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The Root and Bell's disk iteration methods are of the same error propagation characteristics in the simultaneous determination of the zeros of a polynomial, Part I: Correction methods

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Abstract

In this paper we consider the error propagation of the Root and Bell's disk iteration methods enhanced by incorporating a correction term and a choice of a disk inversion formula in the methods, for the simultaneous computation of the zeros of a polynomial. The asymptotic error propagation is proved to be the same in both methods. This result is important considering the fact that these methods are in popular usage in the simultaneous computation of the zeros of a polynomial. The proof of the results herein follows the ideas of [M.S. Petkovic, C. Carstensen, Some improved inclusion methods for polynomial roots with Weierstrass corrections, Comput. Math. Appl. 25 (3) (1993) 59–67]. When the refinement process of correction is efficient, it is this mode of correction we have desired to propose.

Keywords: Interval methods; Correction; R-order of convergence; Zeros of a polynomial

1. Introduction: The Root iteration and Bell's disk iteration methods

This paper is concerned with the simultaneous numerical computation of the simple zeros $\{\lambda_j\}_{j=1}^{n}$ of the polynomial

$$P_n(z) = \sum_{j=0}^{n} a_j z^j = \prod_{j=1}^{n} (z - \lambda_j); \quad n > 2 \tag{1.1}$$

with degree n where the variable z and the constants $\{a_j\}_{j=0(1)n}$ can take real or complex values, by any of the members in the family of the simultaneous root iteration methods

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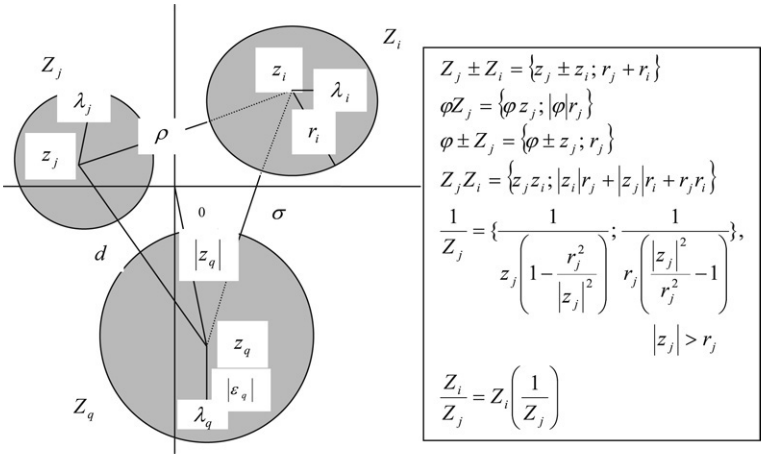

Fig. 1.1. Disks separation and arithmetic operations on circular intervals.

$$z_j^{(s+1)} = z_j^{(s)} - \frac{1}{\left(h_k(z_j^{(s)}) - \sum\limits_{\substack{i=1\\ j\neq i}}^{N}\left(\frac{1}{z_j^{(s)}-z_i^{(s)}}\right)^k\right)^{\frac{1}{k}}}; \quad j = 1(1)n,\ k = 1, 2, 3, \ldots \tag{1.2}$$

in point arithmetic for a fixed k. The order of convergence is $k+2$, and $s = 0, 1, 2, \ldots$ is the iteration index and $h_k(z_j^{(s)})$ is

$$h_k(z) = \frac{(-)^{k-1}}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}\left(\frac{P_n'(z)}{P_n(z)}\right) = \sum_{j=1}^{n}\left(\frac{1}{z-\lambda_j}\right)^k; \quad k = 1, 2, 3, \ldots. \tag{1.3}$$

The conversion of (1.2) into an interval iteration process is achieved by the application of the inclusion relation

$$\sum_{i=1, i\neq j}^{N}\left(\frac{1}{z_j - z_i}\right) \in \sum_{i=1, i\neq j}^{N}\left(\frac{1}{z_j - Z_i}\right); \quad Z_i \cap Z_j = \phi\ (empty),\ \ \left|z_j\right| > r_j \tag{1.4}$$

that holds with the set of circular disks $\{Z_j\}_{j=1(1)N}$. This is a consequence of the isotonicity inclusion theorem, with the circular disks $Z_j = \{z_j; r_j\}$ defined as $Z_j = \{w : |w - z_j| \le r_j; z_j \in \not\subset, r_j \in R, r_j > 0\} = \{z_j; r_j\}$. This is a circular region of centre $z_j = \mathrm{Mid}(Z_j)$ and radius $r_j = \mathrm{Rad}(Z_j)$, obeying arithmetic operations defined in [2] and also stated in Fig. 1.1, but note that the interior of the disks have been shaded and φ is a constant.

Consequently, the point iteration in (1.2) therefore suggests the interval iteration procedure

$$Z_j^{(s+1)} = z_j^{(s)} - \frac{1}{\left(h_k(z_j^{(s)}) - \sum\limits_{\substack{i=1\\ j\neq i}}^{n}\left(\frac{1}{z_j^{(s)}-Z_i^{(s)}}\right)^k\right)^{\frac{1}{k}}}; \quad j = 1(1)n,\ k = 1, 2, 3, \ldots. \tag{1.5}$$

This interval iteration procedure includes the zeros of the polynomial in (1.1) in the generated disks. The initial approximate points $\{z_j^{(0)}\}_{j=1}^{n}$ and circular inclusion disks $\{Z_j^{(0)}\}_{j=1}^{n}$ of the simples zeros $\{\lambda_j\}_{j=1}^{n}$ of (1.1) are supplied to commence the iterations in (1.2) and (1.5) respectively. The implementation of (1.5) require the kth root of a disk, when for example the disk is $Z = \{z; r\}$, then the kth root is given by

$$Z^{\frac{1}{k}} = \bigcup_{m=0}^{k-1} \left\{ |z|^{\frac{1}{k}} e^{\frac{\theta+2\pi m}{k}}; \frac{r}{\sum_{j=0}^{k-1} |z|^{\frac{k-j-1}{k}} (|z|-r)^{\frac{j}{k}}} \right\}; \quad \theta = \arg(z); \; k = 2, 3, 4, \ldots; \; |z| > r. \tag{1.6}$$

The choice of the appropriate kth root of the disk in the denominator of (1.5) is made possible by a theorem due to [3] for the specific case of $k = 2$, the general case is resolved in [4]. The popularity of this class of methods despite their high attainable order which increases with increasing k is limited by their implementation difficulties which arises from the requirement to compute the square root of the denominator and the problems created by the need to choose the appropriate kth root in (1.2) and root of the disk as in the case of (1.5). In this regard, their computational complexity deteriorates with increasing k and for $k = 2$ is seem to be worse than the method of Newton (same as $k = 1$, also known as Maehly's method), as observed in [3]. By this reason, the Borsch–Supan's, Maehly's or Halley-like methods or some others of comparable convergence and without need to compute radicals may be preferred, see [7–28]. The difficulties not withstanding, these methods are no doubt found useful in the numerical inclusion of polynomial zeros. In fact, their accuracy improves in the order of increasing k, though this is most often than not inhibited by round-offs, a default arising from the limited precision of the computing device currently available. Next, consider another class of methods known as the Bell's iteration methods for the simultaneous computation of the zeros of the polynomial in (1.1) from [4, p. 163] in chapter five, the point iteration methods

$$z_j^{(s+1)} = z_j^{(s)} - \frac{\Delta_{k-1,j}\left(z_j^{(s)}\right)}{\Delta_{k,j}\left(z_j^{(s)}\right) - B_k\left(S_{1,j}^{(s)}(z), S_{2,j}^{(s)}(z), \ldots, S_{k,j}^{(s)}(z)\right)}; \quad k = 1, 2, 3, \ldots \tag{1.7}$$

derived from the Bell's polynomials $B_k = B_k(z_1, z_2, z_3, \ldots, z_k)$ of degree k, given by the recursive relation

$$B_k(z_1, z_2, z_3, \ldots, z_k) = \begin{cases} 1; & k = 0 \\ z_1; & k = 1 \\ \frac{1}{2}\left(z_2 + z_1^2\right); & k = 2 \\ \frac{1}{3}z_3 + \frac{1}{2}z_2 z_1 + \frac{1}{6}z_1^3; & k = 3 \\ \frac{1}{k}\sum_{v=1}^{k} z_v B_{k-v}(z_1, z_2, z_3, \ldots, z_k); & k \geq 4 \end{cases} \tag{1.8}$$

where

$$S_{v,j}^{(s)}(z) = \sum_{\substack{i=1 \\ j\neq i}}^{n} \left(\frac{1}{z_j^{(s)} - z_i^{(s)}}\right)^v; \quad v = 1(1)k$$

and since the zeros are simple,

$$\Delta_{k,j}\left(z_j^{(s)}\right) = \sum_{v=1}^{k} \frac{(-)^{v+1}}{v!} \frac{P_n^{(v)}\left(z_j^{(s)}\right)}{P_n\left(z_j^{(s)}\right)} \Delta_{k-v,j}\left(z_j^{(s)}\right); \quad j = 1(1)n. \tag{1.9}$$

The analysis to be given is easily extended to cases of multiplicity of the zeros of (1.1). Similarly, the interval arithmetic version of these methods in (1.7) by (1.4) is

$$Z_j^{(s+1)} = z_j^{(s)} - \frac{\Delta_{k-1,j}\left(z_j^{(s)}\right)}{\Delta_{k,j}\left(z_j^{(s)}\right) - B_k\left(S_{1,j}^{(s)}(Z), S_{2,j}^{(s)}(Z), \ldots, S_{k,j}^{(s)}(Z)\right)}; \quad k = 1, 2, 3, \ldots \tag{1.10}$$

proposed by Wang and Zheng, see [4, p. 167] for numerically isolating the zeros of (1.1) in the respective disks $\{Z_j\}_{j=1}^n$, where

$$S_{v,j}^{(s)}(Z) = \sum_{\substack{i=1\\ j\neq i}}^{N} \left(\frac{1}{z_j^{(s)} - Z_i^{(s)}}\right)^v; \quad v = 1(1)k. \tag{1.11}$$

