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# Noether and some other dynamical symmetries in Kantowski–Sachs model

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**Abstract**

The forms of coupling of the scalar field with gravity, appearing in the induced theory of gravity, and that of the potential are found in the Kantowski–Sachs model under the assumption that the Lagrangian admits Noether symmetry. The form thus obtained makes the Lagrangian degenerate. The constrained dynamics thus evolved due to such degeneracy has been analysed and a solution has also been presented which is inflationary in behaviour. It has further been shown that there exists other technique to explore the dynamical symmetries of the Lagrangian simply by inspecting the field equations. Through this method Noether along with some other dynamical symmetries are found which do not make the Lagrangian degenerate.


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## 1. Introduction

The theory of induced gravity has been introduced [1] in an attempt to construct a theory of gravity consistent with quantum field theory in curved space time. Later it was found to be a strong candidate in several unified theories [2]. There, in the weak energy limit, the Einstein–Hilbert action appears as an effective action induced by the quantum properties of the vacuum state of the matter field. The beauty of the theory lies in the fact that it identifies the inflaton with the scalar field inducing the Newtonian gravitational constant ($G_N$) and the cosmological constant ($\Lambda$). On the other hand in the context of nonminimally coupled scalar field theory, it has been observed [3] that inflation is impossible for large positive coupling constant $\xi$. Further It has been found to overcome the shortcomings of the old inflationary theory viz., the graceful exit problem [4] and the long-standing problem of density perturbation [5]. For this $\xi$ is allowed to take arbitrarily large negative value keeping the Newton's gravitational constant positive all along the evolution of the universe. The theory has also been found [6] to preserve the generic features of Vilenkin [7] and Hartle–Hawking [8] wave functions. Finally, it has also been shown that [9,10] the theory admits wormhole solutions both for real and imaginary fields. Kalara et al. [11] have shown that the action for a nonminimally coupled theory can generate the the-

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ory of induced gravity (along with the $R2$ theory of gravity and the Brans–Dicke theory) under a conformal transformation.

The action for the induced theory of gravity and that of nonminimally coupled theory can be written together in the following form.

$$A = \int d^4X \sqrt{-g}\,\bigl[f(\phi)R - \tfrac{1}{2}\phi_{,\mu}\phi^{,\mu} - V(\phi)\bigr]. \quad (1)$$

For induced theory $f(\phi)$ is usually chosen to be $\epsilon\phi2$ while the usual choice is $(1 - \xi\phi2)$ for nonminimal coupling. Despite such a wide range of successful applications of both the theories, the actual form of coupling of the matter field with gravity $(f(\phi))$ is not known a priori. The above choices are rather ad hoc. Capozziello et al. [12] made an attempt to find the form of such coupling under the assumption that the Lagrangian admits Noether symmetry. However, it further restricts the form of the potential for the scalar field. In the Robertson–Walker model they have observed that [12] $k \neq 0$ imposes strong constraints in the form of coupling and the potential. The form of coupling thus obtained makes the Lagrangian degenerate and the Newton's gravitational constant negative. Further the form of the potential was found to be sixth order in the scalar field $\phi$, for which only trivial solution is admissible.

When the Hessian determinant

$$W = \sum \left| \frac{\partial 2L}{\partial \dot{q}_i \partial \dot{q}_j} \right|$$

