The $D$-property of monotone covering properties and related conclusions

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**Abstract**

In this note, we show that every monotonically (countably) metacompact space is hereditarily a $D$-space and every monotonically meta-Lindelöf space is hereditarily dually $\sigma$-closed discrete. As a corollary, we show that if $X$ is a monotonically meta-Lindelöf (or monotonically (countably) metacompact) monotonically normal space then $X$ is hereditarily paracompact. In the second part of this note, we show that every scattered partition of a hereditarily almost thickly covered space is almost thick, and hence a hereditarily almost thickly covered space is $aD$ and linearly $D$. This answers a question of Guo and Jumnila. We also show that every monotonically $\omega$-monolithic compact space is monotonically monolithic. This answers a question of Alas, Tkachuk and Wilson.

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1. Introduction

A neighborhood assignment for a space $X$ is a function $\phi$ from $X$ to the topology of the space $X$ such that $x \in \phi(x)$ for any $x \in X$. A space $X$ is called a $D$-space if for any neighborhood assignment $\phi$ for $X$ there exists a closed discrete subspace $D$ of $X$ such that $X = \bigcup \{\phi(d) : d \in D\}$ [8]. We denote $\bigcup \{\phi(d) : d \in D\}$ by $\phi(D)$. Many classes of spaces are known to be $D$-spaces [13]. For example, every space with a point-countable base is a $D$-space [2], and every space with a point-countable weak base is a $D$-space [20]. But it is not known whether every regular Lindelöf space is a $D$-space. It is also not known if a regular metacompact or even meta-Lindelöf space is a $D$-space.

Monotone normality was the property motivating all the other monotone covering properties. A space $X$ is monotonically normal [18] if there is a function $G$ which assigns to each ordered pair $(H, K)$ of disjoint closed subsets of $X$ an open set $G(H, K)$ such that: (1) $H \subseteq G(H, K) \subseteq \overline{G(H, K)} \subseteq X \setminus K$; (2) if $(H', K')$ is a pair of disjoint closed subsets having $H \subseteq H'$ and $K \supseteq K'$, then $G(H, K) \subseteq G(H', K')$. A topological space $X$ is monotonically Lindelöf [3] if for each open cover $\mathcal{U}$ of $X$ there is a countable open cover $r(\mathcal{U})$ of $X$ that refines $\mathcal{U}$ and has the property that if an open cover $\mathcal{U}$ refines an open cover $\mathcal{V}$, then $r(\mathcal{U})$ refines $r(\mathcal{V})$. Some properties of monotonically Lindelöf can be found in [3] and [14]. For two collections $\mathcal{U}$ and $\mathcal{V}$ of subsets of a space $X$, we write $\mathcal{U} \prec \mathcal{V}$ to mean that for each $U \in \mathcal{U}$ there is some $V \in \mathcal{V}$ with $U \subseteq V$.

In [24] and [4], the concepts of monotonically countably metacompact and monotonically metacompact were introduced. A space $X$ is monotonically (countably) metacompact if there is a function $r$ that associates with each (countable) open
cover \( U \) of \( X \) an open point-finite refinement \( r(U) \) that covers \( X \), where \( r \) has the property that if \( U \) and \( V \) are open covers with \( U < V \) then \( r(U) < r(V) \) [24,4]. In [12], Good, Knight and Stares have given another definition of monotone countable metacompactness property. In [4], it was pointed out that the Good–Knight–Stares and Popvassilev definitions of monotone countable metacompactness are not equivalent, and neither implies the other. In this note, the definition of monotone countably metacompactness is in the sense of Popvassilev. In [4], it was proved that if \((X, \tau, <)\) is a GO-space that is monotonically countably metacompact then \((X, \tau)\) is hereditarily paracompact, and it was remarked that a monotonically normal space that is monotonically countably metacompact must be paracompact. In [23], it was proved that a monotonically normal space that is monotonically countably metacompact must be hereditarily paracompact. So \(\omega_1 + 1\) is not monotonically countably metacompact [24], hence monotone normal does not imply monotone countably metacompact. We know that monotone metacompactness (meta-Lindelöfness) implies metacompactness (meta-Lindelöfness). So we would like to know whether monotone metacompact (meta-Lindelöf) regular spaces are \( T_3 \)-spaces.

In the first part of this note, we show that every monotonically (countably) metacompact space is hereditarily a \( T_3 \)-space and every monotonically meta-Lindelöf space is hereditarily \( \sigma \)-closed discrete. As a corollary, we show that if \( X \) is a monotonically meta-Lindelöf (or monotonically (countably) metacompact) monotonically normal space then \( X \) is hereditarily paracompact. This gives a positive answer to a question of H.R. Bennett, K.P. Hart and D.J. Lutzer in [4]. This question was also answered by Peng and Li in [23].

Dow, Junnila and Pelant [9] introduced the notion of a thick cover. In [16] Guo and Junnila considered thickness properties related with \( T_3 \)-spaces and raised the following question. Is every scattered partition of a hereditarily almost thickly covered space almost thick? In the second part of this note, we show that every scattered partition of a hereditarily almost thickly covered space is almost thick. This answers the question of Guo and Junnila. A corollary, we have that a hereditarily almost thickly covered space is \( AD \) and linearly \( D \).

In [25], it was proved that every monotonically monolithic space is hereditarily \( D \). In [1], the notion of monotonically \( \kappa \)-monolithic was introduced and its properties were discussed. The following question appears in [1]. Must every monotonically \( \omega \)-monolithic compact space be monotonically monolithic? In the second part of this note, we give a positive answer to the question.

All the spaces in this note are assumed to be \( T_3 \)-spaces. The set of all positive integers is denoted by \( \mathbb{N} \) and \( \omega \) is \( \mathbb{N} \cup \{0\} \). In notation and terminology we will follow [7] and [10].

