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Recursion Theory on Fields and Abstract Dependence

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1. INTRODUCTION

The theory of r.e. (recursively enumerable) vector spaces was introduced in [9] by us. The object of study there was the lattice $\mathscr{L}(V_\infty)$ of r.e. subspaces of a countably infinite-dimensional vector space V_∞ such that V_∞ and its field of scalars were sufficiently effective. Inspired by this several authors have published interesting further results on $\mathscr{L}(V_\infty)$.

In particular we point out Kalantari–Retzlaff [8], Remmel [11], and Shore [16]. We were then interested in whether a similar theory could be developed for the lattice $\mathscr{L}(F_\infty)$ of all r.e. algebraically closed subfields of an algebraically closed fild F_∞ of countably infinite transcendence degree such that F_∞ was sufficiently effective. The major difficulty was that a key lemma which supplied the "punch line" for many priority arguments in $\mathscr{L}(V_\infty)$ was simply false for $\mathscr{L}(F_\infty)$. If $A \subseteq V_\infty$, let $\mathrm{cl}(A)$ be the subspace A spans. If $A \subseteq F_\infty$, let $\mathrm{cl}(A)$ be the algebraically closed subfield of F_∞ that A generates. Let $B = \{b_0, b_1, \ldots\}$ be a vector space basis for V_∞, let V be an infinite-dimensional subspace of V_∞, and let $m \geqslant 0$ be an integer. The lemma alluded to above asserts $(V \cap \mathrm{cl}\{b_m, b_{m+1}, \ldots\}) - \mathrm{cl}\,\varnothing \neq \varnothing$, i.e., there is a nonzero $v \in V \cap \mathrm{cl}\{b_m, b_{m+1}, \ldots\}$. Now let $B = \{b_0, b_1, \ldots\}$ be a transcendence base for F_∞ over its prime subfield, and let F be the infinite-dimensional algebraically closed subfield of F_∞ generated by $\{b_0, b_1 + b_0 b_2, b_2 + b_0 b_3, \ldots\}$. Then $F \cap \mathrm{cl}\{b_1, b_2, \ldots\} - \mathrm{cl}\,\varnothing = \varnothing$, i.e., every element of $F \cap \mathrm{cl}\{b_1, b_2, \ldots\}$ is algebraic. So the obvious corresponding lemma fails for $\mathscr{L}(F_\infty)$. (This example is due to Ash and may be verified by using Jacobians (see [6]).)

In a way this is a manifestation of the nonmodularity of $\mathscr{L}(F_\infty)$, in contrast to the modularity of $\mathscr{L}(V_\infty)$. In a modular lattice an element cannot have two distinct comparable complements. But in $\mathscr{L}(F_\infty)$ if we let $H = \mathrm{cl}\{b_1, b_2, \ldots\}$,

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$G = \text{cl}\{b_0\}$, define F as above, we see that F and G are distinct complements of H which are not comparable.

With the development of new techniques which bypass such lemmas and work for $\mathscr{L}(F_\infty)$, the central role of the dependence relation became apparent. Indeed the operations (vector addition and scalar multiplication for V_∞, field operations for F_∞) play no direct role. Only the relation of dependence occurs. It turns out to be clearer and cleaner to develop the subject for abstract dependence relations as defined by Van den Waarden [17, p. 200].[1] Other well-known equivalents are transitive dependence relations [1, p. 254] and matroids. We use the fully equivalent notion of a closure operation obeying the Steinitz exchange principle. This fits the arguments best.

Let $P(U)$ be the power set of U.

Definition 1.1. A Steinitz closure system (U, cl) consists of a set U and an operation cl: $P(U) \to P(U)$ such that for all $A, B \in P(U)$,

(i) $A \subseteq \text{cl}(A)$,

(ii) $A \subseteq B$ implies $\text{cl}(A) \subseteq \text{cl}(B)$,

(iii) $\text{cl}(\text{cl}(A)) = \text{cl}(A)$,

(iv) $x \in \text{cl}(A)$ implies that there is a finite $A^1 \subseteq A$ such that $x \in \text{cl}(A^1)$,

(v) $x \in \text{cl}(A \cup \{y\}) - \text{cl}(A)$ implies that $y \in \text{cl}(A \cup \{x\})$.

Here (i)–(iv) are Moore's axioms for a closure operation; (v) is the Steinitz exchange principle. Elementary properties are developed in Cohn [1, pp. 252–262], and used here. For us the most important examples are (ω, cl), (V_∞, cl). and (F_∞, cl). (Here $\omega = \{0, 1, 2, \ldots\}$, $\text{cl}(A) = A$ for $A \subseteq \omega$.) We call $A \subseteq U$ closed if $\text{cl}(A) = A$. Every closed set has a well-defined dimension. The key new notion we introduce is *regularity*.

Definition 1.2. A finite-dimensional closed set $C \subseteq U$ is *regular* if it is not the union of a finite number of its proper closed subsets. We call (U, cl) regular if all its finite-dimensional closed subsets are regular.

It can be verified (see Section 2) that (F_∞, cl) is always regular, and that (V_∞, cl) is regular if and only if the scalar field is not finite; while (ω, cl) is obviously not regular. In Section 3 we give a definition of recursively presented Steinitz closure systems. It will follow that a regular Steinitz closure system is recursively presented if and only if

[1] The referee of Crossley–Nerode [2] asked whether Van der Waarden's dependence relations (familiar to algebraists) could be used for effective dimension theory instead of minimal formulas (familiar to logicians) as in [2]. This paper is a partial answer to that question.

(i) U is a recursive set of integers,

(ii) for any a, $b_1, \ldots, b_m$ in U it can be effectively determined whether or not $a \in \mathrm{cl}\{b_1, \ldots, b_m\}$.

We may use Godel numberings to regard V_∞ and F_∞ as having domain ω and to regard (ω, cl), (V_∞, cl) and (F_∞, cl) as recursively presented.

We believe that recursion theory over infinite-dimensional, recursively presented regular Steinitz closure systems (U, cl) is natural and has depth, and that virtually all results previously obtained for $\mathscr{L}(V_\infty)$ can be formulated and proved for such (U, cl). We support this contention by formulating and proving generalizations to regular Steinitz systems of the theorems listed below which are from the above-mentioned papers on $\mathscr{L}(V_\infty)$:

(i) Maximal spaces via e-states (Metakides–Nerode [9, p. 158], Theorem 4.1).

(ii) Maximal spaces generated by maximal subsets of bases (Metakides–Nerode [9, p. 160], Theorem 4.8).

(iii) Maximal spaces with no extendible bases (Metakides–Nerode [9, p. 161], Theorem 4.8; Remmel [11, Theorem 1, p. 402]).

(iv) Supermaximal spaces (Kalantari–Retzlaff [7, p. 486], Theorem 3.1).

(v) Dependence degrees (Shore [16, p. 19], Theorem 2.2).

The generalizations here are respectively Theorem 4.2, 4.8, 5.1, 6.2, and 7.1 for (i)–(v).

Far weaker hypotheses than regularity may be used to get any one of these theorems individually; a different algebraic condition for each theorem. These will be dealt with in a sequel by Nerode and Remmel. Classes of matroids arise in combinatorial theory which satisfy such weaker hypotheses, but these are very much less known to the working mathematician or logician than V_∞ or F_∞.

2. Steinitz Systems

Throughout this section (U, cl) will be a Steinitz closure system.

Proposition 2.1. *If $B \subseteq U$ and $\mathrm{cl}_B(A) = \mathrm{cl}(A \cup B)$, then (U, cl_B) is a Steinitz closure system. We refer to $\mathrm{cl}_B(A)$ as the closure of A* over B.

Definition 2.2. Suppose $A, B, C, I \subseteq U$.

(i) A is closed (over B) if $A = \mathrm{cl}_B(A)$.

(ii) A is independent (over B) if $A \neq \varnothing$ and for all $a \in A$, we have $a \notin \mathrm{cl}_B(A - \{a\})$.

(iii) A spans C (over B) if $C \subseteq \mathrm{cl}_B(A)$.

(iv) $I \subseteq A$ is a basis for A (over B) if I spans A (over B) and I is independent (over B). In case B is empty, omit the phrase "(over B)."

PROPOSITION 2.3. *Let A be closed. Suppose I, $S \subseteq A$, and I is independent and S spans A. If $I \subseteq S$, then there is a basis X for A such that $I \subseteq X \subseteq S$.*

Proof. Theorem 2.4 of [1, p. 256].

PROPOSITION 2.4. *Suppose B and A are closed, $B \subseteq A$. Let B_1 be a basis for B. Let A_1 be a basis for A (over B). Then $A_1 \cup B_1$ is a basis for A.*

Proof. B_1 spans B, A_1 spans A (over B), so $B \subseteq \mathrm{cl}(B_1)$, $A \subseteq \mathrm{cl}(B \cup A_1)$, or $A \subseteq \mathrm{cl}(B_1 \cup A_1)$, or $B_1 \cup A_1$ spans A. Since B_1 is independent and $B_1 \cup A_1$ spans A, Proposition 2.3 yields a basis X for A such that $B_1 \subseteq X \subseteq A_1 \cup B_1$. It suffices to show $X = A_1 \cup B_1$. Otherwise there would be an a in A_1, $a \notin X$; then $X \subseteq (A_1 - \{a\}) \cup B_1$, so $\mathrm{cl}(X) \subseteq \mathrm{cl}((A_1 - \{a\}) \cup B_1)$. Since A_1 is independent over B, $a \notin \mathrm{cl}((A_1 - \{a\}) \cup B_1)$, so $a \notin \mathrm{cl}(X)$, so X does not span A, contrary to hypothesis.

DEFINITION 2.5. Let $B \subseteq A$, B, A both closed. The dimension of A (over B) is the cardinality of any basis of A (over B), denoted by $\dim[A/B]$.

PROPOSITION 2.6. *Suppose $X_1 \cup X_2$ is independent, $X_1, X_2 \subseteq U$. Then $\mathrm{cl}(X_1) \cap \mathrm{cl}(X_2) = \mathrm{cl}(X_1 \cap X_2)$.*

Proof. This is proved exactly as in Corollary 6.7 ([1, p. 28]).

PROPOSITION 2.7. *Let B, $I \subseteq U$, $x \in U$. Suppose B is closed, I is independent (over B), and $x \in \mathrm{cl}_B(I)$. Then there is a smallest finite set $I' \subseteq I$ with $x \in \mathrm{cl}_B(I')$, denoted as* $\mathrm{supp}_I\, x$ *(over B).*

Proof. The fourth clause in the definition of a Steinitz closure system in Section 1 shows a finite I' exists. By Proposition 2.6 we may intersect all such and get *a* smallest.