vanishes the Lagrangian becomes degenerate. It imposes a constraint in the sense that the Legendre transformation does not exist and hence the Hamiltonian of the system cannot be defined unless such constraints are analysed properly. In the domain of Lagrangian dynamics the constraint implies more number of degrees of freedom than the number of field equations. This means that one has to make certain assumptions to obtain exact solutions. However, such degeneracy does not in any case lead to trivial solutions. In a recent communication [13], it has been shown that the existence of 'only trivial solutions' is not due to the presence of degeneracy in the Lagrangian, rather it is due to the existence of a potential in the form $V(\phi) = \Lambda\phi6$ (obtained under the assumption that the Lagrangian admits Noether symmetry) that does not satisfy the field equations. This is a striking feature and perhaps not been encountered earlier. The fact that the Noether symmetry of the Lagrangian restricts the form of the potential in such a manner that it does not satisfy the field equations, is yet to be explained. However, for a particular choice of the coupling parameter in the usual form of the induced theory of gravity, viz., $f(\phi) = \epsilon\phi2$, Lagrangian remains nondegenerate for $\epsilon \neq -1/12$ and the Newton's gravitational constant is positive for $\epsilon > 0$. This choice along with a quartic form of the potential leads to [13] certain dynamical symmetries of the Lagrangian along with a conserved current, which are not Noether symmetries. We like to emphasize that the symmetry thus obtained cannot be explored by the standard technique of finding dynamical symmetries via Noether theorem. Rather, it is found simply from a combination of the field equations. This is in sharp construct with the longstanding claim that all the dynamical symmetries of a physical system are Noether symmetries.

Motivated by the above result, our attempt is now to find the form of coupling $f(\phi)$ in anisotropic models, under the same assumption that the Lagrangian admits Noether symmetry. In the present Letter the Kantowski–Sachs metric has been taken under consideration. In this model, once again, we observe that Noether symmetry exists at the cost of imposing degeneracy in the Lagrangian. Further the Newton's gravitational constant becomes negative which of course is not a desirable feature of any theory. However, the potential this time turns out to be quartic in the scalar field $\phi$ which satisfies the field equations, in contrast to the Robertson–Walker model [13]. It has been pointed out that the nonminimal form of coupling does not admit Noether's theorem. The constraint imposed by the degeneracy has been analysed in the domain of Lagrangian dynamics, which has been found to yield an excess number of the degrees of freedom to the field equations. The most interesting aspect of the present work is that, instead of going through the detailed lengthy calculation of finding Noether symmetry, it has been shown to obtain the same, just by inspecting the field equations. In this second method the coupling parameter has been chosen in the usual form $f(\phi) = \epsilon\phi2$ of induced gravity as in the Robertson–Walker model [13]. It has been observed that the quartic form of the potential yields Noether symmetry for $\epsilon = -1/12$. However, for any other arbitrary $\epsilon$ there exists yet another symmetry along with a conserved current, which keeps the Lagrangian nondegenerate. This is surprising that such an inherent symmetry of the system cannot be explored via Noether theorem. Hence we conclude that there exists dynamical symmetries of a system other than Noether symmetry. It has further been emphasized that even if one chooses the form of the coupling as that of nonminimally coupled scalar fields viz., $(1 - \xi f(\phi))R$ the result is the same, i.e., Noether symmetry leads to identical form of coupling in both the cases. In a nutshell, Noether symmetry transforms a nonminimally coupled theory to Induced theory of gravity that makes the Lagrangian degenerate and Newton's gravitational constant negative.

This Letter is organised as follows. In Section 2, the field equations are obtained from the action principle and the condition for which the Hessian determinant vanishes yielding a degenerate Lagrangian has been found. The form of coupling $f(\phi)$, the potential $V(\phi)$ and the conserved current are then obtained by studying the Noether symmetry. It has been found that for the existence of such symmetry, the Lagrangian turns out to be degenerate. It has also been noticed that solution in the form corresponding to the nonminimally coupled scalar fields viz.,

$$f(\phi) = 1 - \xi\phi 2$$

is not admissible by the Noether symmetry. In Section 3, the constraint imposed by the degenerate system is analysed in the domain of Lagrangian dynamics, whose outcome is a pair of field equations in first order for three degrees of freedom. This implies that one has to make one physically reasonable assumption to obtain nontrivial solutions. A solution has also been presented at the end of this section. In Section 4, it has been shown that the above mentioned symmetry could have been obtained quite easily just by inspecting the field equations. Further it has also been found that some other dynamical symmetry for the system still exists, that cannot be obtained by applying the Noether theorem and that does not make the Lagrangian degenerate. Thus it is confirmed that not all the dynamical symmetries hidden in a Lagrangian could be obtained by the application of Noether's theorem. Concluding remarks are presented in Section 5.