2. Monotone covering properties and \( D \)

**Lemma 1.** Let \( X \) be a monotonically countably metacompact regular space and let \( Y \subset X \). If \( \phi = \{\phi(y); y \in Y\} \) is a neighborhood assignment for \( Y \), then there are open sets \( V_y \) of \( Y \) such that \( y \in V_y \subset \bigcap_{y \in Y} \phi(y) \) for each \( y \in Y \) such that for any \( z \in Y \) and for any \( P_z \subset E_z = \{y; z \in V_y\} \) there is a finite subset \( F_z \subset P_z \) such that \( P_z \subset \bigcup_{y \in F_z} \phi(y) \). \( y \in Y \).

**Proof.** Let \( r \) be a monotone countable metacompactness operator for \( X \). For each \( y \in Y \) there is an open subset \( \phi'(y) \) of \( X \) such that \( \phi'(y) \cap Y = \phi(y) \). Since the space \( X \) is regular, there is an open set \( V'(y) \) of \( X \) such that \( y \in V'(y) \subset \bigcap_{y \in Y} \phi(y) \). If \( U_y = \{\phi'(y); X \setminus V'(y)\} \), then \( U_y \) is an open cover of \( X \) and \( |U_y| < \omega \). Thus there is a point-finite open refinement \( r(U_y) \) of \( U_y \). Let \( O_y \subset r(U_y) \) such that \( y \in O_y \). Thus \( O_y \subset \phi(y) \). If \( V_y = \{O_y \cap Y \cap V'(y)\} \cap Y \subset \bigcap_{y \in Y} \phi(y) \), then \( V_y \) is an open subset of \( Y \) and \( y \in V_y \subset \bigcap_{y \in Y} \phi(y) \).

Suppose there are some \( z \in Y \) and some \( P_z \subset E_z = \{y; z \in V_y\} \) such that for any finite subset \( F \subset P_z \) the set \( P_z \subset \bigcap_{\phi(y) \subset F(y)} \). Thus there is \( y_n \in P_z \) for each \( n \in \mathbb{N} \) such that \( y_n \in P_z \\setminus \bigcup_{\phi(y)} \) for each \( i \leq n \). The cover \( U = \{\phi(y_n); n \neq \phi(y)\} : y \in Y \} \) of \( X \) is countable and \( U_{n+1} \subset U \) for each \( n \in \mathbb{N} \) such that \( r(U_{n+1}) \subset r(U) \). Since \( z \in \bigcap_{y \in Y} \phi(y) \), there is some \( U_n \subset r(U) \) such that \( O_{y_n} \subset U_n \). Since \( U_n \in r(U) \) and \( z \in U_n \) for each \( n \in \mathbb{N} \), we have \( |U_n| = \omega \) by point-finite property of \( r(U) \). We denote \( U_n = \{n \in \mathbb{N} \cap |U_n| < \omega \} \) for each \( n \in \mathbb{N} \). Hence \( y_n \in \bigcup_{U_n} \) \( \bigcup_{i \leq n} \bigcup_{y \in \phi(y)} \). Note that \( \max \{y_n \} \) is a cardinal. Suppose for any \( z \in Y \) and for any \( P_z \subset E_z = \{y; z \in V_y\} \) there is a finite subset \( F_z \subset P_z \) such that \( P_z \subset \bigcup_{y \in F_z} \phi(y) \). \( y \in F_z \). \( y \in Y \).

**Theorem 2.** Let \( X \) be a topological space. If \( \phi = \{\phi(x); x \in X\} \) is a neighborhood assignment for \( X \) and there is an open set \( V_x \) of \( X \) such that \( x \in V_x \) \( \subset \phi(x) \) for each \( x \in X \) and for any \( z \in x \) the set \( E_z = \{x; z \in V_x\} \) is covered by a countable family \( \{E_m; n \in \mathbb{N}\} \) of subsets of \( X \) such that for any open set \( U \) of \( X \) and for each \( n \in \mathbb{N} \) there is a countable closed discrete subspace \( D_m \subset X \setminus U \) such that \( D_m \subset \phi(D_{m+1}) \), then there is a closed discrete subspace \( D \) of \( X \) such that \( X = \bigcup \{\phi(x); d \in D\} \).

**Proof.** We can assume \( X = \{x_\alpha; \alpha < \gamma\} \), where \( \gamma \) is a cardinal. Suppose for each \( \gamma < \beta < \gamma \), we have chosen a countable closed discrete subspace \( \alpha \) satisfying the following conditions:

1. \( x_\alpha \in \bigcup \{\phi(D_\alpha); \alpha = \alpha\} \).
2. \( \bigcup \{D_\alpha; \alpha < \gamma\} \) is a closed discrete subspace of \( X \);
3. \( D_\alpha \cap \bigcup \{\phi(D_\alpha); \alpha < \alpha\} = \emptyset \);
4. For any \( x \in X \setminus \bigcup \{\phi(D_\alpha); \alpha < \alpha\} \), \( V_x \cap \bigcup \{D_\alpha; \alpha < \alpha\} = \emptyset \).
Denote $D'_{\beta} = \bigcup\{D_{\alpha}: \alpha < \beta\}$ and denote $U_{\beta} = \bigcup\{\phi(D_{\alpha}): \alpha < \beta\}$.

If $\beta = \delta + 1$ for some ordinal $\delta$, then $D'_{\beta} = (\bigcup\{D_{\alpha}: \alpha < \delta\}) \cup D_{\delta}$. By the condition (2) the set $\bigcup\{D_{\alpha}: \alpha < \delta\}$ is closed discrete and hence $D'_{\beta}$ is closed discrete in $X$. So we assume that the ordinal $\beta$ is a limit ordinal.

