folgen linearer Operationen, *Monatshefte für Mathematik und Physik* **32** (1922), 3–88.

46. Hans Hahn, Über lineare Gleichungen in linearen Räumen, *Journal für die reine und angewandte Mathematik* **157** (1927), 214–229.

47. Georg Hamel, Über die Zusammensetzung von Vektoren, *Zeitschrift für Mathematik und Physik* **49** (1903), 362–371.

48. Georg Hamel, Eine Basis aller Zahlen und die unstetigen Lösungen der Funktionalgleichung: $f(x + y) = f(x) + f(y)$, *Mathematische Annalen* **60** (1905), 459–462.

49. Felix Hausdorff, *Grundzüge der Mengenlehre*, Leipzig: Veit, 1914.

50. Felix Hausdorff, *Mengenlehre*, Berlin: De Gruyter, 1927.

51. Felix Hausdorff, Zur Theorie der linearen metrischen Räume, *Journal für die reine und angewandte Mathematik* **167** (1932), 294–311.

52. Thomas Hawkins, Another Look at Cayley and the Theory of Matrices, *Archives internationales d'histoire des sciences* **27** (1977), 82–112.

53. Eduard Helly, Über Systeme linearer Gleichungen mit unendlich vielen Unbekannten, *Monatshefte für Mathematik und Physik* **31** (1921), 60–91.

54. Kurt Hensel and Georg Landsberg, *Theorie der algebraischen Funktionen einer Variabeln und ihre Anwendung auf algebraische Kurven und abelsche Integrale*, Leipzig: Teubner, 1902.

55. David Hilbert, Die Theorie der algebraischen Zahlkörper, *Jahresbericht der deutschen Mathematiker-Vereinigung* **4** (1897), 175–546.

56. David Hilbert, Wesen und Ziele einer Analysis der unendlichvielen unabhängigen Variablen, *Rendiconti del Circolo matematico di Palermo* **27** (1909), 59–74.

57. D. H. Hyers, *Integrals and Functional Equations in Linear Topological Spaces*, doctoral dissertation, California Institute of Technology, 1937.

58. Pascual Jordan and John von Neumann, On Inner Products in Linear Metric Spaces, *Annals of Mathematics* **36** (1935), 719–723.

59. Leonid V. Kantorovich, Lineare halbgeordnete Räume, *Matematicheskii Sbornik (2)* **44** (1937), 121–168.

60. Andrei Kolmogorov, Zur Normierbarkeit eines allgemeinen topologischen linearen Raumes, *Studia Mathematica* **5** (1934), 29–34.

61. Wolfgang Krull, Über verallgemeinerte endliche Abelsche Gruppen, *Mathematische Zeitschrift* **23** (1925), 161–196.

62. Edmond Laguerre, Sur le calcul des systèmes linéaires, extrait d'une lettre adressée à M. Hermite, *Journal de l'Ecole polytechnique*, **25** (1867), 215–264.

63. J. P. LaSalle, *Pseudo-Normed Linear Sets over Valued Rings*, doctoral dissertation, California Institute of Technology, 1941.

64. Franciszek Leja, Sur les groupes abstraits continus, *Annales de la Société polonaise de Mathématiques* **3** (1925), 153.

65. Franciszek Leja, Sur la notion du groupe abstrait topologique, *Fundamenta Mathematicae* **9** (1927), 37–44.

66. Alfred Lotze, Systeme geometrischer Analyse. Zweiter Teil. III. Die Graßmannsche Ausdehnungslehre, *Enzyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen*, 1923, III AB 11, pp. 1425–1550.

67. Heinrich Löwig, Komplexe euklidische Räume von beliebiger endlicher oder transfiniter Dimensionszahl, *Acta literarum ac scientiarum regiae Universitatis Hungaricae Francisco-Josephinae, sectio scientiarum mathematicarum* **7** (1934), 1–33.

68. Saunders Mac Lane, History of Abstract Algebra: Origin, Rise, and Decline of a Movement, in *American Mathematical Heritage: Algebra and Applied Mathematics*, Texas Tech University, Mathematics Series, No. 13 (1981), 3–35.

69. Saunders Mac Lane, *Mathematics: Form and Function*, New York: Springer-Verlag, 1986.

70. Mathematical Society of Japan, *Encyclopedic Dictionary of Mathematics*, Cambridge, MA: MIT Press, 1977.

71. George W. Mackey, On Convex Topological Linear Spaces, *Proceedings of the National Academy of Sciences (U.S.A.)* **29** (1943), 315–319.

72. Stanislaw Mazur, Über die Nullstellen linearer Operationen, *Studia Mathematica* **2** (1930), 11–20.

73. A. F. Monna, *Functional Analysis in Historical Perspective*, New York: Wiley, 1973.

74. Emmy Noether, Idealtheorie in Ringbereichen, *Mathematische Annalen* **83** (1921), 24–66.

75. Emmy Noether, Abstrakter Aufbau der Idealtheorie in algebraischen Zahl- und Funktionskörpern, *Mathematische Annalen* **96** (1927), 26–61.