It is instructive to emphasise that the inequalities

$$\begin{aligned} &\text{Rad}\left(\frac{1}{Z^{\frac{1}{k}}}\right) \le \text{Rad}\left(\left(\frac{1}{Z}\right)^{\frac{1}{k}}\right); \quad 0 \notin Z = \{z; r\} \\ &\text{and} \\ &\sum_{j=1; j\neq i}^{m} Rad\left(\left(\frac{1}{z_i - Z_j}\right)^k\right) \le \sum_{j=1; j\neq i}^{m} \text{Rad}\left(\frac{1}{(z_i - Z_j)^k}\right); \quad 0 \notin Z_j = \{z_j; r_j\} \end{aligned} \tag{1.12}$$

influences the way of implementation of (1.5) and (1.10), the methods are implemented as presented. By these, the methods will generate disks of smaller radii than if the computation is reversed with respect to inversion of the power of a disk. In both approaches the convergence rate stays the same. These classes of methods require the need to find the power of a disk,

$$Z^k = \{z; r\}^k = \left\{z^k; \sum_{j=1}^{k} \binom{k}{j} |z|^{k-j} r^j\right\} = \left\{z^k; (|z| + r)^k - |z|^k\right\}; \quad k = 1, 2, \ldots. \tag{1.13}$$

The order of convergence of the methods in (1.2), (1.5), (1.7) and (1.10) is $k + 2$. In particular, when $k = 1$ in (1.2), (1.5), (1.7) and (1.10) gives the Maehly's method, while $k = 2$ in (1.2) and (1.5) is the square root method and for $k = 2$ in (1.7) and (1.10) is the Halley-like method. The explicit nature of these methods becomes robust with a growing k. In what follows, we consider enhancing the convergence rate of the simultaneous root iteration methods in (1.2) and (1.5) and Bell's disk iteration methods in (1.7) and (1.10) by means of application of corrections and in the case of the inclusion methods we use the inversion formulae from [1],

$$\text{INV}(Z) = \begin{cases} \left\{\frac{\bar{z}}{|z|^2 - r^2}; \frac{r}{|z|^2 - r^2}\right\} = \left\{\frac{1}{z\left(1 - \frac{r^2}{|z|^2}\right)}; \frac{r}{|z|^2 - r^2}\right\} & ; (\,)^{-1}, \quad \beta = 1 \\ \left\{\frac{1}{z}; \frac{r}{|z|\,(|z| - r)}\right\} & ; (\,)^{I_1}, \quad \beta = 0 \\ \left\{\frac{1}{z}; \frac{2r}{|z|^2 - r^2}\right\} & ; (\,)^{I_2}, \quad \beta = 0. \end{cases} \tag{1.14}$$

The β is to distinguish the choices with respect to the inversion formulae above. The point $\bar{z}$ is the conjugate of the complex number z. It can be shown, see [1] again, that

$$|\text{Mid}(\text{INV}(Z))| \le \frac{|z|}{|z|^2 - r^2}; \qquad \text{Rad}(\text{INV}(Z)) \le \frac{2r}{|z|^2 - r^2}; \quad |z| > r \tag{1.15}$$

and also $Z^{-1} \subseteq Z^{I_1} \subseteq Z^{I_2}$. This inclusion relation tells which of the inversion will give best zero inclusion in the methods (2.1) and (2.2) in the section ahead.

2. Simultaneous iteration methods with corrections and a choice of disk inversion

Starting with the root iteration (1.2) and (1.5), an enhancement in mind of these methods which incorporates corrections in the process and a choice for inversion of the disk is

$$Z_j^{(s+1)} = z_j^{(s)} - \frac{1}{\left(h_k(z_j^{(s)}) - \sum_{\substack{i=1\\ j\neq i}}^{n} \left(\mathrm{INV}\left(z_j^{(s)} - Z_i^{(s)} + C_i^{(s)}\right)\right)^k\right)^{\frac{1}{k}}}; \quad k = 1, 2, \ldots \tag{2.1}$$

and that of an improvement of Wang and Zheng's method (1.7) and (1.10), is now

$$Z_j^{(s+1)} = z_j^{(s)} - \frac{\Delta_{k-1,j}\left(z_j^{(s)}\right)}{\Delta_{k,j}\left(z_j^{(s)}\right) - B_k\left(SI_{1,j}^{(s)}(Z), SI_{2,j}^{(s)}(Z), \ldots, SI_{k,j}^{(s)}(Z)\right)}; \quad k = 1, 2, 3, \ldots \tag{2.2}$$

where

$$SI_{v,j}^{(s)}(Z) = \sum_{i=1, j\neq i}^{n} \left(\mathrm{INV}\left(z_j^{(s)} - Z_i^{(s)} + C_i^{(s)}\right)\right)^v; \quad v = 1(1)k \tag{2.3}$$

for a fixed k. Here, $C_i^{(s)}$ is the correction term taken from the point method $z_i^{(s+1)} = z_i^{(s)} - C_i^{(s)}$ of order p. The application of correction in (2.1) and (2.2) is motivated by the fact that the approximation $z_j^{(s)} - C_j^{(s)}$ is closer to the root λ_j than the point $z_j^{(s)}$ and therefore enhances convergence rate than the corresponding basic methods (1.2), (1.5), (1.7) and (1.10) respectively as will be seen. The methods (2.1) and (2.2) offer improved convergence than their corresponding basic methods (1.5) and (1.10) respectively, as would be seen from their R-order shortly. The point arithmetic cases $k = 1, 2$ considered in [4] form special instances of (2.1) and (2.2), we herein provide a generalised study of these class of methods. When the correction term $C_i^{(s)}$ is efficient, it is this mode of correction we have desired to propose. Since

$$\mathrm{Rad}\left(\sum_{j=1; j\neq i}^{m} \left(\mathrm{INV}\left(z_i - Z_j\right)\right)^k\right) \leq \mathrm{Rad}\left(\sum_{j=1; j\neq i}^{m} \mathrm{INV}\left(\left(z_i - Z_j\right)^k\right)\right); \quad 0 \notin Z_j = \{z_j; r_j\} \tag{2.4}$$

the implementation of the methods will be as it is above and also during this, sources of zero divide shall be factored out. The convergence requirement of these methods will be given with respect to the initial disk separation ρ in the form $\rho \geq \theta(n-1)r$ where θ is a positive constant, n the degree of the polynomial, r is the maximum radius of the initial starting disks. The point version of (2.1) and (2.2) are given as

$$z_j^{(s+1)} = z_j^{(s)} - \frac{1}{\left(h_k(z_j^{(s)}) - \sum_{\substack{i=1\\ j\neq i}}^{n} \left(\frac{1}{z_j^{(s)} - z_i^{(s)} + C_i^{(s)}}\right)^k\right)^{\frac{1}{k}}} \tag{2.5}$$

and

$$z_j^{(s+1)} = z_j^{(s)} - \frac{\Delta_{k-1,j}\left(z_j^{(s)}\right)}{\Delta_{k,j}\left(z_j^{(s)}\right) - B_k\left(SC_{1,j}^{(s)}(z), SC_{2,j}^{(s)}(z), \ldots, SC_{k,j}^{(s)}(z)\right)} \tag{2.6}$$

where

$$SC_{v,j}^{(s)}(z) = \sum_{\substack{i=1\\ j\neq i}}^{n} \left(\frac{1}{z_j^{(s)} - z_i^{(s)} + C_i^{(s)}} \right)^{v}; \quad v = 1(1)k \tag{2.7}$$

respectively. The convergence of these point methods can be deduced from its interval version, this will be seen in the next section, the hint is that when the radii of the inclusion disks in (2.1) and (2.2) are zeros we have the point methods in (2.5) and (2.6) accordingly.

3. The convergence of the corrected families of inclusion methods (2.1) and (2.2)

The convergence of the modified methods (2.1) and (2.2) is considered in what follows. However, in the analysis to be provided we are supposing that the zeros $\{\lambda_j\}_{j=1}^n$ of the polynomial $P_n(z)$ are simple. The result from this finds simple extension to the case of multiple zeros of the polynomial. For this purpose, set

$$\begin{aligned} &z_j = \mathrm{Mid}(Z_j); \quad r_j = \mathrm{Rad}(Z_j); \quad \varepsilon_j = z_j - \lambda_j; \quad \varepsilon = \max_{j=1(1)n} |\varepsilon_j| \\ &r = \max_{j=1(1)n} \{r_j\}; \quad v_{ij} = z_i - z_j + C_j, \quad m_c = \min_{j=1(1)n} |C_j|, \quad i \neq j = 1(1)n. \end{aligned} \tag{3.1}$$

A measure of separateness of the disks will be $\rho = \mathrm{Min}_{1\le i,j\le n} \{|z|\,;\, z \in z_i - Z_j\} = \mathrm{Min}_{1\le i,j\le n} \{|z_i - z_j| - r_j\}$, though the other alternatives $d = \mathrm{Min}_{1\le i,j\le n} \{|z_i - z_j|\}$ employed in [1] or $\sigma = \mathrm{Min}_{1\le i,j\le n} \{|z_i - z_j| - r_i - r_j\}$ could also well be found useful. In fact, $\sigma > 0$ guarantee non-overlap of the disks $\{Z_j\}_{j=1}^n$, see Fig. 1.1. More importantly, let us start from the following result. It is instructive to draw attention to the fact that the iteration index s has been dropped where convenient.

Theorem 3.1. *Let $\rho \ge \theta(n-1)r$ and assume that*

$$|C_i| \le \alpha\, |\varepsilon_i| < \alpha r; \quad \alpha = 1 + \eta, \quad 0 < \eta < 1, \quad \theta \ge 2^{k+3} + \frac{\eta}{n-1},$$

if
(a) $\lambda_i \in Z_i$ *then* $\lambda_i \in Z_i - C_i$ *for* $i = 1(1)n$
(b) $\lambda_i \in Z_i$ *for* $i = 1(1)n$ *then the inversions in* (2.1) *and* (2.2) *exist, this thus implies*

(1)

$$0 \notin z_j - Z_i + C_i \quad \textit{and}$$

(2)

$$0 \notin H_{1,j} = 1 - \frac{1}{h_k(z_j)} \sum_{\substack{i=1;\\ i\neq j}}^{n} \left(\mathrm{INV}\left(z_j - Z_i + C_i\right)\right)^k \tag{3.2}$$

(3)

$$0 \notin H_{2,j} = 1 - \frac{1}{\Delta_{k,j}(z_j)} B_k\left(SI_{1,j}(Z), SI_{2,j}(Z), \ldots, SI_{k,j}(Z)\right) \tag{3.3}$$

according to the method in use from (2.1) *and* (2.2). *The inclusion methods in* (2.1) *and* (2.2) *therefore defines a feasible interval process. Let $Z_j^{(0)} = \{z_j^{(0)}; r_j^{(0)}\}$ be the initial non-overlapping disks isolating the simple zeros, that is $\lambda_j \in Z_j^{(0)}$ and $Z_j^{(0)} \cap Z_i^{(0)} = \phi$(empty); $i \neq j = 1(1)n$ respectively. If $\rho^{(0)} \ge \theta(n-1)r^{(0)}$ is the initial disk separation, where we let*

$$r^{(s)} = \max_{j=1(1)n} \left\{r_j^{(s)}\right\} \quad \textit{and} \quad \rho^{(s)} = \min_{\substack{i,j=1(1)n\\ i\neq j}} \left\{\left|z_i^{(s)} - z_j^{(s)}\right| - r_j^{(s)}\right\}; \quad s = 0, 1, 2, \ldots \tag{3.4}$$

then the sequences of disks $\{Z_j^{(s)}\}_{j=1,s=0}^{n,\infty}$ generated by the algorithms (2.1) *and* (2.2) *are such that*
(c) $\lambda_j \in Z_j^{(s)}$; $j = 1(1)n$, $s = 1, 2, \ldots$ *and*

(d) *the sequence of radii* $\{r^{(s)}\}_{s=0}^{\infty} \to 0$ *as* $s \to \infty$.