If $x \in X \setminus \bigcup\{\phi(D_{\alpha}): \alpha < \beta\}$, then $V_x \cap (\bigcup\{D_{\alpha}: \alpha < \eta\}) = \emptyset$ for all $\eta < \beta$ by the condition (4). So $V_x \cap (\bigcup\{D_{\alpha}: \alpha < \beta\}) = \emptyset$. If $x \in \bigcup\{\phi(D_{\alpha}): \alpha < \beta\}$, then there is some $\alpha < \beta$ such that $x \in \phi(D_{\alpha})$ and $x \notin \phi(D_{\beta})$ for each $\eta < \alpha$. The set $\bigcup\{D_{\eta}: \eta < \alpha\}$ is a closed discrete subspace of $X$ by the condition (2). Since the set $D_{\alpha}$ is closed discrete in $X$, there is an open neighborhood $M_x$ of $x$ such that $|M_x \cap D_{\alpha}| \leq 1$. If $O_x = (\phi(D_{\alpha}) \cap (X \setminus \bigcup\{D_{\eta}: \eta < \alpha\})) \setminus M_x$, then the set $O_x$ is an open neighborhood of $x$ and $|O_x \cap D_{\beta}| \leq 1$. Thus the set $D'_{\beta}$ is a closed discrete subspace of $X$.

If $x \notin U_{\beta}$, then we let $D_{\beta} = \emptyset$, otherwise we let $A_0 = \{x\}$ and let $E_{y_0} = \{x: V_x \cap A_0 = \emptyset\}$. Thus $E_{y_0} \subset \bigcup\{E_{y_n}: n \in \mathbb{N}\}$ such that for any open set $U$ of $X$ and for each $n \in \mathbb{N}$ there is a countable closed discrete set $D_{y_n} \subset X \setminus U$ such that $E_{y_n} \cup \bigcup\{D_{y_n}: n \in \mathbb{N}\}$. Enumerate $\{E_{y_n}: n \in \mathbb{N}\}$ by prime numbers. So $\{E_{y_n}: n \in \mathbb{N}\} = \{E_n: p$ is a prime number$\}$. If $\bigcup\{E_{y_n}: n \in \mathbb{N}\} = \bigcup\{E_{y_n}: n \in \mathbb{N}\} = \emptyset$, then we let $D_{\beta} = A_0$. So we assume that $\bigcup\{E_{y_n}: n \in \mathbb{N}\} = \bigcup\{E_{y_n}: n \in \mathbb{N}\} = \emptyset$. By Lemma 1 and Theorem 2, we can get the following theorem.

If $\beta < \eta < \alpha$ such that $\bigcup\{D_{\eta}: \beta < \eta\}$ is closed and discrete and $D_{\eta} \subset X \setminus \bigcup\{\phi(D_{\eta}): \beta < \eta\}$ for each $\alpha < \kappa$ and $X = \bigcup\{\phi(d): d \in D\}$, where $D = \bigcup\{D_{\alpha}: \alpha < \kappa\}$. We can see that the set $D$ is closed discrete in $X$. □

By Lemma 1 and Theorem 2, we can get the following theorem.

**Theorem 3.** If $X$ is a monotonically countably metacompact regular space, then $X$ is hereditarily a $D$-space.

**Corollary 4.** If $X$ is a monotonically metacompact regular space, then $X$ is hereditarily a $D$-space.

By Theorem 2 and a short proof we can get Lemma 25 in [22].

**Proposition 5.** ([22, Lemma 25]) If $X$ has a point-countable open cover $V$ such that $\overline{V}$ is Lindelöf for each $V \in \mathcal{V}$, then $X$ is a $D$-space.

**Proof.** Let $\phi$ be any neighborhood assignment for $X$. For each $x \in X$ there is some $V'_x \in \mathcal{V}$ such that $x \notin V'_x$. If $V_x = V'_x \cap (\phi(x))$, then $x \in V_x \subset \phi(x)$. For any $z \in X$ we denote $E_z = \{x: z \in V_x\}$. Thus $E_z \subset \bigcup\{V: z \in V, V \in \mathcal{V}\} \subset \bigcup\{V': z \in V, V \in \mathcal{V}\}$. Since

A space $X$ is locally Lindelöf if every point $x$ of $X$ has a neighborhood $V_x$ which is a Lindelöf $D$-space.

Corollary 6. If $X$ is regular meta-Lindelöf locally Lindelöf $D$, then $X$ is $D$.

A space $X$ is monotonically normal if there is a function $G$ which assigns to each ordered pair $(H, K)$ of disjoint closed subsets of $X$ an open set $G(H, K)$ such that:

1. $H \subset G(H, K) \subset G(H, K) \subset X \setminus K$;
2. if $(H', K')$ is a pair of disjoint closed subsets having $H \subset H'$ and $K \supset K'$, then $G(H, K) \subset G(H', K')$ [18].

Lemma 7. ([5]) A space $X$ is monotonically normal if and only if for each open set $U \subset X$ and $x \in U$, one can assign an open set $U_x$ containing $x$ satisfying the following condition: $U_x \cap V_y \neq \emptyset$ implies $x \in V$ or $y \in U$.

Proposition 8. ([6]) If $X$ is a monotonically normal $D$-space, then $X$ is paracompact.

By Theorem 3 and Proposition 8, we have the following corollary.

Corollary 9. ([23, Theorem 3]) Suppose $X$ is a monotonically (countably) metacompact space that is monotonically normal. Then $X$ is hereditarily paracompact.