PROPOSITION 2.8. *Let B, $I \subseteq U$, B closed, I independent (over B).*

(i) *For $x \in \mathrm{cl}_B(I)$, $J \subseteq I$, we have $x \in \mathrm{cl}_B J \leftrightarrow$* $\mathrm{supp}_I\, x$ *(over B) $\subseteq J$.*

(ii) *Let $x_0, x_1, \ldots$ be a sequence from $\mathrm{cl}_B(I)$. Suppose* $\mathrm{supp}_I\, x_0$ *(over B) $\not\subseteq$* $\bigcup_{i=1}^{\infty} \mathrm{supp}_I\, x_i$ *(over B). Then $x_0 \notin \mathrm{cl}_B\{x_1, x_2, \ldots\}$.*

(iii) *Suppose I is infinite, $F \subseteq U$ is finite. Then I is infinite dimensional over $B \cup F$.*

Proof. Note that (i) is immediate from Proposition 2.7. As for (ii), by (i) we get $x_0 \notin \mathrm{cl}_B \bigcup_{i=1}^{\infty} [\mathrm{supp}_I\, x_i$ (over B)]. But $\mathrm{cl}_B\{x_1, x_2, ...\} \subseteq \mathrm{cl}_B \bigcup_{i=1}^{\infty} [\mathrm{supp}_I\, x_i$ (over B)]. For (iii) note that were I finite dimensional over $F \cup B$, we would get $\mathrm{cl}(G \cup F \cup B) = \mathrm{cl}(I \cup F \cup B)$ for a finite $G \subseteq I$, and $\mathrm{cl}_B I \subseteq \mathrm{cl}(B \cup F \cup G) = \mathrm{cl}_B(F \cup G)$. So $\mathrm{cl}_B(I)$ is contained in a finite-dimensional closed set over B, hence is itself finite dimensional over B.

PROPOSITION 2.9. *Let $V \subseteq U$ be finite dimensional and closed. Then V is regular if and only if whenever $V \subseteq U_1 \cup \cdots \cup U_n$ with $U_1, ..., U_n \subseteq U$ closed, we get that for some i, $V \subseteq U_i$.*

Proof. If V is regular, and $V \subseteq U_1 \cup \cdots \cup U_n$, obviously, $V = (V \cap U_1) \cup \cdots \cup (V \cap U_n)$. All terms are closed. By regularity, for some i $V = V \cap U_i$; so $V \subseteq U_i$. If conversely the condition holds, then $V = U_1 \cup \cdots \cup U_n$ implies $V \subseteq U_i$ for some i. But $U_i \subseteq V$, so $V = U_i$, hence V is regular.

LEMMA 2.10. *Let $C, D_1, ..., D_n$ be subspaces of a vector space over an infinite field. Then $C \subseteq D_1 \cup \cdots \cup D_n$ implies for some i, $C \subseteq D_i$.*

Proof. Otherwise there exists a $v_i \in C - D_i$ for each $i = 1,..., n$. Suppose we were given a set S of n-tuples $(\lambda_1, ..., \lambda_n)$ from the field of scalars and were told that for every i, S has at most $n - 1$ members $(\lambda_1, ..., \lambda_n)$ with $\lambda_1 v_1 + \cdots + \lambda_n v_n \in D_i$. Then from $C \subseteq D_1 \cup \cdots \cup D_n$ we would conclude S has at most $n(n-1) = n^2 - n$ members. Thus if S is any set of n-tuples $(\lambda_1, ..., \lambda_n)$ from the field of scalars with at least $n^2 - n + 1$ members, then for some i there are at least n members $(\lambda_1, ..., \lambda_n)$ of S such that $\lambda_1 v_1 + \cdots + \lambda_n v_n \in D_i$. Since the scalar field is infinite, we can easily find a set S of $n^2 - n + 1$ n-tuples $(\lambda_1, ..., \lambda_n)$ such that any n of them are independent. Apply the observation above and obtain n n-tuples $(\lambda_{i1}, ..., \lambda_{in})$, $i = 1,..., n$ such that these n-tuples are independent and for a single i, $\lambda_{11} v_1 + \cdots + \lambda_{1n} v_n \in D_i, ..., \lambda_{n1} v_1 + \cdots + \lambda_{nn} v_n \in D_i$. Since D_i is a subspace and the matrix is invertible, all $v_1, ..., v_n$ are in D_i, contrary to hypothesis.

PROPOSITION 2.11. *Any (V_∞, cl) is regular over any infinite scalar field. Also (F_∞, cl) is regular.*

Proof. The regularity of (V_∞, cl) is just Lemma 2.10. For (F_∞, cl) suppose $C, D_1, ..., D_n$ are algebraically closed subfields of F_∞ and $C \subseteq D_1 \cup \cdots \cup D_n$. Regard F_∞ as a vector space over its subfield of algebraic elements. Then Lemma 2.10 again yields the desired result.

PROPOSITION 2.12. *If (U, cl) is regular and $V \subseteq U$ is closed, then (U, cl_V) is regular.*

Proof. Let $C, D_1, ..., D_n$ be finite dimensional and closed in (U, cl_V).

Suppose each of $D_1, \ldots, D_n$ is smaller than C. Then if $k = \dim[C/V]$, we know $k > \dim[D_1/V], \ldots, \dim[D_n/V]$. Let $b_1, \ldots, b_k$ be independent over V and in C. For any given i, it cannot be that $b_1, \ldots, b_k$ are all in D_i, for then $\dim[D_i/V] \geqslant k$. So for all i, $\mathrm{cl}\{b_1, \ldots, b_k\} \cap D_i \subseteq \mathrm{cl}\{b_1, \ldots, b_k\}$. By regularity there is a y in $\mathrm{cl}\{b_1, \ldots, b_k\} - \bigcup_{i=1}^{n}[\mathrm{cl}\{b_1, \ldots, b_k\} \cap D_i]$. Since $y \in \mathrm{cl}\{b_1, \ldots, b_k\} \subseteq C$, we get $y \in C - (D_1 \cup \cdots \cup D_n)$ as desired.

Proposition 2.13 (Wagner). *(U, cl) is regular if and only if all closed sets of dimension 2 are regular.*

Proof. Every Steinitz system is regular in dimension 0, 1. Suppose C has dimension $t + 1 > 2$ and $D_1, \ldots, D_n$ have dimension $\leqslant t$. We show that $C - (D_1 \cup \cdots \cup D_n) \neq \varnothing$. For this purpose a definition of Shore [16, p. 19] is used. Let $b_0, \ldots, b_t$ be a basis for C. Call $z \in \mathrm{cl}\{b_0, \ldots, b_{t-1}\}$ t-bad for D_i if z, b_t are independent and $\mathrm{cl}\{z, b_t\} \subseteq D_i$. For $k < t$, call a $z \in \mathrm{cl}\{b_0, \ldots, b_{k-1}\}$ k-bad for D_i if there exist independent y, w in $\mathrm{cl}\{z, b_k\}$ with both y and w $k+1$-bad for D_i.

Lemma 2.14. *If z is k-bad for D_i, then $z, b_k, \ldots, b_t \in D_i$.*

Proof. If z is t-bad for D_i, then $z, b_t \in \mathrm{cl}\{z, b_t\} \subseteq D_i$. If $k < t$ and we assume the lemma holds for all $k + 1$-bad z for D_i, proceed as follows. Let z be k-bad for D_i. There are independent y, w in $\mathrm{cl}\{z, b_k\}$ both $k + 1$-bad for D_i. By inductive hypothesis, $y, w, b_{k+1}, \ldots, b_t$ are all in D_i. So $z, b_k \in \mathrm{cl}\{z, b_k\} = \mathrm{cl}\{y, w\} \subseteq D_i$. Thus $z, b_k, \ldots, b_t \in D_i$.

Now to conclude the proof of Proposition 2.13 we produce a sequence $z_0, \ldots, z_t$ such that z_k is not $k + 1$-bad for any D_i as follows. Let z_0 be b_0. If b_0 were 1-bad for D_i, Lemma 2.14 shows $b_0, \ldots, b_t \in D_i$ and $\dim D_i \geqslant t + 1$, contrary to supposition. So now assume $z_k \in \mathrm{cl}\{b_0, \ldots, b_k\}$ has been chosen with z_k not $k + 1$ bad for any D_i.

Case 1. $k < t - 1$. Since z_k is not $k + 1$ bad for D_i, there cannot exist an independent pair of elements of $\mathrm{cl}\{z_k, b_{k+1}\}$ each of which is $k + 2$-bad for D_i. This says that the following set T_i is $\leqslant$ one dimensional.

$T_i = \mathrm{cl}\{y \in \mathrm{cl}\{z_k, b_{k+1}\} \mid y$ is $k + 2$-bad for $D_i\}$. But $\mathrm{cl}\{z_k, b_{k+1}\}$ is two dimensional. Since by hypothesis all two-dimensional closed sets are regular, there is a z_{k+1} in $\mathrm{cl}\{z_k, b_{k+1}\} - (T_1 \cup \cdots \cup T_n)$. Since $z_k \in \mathrm{cl}\{b_0, \ldots, b_k\}$, we get $z_{k+1} \in \mathrm{cl}\{b_0, \ldots, b_{k+1}\}$. Since z_{k+1} is chosen outside $T_1, \ldots, T_n$, z_{k+1} is not $k + 2$-bad for any D_i.

Case 2. $k = t - 1$. Then $z_{t-1} \in \mathrm{cl}\{b_0, \ldots, b_{t-1}\}$ is not t-bad for D_i, so $T_i = \mathrm{cl}\{z_{t-1}, b_t\} \cap D_i$ is a closed set of dimension $\leqslant 1$. By regularity of dimension-two closed sets there is a z_t in

$$\mathrm{cl}\{z_{t-1}, b_t\} - (T_1 \cup \cdots \cup T_n).$$

Now $z_t \in \text{cl}\{b_0, ..., b_t\} \subseteq C$. Since $z_t \notin T_i$ and $z_t \in \text{cl}\{z_{t-1}, b_t\}$, we get $z_t \notin D_i$. So $z_t \in C - D_1 \cup \cdots \cup D_n$.

In the lattice of closed sets of U the operations are $A \wedge B = A \cap B$, $A \vee B = \text{cl}(A \cup B)$. Then B is a complement of A if $A \vee B = U$, $A \wedge B = \text{cl}\, \varnothing$. This is not useful for Steinitz systems whose lattice of closed sets fails to be modular. The missing ingredient is the following definition.

DEFINITION 2.15. Closed A, B are *independent* if any independent set in A is independent over B.

This apparently asymmetric definition is actually symmetric, as the following proposition demonstrates.

PROPOSITION 2.16. *Let A, B be closed sets such that $A \wedge B = \text{cl}\, \varnothing$. The following are equivalent:*

(i) *There is a basis X for $A \vee B$ such that $A \cap X$ is a basis for A, $B \cap X$ is a basis for B.*

(ii) *For all independent sets $A_1 \subseteq A$, $B_1 \subseteq B$, $A_1 \cup B_1$ is independent.*

(iii) *Every basis for A is a basis for $A \vee B$ over B.*

(iv) *Some basis for A is a basis for $A \vee B$ over B.*

Proof. (iii) → (iv) is immediate; (iv) → (i) is Proposition 2.4. For (i) → (ii), suppose X is a basis for $A \vee B$, $A \cap X$ a basis for A, $B \cap X$ a basis for B. Suppose $A_1 \subseteq A$, $B_1 \subseteq B$, A_1, B_1 are independent, $A_1 \cup B_1$ dependent. Without loss of generality assume A_1, B_1 are finite. Since $A \cap X$ is a basis for A, $A_1 \subseteq \text{cl}(A_1')$, A_1' finite with m elements, $A_1' \subseteq A \cap X$; similarly $B_1' \subseteq \text{cl}(B_1')$, B_1' finite with n elements, $B_1' \subseteq B \cap X$. By hypothesis $A_1' \cup B_1' \subseteq X$ is independent, hence $\text{cl}(A_1' \cup B_1')$ is of dimension $m + n$. Extend A_1 to a basis $\tilde{A}_1$ for $\text{cl}(A_1')$, B_1 to a basis $\tilde{B}_1$ for $\text{cl}(B_1')$, so $\text{cl}(\tilde{A}_1 \cup \tilde{B}_1) = \text{cl}(A_1' \cup B_1')$. Now $\tilde{A}_1$ must have m elements, $\tilde{B}_1$ must have n elements, so since $\tilde{A}_1 \cup \tilde{B}_1$ spans $m + n$-dimensional $\text{cl}(A_1' \cup B_1')$, we conclude $\tilde{A}_1 \cup \tilde{B}_1$ is independent and that $A_1 \cup B_1$ is independent as required. Now to see (ii) → (iii). Let A_1 be a basis for A, B_1 a basis for B. By (ii), $A_1 \cup B_1$ is independent so for $a \in A_1$, $a \notin \text{cl}(A_1 \cup B_1 - \{a\}) = \text{cl}_B(A_1 - \{a\})$, so A_1 is independent over B.