This consequently implies that the enhanced methods (2.1) and (2.2) are convergent.

Proof. First is with the family of root iteration methods (2.1), to this end re-arrange this to be

$$Z_i^{(s+1)} = z_i^{(s)} - \left(\frac{1}{h_k(z_i^{(s)})}\right)_*^{\frac{1}{k}} \left(\frac{1}{1 - \frac{1}{h_k(z_i^{(s)})}\left(\sum_{\substack{j=1\\ j\neq i}}^{n}\left(\mathrm{INV}(z_i^{(s)} - Z_j^{(s)} + C_j^{(s)})\right)\right)^k}\right)_*^{\frac{1}{k}}. \tag{3.5}$$

It is assumed that, in the kth root required above, our choice out of the k possibilities is the appropriate one, this is indicated by the *. This understanding has necessitated the dropping of this indicator in our subsequent considerations.
(a) The task of proving (a) is same as proving that $|z_i - \lambda_i| = |\varepsilon_i| \le r$ implies $|z_i - C_i - \lambda_i| \le r_i$, see [1]. Now, since $z_i^{(s+1)} = z_i^{(s)} - C_i^{(s)}$ and therefore $z_i - \lambda_i = C_i$ where the C_i depends on the roots, we can write therefore that $C_i = o(\varepsilon_i)$ which is same as $|C_i| \le \alpha|\varepsilon_i|$ with α a positive constant. Then $|\varepsilon_i - C_i| < \eta|\varepsilon_i| < \varepsilon \le r_i \le r$, therefore if $\lambda_i \in Z_i$ then $\lambda_i \in Z_i - C_i$ for $i = 1(1)n$ which in fact, means that $z_i^{(s)} - C_i^{(s)} \to \lambda_i$ faster than $z_i^{(s)} \to \lambda_i$; $s = 0, 1, 2, \ldots$. We have thus established by monotone inclusion that if

$$\lambda_j \in \frac{1}{\left(h_{k,j}^{(s)} - \sum_{\substack{i=1\\ j\neq i}}^{n}\left(\frac{1}{z_j^{(s)} - Z_i^{(s)}}\right)^k\right)^{\frac{1}{k}}} \qquad \text{then } \lambda_j \in \frac{1}{\left(h_{k,j}^{(s)} - \sum_{\substack{i=1\\ j\neq i}}^{n}\left(\mathrm{INV}\left(z_j^{(s)} - Z_i^{(s)} + C_i^{(s)}\right)\right)^k\right)^{\frac{1}{k}}}$$

$s = 0, 1, 2, \ldots; k = 1, 2, \ldots$.

(b1) To prove (b1) we observe that for a given disk $Z = \{z; r\}$ then $0 \notin Z = \{z; r\}$ if $|z| > r$. To this regard

$$|v_{ji}| = |z_j - z_i + C_i| \ge |z_j - z_i| - |C_i| > \rho - (1+\eta)r > r(\theta(n-1) - \alpha) > r > r_j \tag{3.6}$$

then $0 \notin \{v_{ji}; r_i\} = z_j - Z_i + C_i$.
(b2) To prove (b2) we require a bound on the absolute value of

$$\frac{1}{h_k(z_j)} = (z_j - \lambda_j)^k \left(1 - \frac{1}{h_k(z_j)} \sum_{i=1; i\neq j}^{n} \left(\frac{1}{z_j - \lambda_i}\right)^k\right). \tag{3.7}$$

This can be obtained from

$$\left|\frac{1}{h_k(z_j)}\right| < \left|z_j - \lambda_j\right|^k \left(1 + \left|\frac{1}{h_k(z_j)}\right| \sum_{i=1; i\neq j}^{n} \left|\frac{1}{z_j - \lambda_i}\right|^k\right) < \varepsilon^k \left(1 + \left|\frac{1}{h_k(z_j)}\right| \left(\frac{n-1}{\rho^k}\right)\right). \tag{3.8}$$

From where

$$\left|\frac{1}{h_k(z_j)}\right| < \frac{\varepsilon^k}{\left(1 - \frac{\varepsilon^k(n-1)}{\rho^k}\right)} < \frac{1}{1 - \left(\frac{1}{\theta^k (n-1)^{k-1}}\right)} \varepsilon^k; \quad n \ge 3. \tag{3.9}$$

Here

$$H_{1,j} = 1 - \frac{1}{h_k(z_j)} \sum_{\substack{i=1 \\ i \neq j}}^{n} \begin{cases} \left\{ \dfrac{1}{v_{ji}\left(1 - \frac{\beta\, r_i^2}{|v_{ji}|^2}\right)}; \dfrac{r_i}{|v_{ji}|^2 - r_i^2} \right\}^k & ;\,(\,)^{-1}, \quad \beta = 1 \\ \left\{ \dfrac{1}{v_{ji}}; \dfrac{r_i}{|v_{ji}|\,(|v_{ji}| - r_i)} \right\}^k & ;\,(\,)^{I_1}, \quad \beta = 0 \\ \left\{ \dfrac{1}{v_{ji}}; \dfrac{2r_i}{|v_{ji}|^2 - r_i^2} \right\}^k & ;\,(\,)^{I_2}, \quad \beta = 0 \end{cases}$$

$$= \left\{ u_j; \rho_j \right\}. \tag{3.10}$$

Now,

$$\rho_j = \frac{1}{|h_{k,j}|} \mathrm{Rad} \left(\sum_{\substack{i=1 \\ i \neq j}}^{n} \left(\mathrm{INV}(z_j - Z_i + C_i) \right)^k \right) = \frac{1}{|h_{k,j}|} \sum_{\substack{i=1 \\ i \neq j}}^{n} Rad \left(\left(\mathrm{INV}(z_j - Z_i + C_i) \right)^k \right)$$

setting $h_{k,j} = h_k(z_j)$ for convenience. So,

$$\begin{aligned} \rho_j &\le \frac{\varepsilon^k}{1 - \frac{1}{\theta^k (n-1)^{k-1}}} \sum_{i=1; i \neq j}^{n} \left(\frac{2r_i}{|v_{ji}|^2 - r_i^2} \right)^k \le \frac{\varepsilon^k}{1 - \frac{1}{\theta^k (n-1)^{k-1}}} \sum_{i \neq j} \left(\frac{2r_i}{(\rho - (1+\eta)r)^2 - r_i^2} \right)^k \\ &< \frac{\varepsilon^k}{1 - \frac{1}{\theta^k (n-1)^{k-1}}} \frac{(n-1)2^k r^k}{[(\rho - (1+\eta)\, r)^2 - r^2]^k} = \frac{\varepsilon^k}{1 - \frac{1}{\theta^k (n-1)^{k-1}}} \frac{(n-1)2^k r^k}{[\rho - (1+\eta)\, r + r]^k [\rho - (1+\eta)\, r - r]^k} \\ &< \frac{\varepsilon^k \theta^k}{\theta^k (n-1)^{k-1} - 1} \frac{(n-1)^k 2^k r^k}{[\theta(n-1) - (1+\eta) + 1]^k [\theta(n-1) - (1+\eta) - 1]^k r^{2k}} \\ &= \frac{\theta^k}{\theta^k (n-1)^{k-1} - 1} \frac{2^k}{\left[\theta - \frac{1+\eta}{n-1} + \frac{1}{n-1}\right]^k \left[\theta - \frac{1+\eta}{n-1} - \frac{1}{n-1}\right]^k} < \frac{1}{80}; \quad \theta \ge 2^{k+3} + \frac{1+\eta}{n-1} - \frac{1}{n-1}. \end{aligned} \tag{3.11}$$

Now, from (3.10)

$$u_j = 1 - T_{1,j}(\beta); \quad T_{1,j}(\beta) = \frac{1}{h_k(z_j)} \sum_{\substack{i=1 \\ i \neq j}}^{n} \left(\frac{1}{v_{ji}\left(1 - \frac{\beta\, r_i^2}{|v_{ji}|^2}\right)} \right)^k. \tag{3.12}$$

So that

$$\begin{aligned} |u_j| &= \left| 1 - \frac{1}{h_k(z_j)} \sum_{\substack{i=1 \\ i \neq j}}^{n} Mid \left(\left(\mathrm{INV}\left(z_j - Z_i + C_i\right) \right)^k \right) \right| \\ &\ge 1 - \left| \frac{1}{h_k(z_j)} \right| \sum_{\substack{i=1 \\ i \neq j}}^{n} \left| Mid \left(\left(\mathrm{INV}\left(z_j - Z_i + C_i\right) \right)^k \right) \right| \\ &> 1 - \frac{\varepsilon^k}{1 - \frac{1}{\theta^k (n-1)^{k-1}}} \sum_{\substack{i=1 \\ i \neq j}}^{n} \left(\frac{|v_{ji}|}{\left| |v_{ji}|^2 - r_i^2 \right|} \right)^k \ge 1 - \frac{\varepsilon^k}{1 - \frac{1}{\theta^k (n-1)^{k-1}}} \sum_{\substack{i=1 \\ i \neq j}}^{n} \left(\frac{(\rho - (1+\eta)r)}{(\rho - (1+\eta)r)^2 - r^2} \right)^k \\ &\ge 1 - \frac{\varepsilon^k (n-1)}{1 - \frac{1}{\theta^k (n-1)^{k-1}}} \left(\frac{\rho - (1+\eta)r}{(\rho - (1+\eta)r + r)(\rho - (1+\eta)r - r)} \right)^k. \end{aligned} \tag{3.13}$$