76. Emmy Noether, Hyperkomplexe Größen und Darstellungstheorie, *Mathematische Zeitschrift* **30** (1929), 641–692.

77. Wladislaw Orlicz, Beiträge zur Theorie der Orthogonalentwicklungen, *Studia Mathematica* **1** (1929), 1–39.

78. Karen V. H. Parshall, Joseph H. M. Wedderburn and the Structure Theory of Algebras, *Archive for History of Exact Sciences* **32** (1985), 223–349.

79. Karen V. H. Parshall, New Light on the Life and Work of Joseph Henry Maclagan Wedderburn (1882–1948), in *Amphora: Festschrift for Hans Wussing on the Occasion of His 65th Birthday*, ed. Sergei S. Demidov *et al.*, 1992, Basel: Birkhäuser, pp. 523–537.

80. Giuseppe Peano, Integrazione per serie delle equazioni differenziali lineari, *Atti della Accademia delle Scienze di Torino, Classe di scienze fisiche, matematiche e naturale* **22** (1887), 437–446.

81. Giuseppe Peano, *Calcolo geometrico secondo l'Ausdehnungslehre di H. Grassmann, preceduto dalle operazioni della logica deduttiva*, Turin: Bocca, 1888.

82. Giuseppe Peano, Intégration par séries des équations différentielles, *Mathematische Annalen* **32** (1888), 450–456.

83. Giuseppe Peano, *I principii di geometria logicamente esposti*, Turin: Bocca, 1889.

84. Giuseppe Peano, Démonstration de l'intégrabilité des équations différentielles ordinaires, *Mathematische Annalen* **37** (1890), 182–228.

85. Giuseppe Peano, *Gli elementi di calcolo geometrico*, Turin: Candeletti, 1891.

86. Giuseppe Peano, Sur les systèmes linéaires, *Monatshefte für Mathematik und Physik* **5** (1894), 136.

87. Giuseppe Peano, *Formulaire de mathématiques*, Vol. 1, Turin: Bocca, 1895.

88. Giuseppe Peano, Trasformazioni lineari dei vettori di un piano, *Atti della Accademia delle scienze di Torino, Classe di scienze fisiche, matematiche e naturale* **31** (1895), 157–166.

89. Giuseppe Peano, Saggio di calcolo geometrico, *Atti della Accademia delle Scienze di Torino, Classe di scienze fisiche, matematiche e naturale* **31** (1896), 952–975.

90. Giuseppe Peano, Analisi della teoria dei vettori, *Atti della Accademia delle Scienze di Torino, Classe di scienze fisiche, matematiche e naturale* **33** (1898), 513–534.

91. Giuseppe Peano, *Formulaire de mathématiques*, Vol. 2, Sect. 3, Turin: Bocca, 1899.

92. Giuseppe Peano, *Formulaire de mathématiques*, Vol. 3, Turin: Bocca, 1901.

93. Giuseppe Peano, *Formulaire de mathématiques*, Vol. 5, Turin: Bocca, 1908.

94. E. W. Paxson, *Analysis in Linear Topological Spaces*, doctoral dissertation, California Institute of Technology, 1937.

95. Salvatore Pincherle, Mémoire sur le calcul fonctionnel distributif, *Mathematische Annalen* **47** (1897), 325–382.

96. Salvatore Pincherle, *Le operazioni distributive e le loro applicazioni all'analisi*, Bologna: Zanichelli, 1901.

97. Salvatore Pincherle, Equations et opérations fonctionnelles, *Encyclopédie des sciences mathématiques pures et appliquées,* tome 2, vol. 5 (1912), 1–81.

98. Frigyes (Frédéric, Friedrich) Riesz, Sur les ensembles de fonctions, *Comptes rendus hebdomadaires des séances de l'Académie des Sciences, Paris* **143** (1906), 738–741.

99. Frigyes Riesz, *Les systèmes d'équations linéaires à une infinité d'inconnues*, Paris: Gauthier–Villars, 1913.

100. Bertrand Russell, *The Principles of Mathematics*, Cambridge: Cambridge Univ. Press, 1903.

101. Michael Scanlan, Who Were the American Postulate Theorists?, *Journal of Symbolic Logic* **56** (1991), 981–1002.

102. Juliusz Schauder, Über die Umkehrung linearer, stetiger Funktionaloperationen, *Studia Mathematica* **2** (1930), 1–6.

103. Rudolf Schimmack, Ueber die axiomatische Begründung der Vektoraddition, *Nachrichten von der königliche Gesellschaft der Wissenschaften zu Göttingen. Mathematisch-physicalische Klasse* (1903), 317–325.

104. Rudolf Schimmack, Ueber die axiomatische Begründung der Vektoraddition, *Abhandlungen der kaiserlichen Leopoldinisch-Carolinischen deutschen Akademie der Naturforscher* **90** (1908), 5–104.

105. Erhard Schmidt, Über die Auflösung linearer Gleichungen mit unendlich vielen Unbekannten, *Rendiconti del Circolo matematico di Palermo* **25** (1908), 53–77.