DEFINITION 2.17. If A, B are closed, then B is an *independent* complement of A if $A \vee B = U$, and $\dim A = \dim[A/B]$.

Note that an independent complement is indeed a complement: if A_1 is a basis for A, B_1 a basis for B, by Proposition 2.16, $A_1 \cup B_1$ is a basis for U, so $A \cap B = \text{cl}(A_1) \cap \text{cl}(B_1) = \text{cl}(A_1 \cap B_1) = \text{cl}(\varnothing)$ by Proposition 2.6.

Of course every closed B has an independent complement, namely, take any basis A_1 for U over B and let $\text{cl}(A_1) = A$. Finally, any two comparable independent complements B, C of A are equal. Suppose $B \subseteq C$. Let A_1, B_1 be bases for A, B, and let C_1 be a basis for C extending B_1. Since B, C are independent complements for A, $A_1 \cup B_1$ and $A_1 \cup C_1$ are both bases of U, and $A_1 \cup B_1 \subseteq A_1 \cup C_1$. So, $A_1 \cup B_1 = A_1 \cup C_1$. But $A_1 \cap B_1 = A_1 \cap C_1 = \varnothing$, so $B_1 = C_1$.

3. Recursive Presentations

Definition 3.1. A Steinitz closure system (U, cl) has *recursive dependence* if:

(i) U is a recursive set of integers;

(ii) there is a uniform effective procedure which, applied to $a, b_1, \ldots, b_n \in U$, determines in a finite number of steps whether or not $a \in \mathrm{cl}\{b_1, \ldots, b_n\}$.

Proposition 3.2. *Suppose* (U, cl) *has recursive dependence. Then there are uniform effective procedures which*:

(i) *from explicit indices for finite sets* A, B *determine whether or not* $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$;

(ii) *from an explicit index of a finite set* A *determine whether or not* A *is independent*;

(iii) *from an explicit index of a finite set* A *compute an explicit index for each subset of* A *which is a basis for* $\mathrm{cl}(A)$;

(iv) *from a recursive enumeration of* A *yield a recursive enumeration of a basis for* $\mathrm{cl}(A)$;

(v) *from an explicit index of a finite independent set* A *yield a recursive enumeration of a basis* B *for* U *which contains* A;

(vi) *from a recursive enumeration of an independent set* I (*over finite set* F), *from an explicit index of* F, *and from an* $x \in \mathrm{cl}_F(I)$, *yields an explicit index for* $\mathrm{supp}_I\, x$ (*over* F).

Proof. An explicit index for a finite set is of course one that yields both an effective listing of the set and a computation of its cardinality. For (i) observe $\mathrm{cl}(A) \subseteq \mathrm{cl}(B) \leftrightarrow A \subseteq \mathrm{cl}(B) \leftrightarrow$ for each $a \in A$, we have $a \in \mathrm{cl}(B)$. This can be determined because of recursive dependence.

For (ii) note that from an explicit index of A we can determine whether $A = \varnothing$, and if not whether any $a \in A$ has the property that $a \in \mathrm{cl}(A - \{a\})$ using recursive dependence. For (iii) note that by (i) and (ii) we can check each $A' \subseteq A$ for independence and also check $\mathrm{cl}(A') = A$. For (iv) list A as $a_0, a_1, \ldots$ effectively. Drop a_i from the list if $a_i \in \mathrm{cl}(a_0, \ldots, a_{i-1})$ using recursive dependence. For (v) list A as $a_0, \ldots, a_n$, and let $a_0, \ldots, a_n, u_1, u_2, \ldots$ be a list of all of U that is effective, then by (iv) to a basis. For (vi) observe that since I can be enumerated as $i_0, i_1, \ldots$ and $x \in \mathrm{cl}_F(I)$, recursive dependence computes an n with

$x \in \mathrm{cl}_F\{i_0, \ldots, i_n\}$. Then we test to find a smallest $I' \subseteq \{i_0, \ldots, i_n\}$ with $x \in \mathrm{cl}_F(I')$, and this is $\mathrm{supp}_I\, x$ (over F).

The definition given below for "recursively presented" is more general than is actually used in this paper. It appears to be the correct requirement for recursion-theoretic arguments in the nonregular case.

DEFINITION 3.3. A Steinitz system (U, cl) is *recursively presented* if:

(i) U is a recursive set of integers;

(ii) there is a uniform effective procedure which, applied to explicit indices of finite sets A, $B_1, \ldots. B_n \subseteq U$, determines whether or not $\mathrm{cl}(A) \subseteq (\mathrm{cl}(B_1)) \cup \cdots \cup (\mathrm{cl}(B_n))$.

PROPOSITION 3.4. *If* (U, cl) *is recursively presented, then* (U, cl) *has recursive dependence.*

Proof. $a \in \mathrm{cl}\{b_1, \ldots, b_n\} \leftrightarrow \mathrm{cl}\{a\} \subseteq \mathrm{cl}\{b_1, \ldots, b_n\}$.

PROPOSITION 3.5. *If* (U, cl) *has recursive dependence and is regular, then* (U, cl) *is recursively presented.*

Proof. Let $A, B_1, \ldots, B_n$ be finite subsets of U given by explicit indices. By regularity $\mathrm{cl}(A) \subseteq (\mathrm{cl}(B_1)) \cup \cdots \cup (\mathrm{cl}(B))$ if and only if $\mathrm{cl}(A) \subseteq \mathrm{cl}(B_i)$ for some i. This can be tested effectively (Proposition 3.2(i)).

PROPOSITION 3.6. *The Steinitz closure systems* (ω, cl), (V_∞, cl), (F_∞, cl) *are recursively presented.*

Proof. We do only the cases of (V_∞, cl) with scalar field infinite and (F_∞, cl). By Propositions 2.11 and 3.5 we need only show recursive dependence. This is classical (Row reduction for (V_∞, cl), Jacobians for (F_∞, cl)); see [6, p. 58; 4, 5] for the field case.

DEFINITION 3.7. Let V be a closed subset of U. For $k \geqslant 1$ let $D(V)_k$ be the set of all k-tuples $y = (y_1, \ldots, y_k)$ such that y is dependent over V. Put $D(V) = \bigcup_{k=1}^{\infty} D(V)$. (Of course, if (U, cl) has recursive dependence and V is r.e., then $D(V)_k$, $D(V)$ are r.e. with r.e. Turing degrees $d(D(V)_k)$, $d(D(V))$.)

Let $\mathscr{L}(U)$ be the lattice of r.e. closed subsets of U.

PROPOSITION 3.8. *Suppose* (U, cl) *is a Steinitz closure system with recursive dependence. Suppose* $V, W \in \mathscr{L}(U)$, $V \subseteq W$, $\dim[W/V] < \infty$. *Then* $d(D(V)) = d(D(W))$.

Proof. Let $w_1, \ldots, w_t$ be a basis for W over V. Then for $y = (y_1, \ldots, y_k)$, $y \in D(W) \leftrightarrow (y_1, \ldots, y_k, w_1, \ldots, w_t) \in D(V)$, so $d(D(W)) \leqslant d(D(V))$. To demonstrate the opposite inequality we show by induction on k that $D(V)_k$ is recursive in $D(W)$ uniformly. Let $v_0, v_1, \ldots$ be an r.e. basis for V.

We can find a basis for U over W, recursive in $D(W)$. Since $\dim[W/V] < \infty$, there is a finite basis of W over V which, together with the aforementioned basis of U over W, yields a basis $u_0, u_1, \ldots$ for U over V which is recursive in $D(W)$. Let $B = \{v_0, v_1, \ldots, u_0, u_1, \ldots\}$. For $k = 1$, $\underline{y} = y_1$, note that $y_1 \in V$ if and only if $\text{supp}_B\, y_1$ is a subset of $\{v_0, v_1, \ldots\}$. This can be determined recursive in $D(W)$. Suppose the proposition is known for k and $(y_1, \ldots, y_{k+1})$ is given. If $(y_1, \ldots, y_k) \in D(V)_k$, certainly $(y_1, \ldots, y_k, y_{k+1}) \in D(V)_{k+1}$. Otherwise $(y_1, \ldots, y_k)$ is independent over V. Look at the list $y_1, \ldots, y_k, u_0, u_1, \ldots$. Since V is r.e., we may effectively drop an element from this list as soon as it is determined that it is in cl_V of the preceding elements of the list. After precisely k of the u_i have been dropped, we drop no more, having guaranteed a list $y_1, \ldots, y_k u_0', u_1', \ldots$ which is a basis for U over V recursive in $D(W)$. Note $y_{k+1} \in \text{cl}\{y_1, \ldots, y_k\}$ if and only if $\text{supp}_B\, y_{k+1} \subseteq \{y_1, \ldots, y_k\}$, where $B = \{v_0, v_1, \ldots, y_1, \ldots, y_k, u_0', u_1', \ldots\}$, and this can be determined recursive in $D(W)$.

COROLLARY 3.9. *Suppose (U, cl) is a Steinitz closure system with recursive dependence.*

(i) *Suppose $V \in \mathscr{L}(U)$, $\dim[U/V] < \infty$. Then $d(D(V)) = 0$.*
(ii) *Suppose $V \in \mathscr{L}(U)$, $\dim V < \infty$. Then $d(D(V)) = 0$.*

Proof. For (i), $d(D(V)) = d(D(U)) = 0$. For (ii), $d(D(V)) = 0$ because (U, cl) has recursive dependence.

PROPOSITION 3.10. *Let (U, cl) be a Steinitz closure system with recursive dependence. Let $V \in L(U)$. Then*

(i) *$D(V)_i$ is r.e. uniformly in $i > 0$.*
(ii) *$d(D(V)_i) \leqslant d(D(V))$ uniformly in $i > 0$.*
(iii) *$d(D(V_i)) \leqslant d(D(V)_{i+1})$ for all $i > 0$.*

Proof. For (i) note $(y_1, \ldots, y_i) \in D(V)_i$ means one of $y_1 \in \text{cl}_V\, \varnothing$, $y_2 \in \text{cl}_V\{y_1\}$, $y_3 \in \text{cl}_V\{y_1, y_2\}, \ldots, y_i \in \text{cl}_V\{y_1, \ldots, y_{i-1}\}$ holds, while V is r.e. (ii) is immediate. For (iii) look at two cases.