Finally,

$$\left|u_j\right| > 1 - \frac{\theta^k}{\theta^k(n-1)^{k-1}-1}\left(\frac{\theta - \frac{1+\eta}{n-1}}{[\theta - \frac{1+\eta}{n-1} + \frac{1}{n-1}][\theta - \frac{1+\eta}{n-1} - \frac{1}{n-1}]}\right)^k > 1 - \frac{3}{32} = \frac{29}{32}. \quad (3.14)$$

Thus $0 \notin \{u_j; \rho_j\} = H_{1,j}$, from which the conclusion is that (2.1) defines a feasible interval iteration method.
(c) Consider now the convergence of the class of methods (2.1), and to prove this we may let

$$Z_j, \hat{Z}_j, r_j, \hat{r}_j, z_j, \hat{z}_j \quad \text{denote } Z_j^{(s)}, Z_j^{(s+1)}, r_j^{(s)}, r_j^{(s+1)}, z_j^{(s)}, z_j^{(s+1)}$$

and also set

$$\hat{r} = \operatorname*{Max}_{j=1(1)n}\left\{\hat{r}_j\right\}; \qquad \hat{\rho} = \operatorname*{Min}_{\substack{i,j=1(1)n\\ i\neq j}}\left\{\left|\hat{z}_i - \hat{z}_j\right| - \hat{r}_j\right\}$$

when the need arises. Therefore, (2.1) is now of the form

$$\hat{Z}_j = z_j - \left(\frac{1}{h_{k,j}}\right)^{\frac{1}{k}}\left(\frac{1}{H_{1,j}}\right)^{\frac{1}{k}} = z_j - \left(\frac{1}{h_{k,j}}\right)^{\frac{1}{k}}\frac{1}{\left\{u_j;\rho_j\right\}^{\frac{1}{k}}}. \quad (3.15)$$

This will be reposed as

$$\hat{Z}_j = z_j - \left(\frac{1}{h_{k,j}}\right)^{\frac{1}{k}}\left(\frac{1}{\left\{u_j;\rho_j\right\}}\right)^{\frac{1}{k}} \quad (3.16)$$

the convergence characteristics is still the same as that in (3.15) the difference is in (1.12). By this arrangement, the method is resolved into the two components

$$\hat{r}_j = Rad\left(\hat{Z}_j\right) = \left|\frac{1}{h_{k,j}}\right|^{\frac{1}{k}}\left(\frac{\rho_j}{\left|u_j\right|^2 - \rho_j^2}\right)\frac{1}{\sum_{r=0}^{k-1}\left(\frac{|\bar{u}_j|}{|u_j|^2-\rho_j^2}\right)^{\frac{k-r-1}{k}}\left(\frac{|\bar{u}_j|-\rho_j}{|u_j|-\rho_j^2}\right)^{\frac{r}{k}}} \quad (3.17)$$

and

$$\hat{z}_j = Mid\left(\hat{Z}_j\right) = z_j - \left(\frac{1}{h_{k,j}}\right)^{\frac{1}{k}}\frac{1}{\left[u_j\left(1-t_j\right)\right]^{\frac{1}{k}}}; \quad t_j = \frac{\rho_j^2}{\left|u_j\right|^2}. \quad (3.18)$$

Recalling the bounds in (3.11) and (3.14),

$$\left|u_j\right| > \frac{29}{32} > \rho_j < \frac{1}{80}; \quad \left|u_j\right|^2 - \left(\rho_j\right)^2 > \left(\frac{29}{32}\right)^2 - \left(\frac{1}{80}\right)^2 = \frac{21\,021}{25\,600} > 0 \quad (3.19)$$

then

$$\hat{r}_j < \frac{\left|\varepsilon_j\right|}{\left(1 - \frac{1}{\theta^k(n-1)^{k-1}}\right)^{\frac{1}{k}}}\frac{\theta^k\left|\varepsilon_j\right|^k}{\theta^k(n-1)^{k-1}-1}QW \quad (3.20)$$

where

$$Q = \frac{2^k(n-1)^k r_j^k}{[\theta(n-1)-(1+\eta)-1]^k[\theta(n-1)-(1+\eta)+1]^k r^{2k}\left(\left|u_j\right|^2 - \left(\rho_j\right)^2\right)}$$

$$W = \frac{1}{\sum_{r=0}^{k-1}\left(\frac{|\bar{u}_j|}{|u_j|^2-\rho_j^2}\right)^{\frac{k-r-1}{k}}\left(\frac{|\bar{u}_j|-\rho_j}{|u_j|^2-\rho_j^2}\right)^{\frac{r}{k}}}.$$

That is (3.20) is now

$$\hat{r}_j < |\varepsilon_j| \left(\frac{\theta}{\theta-1}\right)^2 \frac{2(n-1)}{[\theta(n-1)-(1+\eta)-1][\theta(n-1)-(1+\eta)+1](|u_j|^2-(\rho_j)^2)}. \tag{3.21}$$

So that,

$$\begin{aligned}\hat{r}_j &< \frac{25\,600(n-1)\,2r_j}{21\,021[\theta(n-1)-(1+\eta)-1][\theta(n-1)-(1+\eta)+1]} \\ &< \frac{51\,200\,r_j}{21\,021\left[\theta-\left(\frac{1+\eta}{n-1}\right)-\frac{1}{n-1}\right]\left[\theta-\frac{1+\eta}{n-1}+\frac{1}{n-1}\right]} < r_j\frac{51\,200}{21\,021\times 16\times 16} < \frac{200}{21\,021}r_j.\end{aligned} \tag{3.22}$$

This way, $r_j^{(s)} < \left(\frac{200}{21\,021}\right)^s r_j^{(0)}$ and the generated radii from the method converges to zero. Now that $\lambda_j \in \hat{Z}_j$, that is

$$|\hat{z}_j - \lambda_j| < \hat{r}_j < \frac{200}{21\,021}r \tag{3.23}$$

and thus

$$|(\hat{z}_j - z_j) - (\lambda_j - z_j)| \le |\hat{z}_j - z_j| + r_j < \frac{200}{21\,021}r + r \tag{3.24}$$

is the equivalent inequality

$$|\hat{z}_j - z_j| < \frac{21\,221}{21\,021}r. \tag{3.25}$$

Also

$$\hat{\rho} = |\hat{z}_j - \hat{z}_i| - \hat{r}_k \ge |z_j - z_i| - r_i - \left(|z_j - \hat{z}_j| + |z_i - \hat{z}_i|\right) > \rho - \frac{42\,442}{21\,021}r. \tag{3.26}$$

By these inequalities,

$$\frac{\hat{r}}{\hat{\rho}} < \frac{\left(\frac{200}{21\,021}\right)r}{\rho - \left(\frac{42\,442}{21\,021}\right)r} = \left(\frac{r}{\rho}\right)\frac{200}{21\,021 - 42\,442\left(\frac{r}{\rho}\right)} < \left(\frac{r}{\rho}\right)\frac{640}{63\,023}; \quad \frac{r}{\rho} < 1; \quad n \ge 3. \tag{3.27}$$

Therefore the initial disk separation $\rho^{(0)} \ge \theta(n-1)r^{(0)}$ is seen to be maintained throughout the iterations in (2.1), that is $\rho^{(s)} \ge \theta(n-1)r^{(s)}$; $s = 0, 1, 2, \ldots$. Then $\lambda_i \in Z_i^{(s)} - C_i^{(s)}$; $i = 1(1)n$ and thus by inclusion monotonicity $\lambda_i \in Z_i^{(s+1)}$ subsequently.

(d) From the foregoing, since $\lambda_i \in Z_i^{(0)}$ the initial iteration disk, then by induction $\lambda_i \in Z_i^{(s)}$; $s = 0, 1, 2, \ldots$ Conclusively, the sequence of radii $\left\{r_j^{(s)}\right\}_{s=0,j=1}^{\infty,n}$ generated by the methods (2.1) tend to zero and defines the disk contraction $\ldots \subset Z_j^{(s+1)} \subset Z_j^{(s)} \subset \cdots \subset Z_j^{(1)} \subset Z_j^{(0)}$, therefore (2.1) defines a feasible convergent iteration algorithm. The case of the convergence of method (2.2) is proved analogously and we therefore do not wish to duplicate the efforts. The proof of the theorem is concluded. □

As an example, using the corrector $C_i^{(s)} = N_j^{(s)}$ from the method $z_i^{(s+1)} = z_j^{(s)} - N_j^{(s)}$ of order $p = 2$ when $k = 1$ in (2.1) and (2.2) it can be shown that $|N_i| < \alpha\,|\varepsilon_i| \le \alpha\, r$; $\alpha = \frac{5}{4}$, $\rho \ge 5(n-1)r$. Similar result can be found for other correctors and also realise that the inequality $\rho \ge \theta(n-1)r$ gives the initial disk separation for the convergence of the methods for a fixed k.