## 2. Noether symmetry in Kantowski–Sachs model

We start with the action (1), which for the Kantowski–Sachs metric

$$ds\,2 = -dt\,2 + a2\,dr\,2 + b2(d\theta\,2 + \sin 2\theta\,d\phi\,2) \quad (2)$$

reduces to

$$A = 4\pi \int \big[-4f'ab\dot{b}\dot{\phi} - 2f'b2\dot{a}\dot{\phi} - 4fb\dot{a}\dot{b} - 2fa\dot{b}^2 + 2fa + \tfrac{1}{2}ab2\dot{\phi}^2 - ab2V(\phi)\big]\,dt + \text{surface term}. \quad (3)$$

Field equations are

$$2\frac{\ddot{b}}{b} + \frac{f'}{f}\ddot{\phi} + \frac{f''}{f}\dot{\phi}^2 + 2\frac{f'}{f}\frac{\dot{b}}{b}\dot{\phi} + \frac{\dot{b}^2}{b2} + \frac{\dot{\phi}^2}{4f} + \frac{1}{b2} - \frac{V(\phi)}{2f} = 0, \quad (4)$$

$$\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{f'}{f}\ddot{\phi} + \frac{\dot{a}\dot{b}}{ab} + \frac{f'\dot{a}}{fa}\dot{\phi} + \frac{f'\dot{b}}{fb}\dot{\phi} + \frac{f''}{f}\dot{\phi}^2 + \frac{\dot{\phi}^2}{4f} - \frac{V(\phi)}{2f} = 0, \quad (5)$$

$$\frac{\ddot{a}}{a} + 2\frac{\ddot{b}}{b} + 2\frac{\dot{a}\dot{b}}{ab} + \frac{\dot{b}^2}{b2} - \frac{\ddot{\phi}}{2f'} - \left(\frac{\dot{a}}{a} + 2\frac{\dot{b}}{b}\right)\frac{\dot{\phi}}{2f'} + \frac{1}{b2} - \frac{V'(\phi)}{2f'} = 0, \quad (6)$$

$$\frac{\dot{b}^2}{b2} + \frac{f'\dot{a}}{fa}\dot{\phi} + 2\frac{f'\dot{b}}{fb}\dot{\phi} - \frac{\dot{\phi}^2}{4f} + 2\frac{\dot{a}\dot{b}}{ab} + \frac{1}{b2} - \frac{V(\phi)}{2f} = 0, \quad (7)$$

where overdot and prime represent derivatives with respect to time and $\phi$, respectively. The Hessian determinant,

$$W = \sum \left| \frac{\partial 2L}{\partial \dot{q}_i \partial \dot{q}_j} \right|$$

turns out to be,

$$W = -16\pi f a b4\big(3f'^2 + f\big). \quad (8)$$

Hence, for $3f'^2 + f = 0$, whose exact solution is

$$f = -\frac{1}{12}(\phi - \phi_0)2 \quad (9)$$

the Hessian determinant vanishes and the Lagrangian (3) becomes degenerate as in the Robertson–Walker case [12] and [13].