Case 1. $\dim[U/V] < \infty$. By Corollary 3.9 $d(D(V)) = 0$, $d(D(V)_i) = 0$.

Case 2. There exist $b_0, \ldots, b_i$ independent over V. Then

$$(y_1, \ldots, y_i) \in D(V)_i \leftrightarrow \text{for all } j,\ (b_j, y_1, \ldots, y_i) \in D(V)_{i+1}.$$

One direction is obvious. For the other suppose both that $(y_1, \ldots, y_i) \notin D(V)_i$ and for all j, $(b_j, y_1, \ldots, y_i) \in D(V)_{i+1}$. Since $y_1, \ldots, y_i$ are independent over V while $y_1, \ldots, y_i, b_j$ is dependent over V, we get $b_j \in \text{cl}_V\{y_1, \ldots, y_i\}$ for all i. Since

$b_0, \ldots, b_i$ are $i+1$ in number and independent over V, $\mathrm{cl}_V\{y_1, \ldots, y_i\}$ is $\geqslant i+1$ dimensional, a contradiction.

PROPOSITION 3.11. *Suppose that (U, cl) is a Steinitz closure system of infinite dimension with recursive dependence. Let $B = \{b_i \mid i < \omega\}$ be a recursive basis for U. Then there is a 1–1 recursive function $\#: U \to \omega$ such that*

(i) $\# b_i \geqslant i$ *for all* i,

(ii) $b \in \mathrm{supp}_B\, u \to \# u \geqslant \# b$ *for all* $b \in B$, $u \in U$.

Proof. Let $B_0, B_1, \ldots$ be a recursive list of all explicit finite subsets of B. Define $\mathrm{cl}^e B_i = \mathrm{cl}\, B_i - \bigcup\{\mathrm{cl}\, B' \mid B' \subsetneq B_i\}$. Due to Proposition 2.6, $B_i \neq B_j$ implies $\mathrm{cl}^e B_i \cap \mathrm{cl}^e B_j = \varnothing$. We get U is the disjoint union of all $\mathrm{cl}^e B_i$. Let $R_0, R_1, \ldots$ be a recursive list of disjoint infinite recursive sets. Let $\#$ map B 1–1 recursively to R_0 so that $\# b_i \in [x \in R_0 \mid x \geqslant i]$. Let $\#$ map $(\mathrm{cl}^e B_i) - B_i$ 1–1 recursively to $[x \in R_{i+1} \mid$ for all $b \in B_i, x \rangle \# b]$. Then (i) is clear; for (ii), $b \in \mathrm{supp}_B\, s = B_i$ implies $\# x > \# b$ if $x \notin B_i$,

$$\# x = \# b \qquad \text{if} \quad x = b \in B_i .$$

If V is a subset of U, let $\mathrm{supp}_B\, V$ be the union of all $\mathrm{supp}_B\, v$ with v in V.

COROLLARY 3.12. *Suppose j-tuples $\underset{\sim}{x} = (x_1, \ldots, x_j)$ from U are numbered effectively so that for all i, $\# \underset{\sim}{x} > \# x_i$. Suppose V is a closed set in U and $x \in U$ and $\underset{\sim}{x} \in D(\mathrm{cl}(V \cup \{x\})) - D(V)$. Then for all $b \in \mathrm{supp}_B\, x - \mathrm{supp}_B\, V$ we have $\# \underset{\sim}{x} > \# b$.*

Proof. Since $\underset{\sim}{x}$ is dependent on $\mathrm{cl}(V \cup \{x\})$, it follows that $(\underset{\sim}{x}, x)$ is a dependent sequence over V. So we have $x \in \mathrm{cl}_V\{x_1, \ldots, x_j\}$. This yields $\mathrm{supp}_B\, x \leqslant \bigcup_{i=1}^{j} \mathrm{supp}_B\, x_i \cup \mathrm{supp}_B\, V$. For the b specified above we may then conclude that $b \in \mathrm{supp}_B\, x_i$ for some i. Then Proposition 3.11 yields $\# x_i \geqslant \# b$. The hypotheses $\# \underset{\sim}{x} > \# x_i$ then yields $\# x > \# b$.

DEFINITION 3.13. Suppose (U, cl) is a Steinitz closure system with recursive dependence. Then $V \in \mathscr{L}(U)$ is *decidable* if $D(V)$ is a recursive set.

PROPOSITION 3.14. *Let (U, cl) be a Steinitz system with recursive dependence and $V \in \mathscr{L}(U)$. Then the following are equivalent.*

(i) *V is decidable.*

(ii) *V has an independent complement $W \in \mathscr{L}(U)$.*

(iii) *V has a basis which is a recursive subset of a recursive basis of U.*

Proof. For (i) $\to$ (ii), $D(V)$ recursive gives a procedure for taking an r.e. enumeration of U and omitting an element if and only if dependent over V on preceding elements, getting an r.e. basis $u_0, u_1, \ldots$ for U over V and by Proposition 2.16 an independent complement $W = \mathrm{cl}(u_0, u_1, \ldots)$ for V in $\mathscr{L}(U)$. For (ii) implies (iii) let $u_0, u_1, \ldots$ be an r.e. basis for W and let $v_0, v_1, \ldots$ be an r.e. basis for V, then by Proposition 2.16, $\{u_0, u_1, \ldots, v_0, v_1, \ldots\}$ is an r.e. basis for U. Every r.e. basis for U is recursive (exercise), so since $\{u_0, u_1, \ldots\} \cap \{v_0, v_1, \ldots\} = \varnothing$ we have (iii).

We show that (iii) implies (i). By assumption there are disjoint r.e. independent sets $v_0, v_1, \ldots, u_0, u_1, \ldots$ such that $v_0, v_1, \ldots$ is a basis for V and $v_0, v_1, \ldots, u_0, u_1, \ldots$ is a basis for U. Apply the argument in the proof of Proposition 3.8 (for $d(D(V)) \leqslant d(D(W))$) to show $d(V) = 0$ as required.

PROPOSITION 3.15. *Let* (U, cl) *be a Steinitz closure system with recursive dependence. Then for every infinite-dimensional* $V \in \mathscr{L}(U)$ *there is an infinite-dimensional decidable* $W \subseteq V$.

Proof. Let $u_0, u_1, \ldots$ be an effective enumeration of U. Let $v_0, v_1, \ldots$ be an effective enumeration of V. Define a sequence $a_0, a_1, \ldots$ inductively as follows. Let $a_0 = v_m$ with m least such that $v_m \notin \mathrm{cl}\, \varnothing$. For $n > 0$, let $a_{2n} = v_m$ with m least such that $v_m \notin \mathrm{cl}\{a_0, \ldots, a_{2n-1}\}$. For $n \geqslant 0$, let $a_{2n+1} = u_m$ with m least such that $u_m \notin \mathrm{cl}\{a_0, \ldots, a_{2n}\}$. By construction $a_0, a_1, \ldots$ is a recursive basis for U, while $a_0, a_2, a_4, \ldots$ is an r.e. basis for an infinite-dimensional closed subset of V. Apply Proposition 3.14(iii) to conclude that $W = \mathrm{cl}\{a_0, a_2, a_4, \ldots\}$ is decidable.

There are lots of $V \in \mathscr{L}(U)$ which are recursive sets but not decidable. See Theorem 7.1.

4. Maximal Elements

Metakides and Nerode [9] and Remmel [11] used e-state arguments to produce maximal elements of $\mathscr{L}(V_\infty)$. These proofs used algebraic lemmas true for $\mathscr{L}(V_\infty)$ but false for other Steinitz closure systems such as $\mathscr{L}(F_\infty)$. We give a proof for the existence of maximal elements here which uses a new definition of e-state entirely avoiding those lemmas (Theorem 4.2). Remmel has subsequently used our new definition of e-state to handle problems arising from dependence relations which fail to obey the exchange principle. We further modify Shore's argument for $\mathscr{L}(V_\infty)$ to show that a maximal subset of a basis yields a maximal space so as to avoid the algebraic lemmas (Theorem 4.8). The theorems of this section depend on (U, cl) having recursive dependence, but do not depend on regularity.

DEFINITION 4.1. A $V \in \mathscr{L}(U)$ is *maximal* if (i) and (ii) below hold.

(i) $\dim[U/V] = \infty$.

(ii) For any $W \in \mathscr{L}(U)$ such that $W \supseteq V$, either $\dim[W/V] < \infty$ or $\dim[U/W] < \infty$.

If (U, cl) has recursive dependence we let $\mathscr{L}(U)$ be the lattice of r.e. closed subsets of U.

THEOREM 4.2. *Suppose* (U, cl) *is infinite dimensional and has recursive dependence. Then* $\mathscr{L}(U)$ *contains maximal elements* V.

Proof. Let $W_0, W_1, \ldots$ be a standard Kleene style recursive enumeration of $\mathscr{L}(U)$. Let W_k^s be the explicit finite-dimensional closed subset of W_k constructed by stage s, so $W_k^0 \subseteq W_k^1 \subseteq \cdots$ and $W_k = \bigcup_s W_k^s$. At stage s we construct an explicit finite independent subset M^s of U and an infinite recursive sequence of distinct independent elements $a_0^s, a_1^s, \ldots$ disjoint from M^s such that $M^s \cup \{a_0^s, a_1^s, \ldots\}$ is a basis for U. Here $M^0 \subseteq M^1 \subseteq \ldots$, $V = \mathrm{cl}(\bigcup_s M^s)$. It will be clearest to use a tower of windows as a visual aid. At stage s, a_j^s will be the content of the jth window from the bottom. At stage $s+1$, a finite number of windows will have their contents removed. The remaining window contents are then allowed to fall to occupy all windows. Then a_j^{s+1} is the resulting content of the jth window from the bottom. The removed contents are added to M^s to form M^{s+1}. The new feature is the definition of e-state below.

DEFINITION 4.3. The e-state of a_j^s at stage s is the $e+1$-tuple $\alpha = (\alpha_0, \ldots, \alpha_e)$ where α_n is 1 or 0 according as to whether or not

$$a_j \in \mathrm{cl}(W_n^s \cup M^s \cup \{a_0^s, \ldots, a_{j-1}^s\}).$$

These e-states are lexicographically ordered as is usual for e-states. Let P_e be the requirement that if $W_e \supseteq V$, then either $\dim[W/V] < \infty$ or $\dim[U/W] < \infty$, Let N_e be the requirement that $\lim_s a_0^s = a_0, \ldots \lim_s a_{s-1}^e = a_{e-1}$ exist (i.e., that $\dim[U/W] \geqslant e$). The priority ordering of requirements is of course N_0, P_0, N_1, $P_1, \ldots$ which reflects itself in the lexicographic ordering of e-states.

DEFINITION 4.4. P_e requires attention at stage $s+1$ if $e \leqslant s+1$ and there exists a $j > e$ such that $a_j^e, a_e^s \leqslant s+1$ and the e-state of a_e^s is less than the e-state of a_j^s.

CONSTRUCTION.

Stage 0. Let M^0 be the empty set. Let $a_0^0, a_1^0, \ldots$ be a recursive base $b_0, b_1, \ldots$ for U.