4. The R-order of convergence of the Root and Bell's modified methods

The R-order of convergence of the modified methods (2.1) and (2.2) will be determined as follows. Let us commence our consideration with the case of (2.1), that of (2.2) will be highlighted. By now (2.1) has become

$$\begin{aligned}\hat{z}_j-\lambda_j &= z_j-\lambda_j-\left(\frac{1}{h_k(z_j)}\right)^{\frac{1}{k}}\frac{1}{\left[\left(1-T_{1,j}(\beta)\right)\left(1-t_j\right)\right]^{\frac{1}{k}}}\\ &= \frac{\varepsilon_j\left[\left(1-T_{1,j}(\beta)\right)\left(1-t_j\right)\right]^{\frac{1}{k}}-\left(\frac{1}{h_k(z_j)}\right)^{\frac{1}{k}}}{\left[\left(1-T_{1,j}(\beta)\right)\left(1-t_j\right)\right]^{\frac{1}{k}}}.\end{aligned}\tag{4.1}$$

This is the same as

$$\hat{\varepsilon}_j=\frac{\varepsilon_j\left(T_{1,j}(\beta)t_j-t_j+\left(\frac{1}{h_k(z_j)}\sum_{\substack{i=1\\ i\neq j}}^{n}\left(\frac{1}{z_j-\lambda_i}\right)^k-T_{1,j}(\beta)\right)\right)}{\left[\left(1-T_{1,j}(\beta)\right)\left(1-t_j\right)\right]^{\frac{1}{k}}\sum_{r=0}^{k-1}\left(\left[\left(1-T_{1,j}(\beta)\right)\left(1-t_j\right)\right]^{\frac{k-r-1}{k}}\left[1-\frac{1}{h_k(z_j)}\sum_{\substack{i=1\\ i\neq j}}^{n}\left(\frac{1}{z_j-\lambda_i}\right)^k\right]^{\frac{r}{k}}\right)}.\tag{4.2}$$

The identities

$$A^{\frac{1}{k}}-B^{\frac{1}{k}}=\frac{A-B}{\sum_{j=0}^{k-1}A^{\frac{k-j-1}{k}}B^{\frac{j}{k}}}\quad\text{and}\quad A^k-B^k=(A-B)\sum_{j=0}^{k-1}B^jA^{k-j-1};\quad k=1,2,\ldots\tag{4.3}$$

reduces part of the expression in the numerator of (4.2) into

$$\begin{aligned}T_1&=\frac{1}{h_k(z_j)}\sum_{\substack{i=1\\ i\neq j}}^{n}\left(\frac{1}{z_j-\lambda_i}\right)^k-T_{1,j}(\beta)=\frac{1}{h_k(z_j)}\sum_{\substack{i=1\\ j\neq i}}^{n}\left[\left(\frac{1}{z_j-\lambda_i}-\frac{1}{v_{ji}\left(1-\frac{\beta r_i^2}{|v_{ji}|^2}\right)}\right)U_{ji}(\beta)\right]\\ &=\frac{1}{h_k(z_j)}\sum_{\substack{i=1\\ j\neq i}}^{n}\left[\left(\frac{\left(z_j-z_i+C_i\right)\left(1-\frac{\beta r_i^2}{|v_{ji}|^2}\right)-\left(z_j-\lambda_i\right)}{\left(z_j-\lambda_i\right)v_{ji}\left(1-\frac{\beta r_i^2}{|v_{ji}|^2}\right)}\right)U_{ji}(\beta)\right]\end{aligned}$$

where $U_{ji}(\beta)$ is

$$U_{ji}(\beta)=\sum_{r=0}^{k-1}\left[\left(\frac{1}{z_j-\lambda_i}\right)^{k-r-1}\left(\frac{1}{v_{ji}\left(1-\frac{\beta r_i^2}{|v_{ji}|^2}\right)}\right)^r\right].$$

A further simplification of T_1 yields the more revealing expression,

$$T_1=\frac{1}{h_k(z_j)}\sum_{\substack{i=1,\\ j\neq i}}^{n}\left[\left(\frac{\left(-z_i+\lambda_i+C_i\right)-v_{ji}\frac{\beta r_i^2}{|v_{ji}|^2}}{\left(z_j-\lambda_i\right)v_{ji}\left(1-\frac{\beta r_i^2}{|v_{ji}|^2}\right)}\right)U_{ij}(\beta)\right].\tag{4.4}$$

It is possible to find a constant D (> 0) such that

$$|T_1| \leq D \left|\frac{1}{h_k(z_j)}\right| \cdot \left(|-z_i + \lambda_i + C_i| + \frac{\beta\, r_i^2}{|v_{ji}|^2}\right).$$

Now, by (1.13), it follows that

$$\left(\mathrm{INV}\left(z_j - Z_i + C_i\right)\right)^k = \begin{cases} \left\{\dfrac{1}{v_{ji}^k\left(1 - \frac{\beta\, r_i^2}{|v_{ji}|^2}\right)^k};\ \dfrac{\left(|v_{ji}| + r_i\right)^k - |v_{ji}|^k}{\left(|v_{ji}|^2 - r_i^2\right)^k}\right\} & ;(\,)^{-1}, \quad \beta = 1 \\ \left\{\dfrac{1}{v_{ji}^k};\ \left(\dfrac{1}{|v_{ji}|} + \dfrac{r_i}{|v_{ji}|\left(|v_{ji}| - r_i\right)}\right)^k - \dfrac{1}{|v_{ji}|^k}\right\} & ;(\,)^{I_I}, \quad \beta = 0 \\ \left\{\dfrac{1}{v_{ji}^k};\ \left(\dfrac{1}{|v_{ji}|} + \dfrac{2r_i}{|v_{ji}|^2 - r_i^2}\right)^k - \dfrac{1}{|v_{ji}|^k}\right\} & ;(\,)^{I_2}, \quad \beta = 0. \end{cases}$$

Because of the inequalities,

$$\begin{cases} \left(|v_{ji}| + r_i\right)^k - |v_{ji}|^k < k\, r_i \left(|v_{ji}| + r_i\right)^{k-1} \\ \left(\dfrac{1}{|v_{ji}|} + \dfrac{r_i}{|v_{ji}|\left(|v_{ji}| - r_i\right)}\right)^k - \dfrac{1}{|v_{ji}|^k} < k \dfrac{r_i}{|v_{ji}|\left(|v_{ji}| - r_i\right)} \left(\dfrac{1}{|v_{ji}|} + \dfrac{r_i}{|v_{ji}|\left(|v_{ji}| - r_i\right)}\right)^{k-1} \\ \left(\dfrac{1}{|v_{ji}|} + \dfrac{2r_i}{|v_{ji}|^2 - r_i^2}\right)^k - \dfrac{1}{|v_{ji}|^k} < k \dfrac{2r_i}{|v_{ji}|^2 - r_i^2} \left(\dfrac{1}{|v_{ji}|} + \dfrac{2r_i}{|v_{ji}|^2 - r_i^2}\right)^{k-1} \end{cases}$$

then

$$Rad\left(\left(\mathrm{INV}(z_j - Z_i + C_i)\right)^k\right) < k \frac{2r_i}{|v_{ji}|^2 - r_i^2} \left(\frac{1}{|v_{ji}|} + \frac{2r_i}{|v_{ji}|^2 - r_i^2}\right)^{k-1}. \tag{4.5}$$

This induces the inclusion relation,

$$\sum_{\substack{i=1 \\ i \neq j}}^{n} \left(\mathrm{INV}(z_j - Z_i + C_i)\right)^k \subseteq \left\{\sum_{\substack{i=1 \\ j \neq i}}^{n} \left(\frac{1}{z_j - z_i + C_i}\right)^k;\ \ 2k \sum_{\substack{i=1 \\ i \neq j}}^{n} \frac{r_i}{|v_{ji}|^2 - r_i^2} \left(\frac{1}{|v_{ji}|} + \frac{2r_i}{|v_{ji}|^2 - r_i^2}\right)^{k-1}\right\}$$
$$\subset \left\{\sum_{\substack{i=1 \\ j \neq i}}^{n} \frac{1}{v_{ji}^k};\ \frac{2(n-1)k\, r}{(\rho - (1+\eta)r)^2 - r^2} \left(\frac{1}{\rho - (1+\eta)r} + \frac{2r}{(\rho - (1+\eta)r)^2 - r^2}\right)^{k-1}\right\}.$$

From this we are therefore sure of the inclusion

$$\sum_{i=1, i \neq j}^{n} \left(\mathrm{INV}(z_j - Z_i + C_i)\right)^k \subset \left\{\sum_{\substack{i=1 \\ j \neq i}}^{n} \frac{1}{v_{ji}^k};\ \ \frac{2(n-1)k\, r}{(\rho - m_c)^2} \left(\frac{1}{r} + \frac{2r}{(\rho - m_c)^2}\right)^{k-1}\right\} \tag{4.6}$$

for which now,

$$\rho_j \leq \frac{1}{1 - \left(\frac{1}{\theta^k (n-1)^{k-1}}\right)} \frac{2(n-1)k\, r \varepsilon^k}{(\rho - (1+\eta)r)^2 - r^2} \left(\frac{1}{\rho - (1+\eta)r} + \frac{2r}{(\rho - (1+\eta)r)^2 - r^2}\right)^{k-1} \tag{4.7}$$

from (3.11). In the notations of [1], write, $\rho_j = o\left(\varepsilon^k r\right)$. This notation suppresses the constants whose explicit nature does not obscure the qualitative behaviour of the sequences of interest, however this is just only a matter of convenience. Arising from the foregoing analysis so far we shall have

$$\begin{aligned} & z_j - \lambda_j - C_j = o\left(\varepsilon^p\right); \quad u_j = o(1); \quad v_{ji} = o(1); \quad t_j = o\left(\varepsilon^{2k} r^2\right); \\ & T_{1,j}(\beta) = o\left(\varepsilon^k\right); \quad \left(1 - T_{1,j}(\beta)\right)\left(1 - t_j\right) = o(1); \quad z_j - z_i + C_i = o(1); \\ & \frac{1}{h_k} = o\left(\varepsilon^k\right); \quad z_i - \lambda_j = o(1); \quad P_n(z) = o(\varepsilon) \end{aligned} \tag{4.8}$$

and which similarly

$$\left|u_j\right|^2 - \rho_j^2 = o(1); \quad \left(\frac{\left|\bar{u}_j\right|}{\left|u_j\right|^2 - \rho_j^2}\right)^{\frac{k-r-1}{k}} \left(\frac{\left|\bar{u}_j\right| - \rho_j}{\left|u_j\right|^2 - \rho_j^2}\right)^{\frac{r}{k}} = o(1); \quad r = 0(1)k-1$$

and

$$\left(\frac{1}{z_j - \lambda_i}\right)^{k-r-1} \left(\frac{1}{v_{ij}\left(1 - \frac{\beta\, r_i^2}{\left|v_{ij}\right|^2}\right)}\right)^r = o(1), \qquad U_{ji}(\beta) = o(1).$$

Also

$$\left(\left(1 - S_{1,j}(\beta)\right)\left(1 - t_j\right)\right)^{\frac{k-r-1}{k}} \left(1 - \frac{1}{h_k} \sum_{i=1, i\neq j}^{n} \left(\frac{1}{z_j - \lambda_i}\right)^k\right)^{\frac{r}{k}} = o(1) \tag{4.9}$$

and more so for the other similar terms. Invoking these in

$$\hat{r}_j = \left(\left|\frac{1}{h_{k,j}}\right|^{\frac{1}{k}} \rho_j\right) \frac{1}{\left|u_j\right|^2 - \rho_j^2} \cdot \frac{1}{\sum_{r=0}^{k-1} \left(\frac{\left|\bar{u}_j\right|}{\left|u_j\right|^2 - \rho_j^2}\right)^{\frac{k-r-1}{k}} \left(\frac{\left|\bar{u}_j\right| - \rho_j}{\left|u_j\right|^2 - \rho_j^2}\right)^{\frac{r}{k}}}$$

from (3.17) and bearing in mind that $\rho_j = o\left(\varepsilon^k r\right)$, therefore $\hat{r}_j = o\left(\varepsilon^{k+1} r\right)$. For the case of the family of methods (2.2), the re-arrangement is

$$Z_j^{(s+1)} = z_j^{(s)} - \left(\frac{\Delta_{k-1,j}\left(z_j^{(s)}\right)}{\Delta_{k,j}\left(z_j^{(s)}\right)}\right) \frac{1}{1 - \frac{1}{\Delta_{k,j}\left(z_j^{(s)}\right)} B_k\left(SI_{1,j}^{(s)}(Z), SI_{2,j}^{(s)}(Z), \ldots, SI_{k,j}^{(s)}(Z)\right)}. \tag{4.10}$$