Lemma 10. ([23, Theorem 10]) Suppose $X$ is a monotonically meta-Lindelöf regular space. Let $Y \subset X$, and for each $y \in Y$, let $\phi(y)$ be an open neighborhood of $y$ in $X$. Then there is an open neighborhood $V_y$ of $y$ such that $x \in V_y$ for each $y \in Y$ and $V_y \subset \phi(y)$ and satisfies that if $p \in \bigcap \{V_x: x \in Y'\}$ for some subset $Y' \subset Y$, then there is a countable subset $Y'' \subset Y'$ such that $Y'' \subset \bigcup \{\phi(y): y \in Y''\}$.

By Lemma 10, we can get the following corollary.

Corollary 11. Let $X$ be a monotonically meta-Lindelöf regular space and $Y \subset X$. If $\phi = \{\phi(y): y \in Y\}$ is a neighborhood assignment for the subspace $Y$ of $X$ (i.e. $\phi(y)$ is open in $Y$ for each $y \in Y$), then there is an open set $V_y$ for each $y \in Y$ such that $y \in V_y \subset \phi(y)$ for each $y \in Y$ such that for any $z \in Y$ and for any $P_z \subset E = \{y: z \in V_y\}$ there is a countable subset $F_z \subset P_z$ such that $P_z \subset \bigcup \{\phi(y): y \in F_z\}$.

Theorem 12. Let $X$ be a topological space. If $\phi = \{\phi(x): x \in X\}$ is a neighborhood assignment for $X$ and there is an open set $V_x$ of $X$ such that $x \in V_x \subset \phi(x)$ for each $x \in X$ such that for any $x \in X$ for any open set $U$ of $X$ there is a countable subset $U_x \subset X \setminus U$ such that $\{x: z \in V_x \setminus U \subset \bigcup \{\phi(y): y \in U_x\}\}$, then there is an $\omega$-discrete subspace $D$ of $X$ such that $X = \bigcup \{\phi(d): d \in D\}$.

Proof. We can assume $X = \{\alpha: \alpha < \gamma\}$, where $\gamma$ is a cardinal. Suppose for each $\alpha < \beta < \gamma$, we have chosen a countable set $D_{\alpha}$ satisfying the following conditions:

1. $x_0 \in \bigcup \{\phi(D_{\alpha})\}: \eta \leq \alpha$;
2. $D_{\alpha} \cap \bigcup \{\phi(D_{\eta})\}: \eta < \alpha$ is $\emptyset$;
3. if $x \in X \setminus \bigcup \{\phi(D_{\eta})\}: \eta \leq \alpha$, then $V_x \cap \bigcup \{\phi(D_{\eta})\}: \eta \leq \alpha$ is $\emptyset$.

By the condition (3), we have $\bigcup \{\phi(D_{\eta})\}: \eta \leq \alpha \subset \bigcup \{\phi(D_{\alpha})\}: \eta \leq \alpha$ for each $\alpha < \beta$. Denote $D_{\beta} = \bigcup \{\phi(D_{\alpha}): \alpha < \beta\}$ and denote $U_\beta = \bigcup \{\phi(D_{\alpha})\}: \alpha < \beta$. In what follows, we show that $D_{\beta} \subset U_\beta$.

If $\beta = \delta + 1$ for some ordinal $\delta$, then $D_{\beta} = (\bigcup \{D_{\alpha}: \alpha < \delta\}) \cup D_\delta = \bigcup \{D_{\alpha}: \alpha \leq \delta\}$. By the condition (3), we have $D_{\beta} \subset U_\beta$.

Let $\beta$ be a limit ordinal. If $V_x \cap D_{\beta} \neq \emptyset$, then there is some $\alpha < \beta$ such that $V_x \cap D_{\alpha} \neq \emptyset$. Thus $x \in \bigcup \{\phi(D_{\eta})\}: \eta \leq \alpha$ by the condition (3). So $D_{\beta} \subset U_\beta$.

If $x_0 \in D_{\beta}$, then we let $D_\beta = \emptyset$, otherwise we let $A_0 = \{x_\beta\}$ and let $F_0 = \{x: V_x \setminus A_0 \neq \emptyset\}$. If $F_0 \setminus (\phi(A_0) \cup U_\beta) \neq \emptyset$, then there is a countable set $A_1$ of $X$ such that $A_1 \subset X \setminus (\phi(A_0) \cup U_\beta)$ and $F_0 \setminus (\phi(A_0) \cup U_\beta) \subset \phi(A_1)$.

Assume we have a countable set $A_n$ for each $n \leq \omega$ such that $A_n \subset X \setminus \{\phi(A_{n-1}): n < \omega\}$ and $V_x \cap (\bigcup \{\phi(A_i): i < n\}) \neq \emptyset$ if $x \in \bigcup \{\phi(A_i): i < n\} \setminus U_\beta$. Denote $F_n = \{x: V_x \setminus A_n \neq \emptyset\}$. Since $A_m$ is countable and for each $d \in A_m$ there is a countable set $M_d$ such that $M_d \subset X \setminus (U_\beta \cup \{\phi(A_i): i \leq m\})$ and $x \in \{\phi(A_i): i \leq m\} \setminus U_\beta$, we denote $A_{n+1} = \bigcup \{M_d: d \in A_m\}$. Thus $\bigcup \{A_n: n \leq \omega\} = \phi(A_{\omega})$. We let $D_\beta = \bigcup \{A_n: n \leq \omega\}$. If $x \notin U_\beta \cup \phi(D_{\beta})$, then $V_x \cap (D_{\beta} \cup D_{\beta}) = \emptyset$.
Thus we have a countable set $D_\alpha$ for each $\alpha < \kappa$ such that if $\beta < \kappa$ and $x \notin \bigcup(\phi(D_\alpha): \alpha \leq \beta)$ then $V_x \cap \bigcup(D_\alpha: \alpha \leq \beta) = \emptyset$. Thus $\bigcup(D_\alpha: \alpha \leq \beta) \subset \bigcup(\phi(D_\alpha): \alpha \leq \beta)$.