Stage $s+1$. If no e requires attention, let $M^{s+1} = M^s$ and $a_i^{s+1} = a_i$ for all i. Otherwise let $e(s+1)$ be the least e requiring attention. For that $e = e(s+1)$ let $j(s+1)$ be the least j. Remove the contents of windows numbered $e(s+1)$, $e(s+1)+1, \ldots, j(s+1)-1$, add these to M^s to get M^{s+1}, and let contents of remaining windows drop to form the a_k^{s+1}. More formally, $M^{s+1} = M^s \cup \{a_{e(s+1)}^s, \ldots, a_{j(s+1)-1}^s\}$ and $a_i^{s+1} = a_i^s$ for $i < e(s+1)$ and $a_{e(s+1)+i}^{s+1} = a_{j(s+1)+i}^s$ for all i.

LEMMA 4.5. *N_e is satisfied for all e.*

Proof. $a_0{}^s = a_0$ for all s, so N_0 is satisfied. If N_e is satisfied, let s_0 be such that $a_0 = a_0{}^s, \ldots, a_{e-1} = a_{e-1}^s$ for all $s \geqslant s_0$. Then $a_e{}^s \neq a_e^{s+1}$ for $s \geqslant s_0$ only due to its replacement by an element of higher e-state (examine the definition of e-state and of requiring attention). There are only a finite number of e-states, 2^{e+1}.

The final e-state of a_n is the e-state of $a_n{}^s$ for large s.

LEMMA 4.6. *All but a finite number of a_n have the same final e-state.*

Proof. Otherwise there is at least final e-state α possessed by infinitely many a_n and at least final e-state $\beta > \alpha$ possessed by infinitely many a_n. So there are $j > i > e$ such that the final e-state of a_i is α and the final e-state of a_j is β. Choose by Lemma 4.5 an s_0 such that for $s \geqslant s_0$, we have $a_0^{s_0} = a_0{}^s = a_0, \ldots, a_j^{s_0} = a_j{}^s = a_j$, and for all $t \leqslant j$ the e-state of $a_t{}^s$ is the final e-state of a_t, and $i \leqslant s_0 + 1$ and a_i, $a_j \leqslant s_0 + 1$. Then P_e requires attention at stage $s + 1$. So $e(s+1)$ is defined, $e(s+1) \leqslant i$, $a_{e(s+1)}^{s+1} \neq a_{e(s+1)}^s$. Since $e(s+1) \leqslant i < j$, this contradicts the choice of s_0.

LEMMA 4.7. *All P_e are satisfied.*

Proof. Let $\alpha = (\alpha_0, \ldots, \alpha_e)$ be the final e-state of all but a finite number of a_i, let k be such that for all $j > k$, a_j has e-state α. We may suppose $W_e \supseteq V$, where $V = \mathrm{cl}(\bigcup_s M^s)$.

Case 1. α_e is 1. Then for $j > k$, $a_j \in \mathrm{cl}(W_e \cup \bigcup_s M^s \cup \{\alpha_0, \ldots, a_{j-1}\})$. So $U = \mathrm{cl}(W_e \cup V \cup \{a_0, \ldots, a_k\}) = \mathrm{cl}(W_e \cup \{a_0, \ldots, a_k\})$. So $\dim[U/W_e] < \infty$.

Case 2. α_e is 0. For all $j > k$,

$$a_j \notin \mathrm{cl}(W_e \cup \textstyle\bigcup_s M^s \cup \{a_0, \ldots, a_{j-1}\}.$$

Now $a_{k+1}, a_{k+2}, \ldots$ certainly span U over $\mathrm{cl}(V \cup \{a_0, \ldots, a_k\})$. If $\dim[W_e/V] = \infty$, there would surely be a $j > k$ and a $w \in W_e$ such that $w \in \mathrm{cl}(V \cup \{a_0, \ldots, a_j\}) - \mathrm{cl}(V \cup \{a_0, \ldots, a_{j-1}\})$. By the exchange principle we get

$$a_j \in \mathrm{cl}(V \cup \{a_0, \ldots, a_{j-1}\} \cup \{w\}) \subseteq \mathrm{cl}(W_e \cup V \cup \{a_0, \ldots, a_{j-1}\}).$$

This is contrary to the choice of $j > k$. So $\dim[W_e/V] < \infty$ as required.

We modify Shore's argument that a maximal subset of a basis generates a maximal space (Metakides and Nerode [9, theorem 4.7]) so that it works for for Steinitz closure operations.

THEOREM 4.8. *Suppose (U, cl) is infinite dimensional and has recursive dependence. Let B be an r.e. basis for U, M a maximal subset of B. Then $\mathrm{cl}(M)$ is maximal in $\mathscr{L}(U)$.*

Proof. Suppose to the contrary there were a $W \in \mathscr{L}(U)$, $M \subsetneq W$, $\dim[W/M] = \dim[U/W] = \infty$. Let $w_0, w_1, \ldots$ be a recursive enumeration of W. Let i be least with $\text{supp}_B\, w_i \neq \varnothing$, put $D_0 = \text{supp}_B\, w_i$. Suppose $D_0, \ldots, D_n$ are defined. It cannot be that for all j, $\text{supp}_B\, w_j \subseteq D_0 \cup \cdots \cup D_n$, for then every w_j is in $\text{cl}(D_0 \cup \cdots \cup D_n)$, $W \subseteq \text{cl}(D_0 \cup \cdots \cup D_n)$ and W is finite dimensional. So there is a least j with $\text{supp}_B\, w_j \nsubseteq D_0 \cup \cdots \cup D_n$. Put $D_{n+1} = \text{supp}_B\, w_j - (D_0 \cup \cdots \cup D_n)$.

We get $\text{supp}_B\, w_j \subseteq \bigcup_{i=0}^{\infty} D_i$ for all j, since by construction $\bigcup_{j'<j} \text{supp}_B\, w_{j'} \subseteq D_0 \cup \cdots \cup D_n$ yields $\text{supp}_B\, w_j \subseteq D_0 \cup \cdots \cup D_{n+1}$.

Case 1. For every finite $B' \subseteq B$ there is an n such that $D_n - (M \cup B')$ has at least two elements. Let $m_0, m_1, \ldots$ be an effective enumeration of M, let $M^s = \{m_0, \ldots, m_s\}$. Let A^s be the union of M^s with the least elements of each of $D_0 - M^s, \ldots, D_s - M^s$, let $A = \bigcup_s A^s$. By construction, A contains the least element of each $D_n - M$ and omits the next to least if it exists. But being in Case 1 implies that for infinitely many n, $D_n - M$ has at least two elements. Since the D_n are disjoint, $B - A$ and $A - M$ are both infinite. This violates the assumption that M is a maximal subset of B.

Case 2. There is a finite $B' \subseteq B$ such that for all n, $D_n - (M \cup B')$ has at most one element. Let $A = B \cap \text{cl}(W \cup B')$. We show that $B - A$ and $A - M$ are both infinite, so that M is not maximal in B, a contradiction.

Suppose $B - A$ were finite. Then $\dim[U/W \cup B']$ is finite. Since B' is finite, $\dim[U/W]$ must be finite, contrary to hypothesis.

To show that $A - M$ is infinite we show (i) every $D_e \subseteq A$, (ii) $(\bigcup_{i=0}^{\infty} D_i) - M$ is infinite. For (ii) note that for all j, $\text{supp}_B\, w_j \subseteq \bigcup_{i=0}^{\infty} D_i$, so $W \subseteq \text{cl}(\bigcup_{i=0}^{\infty} D_i)$. So $\dim[W/M]$ infinite implies $\dim[\bigcup_{i=0}^{\infty} D_i/M]$ is infinite, which implies $\epsilon(\bigcup_{i=0}^{\infty} D_i) - M$ is infinite. For (i) let e be least with $D_e \nsubseteq A$, so that $D_0, \ldots, D_{e-1} \subseteq A$ (this is a possibly empty list). Then for some j, $D_e = (\text{supp}_B\, w_j) - \bigcup_{i=0}^{e-1} D_i$. Choose B' for Case 2. Then there is at most one b in $D_e - (M \cup B')$. For such a, b, $\text{supp}_B\, w_j \subseteq M \cup B' \cup (\bigcup_{i=0}^{e-1} D_i) \cup \{b\}$; so by the exchange lemma $b \in \text{cl}(M \cup B' \cup (\bigcup_{i=0}^{e-1} D_i) \cup \{w_j\}) \subseteq \text{cl}(W \cup B')$. So we always get $D_e \subseteq \text{cl}(W \cup B')$ or $D_e \subseteq B \cap \text{cl}(W \cup B') = A$ as desired.

5. Maximal Elements with No Extendible Base

In [9], [11] maximal elements V of $\mathscr{L}(V_\infty)$ are obtained such that no r.e. basis of V is extendible to an infinitely larger r.e. independent set. We obtain a corresponding result (Theorem 5.1) for recursively presented regular Steinitz closure systems by using the e-state definition for Theorem 4.2 and elaborating the requirements for Theorem 4.2. This section, unlike Section 4, requires regularity. The construction can be modified to yield results not covered by Section 6 on supermaximal elements. We do not do this here. Clause (i) of the proof in [9] is inadequate, we substitute the clause from Remmel [11].[2]

THEOREM 5.1. *Suppose* (U, cl) *is recursively presented, regular, and of infinite dimension. Then there exist maximal* V *in* $\mathscr{L}(U)$ *such that no r.e. basis of* V *can be extended to an infinitely larger r.e. independent set.*

Proof. We adopt the conventions of the proof of Theorem 4.2. In addition, let $I_0, I_1, \ldots$ be a Kleene-style recursive enumeration of all r.e. independent sets of U. Let I_e^s be the explicit finite subset of I enumerated by stage s, so $I^0 \subseteq I^1 \subseteq \cdots$ and $\bigcup_s I^s = I$. The requirements are as follows:

N_e: $\lim_s a_0^s = a_0, \ldots, \lim_s a_e^s = a_e$ exist.

P_e^1: If $W_e \supseteq V$ and $\dim[W_e/V] = \infty$, then for all but a finite number of e, we have

$$a_e \in \text{cl}(M \cup W_e \cup \{a_0, \ldots, a_{e-1}\}).$$

P_e^2: If $\text{cl}(I_e) \supseteq V$ and $\dim[\text{cl}(I_e)/V] = \infty$, then there is a $z \in M$ with

$$(\text{supp}_{I_e} z) - V \neq \varnothing.$$

To satisfy N_e is to obtain $\dim[U/V] > e + 1$. To satisfy all P_e^1 is to show V is maximal in $\mathscr{L}(U)$ (see the proof of Lemma 4.7). Why does P_e^2 imply we cannot have both $I_e \cap V$ a basis for V and $I_e - V$ infinite? Otherwise by P_e^2 there would be a $z \in M \subseteq V$ with $\text{supp}_{I_e} z - V \neq \varnothing$. From $\text{supp}_{I_e} z \nsubseteq V$ get $\text{supp}_{I_e} z \nsubseteq V \cap I_e$, or by the definition of support $z \notin \text{cl}(V \cap I_e) \subseteq V$. So $z \in V$, $z \notin V$, a contradiction.