By then

$$\hat{Z}_j = z_j - \left(\frac{\Delta_{k-1,j}(z_j)}{\Delta_{k,j}(z_j)}\right)\left(H_{2,j}\right)^{-1} = z_j - \left(\frac{\Delta_{k-1,j}(z_j)}{\Delta_{k,j}(z_j)}\right)\left(\frac{1}{\left\{u_j; \rho_j\right\}}\right). \tag{4.11}$$

Here, define similarly that

$$\begin{aligned} & u_j = 1 - T_{2,j}; \\ & T_{2,j} = \frac{1}{\Delta_{k,j}\left(z_j^{(s)}\right)} B_k\left(SI_{1,j}^{(s)}(Z), \ldots, SI_{k,j}^{(s)}(Z)\right) = \frac{1}{\Delta_{k,j}\left(z_j^{(s)}\right)} B_k\left(SI_{1,j}^{(s)}(Z;\beta), \ldots, SI_{k,j}^{(s)}(Z;\beta)\right) \\ & \rho_j = \frac{1}{\left|\Delta_{k,j}\left(z_j^{(s)}\right)\right|} Rad\left(B_k\left(SI_{1,j}^{(s)}(Z), SI_{2,j}^{(s)}(Z), \ldots, SI_{k,j}^{(s)}(Z)\right)\right) \end{aligned} \tag{4.12}$$

and

$$SI_{v,j}(Z;\beta)=\sum_{\substack{i=1\\ i\neq j}}^{n}\left(\frac{1}{v_{ij}\left(1-\frac{\beta\, r_i^2}{|v_{ij}|^2}\right)}\right)^v;\quad \beta=0,1;\; v=1(1)k. \tag{4.13}$$

So that (2.2) is equivalently resolved into the components

$$\hat{z}_j=Mid\left(\hat{Z}_j\right)=z_j-\left(\frac{\Delta_{k-1,j}(z_j)}{\Delta_{k,j}(z_j)}\right)\frac{\bar{u}_j}{|u_j|^2-(\rho_j)^2};$$
$$\hat{r}_j=Rad\left(\hat{Z}_j\right)=\left|\frac{\Delta_{k-1,j}}{\Delta_{k,j}}\right|\frac{\rho_j}{|u_j|^2-(\rho_j)^2}. \tag{4.14}$$

Because

$$\frac{\Delta_{k-1,j}\left(z_j^{(s)}\right)}{\Delta_{k,j}\left(z_j^{(s)}\right)}=(z_j-\lambda_j)\left[1-\frac{1}{\Delta_{k,j}\left(z_j^{(s)}\right)}B_k\left(S_{1,j}^{(s)}(\lambda),S_{2,j}^{(s)}(\lambda),\ldots,S_{k,j}^{(s)}(\lambda)\right)\right] \tag{4.15}$$

where

$$S_{v,j}(\lambda)=\sum_{i=1,i\neq j}^{n}\left(\frac{1}{z_j-\lambda_i}\right)^v;\quad v=1(1)k \tag{4.16}$$

thus

$$\hat{\varepsilon}_j=\frac{\varepsilon_j\left(T_{2,j}(\beta)t_j-t_j+\left(\frac{1}{\Delta_{k,j}(z)}B_k\left(S_{1,j}(\lambda),S_{1,j}(\lambda),\ldots,S_{k,j}(\lambda)\right)-T_{2,j}(\beta)\right)\right)}{\left(1-T_{2,j}(\beta)\right)\left(1-t_j\right)}. \tag{4.17}$$

Using (1.9), establish there that $\frac{1}{\Delta_{k,j}\left(z_j^{(s)}\right)}=o\left(\varepsilon^k\right)$ and by application of (4.3), see that

$$T_2=\frac{1}{\Delta_{k,j}(z)}B_k\left(S_{1,j}(\lambda),\ldots,S_{k,j}(\lambda)\right)-T_{2,j}(\beta)=o\left(\varepsilon^{p+k}\right)+\beta\, o\left(\varepsilon^k r^2\right) \tag{4.18}$$

in which again $\rho_j=O\left(\varepsilon^k\, r\right)$. Finally, in both classes of the methods (2.1) and (2.2)

$$T_t=\begin{cases}o\left(\varepsilon^{p+k}\right); & \beta=0\\ o\left(\varepsilon^{p+k}\right)+\;o\left(r^2\varepsilon^k\right); & \beta=1\end{cases};\quad t=1,2;\quad p\geq 2;\; k=1,2,\ldots \tag{4.19}$$

using (4.4) and (4.18). Conclusively, the error expressions are

$$\hat{r}_j=o\left(\varepsilon^{k+1}r\right);$$
$$\hat{\varepsilon}_j=o\left(r^2\varepsilon^{3k+1}\right)+o\left(r^2\varepsilon^{2k+1}\right)+\begin{cases}o\left(\varepsilon^{p+k+1}\right)+\beta o\left(r^2\varepsilon^{k+1}\right); & \beta=1\\ o\left(\varepsilon^{p+k+1}\right); & \beta=0\end{cases} \tag{4.20}$$

for both cases of the methods in (2.1) and (2.2) for a general k and a corrector of order p. Considering dominant terms, and noting that $|\varepsilon_i|\;<\varepsilon\leq\; r$ when convergence sets in, in the long run on the iteration index s, then we can put that $\varepsilon=o\,(r)$ and thus in both classes of the methods (2.1) and (2.2) the error propagate its effects as in the error relations

$$\hat{r}_j=o\left(r\varepsilon^{k+1}\right);$$

$$\hat{\varepsilon}_j = \begin{cases} \left[o\left(r^2\right) + o\left(\varepsilon^p\right)\right] o\left(\varepsilon^{k+1}\right); & \beta = 1 \\ \left[o\left(r^2\right) + o\left(\varepsilon^{p-k}\right)\right] o\left(\varepsilon^{2k+1}\right); & \beta = 0 \end{cases} \tag{4.21}$$

as we have set out to show. The R-order of convergence is now found by the application of the Schmidt theorem accordingly on the error relations in (4.21). Here is the theorem, see [5,6].

Theorem 4.1 (*Schmidt*). *Given the error relation*

$$\varepsilon_j^{(s+1)} \le \alpha_j \prod_{i=1}^{n} \left(\varepsilon_i^{(s)}\right)^{p_{ji}}; \quad j = 1, 2, \ldots, n; \; s = 0, 1, 2, \ldots \tag{4.22}$$

where $p_{ji} \ge 0$, $\alpha_j > 0$; $1 \le i, j \le n$. Denote the matrix of exponents which appear in (4.22) as $P_n = \left[p_{ji}\right]_{n\times n}$. If the non-negative matrix P_t has the spectral radius $\rho(P_n) > 1$ and a corresponding eigen-vector $X_\rho > 0$, then all sequences $\left\{\varepsilon_j^{(s)}\right\}_{j=1(1)n}$ have the R-order of convergence at least $\rho(P_n)$.

Consider the cases there are: for a start, the case of the exact inversion ($\beta = 1$) with the order of correction $p \ge 2$, the associated P-matrix is

$$P_2 = \begin{bmatrix} 1 & k+1 \\ 2 & k+1 \end{bmatrix}; \quad p \ge 2, \quad \beta = 1 \tag{4.23}$$

which gives that

$$\rho\left(P_2\right) = \frac{k+2+\sqrt{(k+2)^2+4(k+1)}}{2}; \quad X_\rho = \begin{pmatrix} k+1 \\ \dfrac{k+\sqrt{(k+2)^2+4(k+1)}}{2} \end{pmatrix} > 0.$$

This leads to the lower bound

$$O_R((2.1), (2.2)) \ge \rho\left(P_2\right) = \frac{k+2+\sqrt{(k+2)^2+4(k+1)}}{2}; \quad p \ge 2, \quad \beta = 1 \tag{4.24}$$

of the R-order of convergence with the exact inversion procedure in the methods (2.1) and (2.2); since the term r^2 is the dominating term in the expression (4.21) of the parenthesis. For the centred inversion, $\beta = 0$, by Schmidt theorem,

$$P_2 = \begin{bmatrix} 1 & k+1 \\ 0 & p+k+1 \end{bmatrix}; \qquad p = 2, 3, 4, \ldots, k+1; \quad \beta = 0 \tag{4.25}$$

from which

$$\rho\left(P_2\right) = \frac{p+k+2+\sqrt{(p+k+2)^2-4(p+k+1)}}{2} = k+p+1;$$

$$X_\rho = \begin{pmatrix} k+1 \\ \dfrac{p+k+\sqrt{(p+k+2)^2-4(p+k+1)}}{2} \end{pmatrix} = \begin{pmatrix} k+1 \\ p+k \end{pmatrix} > 0.$$

This similarly leads to

$$O_R((2.1), (2.2)) \ge \rho\left(P_2\right) = \frac{p+k+2+\sqrt{(p+k+2)^2-4(p+k+1)}}{2} = k+p+1; \quad p = 2, 3, 4, \ldots, k+1; \quad \beta = 0 \tag{4.26}$$

the lower bound of the R-order of convergence with the centred inversion; since the term ε^{p-k} is the dominating term of the parenthesis in the expression (4.21). Finally, when $p \ge k+2$, $\beta = 0$ the term r^2 again is the dominating term of the parenthesis in the expression (4.21). In the same token, the lower bound of the R-order of convergence with centred inversion, $\beta = 0$; is found as follows: Here

$$P_2 = \begin{bmatrix} 1 & k+1 \\ 2 & 2k+1 \end{bmatrix}; \quad p \geq k+2, \quad \beta = 0. \tag{4.27}$$