For each $x$ there is a minimal ordinal $\alpha_x$ such that $x \in \phi(D_{\alpha_x})$. Thus $x \notin \bigcup(\phi(D_\eta): \eta \leq \alpha_x)$ and hence $V_x \cap \bigcup(D_\alpha: \eta < \alpha_x) = \emptyset$. Thus $\bigcup(\phi(D_\alpha \cap (V_x \cap \phi(D_{\alpha_x})) \neq \emptyset) \leq 1$. So the family $\{D_\alpha: \alpha < \kappa\}$ is a discrete family of countable subsets of $X$. Thus the set $D = \bigcup(D_\alpha: \alpha < \kappa)$ is $\sigma$-closed discrete and $X = \bigcup(\phi(d): d \in D)$. \hfill \Box

A space $X$ is called a $D\sigma$-space if for any neighborhood assignment $\phi$ for $X$ there exists a $\sigma$-closed discrete subspace $D$ of $X$ such that $X = \bigcup(\phi(d): d \in D)$ [21]. In [19], van Mill, Tkachuk, and Wilson developed ideas related to $D$-spaces by defining for a topological property $\mathcal{P}$, a space $X$ to be dually $\mathcal{P}$ if for each neighborhood assignment $\phi = \{\phi(x): x \in X\}$ for $X$, there is a subspace $Y \subset X$ with property $\mathcal{P}$ such that $X = \bigcup(\phi(x): x \in Y)$. So a $D\sigma$-space is also called a dually $\sigma$-closed discrete space in this note. A collection $\mathcal{V}$ of subsets of a space $X$ is said to be cushioned in a collection $\mathcal{U}$ of subsets of $X$ if there exists a function $T: \mathcal{V} \to \mathcal{U}$ such that for any $\mathcal{W} \subset \mathcal{V}$, we have $\bigcup(\mathcal{W}: \mathcal{W} \subset \mathcal{V}) \subset \bigcup(T(\mathcal{W}): \mathcal{W} \subset \mathcal{V})$ [7]. If a family $\mathcal{V} = \bigcup(\mathcal{V}_n: n \in \mathbb{N})$ and $\mathcal{V}_n$ is cushioned in $\mathcal{U}$ for each $n \in \mathbb{N}$, then the family $\mathcal{V}$ is called $\sigma$-cushioned in $\mathcal{U}$ [7].

By Corollary 11 and Theorem 12, we have the following theorem.

**Theorem 13.** If $X$ is a monotonically meta-Lindelöf regular space, then $X$ is hereditarily dually $\sigma$-closed discrete.

**Proposition 14.** ([7, Theorem 2.3]) Let $X$ be a regular space. $X$ is paracompact if and only if every open cover of $X$ has a $\sigma$-cushioned open refinement.

**Lemma 15.** Let $X$ be a monotonically normal space. If $X$ is a $D\sigma$-space, then $X$ is paracompact.

**Proof.** Let $\mathcal{U}$ be any open cover of $X$. For each $x \in X$ there is some $\phi(x) \in \mathcal{U}$ such that $x \in \phi(x)$. Since $X$ is a monotonically normal space, there is an open neighborhood $V_x$ of $x$ such that $x \in V_x \subset \phi(x)$ for each $x \in X$ satisfying that $V_x \cap V_y \neq \emptyset$ implies $x \in \phi(y)$ or $y \in \phi(x)$ by Lemma 7. Thus $\phi' = \{V_x: x \in X\}$ is a neighborhood assignment for $X$. Since the space $X$ is a $D\sigma$-space, there is a $\sigma$-closed discrete subset $D$ of $X$ such that $X = \bigcup(\phi(x): x \in D)$, where $D = \bigcup(D_n: n \in \mathbb{N})$ and $D_n$ is closed discrete for each $n \in \mathbb{N}$. For any $F_n \subset D_n$ and for any $x \in X \setminus \phi(F_n)$ the set $U = X \setminus F_n$ is an open neighborhood of $y$. Since $y \notin \phi(x)$ and $x \notin F_n$ for each $x \in F_n$, the set $U \cap V_x = \emptyset$ for each $x \in F_n$. Thus $\bigcup(V_x: x \in F_n) \subset \phi(F_n)$. So $\{V_x: x \in D\}$ is a $\sigma$-cushioned open refinement of $\mathcal{U}$. Thus $X$ is paracompact. \hfill \Box

By Theorem 13 and Lemma 15 we have the following corollary.

**Corollary 16.** ([23, Theorem 5]) Suppose $X$ is a monotonically normal space that is monotonically meta-Lindelöf. Then $X$ is hereditarily paracompact.

The space $\omega_1$ is monotonically normal, but not a $D$-space and not a $D\sigma$-space. So by Theorems 3 and 13 we have the following corollaries.

**Corollary 17.** ([24]) $\omega_1 + 1$ is not monotonically countably metacompact.

**Corollary 18.** ([3, Example 2.3]) $\omega_1 + 1$ is not monotonically Lindelöf.

**Corollary 19.** ([11, Proposition 2]) $\omega_1 + 1$ is not monotonically meta-Lindelöf.

3. Answers to two questions

In [16], Guo and Junnila considered thickness properties related with $D$-spaces. In [16], there is a question which relates to an almost thick cover. In what follows, we will discuss the question.

If $X$ is a space and $A \subset X$, then we denote $[A]^{<\omega} = \{F: F \subset A \text{ and } |F| < \omega\}$.