Let us now turn our attention to find the condition under which the Lagrangian (3) would admit Noether symmetry. In the Lagrangian under consideration the configuration space is $Q = (a, b, \phi)$, whose tangent space is $TQ = (a, b, \phi, \dot{a}, \dot{b}, \dot{\phi})$. Hence the infinitesimal generator of the Noether symmetry is

$$X = \alpha\frac{\partial}{\partial a} + \beta\frac{\partial}{\partial b} + \gamma\frac{\partial}{\partial \phi} + \dot{\alpha}\frac{\partial}{\partial \dot{a}} + \dot{\beta}\frac{\partial}{\partial \dot{b}} + \dot{\gamma}\frac{\partial}{\partial \dot{\phi}}. \tag{10}$$

The existence of Noether symmetry implies the existence of a vector field $X$ such that the Lie derivative of the Lagrangian with respect to the vector field vanishes, i.e.,

$$£_X L = 0. \tag{11}$$

This yields an expression which is second degree in $\dot{a}, \dot{b}$ and $\dot{\phi}$ and whose coefficients are functions of $a, b$ and $\phi$ only. Thus to satisfy equation (11), we obtain following the set of equations,

$$2f\frac{\partial\beta}{\partial a} + f'b\frac{\partial\gamma}{\partial a} = 0, \tag{12}$$

$$\alpha + 2b\frac{\partial\alpha}{\partial b} + 2a\frac{\partial\beta}{\partial b} + a\frac{f'}{f}\left(\gamma + 2b\frac{\partial\gamma}{\partial b}\right) = 0, \tag{13}$$

$$b\alpha + 2a\beta + 2ab\frac{\partial\gamma}{\partial\phi} - 4f'\left(b\frac{\partial\alpha}{\partial\phi} + 2a\frac{\partial\beta}{\partial\phi}\right) = 0, \tag{14}$$

$$\beta + b\frac{\partial\alpha}{\partial a} + a\frac{\partial\beta}{\partial a} + b\frac{\partial\beta}{\partial b} + b\frac{f'}{f}\left(\gamma + a\frac{\partial\gamma}{\partial a} + \frac{b}{2}\frac{\partial\gamma}{\partial b}\right) = 0, \tag{15}$$

$$f\left(b\frac{\partial\alpha}{\partial\phi} + a\frac{\partial\beta}{\partial\phi}\right) + f'\left(b\alpha + a\beta + \frac{b2}{2}\frac{\partial\alpha}{\partial b} + ab\frac{\partial\beta}{\partial b} + ab\frac{\partial\gamma}{\partial\phi}\right) + f''ab\gamma - \frac{ab2}{4}\frac{\partial\gamma}{\partial b} = 0, \tag{16}$$

$$f\frac{\partial\beta}{\partial\phi} + f'\left(\beta + \frac{b}{2}\frac{\partial\alpha}{\partial a} + a\frac{\partial\beta}{\partial a} + \frac{b}{2}\frac{\partial\gamma}{\partial\phi}\right) + \frac{f''b\gamma}{2} - \frac{ab}{4}\frac{\partial\gamma}{\partial a} = 0, \tag{17}$$

$$\alpha + \frac{f'}{f}a\gamma - \frac{ab2}{2}\left[V\left(\frac{\alpha}{a} + 2\frac{\beta}{b}\right) + V'\gamma\right] = 0. \tag{18}$$

The above set of differential equations can essentially be solved by the method of separation of variables which finally yields a differential equation in $f$ viz.,

$$3f'^2 + f = 0 \tag{19}$$

whose solution is already given in Eq. (9). In addition $\alpha, \beta, \gamma$ and $V$ are also obtained in the process as

$$\alpha = \frac{2l}{ab(\phi + \phi_0)3}, \qquad \beta = \frac{l}{a2(\phi + \phi_0)3},$$
$$\gamma = -\frac{l}{a^2b(\phi + \phi_0)2}, \qquad V = \lambda(\phi + \phi_0)4, \tag{20}$$

where $l, \lambda$ and $\phi_0$ are constants of integrations. So the Lagrangian (3) admits Noether symmetry under the condition that $f$ should have the form given by (9) while $V$ should be quartic in the scalar field $\phi$. However, it is to be noted that the form of $f$ given by (9) makes the Lagrangian degenerate. Thus a constraint has been imposed on the Lagrangian in order that it admits Noether symmetry.