This gives that

$$\rho(P_2) = \frac{2k+2+\sqrt{(2k+2)^2+4}}{2}; \quad X_\rho = \begin{pmatrix} k+1 \\ \dfrac{2k+\sqrt{(2k+2)^2+4}}{2} \end{pmatrix} > 0;$$

from where we have the lower bound

$$O_R((2.1),(2.2)) \geq \rho(P_2) = \frac{2k+2+\sqrt{(2k+2)^2+4}}{2}; \quad p \geq k+2, \quad \beta = 0 \tag{4.28}$$

of the R-order of convergence with respect to the centred inversion formula in the methods (2.1) and (2.2). Summarily, the lower bound of the R-order of convergence of (2.1) and (2.2) is

$$O_R((2.1),(2.2)) \geq \begin{cases} \dfrac{k+2+\sqrt{(k+2)^2+4(k+1)}}{2}; & p \geq 2, \quad \beta = 1 \\ p+k+1; & p = 2,3,4,\ldots,k+1, \quad \beta = 0 \\ k+1+\sqrt{(k+1)^2+1}; & p \geq k+2, \quad \beta = 0. \end{cases} \tag{4.29}$$

The R-order of the resultant point methods in (2.5) and (2.6) for a fixed k, is given therefore by

$$O_R((2.5),(2.6)) \geq \begin{cases} p+k+1; & p = 2,3,\ldots,k+1 \\ k+1+\sqrt{(k+1)^2+1}; & p \geq k+2. \end{cases} \tag{4.30}$$

The methods in (2.5) and (2.6) are consequences of the fact that disk arithmetic is consistent with point arithmetic; this is so when $\beta = 0$ in (1.14). It should be deduced from (4.29) and (4.30) therefore, that the greatest lower bound of the R-order of convergence of these methods (2.1), (2.2), (2.5) and (2.6) cannot be better than $R_k = k+1 \quad + \sqrt{(k+1)^2+1}$ irrespective of increasing order p of the correction, this bound occurs when the correction is of order at least $p = k+2$, which is the order of the corresponding basic method. Thus the greatest lower bound R_k of the R-order of a corrected method occurs when the corrector is of at least the same order as the method to be corrected. This R_k is the greatest lower bound of the R-order attainable by correction in these methods and therefore represents an R-order barrier result. Interestingly, when the correction term $C_i^{(s)}$ is set to zero, it gives rise to the basic methods in (1.2), (1.5), (1.7) and (1.10), imply $p = 1$ in (4.21), confirming the order of $k+2$ of the basic methods accordingly. We have in this paper improved the order $k+2$ of (1.2), (1.5), (1.7) and (1.10) to $O_R = k+p+1; k \geq 1$, $p \geq 2$ by application of the process of correction on these methods in (2.1), (2.2), (2.5) and (2.6) see (4.29) and (4.30) for the results.

5. Efficient correctors versus correctors that lead to maximum lower bound R-order methods

When the refinement process of the correction in (2.1), (2.2), (2.5) and (2.6) is efficient which will be so if the term $C_i^{(s)}$ is already computed or rather available as a term in the method in question, that is without the need to compute it all over again for the purpose of correcting the method, it is this mode of correction we have desired to propose. On the choice of a corrector $C_i^{(s)}$, the use of the corrector term

$$C_{1,i}^{(s)} = N_i^{(s)} = N(z_i^{(s)}) = \frac{P_n(z_i^{(s)})}{P_n'(z_i^{(s)})}; \quad (p = 2) \tag{5.1}$$

with the order of the corresponding point iteration method written beside it, for all $k \geq 1$ is efficient since this term is available in (2.1), (2.2), (2.5) and (2.6) for all k. So also is

$$C_{2,i}^{(s)} = H(z_i^{(s)}) = \frac{1}{\frac{P_n'(z_i^{(s)})}{P_n(z_i^{(s)})} - \frac{P_n''(z_i^{(s)})}{2P_n'(z_i^{(s)})}}; \quad (p = 3)$$

for all $k \geq 2$. Generally,

$$C_{l,j}^{(s)} = \frac{\Delta_{l-1,j}(z_j^{(s)})}{\Delta_{l,j}(z_j^{(s)})}; \quad (p = l + 1), \tag{5.2}$$

is an efficient corrector for a method with fixed k such that $k \geq l$. It should be mentioned that the use of

$$_kC_{l,j}^{(s)} = \frac{\Delta_{l-1,j}\left(z_j^{(s)}\right)}{\Delta_{l,j}\left(z_j^{(s)}\right) - B_l\left(SC_{1,j}^{(s)}(z), SC_{2,j}^{(s)}(z), \ldots, SC_{l,j}^{(s)}(z)\right)}; \quad (p = l + 2),\ l \leq k \tag{5.3}$$

as a corrector is not efficient, because of the extra cost of computing sums of powers of inverses of the points $z_j^{(s)} - z_i^{(s)} + {}_kC_{l,i}^{(s)}$ in $B_l\left(SC_{1,j}^{(s)}(z), SC_{2,j}^{(s)}(z), \ldots, SC_{l,j}^{(s)}(z)\right)$. More so is the fact that the term ${}_kC_{l,j}^{(s)}$ is not available in the basic method and may become expensive to compute, therefore ${}_kC_{l,j}^{(s)}$ may not be recommended as a corrector, although it gives the maximum lower bound R-order of convergence when $l = k$ as can be deduced from (4.29) and (4.30) since the order of the arising point method $z_j^{(s+1)} = z_j^{(s)} - {}_kC_{l,j}^{(s)}$ is the same as the basic method to be corrected, in this regard therefore, their use may not completely be discouraged. For an example, the point arithmetic method $z_j^{(s+1)} = z_j^{(s)} - {}_1C_{1,j}^{(s)}$ from setting $k = 1$ $(= l)$ in (2.1), (2.2), (2.5) and (2.6) is of order three where

$$_1C_{1,i}^{(s)} = \frac{1}{\frac{1}{N_i^{(s)}} - \sum_{j=1, i\neq j}^{n} \frac{1}{z_i^{(s)} - z_j^{(s)}}}, \quad i = 1(1)n \tag{5.4}$$

and we can show that $\left|{}_1C_{1,i}\right| \leq \frac{5}{3}\ |\varepsilon_i|$ provided that $\rho \geq 5(n-1)r$. The R-order with ${}_1C_{1,i}^{(s)}$ as corrector in (2.1), (2.2), (2.5) and (2.6) for $k = 1$ is $R_1 = 2 + \sqrt{5} \approx 4.236$. This enhancement appears to be marginal when compared to that from using the Newton's corrector $N_i^{(s)}$ which gives a convergence order of four.

6. Examples of the methods and numerical experiments

In this section examples of the enhanced methods (2.1), (2.2), (2.5) and (2.6) are given along with efficient correctors. The case of $k = 1$ in (2.1) and (2.2) leads to the interval method

$$Z_j^{(s+1)} = z_j^{(s)} - \frac{1}{\frac{1}{N_j^{(s)}} - \sum_{i=1, i\neq j}^{n} \mathrm{INV}\left(z_j^{(s)} - Z_i^{(s)} + C_i^{(s)}\right)}; \qquad j = 1(1)n \tag{6.1}$$

with R-order

$$O_R((6.1)) \geq \begin{cases} 2 + \sqrt{5} = 4.2361\ldots; & p \geq 3,\ \beta = 0 \\ 4; & p = 2,\ \beta = 0 \\ (3 + \sqrt{17})/2 = 3.5616\ldots; & p \geq 2,\ \beta = 1 \end{cases}$$

with a correction of convergence order p. This is equivalent to that in [1] when $C_i^{(s)} = N_i^{(s)}$ which is an efficient corrector for (6.1) and the point arithmetic equivalent. The point equivalent method is in the sense of (2.5) and (2.6), obtained by replacing the disk $Z_j^{(s)}$ by the point $z_j^{(s)}$ in (2.1) and (2.2), thus has the order

$$O_R \geq \begin{cases} 4; & p = 2 \\ 2 + \sqrt{5} = 4.2361\ldots; & p \geq 3 \end{cases} \tag{6.2}$$

from (4.30). For $k = 2$ in (2.1) is the square root iteration method

$$Z_j^{(s+1)} = z_j^{(s)} - \frac{1}{\left(h_2(z_j^{(s)}) - \sum_{i=1, j\neq i}^{n} \left(\mathrm{INV}(z_j^{(s)} - Z_i^{(s)} + C_i^{(s)})\right)^2\right)^{\frac{1}{2}}}; \qquad m = 1, 2 \tag{6.3}$$

with efficient correctors $C_i^{(s)} = C_{m,i}^{(s)}$ given accordingly in (5.2). That of (2.2) when $k = 2$ is the Halley-like method

$$Z_i^{(s)} = z_i^{(s)} - \frac{1}{\frac{1}{H_i^{(s)}} - \frac{N_i^{(s)}}{2}\left(\sum_{j=1, j\neq i}^{n} \left(\mathrm{INV}(T_{i,j,m}^{(s)}(Z))\right)^2 + \left(\sum_{j=1, j\neq i}^{n} \mathrm{INV}(T_{i,j,m}^{(s)}(Z))\right)^2\right)} \tag{6.4}$$

where the choice of $C_j^{(s)} = C_{m,j}^{(s)}$ are efficient correctors,

$$T_{i,j,m}^{(s)}(Z) = z_i^{(s)} - Z_j^{(s)} + C_{m,j}^{(s)}; \qquad m = 1, 2 \quad \text{and} \quad h_2(z) = \left(\frac{P_n'(z)}{P_n(z)}\right)^2 - \frac{P_n''(z)}{P_n(z)}. \tag{6.5}$$

The error propagates its effect the same way in both methods. Therefore, the R-order of the interval methods is given by

$$O_R((6.3), (6.4)) \geq \begin{cases} 2 + \sqrt{7} = 4.6458\ldots; & p \geq 2,\ \beta = 1 \\ 5; & p = 2,\ \beta = 0 \\ 6; & p = 3,\ \beta = 0 \\ 3 + \sqrt{10} = 6.162\ldots; & p \geq 4,\ \beta = 0. \end{cases} \tag{6.6}$$

The equivalent point method has the R-order given by

$$O_R \geq \begin{cases} 5; & p = 2 \\ 6; & p = 3 \\ 3 + \sqrt{10} = 6.162\ldots; & p \geq 4. \end{cases} \tag{6.7}$$