We add to the apparatus for Theorem 4.2 movable markers $B_0, B_1, \ldots$. At each stage s, a finite number of markers B_e are used to mark elements $\hat{B}_e^s$ in $(\text{supp}_{I_e^s} z) - \text{cl}(M^s)$ for a $z \in M^s \cap \text{cl}(I_e^s)$. We shall say that P_e^s is *satisfied* at stage s if there is at least one z in $M^s \cap \text{cl}(I_e^s)$ such that $\hat{B}_e^s$ is defined and is in $(\text{supp}_{I_e^s} z) - \text{cl}(M^s)$. The intention is that if $\lim_s \hat{B}_e^s$ is defined (i.e., for some s_0, $s \geqslant s_0$ implies $\hat{B}_e^s$ is defined and $\hat{B}_e^s = \hat{B}_e^{s_0}$), then this $\hat{B}_e = \lim_s \hat{B}_e^s$ in $(\text{supp}_{I_e} z) - V$ for a $z \in M \cap \text{cl}(I_e)$, and therefore witnesses the fact that P_e^2 is met. Let supp_s be the support relative to basis $M^s \cup \{a_0^s, a_1^s, \ldots\}$ of U. Let $G^s(x)$ be the largest i such that $i \in \text{supp}_s x$ if $x \notin \text{cl}(M^s)$, $G^s(x) = -1$ if $x \in \text{cl}(M^s)$.

DEFINITION 5.2. (i) P_e^1 requires attention at stage $2s > 0$ if $e \leqslant 2s$ and there exists a $j > e$ such that $a_j^{2s-1}, a_e^{2s-1} \leqslant 2s$ and the e-state of a_e^{2s-1} is less than the e-state of a_j^{2s-1} (as given by Definition 4.3).

(ii) P_e^2 requires attention at stage $2s + 1$ if (ii)(a) and (ii)(b) below hold.

[2] A change in [11, p. 404] is needed to justify $g(x, 3s + 2) = g(y, 3s + 2)$. We incorporate this change.

(a) P_e^2 is not satisfied at stage $2s + 1$.

(b) There exist x, y in I_e^{2s+1} such that the uppermost window occupied by any element of $\text{supp}_{2s}\, x$ is above the eth window and is below the uppermost window occupied by any of the elements of $\text{supp}_{2s}\, y$. In symbols, $G^{2s}(x) > e$ and $G^{2s}(x) < G^{2s}(y)$.

CONSTRUCTION.

Stage 0. Let M^0 be the empty set. Let a_0^i be b_i where $b_0, b_1, \ldots$ is a recursive base for U. No marker B_e is in use at stage 0.

Stage $2s > 0$. If no P_e^1 requires attention at stage $2s$, let $M^{2s} = M^{2s-1}$ and $a_i^{2s} = a_i^{2s-1}$ for all i. A marker B_e is then in use at stage $2s$ if and only if in use at stage $2s - 1$, and then $\hat{B}_e^{2s} = \hat{B}_e^{2s-1}$. Otherwise let $e(2s)$ be the least e such that P_e^1 requires attention at stage $2s$, let $j(2s)$ be the least j for that $e = e(2s)$. Remove the contents of windows $e(2s), \ldots, j(2s) - 1$ (i.e., remove $a_{e(2s)}^{2s-1}, \ldots, a_{j(2s)-1}^{2s}$ from their windows), add these to M^{2s-1} to get M^{2s}, and let remaining contents of windows drop to fill all windows and to define a_k^{2s} for all k. A marker B_e is in use at stage $2s$ if and only if B_e was in use at stage $2s - 1$ and $\hat{B}_e^{2s-1} \notin \text{cl}(M^{2s})$, and then $\hat{B}_e^{2s} = \hat{B}_e^{2s-1}$.

Stage $2s + 1$. If no P_e^2 requires attention at stage $2s + 1$, let $M^{2s+1} = M^{2s}$ and let $a_i^{2s+1} = a_i^{2s}$ for all i. Then B_e is in use at stage $2s + 1$ if and only if B_e was in use at stage $2s$, and then $\hat{B}_e^{2s+1} = \hat{B}_e^{2s}$. Otherwise let $e(2s + 1)$ be the least e such that P_e^2 requires attention at stage $2s + 1$, let $x(2s + 1)$ be the least x for that $e = e(2s + 1)$, and let $y(2s + 1)$ be the least y for those $e = e(2s + 1)$ and $x = x(2s + 1)$. Let $i = G^{2s}(x)$, $j = G^{2s}(y)$. Since $\{x, y\}$ is an independent set, regularity implies $\text{cl}\{x, y\} - \text{cl}\{x\} - \text{cl}(y) \neq \varnothing$. Let $z(2s + 1)$ be the least z in $\text{cl}\{x, y\} - \text{cl}\{x\} - \text{cl}\{y\}$. Remove from the windows all a_k^{2s} such that $a_k^{2s} \in \text{supp}_{2s}\, y$ and $i < k \leqslant j$, and let window contents drop to fill windows and to define the a_k^{2s+1}. Let M^{2s+1} be obtained by adding $z(2s + 1)$ together with all the removed a_k^{2s} other than a_j^{2s}. Formally

$$M^{2s+1} = M^{2s} \cup \{z(2s + 1)\} \cup [a_k^{2s} \in \text{supp}_{2s}\, y \mid i < k < j].$$

A marker B_e is used at stage $2s + 1$ if and only if either e is $e(2s + 1)$ (in which case we put $\hat{B}_{e(2s+1)}^{2s+1} = x(2s + 1)$) or B_e was in use at stage $2s$ and $\hat{B}_e^{2s} \notin \text{cl}(M^{2s+1})$ (in which case we put $\hat{B}_e^{2s+1} = \hat{B}_e^{2s}$). This concludes the construction. We would like to verify two claims, $G^{2s}(z(2s + 1)) = G^{2s}(y(2s + 1))$ and $G^{2s+1}(y(2s + 1)) \leqslant G^{2s+1}(x(2s + 1))$. To see these first note that the exchange principle yields $y(s + 1) \leqslant \text{cl}\{x(s + 1), z(s + 1)\}$, so $\text{supp}_{2s}\, y(s + 1) \subseteq (\text{supp}_{2s}\, x(2s + 1)) \cup (\text{supp}_{2s}\, z(2s + 1))$.

Since $G^{2s}(x(2s + 1)) < G^{2s}(y(2s + 1))$, the first claim follows.

Since $z(2s + 1) \in M^{2s+1}$ and $y(2s + 1)$ is in $\text{cl}\{x(2s + 1), z(2s + 1)\}$ we get $\text{supp}_{2s+1}\, z(2s + 1) \subseteq \text{supp}_{2s+1}\, x(2s + 1)$. This verifies the second claim.

LEMMA 5.3. *N_e is met.*

Proof. Suppose N_{e-1} is met, so there is an s_0 such that $s \geqslant s_0$ implies $a_0^s = a_0, \ldots, a_{e-1} = a_{e-1}^s$, $M \cup \{a_0, \ldots, a_{e-1}\}$ independent. We examine changes $a_e^{2s} \neq a_e^{2s+1}$ at stages $2s > s_0$. By construction the least i with $a_i^{2s} \neq a_i^{2s+1}$ is the least $i > G^{2s}(x(2s+1))$ with $a_i^{2s} \in \text{supp}_{2s}\, y(2s+1)$. So $s > G^{2s}(x(2s+1))$. Also the definition of $P_{e(2s+1)}$ requiring attention (part (ii)(b)) implies that $G^{2s}(x(2s+1)) > e(2s+1)$. Combining these two inequalities yields $e(2s+1) < e$. Note that $z(2s+1) \in M^{2s+1} \cap \text{cl}(I_e^{2s+1})$ and $\hat{B}_{e(2s+1)}^{2s+1} = x(2s+1) \in (\text{supp}_{I_e^{2s+1}} z) - \text{cl}(M^{2s+1})$ so $P_{e(2s+1)}^2$ is satisfied at stage $2s+1$. If $P_{e(2s+1)}^2$ were to become unsatisfied at a stage $s' > 2s+1$, this would be because $B_{e(2s+1)}$ has to be removed as $x(2s+1)$ is in $\text{cl}(M^{s'})$. Then $\text{supp}_{2s}\, x(2s+1)$ is dependent over $M^{s'}$, hence over M. But $e > G^{2s}(x(2s+1))$ says $\text{supp}_{2s}\, x(2s+1) \subseteq M \cup \{a_0^s, \ldots, a_{e-1}\}$ and $s \geqslant s_0$ says $\text{supp}_{2s}\, x(2s+1) \subseteq M \cup \{a_0, \ldots, a_{e-1}^s\}$. So the latter is dependent, contrary to hypothesis. Thus there are at most e values $2s > s_0$ such that $a_e^{2s+1} \neq a_e^{2s}$, one for each value of $e(2s+1) < e$. So there is a stage $s_1 > s_0$ such that for all $2s \geqslant s_1$, $a_e^{2s+1} = a_e^{2s}$. On the other hand, looking at the maximal space construction of Definition 5.2(i) we see that if $2s > s_1$, then $a_e^{2s} \neq a_e^{2s-1}$ only when the e-state of a_e^{2s} exceeds the e-state of a_e^{2s-1}. The e-states are 2^{e+1} in number. So $\lim_s a_e^s = a_e$ exists.

LEMMA 5.4. *P_e^1 is met.*

Proof. Similar to Lemma 4.7.

LEMMA 5.5. *P_e^2 is met.*

Proof. For an induction, assume that for all $i < e$, P_i^2 is met. Then there is a stage s_0 such that for all $2s+1 \geqslant s_0$, if $e(2s+1)$ is defined, then $e(2s+1) \geqslant e$. Let supp x be the support of x relative to $M, a_0, a_1, \ldots$. Let $G(x) = \lim_s G^s(x) =$ least i with $x \in \text{cl}(M \cup \{a_0, \ldots, a_i\})$ for $x \notin \text{cl}(M)$, $G(x) = -1$ if $x \in \text{cl}(M)$. Now suppose (to verify P_e^2) that $\text{cl}(I_e) \supseteq V$ and $\dim[\text{cl}(I_e)/V] = \infty$, where $V = \text{cl}(M)$. Since I_e is infinite dimensional over $V = \text{cl}(M)$, there is an $x \in I_e$ with $x \notin \text{cl}(M \cup \{a_0, \ldots, a_e\})$, so $G(x) > e$. For the same reason there is a $y \in I_e$ with $y \notin \text{cl}(M \cup \{a_0, \ldots, a_{G(x)}\})$, so $G(y) > G(x)$. Then $\{x, y\}$ is independent over M, $G(y) > G(x) > e$. Now choose an $s_1 > s_0$ such that for all $s \geqslant s_1$,

(1) $a_i = a_i^s$ for all $i \leqslant G(y)$,

(2) $G^s(y) = G(y)$, $G^s(x) = G(x)$,

(3) $x, y \in I_e^s$.

Suppose P_e^2 were not satisfied at stage $2s+1 > s_1$. Certainly we have arranged it so that P_e^2 will then require attention at stage $2s+1$. Since $2s+1 > s_0$ certainly $e(2s+1)$ is e, and by construction $a_{G^{2s}(y(2s+1))}^{2s+1} \neq a_{G^{2s}(y(2s+1))}^{2s}$. Since $2s+1 > s_1$, this says $a_{G(y)} \neq a_{G(y)}$, a contradiction. So P_e^2

is satisfied at every stage $2s + 1 > s_1$. There was a last stage $2s + 1$ when B_e was introduced as a marker and $z = z(2s + 1), y = y(2s + 1), x = x(2s + 1) = \hat{B}_e^{2s+1}$ were introduced. By construction we had $\hat{B}_e^{2s+1} = x \in (\mathrm{supp}_{I_e^{2s+1}} z) - \mathrm{cl}(M^s)$. Since B_e is never moved, we get $x \in (\mathrm{supp}_{I_e} z) - V$. By construction $z \in M^{2s+1} \cap \mathrm{cl}(I_e^{2s+1})$, so $z \in V \cap \mathrm{cl}(I_e)$. So P_e^2 is met.