A cover $\mathcal{L}$ of a space $X$ is thick if we can assign $L_H \in [\mathcal{L}]^{<\omega}$ and $L_H = \bigcup L_H$ to each $H \in [X]^{<\omega}$ in such a way that $X \subset \bigcup(L_H: H \in [A]^{<\omega})$ for every $A \subset X$ [9]. A space $X$ is thickly covered if every open cover of $X$ is thick [9]. A cover $\mathcal{L}$ of a space $X$ is almost thick provided that we can assign $L_H \in [\mathcal{L}]^{<\omega}$ and $L_H = \bigcup L_H$ to each $H \in [X]^{<\omega}$ so that for every non-closed subset $A \subset X$, there exists $H \in [A]^{<\omega}$ such that $L_H \cap (A \setminus A) \neq \emptyset$ [16]. A space is almost thickly covered if every open cover of the space is almost thick [16]. A partition $\mathcal{N}$ of a space $X$ is scattered provided that we can write $\mathcal{N} = \{N_\alpha: \alpha < \lambda\}$ for some ordinal $\lambda$ so that, for each $\beta < \lambda$, the set $\bigcup_{\alpha < \beta} N_\alpha$ is open in $X$ [26].

**Proposition 20.** ([16, Proposition 3.6]) A space $X$ is hereditarily thickly covered if and only if every scattered partition of the space is thick.
The following problems appear in [16],
Is every scattered partition of a hereditarily almost thickly covered space almost thick [16, Problem 3.7]?
Is a hereditarily almost thickly covered space an aD and linearly D [16, Problem 3.9]?

A space $X$ is aD provided that, for every open cover $\mathcal{V}$ of $X$ and for each closed subset $F$ of $X$, there exists a closed discrete set $D \subset F$, and for every $x \in D$ there is a set $V_x \in \mathcal{V}$ such that $x \in V_x$ and $\{V_x \mid x \in D\}$ covers $F$ [2]. A space linearly D provided that for every monotone open cover $\mathcal{U}$ of $X$, if $\mathcal{X} \notin \mathcal{U}$, then there exists a closed discrete set $D \subset X$ such that $D$ is not contained in any member of $\mathcal{U}$ [17].

In [16], Guo and Junnila pointed out that a positive solution to the first problem would yield a positive solution to the second problem. So we will give an answer to the first problem. The proof of the following theorem is similar to the second part of the proof of Proposition 3.6 in [16].

**Theorem 21.** Every scattered partition of a hereditarily almost thickly covered space is almost thick.

**Proof.** Let $X$ be a hereditarily almost thickly covered space. We use transfinite induction on the ordinal $\mu$ to show that, for every monotone family $\mathcal{G} = \{G_\alpha : \alpha < \mu\}$ of open subsets of $X$, the family $\{G_\alpha \setminus \bigcup_{\beta < \alpha} G_\beta : \alpha < \mu\}$ is an almost thick partition of the subspace $\bigcup \mathcal{G}$ of $X$. The result is true if $\mu$ is a finite ordinal.

Let $\lambda$ be an ordinal such that the result holds for each $\mu < \lambda$.

To prove the result for $\lambda$, we can assume $\lambda > \omega$.

Let $\mathcal{G} = \{G_\alpha : \alpha < \lambda\}$ be a monotone family of open subsets of $X$. Denote $Z = \bigcup \mathcal{G} = \bigcup \{G_\alpha : \alpha < \lambda\}$. The subspace $Z$ of $X$ is an almost thickly covered space. Thus for each finite subset $H \subset Z$, we can assign an ordinal $\alpha_H < \lambda$ such that $H \subset G_{\alpha_H}$ and if $A \subset Z$ is not closed in $Z$, then there are some $x \in (\overline{A} \cap Z) \setminus A$ and some $H \subset A$, $|H| < \omega$ such that $x \in G_{\alpha_H}$. If $K \subset Z$ and $K$ is finite, then we can let $\alpha_K = \max(\alpha_H : H \subset K)$. Thus $\alpha_K < \alpha_{\lambda}$ if $K \subset \lambda$.

For any finite set $K \subset Z$ and for any $H \subset \lambda$, we have $\alpha_K$ and $\alpha_H$. By induction, the partition $\{S_\alpha : \alpha < \alpha_K\}$ is an almost thick cover of $\bigcup \{G_\alpha \setminus \bigcup_{\beta < \alpha} G_\beta : \alpha < \alpha_K\}$, where $S_\alpha = G_\alpha \setminus \bigcup_{\beta < \alpha} G_\beta$ for each $\alpha < \alpha_K$. For each finite set $L \subset \bigcup \{S_\alpha : \alpha < \alpha_K\}$, we can assign a finite set $E_L^W \subset \alpha_H$ such that $B \subset \bigcup \{S_\alpha : \alpha < \alpha_H\}$ and $B$ is not closed in $\bigcup \{S_\alpha : \alpha < \alpha_H\}$, then there are some finite set $L \subset B$ and some $x \in (B \setminus B) \cap (\bigcup \{S_\alpha : \alpha < \alpha_H\})$ such that $x \in \bigcup \{S_\alpha : \alpha \in E_L^W\}$.