Furthermore, the corrected cube root method

$$Z_j^{(s+1)} = z_j^{(s)} - \frac{1}{\left(h_3(z_j^{(s)}) - \sum_{i=1, j\neq i}^{n} \left(\mathrm{INV}(z_j^{(s)} - Z_i^{(s)} + C_i^{(s)})\right)^3\right)^{\frac{1}{3}}}; \qquad t = 1, 2, 3 \tag{6.8}$$

with $C_i^{(s)} = C_{t,i}^{(s)}$, is obtained when $k = 3$ in (2.1) and the enhanced Wang and Zheng interval method (2.2)

$$Z_i^{(s)} = z_i^{(s)} - \frac{\Delta_{2,j}\left(z_j^{(s)}\right)}{\Delta_{3,j}(z_j^{(s)}) - \left(\frac{1}{3}\sum_{j=1, j\neq i}^{n} (\mathrm{INV}(T_{i,j,t}^{(s)}(Z)))^3 + \frac{1}{2}M_{i,j,t}^{(s)} + \frac{1}{6}\left(\sum_{j=1, j\neq i}^{n} \mathrm{INV}(T_{i,j,t}^{(s)}(Z))\right)^3\right)} \tag{6.9}$$

Table 6.1
Method (6.1)—The maximum radii of inclusion disks on P[1]

$(\ )^{-1}, C_j^{(s)} = 0$	$(\ )^{-1}, C_j^{(s)} = N_j^{(s)}$	$(\ )^{I_1}, C_j^{(s)} = N_j^{(s)}$	$(\ )^{I_2}, C_j^{(s)} = N_j^{(s)}$
6.7036e–014	6.5482e–017	1.3918e–016	2.6513e–011

$a(-e) = a \times 10^{-e}$

where $M_{i,j,t}^{(s)}(Z) = \sum_{j=1,j\neq i}^{n}\left(\mathrm{INV}(T_{i,j,t}^{(s)}(Z))\right)^2 \sum_{j=1,j\neq i}^{n}\left(\mathrm{INV}(T_{i,j,t}^{(s)}(Z))\right)$ is in the same token from (2.6). They are of the R-order O_R given by

$$O_R \geq \begin{cases} \left(5+\sqrt{41}\right)/2 = 5.7016\ldots; & p \geq 2,\ \beta = 1 \\ 6; & p = 2,\ \beta = 0 \\ 7; & p = 3,\ \beta = 0 \\ 8; & p = 4,\ \beta = 0 \\ 4+\sqrt{17} = 8.1231\ldots; & p \geq 5,\ \beta = 0 \end{cases} \tag{6.10}$$

and similarly by

$$O_R \geq \begin{cases} 6; & p = 2 \\ 7; & p = 3 \\ 8; & p = 4 \\ 4+\sqrt{17} = 8.1231\ldots; & p \geq 5 \end{cases} \tag{6.11}$$

for the corresponding point method respectively. Higher-order methods are no doubt robust and are obtained from (2.1), (2.2), (2.5) and (2.6) for higher values of k. We illustrate an application of the method (6.1) by considering the problem of computing the zeros of the polynomials in [1]:

$$\mathrm{P}[1]: \quad P_9(z) = z^9 + 3z^8 - 3z^7 - 9z^6 + 3z^5 + 9z^4 + 99z^3 + 297z^2 - 100z - 300$$

with $\lambda_j = -3, \pm 1, \pm 2i, \pm 2 \pm i$ and $Z_1^{(0)} = \{-3.2 + 0.2i; r\}$; $Z_2^{(0)} = \{-1.1 - 0.2i; r\}$;

$$Z_3^{(0)} = \{0.1 + 1.7i; r\}; \quad Z_4^{(0)} = \{-1.9 + 1.3i; r\}; \quad Z_5^{(0)} = \{-1.8 - 0.8i; r\};$$
$$Z_6^{(0)} = \{2.3 + 1.1i; r\}; \quad Z_7^{(0)} = \{1.9 - 0.7i; r\}; \quad Z_8^{(0)} = \{1.2 + 0.2i; r\};$$
$$Z_9^{(0)} = \{0.2 - 2.2i; r\}; \quad r = 0.35.$$

The numerical results are from codes written in MATLAB using a Dell Laptop of processor clock speed 2.7 Mhz, RAM of 512 Mb and hard disk space of 30 Gb. The results of the maximum radii of the inclusion disks are in Table 6.1.

The method (6.1) using the inversion $(\ )^{I_2}$ produces disks of largest radii and similarly for the other inversion formulae in the order implied by $Z^{-1} \subseteq Z^{I_1} \subseteq Z^{I_2}$. However, with respect to correction, but without updating the iterates, the methods of the inversion $(\ )^{-1}$ give the best results in all cases when compared to the other inversion formulae and by far better than the corresponding basic methods which is without correction, that is when $C_j^{(s)} = 0$ and the inversion is $(\ )^{-1}$. Observe that results in Table 6.1 compares with [1]. A follow-up report to this (Part II, III), hopes to consider the computational complexities and acceleration of convergence of these methods by Gauss–Seidel and Gauss–Jacobi means of updating of the generated iterates in (2.1), (2.2), (2.5) and (2.6).

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References

[1] M.S. Petkovic, C. Carstensen, Some improved inclusion methods for polynomial roots with Weierstrass corrections, Comput. Math. Appl. 25 (3) (1993) 59–67.
[2] I. Gargantini, Comparing parallel Newton's method with parallel Laguere's method, Comput. Math. Appl. 2 (1976) 201–206.
[3] I. Gargantini, Parallel square-root iterations for multiple roots, Comput. Math. Appl. 6 (1980) 279–288.
[4] M. Petkovic, Iterative methods for simultaneous inclusion of polynomial zeros, in: Lecture Notes in Mathematics, Springer-Verlag, 1989.
[5] J.W. Schmidt, On the R-order of coupled sequences, Computing 26 (1981) 333–342.
[6] J. Herzberger, L. Metzner, On the Q-order and R-order of convergence for coupled sequences arising in iterative numerical processes, in: G. Alefeld, J. Herzberger (Eds.), Numerical Methods and Error Bounds, in: Mathematical Research, vol. 89, Akademie Verlag, Berlin, 1996, pp. 120–131.
[7] G. Alefeld, J. Herzberger, On the convergence speed of some algorithms for the simultaneous approximation of polynomial roots, SIAM. J. Numer. 11 (2) (1974) 237–243.
[8] D. Braess, K.P. Hadeler, Simultaneous inclusion of the zeros of a polynomial, Numer. Math. 21 (1973) 161–165.
[9] F. Van Loan Charles, Introduction to Scientific Computing-A Matrix–Vector Approach Using MATLAB, Prentice Hall, New Jersey, 1997.
[10] I. Gargantini, An application of interval mathematics: A polynomial solver with degree four convergence, Freiburger Intervall-Berichte, Institut fur Angewandte Mathematik Universitat Freiburg i. Br., 1981, pp. 15–25.
[11] P. Henrici, Applied and Computational Complex Analysis, vol. I, John Wiley and Sons Inc., New York, 1974.
[12] M.N.O. Ikhile, On the convergence of some interval methods for simultaneous computation of polynomial zeros, Int. J. Comput. Math. 79 (10) (2002) 1099–1111.
[13] M.N.O. Ikhile, An iterative method for simultaneous inclusion of polynomial zeros, J. Nig. Ass. Math. Phys. 5 (2001) 19–36.
[14] M.N.O. Ikhile, On the Weierstrass and some Petkovic-like methods for numerical determination of polynomial zeros, Int. J. Comput. Math. (2003) 1505–1519.
[15] E. Ludwig, A remark on simultaneous inclusions of the zeros of a polynomial by Gershgorin's Theorem, Numer. Math. 21 (1973) 425–427.
[16] R.E. Moore, Interval Analysis, Prentice-Hall, Englewood Cliffs, NJ, 1966.
[17] A.-W. Nourein, An iteration formular for the simultaneous determination of the zeros of a polynomial, J. Comput. Appl. Math. 1 (4) (1975).
[18] M.A. Nourein, An improvement on Nourein's method for the simultaneous determination of the zeros of a polynomial (an algorithm), J. Comput. Appl. Math. 3 (1977) 109–112.
[19] M. Petkovic, Schroder-like algorithms for multiple complex zeros of a polynomial, Computing 45 (1990) 39–50.
[20] M.S. Petkovic, L.D. Petkovic, L.Z. Rancic, Higher order simultaneous methods for the determination of polynomial multiple zeros, Int. J. Comput. Math. 80 (11) (2003) 1407–1427.
[21] L.D. Petkovic, M.S. Petkovic, D. Milosevic, Inclusion Weierstrass-like root-finders with corrections, Filomat 17 (2003) 143–154.
[22] M.S. Petkovic, L.D. Petkovic, D. Zivkovic, Laguerre-like methods for the simultaneous approximation of polynomial zeros, in: G. Alefeld, X. Chen (Eds.), Topics in Numerical Analysis with Special Emphasis on Nonlinear Problems, Springer-Verlag, New York, 2001, pp. 189–210.
[23] M.S. Petkovic, D.M. Milosevic, Ostrowski-like method with corrections for the inclusion of polynomial zeros, Reliable Computing 10 (2004) 437–467.
[24] D.M. Milosevic, L.D. Petkovic, On the improved family of simultaneous methods for the inclusion of multiple zeros of polynomials, Novi Sad J. Math. 35 (1) (2005) 49–57.
[25] L.Z. Rancic, M.S. Petkovic, Square-root families for the simultaneous approximation of polynomial multiple zeros, Novi Sad Math. 35 (1) (2005) 59–70.
[26] H. Ratschek, J. Rokne, New Computer Methods for Global Optimisation, Ellis Horwood Ltd. John Wiley and Sons, NY, 1988.
[27] G.R. Voigt, Rates of convergence for a class of iterative procedures, SIAM J. Numer. Anal. 8 (1) (1971).
[28] M.N.O. Ikhile, Corrected Roots and Bell's Disk Iteration Methods for the Simultaneous Determination of the Zeros of a Polynomial. Part of a Colloquium Talk at *the Department of Applied Mathematics, University of Stellenbosch, Cape Town, South Africa, March (2006)*. Also a Seminar Talk and a Technical Report at the *African Institute for Mathematical Sciences (AIMS), Cape Town, South Africa, March/April (2006).* Program and Abstracts (p.18) of the *30th Annual Conference of the South African Society of Numerical and Applied Mathematics (SANUM), University of Stellenbosch, Cape Town, South Africa, April (2006).*