6. Supermaximal Elements

We extend the construction of supermaximal elements from the vector space case of Kalantari and Retzlaff [8], improving the results so that the supermaximal element is recursive as a set. We require regularity.

Definition 6.1. A $V \in \mathscr{L}(U)$ is supermaximal if (i) $\dim[U/V] = \infty$, (ii) for all $W \in \mathscr{L}(U)$, $W \supseteq V$ and $\dim[W/V] = \infty$ imply $W = U$.

Note that (ii) can be replaced by (ii′): for all $W \in \mathscr{L}(U)$, if

$$\dim[\mathrm{cl}(W \cup V)/V] = \infty,$$

then $\mathrm{cl}(W \cup V) = U$. This is the form we use to translate (ii) into a requirement.

Theorem 6.2. *Suppose* (U, cl) *is recursively presented, of infinite dimension, and regular. Then there are supermaximal* $V \in \mathscr{L}(U)$ *which are recursive as subsets of* U.

(*Note that in* (ω, cl) *where* $\mathrm{cl}(A) = A$ *for all* $A \subseteq \omega$, *there are no supermaximal elements. Of course* (ω, cl) *is not regular.*)

Proof. Let $b_0, b_1, \ldots$ be a recursive basis for U. Let $W_0, W_1, \ldots$ be a recursive enumeration of $\mathscr{L}(U)$ of the standard sort. Let V^s and W^s be the explicit finite-dimensional subspaces of V and W, respectively, constructed by stage s. We keep track of an infinite recursive sequence $a_0^s, a_1^s, \ldots$ independent over V^s at stage s. Then V will be $\bigcup_s V^s$ and the limits $a_k = \lim_s a_k^s$ will be an infinite independent set over V to satisfy (i) of Definition 6.1. The requirements which must be met are as follows.

R: V is a recursive subset of U.
$P_{\langle e,n\rangle}$: If $\dim[\mathrm{cl}\,(W_e \cup V) : V] = \infty$, then $b_n \in \mathrm{cl}(W_e \cup V)$.
$N_{\langle e,n\rangle}$: $\lim_s a_{\langle e,n\rangle} = a_{\langle e,n\rangle}$ exists.

In the usual language, the priority ordering is $R, N_0, P_0, N_1, P_1, \ldots$.

Definition 6.3. $P_{\langle e,n\rangle}$ *requires attention* at stage s if (i) and (ii) below hold.

(i) $b_n \notin \mathrm{cl}(W_e^s \cup V^s)$.

(ii) There is an $x \in W_e{}^s$ such that

$$x \notin \mathrm{cl}[V^s \cup \{a_0{}^s,\ldots, a^s_{\langle e,n\rangle}\} \cup \{b_n\}].$$

CONSTRUCTION OF V

Stage 0. Let V^0 be $\{0\}$, let $a_0{}^s, a_1{}^s,\ldots$ be $b_0, b_1,\ldots$.

Stage $s \geqslant 0$. If no $P_{\langle e,n\rangle}$ requires attention at stage s, let $V^{s+1} = V^s$, let $a_i^{s+1} = a_i{}^s$ for all i. Otherwise, there is a least pair $\langle e, n\rangle$ requiring attention. For that pair $\langle e, n\rangle$ let x be the least one satisfying Definition 6.3(ii). List all those $u \in U$ with $u < s$ which are not in V^s as $u_1,\ldots, u_t$. By combining (i) and (ii) in Definition 6.3 we see that $\{x, b_n\}$ is a two-element set independent over V^s. The assumption (U, cl) regular and Proposition 2.12 imply that (U, cl_{V^s}) is also regular. The definition of regularity implies that there is a y (which we choose least) such that

$$y \in \mathrm{cl}_{V^s}\{x, b_n\} - (\mathrm{cl}_{V^s}\{x\} - \mathrm{cl}_{V^s}\{b_n\} - \mathrm{cl}_{V^s}\{u_1\} - \cdots - \mathrm{cl}_{V^s}\{u_t\}).$$

Define V^{s+1} as $\mathrm{cl}(V^s \cup \{y\})$. It remains to define the a_i^{s+1}. Let a_0^{s+1} be $a^s_{m_0}$ where m_0 is least such that $a^s_{m_0} \notin V^{s+1}$. For an induction, define a^{s+1}_{k+1} as $a_m{}^s$ where m is least such that $a_m{}^s \notin \mathrm{cl}(V^{s+1} \cup \{a_0^{s+1},\ldots, a_k^{s+1}\})$. Finally we say $P_{\langle e,n\rangle}$ received attention at stage s (using x and y). This completes the construction of V.

LEMMA 6.4. *R is met.*

Proof. To conclude V is a recursive subset of U, it suffices to show that for all $u \in U$, $u \in V$ implies $u \in V^{u+1}$; for $V^0 \subseteq V^1 \subseteq \cdots$ and the V^s are explicitly given. So we must show that for $u < s$, if $u \notin V^s$, then $u \notin V^{s+1}$. Suppose indeed $u < s$, $u \notin V^s$. If no $P_{\langle e,n\rangle}$ receives attention at stage s, then $V^s = V^{s+1}$ and so $u \notin V^{s+1}$. If a $P_{\langle e,n\rangle}$ receives attention at stage s (using x and y), by construction the given u is one of $u_1,\ldots, u_t$. So we must show $u_1,\ldots, u_t \notin V^{s+1}$. Were $u_i \in V^{s+1} = \mathrm{cl}(V^s \cup \{y\})$, then since the choice of y in the construction ensures $y \notin V^s$, we may apply the exchange principle and get $y \in \mathrm{cl}(V^s \cup \{x_i\}) = \mathrm{cl}_{V^s}\{x_i\}$. This contradicts the choice of y in the construction. So $u_i \notin V^{s+1}$, $u \notin V^{s+1}$, and R is met.

This proof has little to do with supermaximality and allows one to get recursive *sets* satisfying many different kinds of requirements.

LEMMA 6.5. *Suppose $P_{\langle e,n\rangle}$ receives attention at some stage s. Then $P_{\langle e,n\rangle}$ is met, and $P_{\langle e,n\rangle}$ never receives attention at any stage $s' > s$.*

Proof. Suppose $P_{\langle e,n\rangle}$ received attention at stage s' (using x and y). The choice of y in the construction guarantees that $y \in \mathrm{cl}(V^s \cup \{x\} \cup \{b_n\}) - \mathrm{cl}(V^s \cup \{x\})$. Apply the exchange principle to conclude that $b_n \in \mathrm{cl}(V^s \cup \{x\} \cup \{y\})$. But V^{s+1} is $\mathrm{cl}(V^s \cup \{y\})$ and $x \in W_e{}^s$, so $b_n \in \mathrm{cl}(V^{s+1} \cup W_e^{s+1})$. This gives $b_n \in \text{cl}(V \cup W_e)$, so $P_{\langle e,n\rangle}$ is met. In addition this gives $b_n \in \text{cl}(V^{s'} \cup W_e^{s'})$ for all $s' > s$, so clause (i) in Definition 6.3 is never satisfied for s replaced by an $s' > s$. So $P_{\langle e,n\rangle}$ never receives attention at any stage $s' > s$.

LEMMA 6.6. *If $P_{\langle e,n\rangle}$ receives attention at stage s, then $a_0^{s+1} = a_0^s,\ldots, a_{\langle n,e\rangle}^{s+1} = a_{\langle n,e\rangle}^s$.*

Proof. The definition of a_k^{s+1} shows that we need prove only that $a_0^s,\ldots, a_{\langle n,e\rangle}^s$ is independent over $V^{s+1} = \text{cl}(V^s \cup \{y\})$. If $a_0^s,\ldots, a_{\langle n,s\rangle}^s$ are supposed dependent over $\text{cl}(V^s \cup \{y\})$, then certainly $a_0^s,\ldots, a_{\langle n,e\rangle}^s, y$ is dependent over V^s. But $a_0^s,\ldots, a_{\langle n,e\rangle}^s$ is independent over V^s, so it follows that $y \in \text{cl}(V^s \cup \{a_0^s,\ldots, a_{\langle n,e\rangle}^s\})$. The choice of y in the construction ensures

$$y \in \text{cl}(V^s \cup \{x\} \cup \{b_n\}) - \text{cl}(V^s \cup \{b_n\}).$$

Apply the exchange principle and get

$$x \in \text{cl}(V^s \cup \{y\} \cup \{b_n\}).$$

Since $y \in \text{cl}(V^s \cup \{a_0^s,\ldots, a_{\langle n,e\rangle}^s\})$, we now get $x \in \text{cl}(V^s \cup \{a_0^s,\ldots, a_{\langle n,e\rangle}^s\} \cup \{b_n\})$. This contradicts condition (ii) of Definition 6.3.

LEMMA 6.7. *$N_{\langle e,n\rangle}$ is met.*

Proof. a_0^s never changes. Suppose, for an induction, that for $s \geqslant s_0$ we have $a_0^s = a_0,\ldots, a_{\langle e,n\rangle-1}^s = a_{\langle e,n\rangle-1}$. Then $a_{\langle e,n\rangle}^s \neq a_{\langle e,n\rangle}^{s+1}$ for an $s > s_0$ according to Lemma 6.6 only if a $P_{\langle e',n'\rangle}$ receives attention at stage s and $\langle e', n'\rangle < \langle e, n\rangle$. By Lemma 6.5 this happens at most $\langle e, n\rangle$ times, at most once for each $P_{\langle e',n'\rangle}$ with $\langle e', n'\rangle < \langle e, n\rangle$.

LEMMA 6.8. *$P_{\langle e,n\rangle}$ is met.*

Proof. Otherwise there is a least $\langle e, n\rangle$ such that $\dim[\text{cl}(W_e \cup V)/V] \doteq \infty$ and $b_n \notin \text{cl}(W_e \cup V)$. By Lemma 6.5 we know $P_{\langle e,n\rangle}$ never receives attention at any stage. By Lemma 6.6 there is an s_0 such that for all $s \geqslant s_0$ we have $a_0^s = a_0,\ldots, a_{\langle e,n\rangle} = a_{\langle e,n\rangle}^s$. By Lemma 6.5, s_0 may be chosen so that for no $\langle e', n'\rangle < \langle e, n\rangle$ does $P_{\langle e',n'\rangle}$ receive attention at any stage $s \geqslant s_0$. Since $\dim[\text{cl}(W_e \cup V)/V] = \infty$, we get $\dim[W_e/V] = \infty$, so by Proposition 2.8(iii) we get $\dim[W_e/\text{cl}(V \cup \{a_0,\ldots, a_{\langle e,n\rangle}\} \cup \{b_n\})] = \infty$. All this is to get an $x \in W_e$ such that $x \notin \text{cl}[V \cup \{a_0,\ldots, a_{\langle e,n\rangle}\} \cup \{b_n\}]$.