We denote $F_K = \{\alpha_K \cup (\bigcup \{E_{L}^W : H \subset \lambda \text{ and } L \subset \bigcup \{S_\alpha : \alpha < \alpha_H\} \cap \mathcal{K}\})\}$ for each finite set $K \subset \lambda$. Thus the set $F_K$ is finite and $F_K \subset \lambda$. Let $A \subset \lambda$ and let $A$ be not closed in $\lambda$. Since $X$ is a hereditarily almost thickly covered space, the open cover $\{G_\alpha : \alpha < \lambda\}$ of $Z$ is almost thick. Thus there are some $K \subset A$, $|K| < \omega$ and some $x \in (\overline{A} \cap Z) \setminus A$ such that $x \in G_{\alpha_K}$. Since $\alpha_K \in F_K$, we have $x \in \bigcup \{S_\alpha : \alpha < \alpha_K\}$ if $x \in S_{\alpha_K}$. So we assume that $x \notin S_{\alpha_K}$. Then $x \in \bigcup \{S_\alpha : \alpha < \alpha_K\} = \bigcup \{S_\alpha : \alpha < \alpha_K\} = \mathcal{O}_X \subset Z$. So we can assume that $x \notin S_{\alpha_K}$. Then $x \in \bigcup \{S_\alpha : \alpha < \alpha_K\} = \bigcup \{S_\alpha : \alpha < \alpha_K\} = \mathcal{O}_X \subset Z$. Thus the set $\mathcal{O}_X$ is open. So $x \in (\mathcal{O}_X \cap \mathcal{O}_X) \setminus (\mathcal{O}_X \cap \mathcal{O}_X)$. Thus the set $\mathcal{O}_X \cap \mathcal{O}_X$ is not closed in $\mathcal{O}_X$. The partition $\{S_\alpha : \alpha < \alpha_K\}$ of $\mathcal{O}_X$ is almost thick by induction. Thus there are some $y \in (\mathcal{O}_X \cap \mathcal{O}_X) \setminus (\mathcal{O}_X \cap \mathcal{O}_X)$ and some finite $L \subset \mathcal{O}_X \cap \mathcal{O}_X$ such that $y \in \bigcup \{S_\alpha : \alpha \in E_L^W\}$, where $E_L^W$ is finite. Denote $K' = K \cup L$, thus $|K'| < \omega$ and $K' \subset \lambda$. Thus $K' \subset A$, $L \subset \bigcup \{S_\alpha : \alpha < \alpha_K\}$ and $K' \subset \lambda$ such that $y \in \bigcup \{S_\alpha : \alpha \in E_L^W\}$. So $y \in \bigcup \{S_\alpha : \alpha \in F_{K'}\}$. Thus the partition $\{S_\alpha : \alpha < \lambda\}$ is almost thick. □

By Theorem 21 and a proof which is similar to the proof of Proposition 3.8 in [16], we can get the following theorem. This was also pointed out by Guo and Junnila in [16].

**Theorem 22.** A hereditarily almost thickly covered space is aD and linearly D.

In [25], it was proved that every monotonically monolithic space is hereditarily D. In [1], the notion of monotonically $\kappa$-monolithic was introduced and its properties were discussed. The following question appears in [1]. Must every monotonically $\omega$-monolithic compact space be monotonically monolithic? In what follows, we give a positive answer to the question.

Firstly, let’s recall some notions.

A space $X$ is monotonically monolithic [25] if one can assign to each $A \subset X$ a collection $\mathcal{N}(A)$ of subsets of $X$ such that

1. $|\mathcal{N}(A)| \leq |A| + \omega$;
2. $A \subset B \Rightarrow \mathcal{N}(A) \subset \mathcal{N}(B)$;
3. if $\{A_\alpha : \alpha < \delta\}$ is an increasing collection of subsets of $X$, and $A = \bigcup \{A_\alpha : \alpha < \delta\}$, then $\mathcal{N}(A) = \bigcup \{\mathcal{N}(A_\alpha) : \alpha < \delta\}$;
4. if $U$ is open and $x \in \overline{A} \cup U$, then there is some $B \in \mathcal{N}(A)$ with $x \in B \subset U$.

We call $\mathcal{N}$ a monotonically monolithic operator for $X$.

Further, for an infinite cardinal $\kappa$, $X$ is said to be monotonically $\kappa$-monolithic [1] if $\mathcal{N}(A)$ is defined for all sets $A$ with $|A| < \kappa$ and satisfies the above conditions.

Condition (4) can be rephrased by declaring that $\mathcal{N}(A)$ contains a network at every point of $\overline{A}$, i.e. if $x \in \overline{A}$ and $U$ is an open neighborhood of $x$, then there is some $B \in \mathcal{N}(A)$ such that $x \in B \subset U$ [15].
For example, any space with a point-countable base is monotonically monolithic.

**Lemma 23.** ([15, Theorem 3.2]) A space $X$ is monotonically monolithic if and only if one can assign to each finite subset $F$ of $X$ a countable collection $\mathcal{N}(F)$ of subsets of $X$ such that, for $A \subset X$, $\bigcup_{F \in [A]^{<\omega}} \mathcal{N}(F)$ contains a network at each point of $\overline{A}$.

**Theorem 24.** A space $X$ is monotonically $\omega$-monolithic if and only if one can assign to each finite subset $F$ of $X$ a countable collection $\mathcal{N}(F)$ of subsets of $X$ such that, for every countable set $A \subset X$, $\bigcup_{F \in [A]^{<\omega}} \mathcal{N}(F)$ contains a network at each point of $\overline{A}$.

**Proof.** “$\Rightarrow$” If one can assign to each finite subset $F$ of $X$ a countable collection $\mathcal{N}(F)$ of subsets of $X$ such that $\bigcup_{F \in [A]^{<\omega}} \mathcal{N}(F)$ contains a network at each point of $\overline{A}$ for every countable subset $A$ of $X$, then we denote $\mathcal{N}^*(A) = \bigcup_{F \in [A]^{<\omega}} \mathcal{N}(F)$ for every countable subset $A$ of $X$. Thus the operator $\mathcal{N}^*$ is a monotonically $\omega$-monolithic operator for $X$.