106. Otto Schmidt, Über unendliche Gruppen mit endlicher Kette, *Mathematische Zeitschrift* **29** (1928), 34–41.

107. Otto Schreier, Abstrakte kontinuierliche Gruppen, *Abhandlungen aus dem mathematischen Seminar der Hamburgischen Universität* **4** (1926), 15–32.

108. Wacław Sierpiński, Sur la question de la mesurabilité de la base de M. Hamel, *Fundamenta Mathematicae* **1** (1920), 105–111.

109. Ernst Steinitz, Algebraische Theorie der Körper, *Journal für die reine und angewandte Mathematik* **137** (1910), 167–302.

110. Marshall H. Stone, Linear Transformations in Hilbert Space. I. Geometrical Aspects, *Proceedings of the National Academy of Sciences (U.S.A.)* **15** (1929), 198–200.

111. Marshall H. Stone, *Linear Transformations in Hilbert Space and Their Applications to Analysis*, New York: Amer. Math. Soc., 1932.

112. Otto Toeplitz, Ueber die Auflösung unendlichvieler linearer Gleichungen mit unendlichvielen Unbekannten, *Rendiconti del Circolo matematico di Palermo* **28** (1909), 88–96.

113. David van Dantzig, Zur topologischen Algebra, *Mathematische Annalen* **107** (1933), 587–626.

114. Bartel L. van der Waerden, Zur Nullstellentheorie der Polynomideale, *Mathematische Annalen* **96** (1927), 183–208.

115. Bartel L. van der Waerden, *Moderne Algebra*, Vol. 1, Berlin: Springer-Verlag, 1930.

116. Bartel L. van der Waerden, *Moderne Algebra*, Vol. 2, Berlin: Springer-Verlag, 1931.

117. Bartel L. van der Waerden, *Moderne Algebra*, 2d ed., Vol. 1, Berlin: Springer-Verlag, 1937.

118. Bartel L. van der Waerden, On the Sources of My Book *Moderne Algebra, Historia Mathematica* **2** (1975), 31–40.

119. John von Neumann, Mathematische Grundlagen der Quantenmechanik, *Nachrichten von der königliche Gesellschaft der Wissenschaften zu Göttingen. Mathematisch-physicalische Klasse* (1927), 1–57.

120. John von Neumann, Allgemeine Eigenwerttheorie Hermitescher Funktionaloperatoren, *Mathematische Annalen* **102** (1930), 49–131.

121. John von Neumann, Zur Algebra der Funktionaloperatoren und Theorie der normalen Operatoren, *Mathematische Annalen* **102** (1930), 370–427.

122. John von Neumann, On Complete Topological Spaces, *Transactions of the American Mathematical Society* **37** (1935), 1–20.

123. Heinrich Weber, Die allgemeinen Grundlagen der Galois'schen Gleichungstheorie, *Mathematische Annalen* **43** (1893), 521–549.

124. Joseph H. M. Wedderburn, On Hypercomplex Numbers, *Proceedings of the London Mathematical Society (2)* **6** (1907), 77–118.

125. Joseph H. M. Wedderburn, Algebras Which Do Not Possess a Finite Basis, *Transactions of the American Mathematical Society* **26** (1924), 395–426.

126. Joseph H. M. Wedderburn, *Lectures on Matrices*, New York: Amer. Math. Soc., 1934.

127. John V. Wehausen, Transformations in Linear Topological Spaces, *Duke Mathematical Journal* **4** (1938), 157–169.

128. André Weil, Sul calcolo funzionale lineare, *Rendiconti della reale Accademia nazionale dei Lincei, Classe di scienze fisiche, matematiche e naturali* **6** (1927), 773–777.

129. Hermann Weyl, *Raum, Zeit, Materie. Vorlesungen über allgemeine Relativitätstheorie*, Berlin: Springer-Verlag, 1918; 4th ed. (1921); English trans. (1922) of 4th ed.

130. Hermann Weyl, *Gruppentheorie und Quantenmechanik*, Leipzig: S. Hirzel, 1928.

131. Hermann Weyl, A Half Century of Mathematics, *American Mathematical Monthly* **58** (1951), 523–533.

132. Norbert Wiener, On the Theory of Sets of Points in Terms of Continuous Transformations, *Comptes rendus du Congrès international des Mathèmaticiens* (*Toulouse*), 1921, pp. 312–315.

133. Norbert Wiener, Limit in Terms of Continuous Transformation, *Bulletin de la Société mathématique de France* **50** (1922), 119–134.

134. Norbert Wiener, The Group of the Linear Continuum, *Proceedings of the London Mathematical Society* (2) **20** (1922), 329–346.

135. Norbert Wiener, Note on a Paper of M. Banach, *Fundamenta Mathematicae* **4** (1923), 136–143.

136. Norbert Wiener, *I Am a Mathematician. The Later Life of a Prodigy*, New York: Doubleday, 1956.

137. Hans Wussing, *The Genesis of the Abstract Group Concept*, Cambridge, MA: MIT Press, 1984.

138. Ernst Zermelo, Neuer Beweis für die Möglichkeit einer Wohlordnung, *Mathematische Annalen* **65** (1908), 107–128.