Let $s \geqslant s_0$ be chosen so that $x \in W_e^s$. Certainly by the above $b_n \notin \text{cl}(W_e^s \cup V^s)$ and $x \in W_e^s$ and $x \notin \text{cl}[V^s \cup \{a_0^s,\ldots, a_{\langle e,n\rangle}^s\} \cup \{b_n\}]$. So $P_{\langle e,n\rangle}$ requires attention at stage s. By the choice of $s > s_0$, $P_{\langle e,n\rangle}$ receives attention at stage s, contrary to hypothesis. So $P_{\langle e,n\rangle}$ is met.

7

We show how to lift the main theorem of Shore [16] to regular Steinitz closure systems. This method may be used to control the dependence degree of supermaximal elements of the sort constructed in Section 6, but we omit such development here.

THEOREM 7.1. *Suppose* (U, cl) *is recursively presented, of infinite dimension and regular. Let* $A_0, A_1, \ldots$ *be a sequence of sets of integers such that* (i) A_i *is r.e. uniformly in* i, $i > 0$. (ii) $d(A_i) \leqslant d(A_0)$ *uniformly in* i, $i > 0$. (iii) $d(A_i) \leqslant d(A_{i+1})$ *for* $i > 0$. *Then there is a* $V \in \mathscr{L}(U)$ *such that* $d(D(V)_i) = d(A_i)$ *for all* $i > 0$ *and* $d(D(V)) = A_0$.

Proof. Let B be an r.e. base for U. For each pair (n, k) in $\omega \times \omega$, recursively pick an explicit finite subset $B_k{}^n$ of B, of cardinality k if $k > 0$ and of cardinality $n + 1$ if $k = 0$. Do this in such a way that distinct pairs are assigned disjoint sets. Since (U, cl) is regular and recursively presented, we can compute an $x_k{}^n \in \mathrm{cl}(B_k{}^n) - \bigcup \{\mathrm{cl}\, B' \mid B' \not\subseteq B_k{}^n\}$ and put $V = \mathrm{cl}\{x_k{}^n \mid n \in A_k\}$. Now regard $B_k{}^n$ as a k-tuple of elements of U if $k > 0$, an $n + 1$-tuple of elements of U if $k = 0$. Even this small amount of care yields

$$n \in A_k \leftrightarrow B_k{}^n \in D(V)_k \qquad \text{for} \quad k > 0,$$
$$n \in A_0 \leftrightarrow B_0{}^n \in D(V) \qquad \text{(exercise).}$$

So we get $d(A_k) \leqslant d(D(V_k))$ uniformly in $k > 0$, $d(A_0) \leqslant d(D(V))$. To obtain the opposite inequalities a more careful choice of $x_k{}^n$ is required. First, modify the choice of $B_k{}^n$ if necessary so that whenever $b \in B_k{}^n$, then $\# b > n$. Now we do the actual construction. Each stage s is divided into substages k, $k = 1, \ldots, s$. We will construct a finite explicit subset $I^{s,k}$ of U before stage s, substage k. Let $V^{s,k} = \mathrm{cl}(I^{s,k})$ and finally let $V = \bigcup_{s,k} V^{s,k}$. Let $A_k{}^s$ be the finite subset of A_k enumerated by stage s, arranged so that for all k and s, we have that $A^{s+1} - A_k{}^s$ has at most one member. Then stage s of the construction goes as follows.

Substage 1 *of Stage s.* Suppose that $n_1 \in A_1^{s+1} - A_1{}^s$. Let $x_1^{n_1}$ be the unique member of $B_1^{n_1}$. Put $I^{s,2} = I^{s,1} \cup \{x_1^{n_1}\}$.

Substage k of stage s with $1 < k < s$. Suppose that $n_k \in A_k^{s+1} - A_k{}^s$. Let $\underline{x}^1, \ldots, \underline{x}^t$ be those j-tuples $\underline{x}$ such that $j < k$ and $\# \underline{x} \leqslant n_1$ and $\underline{x} \notin D(V^{s,k})$. Since $\underline{x}$ is a j-tuple it may be written $(\underline{x}_1, \ldots, \underline{x}_j)$. Let $\mathrm{cl}_{V^{s,k}}\, \underline{x}$ be $\mathrm{cl}(V^{s,k} \cup \{\underline{x}_i \mid 1, \ldots, j\})$. Now $B_k^{n_k}$ is a k element set independent over $V^{s,k}$ by construction. Recursive presentability and regularity of (U, cl) imply that we may compute an $x_k^{n_k}$ in

$$\mathrm{cl}_{V^{s,k}}\, B_k^{n_k} - \bigcup \{\mathrm{cl}_{V^{s,k}}\, B' \mid B' \subsetneq B_k^{n_k}\} - \bigcup_{i=1}^{t} \mathrm{cl}_{V^{s,k}}\, \underline{x}^i.$$

Finally define $I^{s,k+1} = I^{s,k} \cup \{x_k^{n_k}\}$.

Substage s of stage s. Suppose that $n_s \in A_0^{s+1} - A_0{}^s$. Let $\underset{\sim}{x}^1, \dots, \underset{\sim}{x}^t$ be those j-tuples $\underset{\sim}{x}$ with $j \leqslant n_s$ and $\# x \leqslant n_1$ and $\underset{\sim}{x} \notin D(V^{s,s})$. Now $B_0^{n_s}$ is an $n_s + 1$-element set independent over $V^{s,s}$ by construction. Recursive presentability and regularity of (U, cl) imply that we can compute an $x_k^{n_s}$ in

$$\text{cl}_{V^{s,s}} B_0^{n_s} - \bigcup \{\text{cl}_{V^{s,s}} B' : B' \subsetneq B_0^{n_s}\} - \bigcup_{i=1}^{t} \text{cl}_{V^{s,s}} \underset{\sim}{x}^i.$$

Finally define $I^{s+1,1} = I^{s,s} \cup \{x_k^{n_s}\}$. This completes the contruction.

LEMMA 7.2. *Suppose that $j \geqslant 1$ and for all $a \leqslant j$, we have $a \in A_0^{s_0} \leftrightarrow a \in A_0$. Suppose that $\underset{\sim}{x}$ is a j-tuple from U. Suppose $s_1 > s_0$ is such that for all $a < \# \underset{\sim}{x}$ and all $i \leqslant j$, we have $a \in A_i^{s_1} \leftrightarrow a \in A_i$. Then for all $s > s_1$, we have that $n_k \in A_k^{s+1} - A_k{}^s$ and $\underset{\sim}{x} \notin D(V^{s,k})$ imply $\underset{\sim}{x} \notin D(\text{cl}(V^{s,k} \cup \{x_k^{n_k}\})$.*

Proof. Since $i = 1$ is an $i \leqslant j$, for all $a < \# x$ we have $a \in A_1^{s_1} \leftrightarrow a \in A_1$. So if $s > s_1$ and $n_1 \in A_1^{s+1} - A_1{}^s$ we may conclude $n_1 \geqslant \# x$.

Case 1. $s > k > j$. At stage s, substage k we have $\underset{\sim}{x}$ a j-tuple with $j < k$, $\# \underset{\sim}{x} \leqslant n_1$, and $x \notin D(V^{s,k})$. So $\underset{\sim}{x}$ is one of $\underset{\sim}{x}^1, \dots, \underset{\sim}{x}^t$. By construction this implies $x_k^{n_k} \notin \text{cl}_{V^{s,k}} \underset{\sim}{x}$. Were $\underset{\sim}{x}$ in $D(\text{cl}(V^{s,k} \cup \{x_k^{n_k}\})$ then $\underset{\sim}{x}$, $x_k^{n_k}$ would be a dependent sequence over $V^{s,k}$. By assumption, $\underset{\sim}{x}$ is independent over $V^{s,k}$, so we could conclude $x_k^{n_k} \in \text{cl}_{V^{s,k}} \underset{\sim}{x}$, contrary to what was proved above.

Case 2. $s = k > j$. The choice of s_0 ensures that for all $a \leqslant j$, $a \in A_0^{s_0} \leftrightarrow a \in A_0$. So $s > s_0$ and $n_s \in A_0^{s+1} - A_0{}^s$ imply $n_s > j$. Combining this with the already known $n_1 \geqslant \# x$ and the assumed $\underset{\sim}{x} \notin D(V^{s,s})$ implies that at stage s, substage s, $\underset{\sim}{x}$ is one of $\underset{\sim}{x}^1, \dots, \underset{\sim}{x}^t$. Just as in Case 1 we can go on to conclude

$$\underset{\sim}{x} \notin D(\text{cl}(V^{s,s} \cup \{x_s^{n_s}\}).$$

Case 3. $k < j$. For $s_1 > s_0$, for all $i \leqslant j$ and all $a < \# x$ we know $a \in A_i{}^s \leftrightarrow a \in A_i$. By assumption, k is an $i \leqslant j$ and $n_k \in A_k^{s+1} - A_k{}^s$ for an $s > s_1$, so we may conclude $n_k \geqslant \# \underset{\sim}{x}$. Now if $b \in \text{supp}_B\, x_k^{n_k} = B_k^{n_k}$, the choice of numbering $\#$ implies $\# b > n_k$. Combining, we get $\# b > \# \underset{\sim}{x}$ for all $b \in \text{supp}_B\, x_k^{n_k}$. If we had $\underset{\sim}{x} \in D(\text{cl}(V^{s,k} \cup \{x_k^{n_k}\})$, then Corollary 3.12 would imply that every $b \in \text{supp}_B\, x_k^{n_k} - \text{supp}_B\, V^{s,k}$ has $\# \underset{\sim}{x} > \# b$. Combining with the above, we would conclude $\text{supp}_B\, x_k^{n_k} \subseteq \text{supp}_B\, V^{s,k}$. This is false since the left-hand side is $B_k^{n_k}$, the right-hand side is the union of certain other $B_{k'}^{n'}$, and the $B_k{}^n$ are disjoint and nonempty. So $\underset{\sim}{x} \notin D(\text{cl}(V^{s,k} \cup \{x_k^{n_k}\}))$ as required. This concludes the proof of Lemma 7.2. We return to complete the proof of Theorem 7.1. Suppose a $j \geqslant 1$ is given. How do we determine for $\underset{\sim}{x}$ a j-tuple from U whether or not $\underset{\sim}{x} \in D(V)_j$, recursive in A_j? For the given j, we may suppose s_0 for Lemma 7.2 given. Now recursive in A_j (since $d(A_0) \leqslant d(A_1) \leqslant \cdots \leqslant d(A_j)$) we can compute $s_1 > s_0$ for Lemma 7.2. Then $\underset{\sim}{x} \in D(V)_j \leftrightarrow \underset{\sim}{x} \in D(V^{s_1+1,1})_j$. But given s, $V^{s_1+1,1}$

is an explicit finite dimensional closed set with $d(D(V^{s_1+1,1})) = 0$. So we have $d(D(V)_j) \leqslant d(A_j)$. To see that $d(D(V)) \leqslant d(A_0)$, observe that above s_1 is computed from A_j uniformly, which can be computed from A_0, so $d(D(V)_j) \leqslant d(A_0)$ uniformly in j, or $d(D(V)) \leqslant d(A_0)$ as required.

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