“$\Leftarrow$” Let $X$ be monotonically $\omega$-monolithic and let $\mathcal{N}$ be a monotonically $\omega$-monolithic operator for $X$. If $F \subset X$ and $|F| < \omega$, then $\mathcal{N}(F)$ is a countable family of subsets of $X$. We show that $\bigcup_{F \in [\omega]} \mathcal{N}(F)$ contains a network at each point of $\overline{\omega}$ for every countable subset $A$ of $X$.

Let $A \subset X$ with $|A| \leq \omega$. If $x \in \overline{A}$ and $U$ is an open neighborhood of $x$, then there is some $B \in \mathcal{N}(A)$ such that $x \in B \subset U$. Denote $A = \{x_n: n \in \mathbb{N}\} = \bigcup_{A_n: n \in \mathbb{N}}$, where $A_n = \{x_i: 1 \leq i \leq n\}$. Thus $\mathcal{N}(A) = \bigcup_{\gamma \in \mathbb{N}}(N(A_\gamma): n \in \mathbb{N})$. There is some $n \in \mathbb{N}$ such that $B \in \mathcal{N}(A_n)$, hence $\bigcup_{F \in [\omega]} \mathcal{N}(F)$ contains a network at each point of $\overline{A}$. □

Recall that if for any $A \subset X$ and for any $x \in \overline{A}$ there is a countable set $C \subset A$ such that $x \in \overline{C}$ then the space $X$ is said to have countable tightness. Denoted it by $t(X) \leq \omega$. If $|C| \leq \kappa$ and $\kappa \geq \omega$ in the above definition, then we say that $t(X) \leq \kappa$.

We know that if $X$ has Fréchet–Urysohn property then $t(X) \leq \omega$.

**Theorem 25.** If $X$ is a monotonically $\omega$-monolithic space with $t(X) \leq \omega$, then $X$ is monotonically monolithic.

**Proof.** Since $X$ is a monotonically $\omega$-monolithic space, one can assign to each finite subset $F$ of $X$ a countable collection $\mathcal{N}(F)$ of subsets of $X$ such that, for every countable set $A \subset X$, $\bigcup_{F \in [A]^{<\omega}} \mathcal{N}(F)$ contains a network at each point of $\overline{A}$ by Theorem 24.

Now let $A$ be any subset of $X$, we denote $\mathcal{N}^*(A) = \bigcup_{F \in [A]^{<\omega}} \mathcal{N}(F)$. To prove that $X$ is monotonically monolithic, we just need to prove $\mathcal{N}^*(A)$ contains a network at every point of $\overline{A}$ by Lemma 23. Since $t(X) \leq \omega$, there is a countable subset $C \subset A$ such that $x \in \overline{C}$ if $x \in \overline{A}$. The set $C$ is countable, thus $\bigcup_{F \in [C]^{<\omega}} \mathcal{N}(F)$ contains a network at each point of $\overline{C}$. If $U$ is an open neighborhood of $x$, then there is some $F_1 \in [C]^{<\omega}$ and some $B \in \mathcal{N}(F_1)$ such that $x \in B \subset U$. So $B \in \bigcup_{F \in [\omega]} \mathcal{N}(F) = \mathcal{N}^*(A)$. Thus $\mathcal{N}^*(A)$ contains a network at every point of $\overline{A}$. Thus $X$ is monotonically monolithic by Lemma 23. □

**Lemma 26.** ([1, Corollary 2.24]) If a countably compact space $X$ is monotonically $\omega$-monolithic, then $X$ is compact and has the Fréchet–Urysohn property.

By Theorem 25 and Lemma 26, we have:

**Theorem 27.** If a countably compact space $X$ is monotonically $\omega$-monolithic, then $X$ is a compact monotonically monolithic space.

Theorem 27 gives a positive answer to Problem 3.3 in [1].

**Corollary 28.** Every compact monotonically $\omega$-monolithic space is monotonically monolithic.

Similar to Theorem 24, we have:

**Theorem 29.** A space $X$ is monotonically $\kappa$-monolithic if and only if one can assign to each finite subset $F$ of $X$ a countable collection $\mathcal{N}(F)$ of subsets of $X$ such that, for every subset $A \subset X$ with $|A| \leq \kappa$, $\bigcup_{F \in [A]^{<\omega}} \mathcal{N}(F)$ contains a network at each point of $\overline{A}$.

**Proof.** “$\Rightarrow$” It is similar to the proof of the first part of Theorem 24.

“$\Leftarrow$” Let $X$ be monotonically $\kappa$-monolithic and let $\mathcal{N}$ be a monotonically $\kappa$-monolithic operator for $X$. If $F \subset X$ and $|F| < \omega$, then $\mathcal{N}(F)$ is a countable family of subsets of $X$. Let $A \subset X$ and $|A| \leq \kappa$. We show that $\bigcup_{F \in [A]^{<\omega}} \mathcal{N}(F)$ contains a network at each point of $\overline{A}$. For any $x \in \overline{A}$ and for any open neighborhood $U$ of $x$ there is some $B \in \mathcal{N}(A)$ such that $x \in B \subset U$. Denote $\gamma = \min(\alpha$: there is a subset $A_\alpha \subset A$ such that $|A_\alpha| = \alpha$ and $B \in \mathcal{N}(A_\alpha))$. We can see that $\gamma < \omega$. Thus $\bigcup_{F \in [\omega]} \mathcal{N}(F)$ contains a network at each point of $\overline{A}$. □

Similar to Theorem 25, we have:
Theorem 30. Let $\kappa$ be an infinite cardinal. If $X$ is a monotonically $\kappa$-monolithic space with $t(X) \leq \kappa$, then $X$ is monotonically monolithic.

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