Now for Cartan one form

$$\theta_L = \frac{\partial L}{\partial\dot{a}}da + \frac{\partial L}{\partial\dot{b}}db + \frac{\partial L}{\partial\dot{\phi}}d\phi, \tag{21}$$

the constant of motion $i_X\theta_L$ is obtained as,

$$F = \frac{\frac{d}{dt}[b(\phi + \phi_0)]}{a(\phi + \phi_0)2} = \frac{\frac{d}{dt}(b\phi)}{a\phi 2} \quad (\text{for } \phi_0 = 0). \tag{22}$$

At this stage we would like to point out the fact that a solution in the form $f(\phi) = 1 - \xi\phi 2$ does not satisfy Eq. (19), implying that nonminimal coupling is not admissible by Noether symmetry. The Noether symmetry thus obtained does not yield a physically desirable feature in the sense that Newton's gravitational constant turns out to be negative. Further nonminimal coupling which has wonderful features as discussed in the introduction, is not admissible. Still we carry out analysing the constraint imposed by the solution of $f(\phi)$ to show that degeneracy does not yield trivial solution, as mentioned in [12].

## 3. Analysing the constraint and presenting a solution

The degeneracy in the Lagrangian imposed by the claim that it should have Noether symmetry, leads to constrained dynamics as mentioned in the introduction. This gives rise to underdetermined situation where the number of the true degrees of freedom exceeds the number of the field equations. To apprehend the situation, let us substitute $f$, $f'$, $f''$ from Eq. (9) and $V$, $V'$ from Eq. (20) in the field Eqs. (4)–(7) to obtain

$$2\frac{\ddot{b}}{b}+2\frac{\ddot{\phi}}{\phi}+\frac{\dot{b}^2}{b2}-\frac{\dot{\phi}^2}{\phi 2}+4\frac{\dot{b}\dot{\phi}}{b\phi}+\frac{1}{b2}+6\lambda\phi 2=0, \tag{23}$$

$$\frac{\ddot{a}}{a}+\frac{\ddot{b}}{b}+2\frac{\ddot{\phi}}{\phi}+\frac{\dot{a}\dot{b}}{ab}+2\frac{\dot{a}\dot{\phi}}{a\phi}+2\frac{\dot{b}\dot{\phi}}{b\phi}-\frac{\dot{\phi}^2}{\phi 2}+6\lambda\phi 2 =0, \tag{24}$$

$$\frac{\ddot{a}}{a}+2\frac{\ddot{b}}{b}+3\frac{\ddot{\phi}}{\phi}+2\frac{\dot{a}\dot{b}}{ab}+3\frac{\dot{a}\dot{\phi}}{a\phi}+6\frac{\dot{b}\dot{\phi}}{b\phi}+\frac{\dot{b}^2}{b2}+\frac{1}{b2} +12\lambda\phi 2=0, \tag{25}$$

$$\frac{\dot{b}^2}{b2}+3\frac{\dot{\phi}^2}{\phi 2}+2\frac{\dot{a}\dot{\phi}}{a\phi}+2\frac{\dot{a}\dot{b}}{ab}+4\frac{\dot{b}\dot{\phi}}{b\phi}+\frac{1}{b2}+6\lambda b2 =0. \tag{26}$$

In addition we have yet another equation viz., Eq. (22), which is actually the constraint that has to be satisfied by the field Eqs. (23)–(26). To check if any new constraint arises from these field equations we have to take time derivative of Eq. (22) and eliminate acceleration terms [14] between the equation thus obtained and the field Eqs. (22)–(26). Time derivative of Eq. (22) is (using the same equation in it)

$$\frac{\ddot{b}}{b}+\frac{\ddot{\phi}}{\phi}=2\frac{\dot{\phi}^2}{\phi 2}+F\frac{\dot{a}\phi}{b}. \tag{27}$$

Now eliminating acceleration terms between Eqs. (23) and (27) one gets back the Hamiltonian constraint equation (26). Hence Eq. (13) is no longer an independent equation. In view of Eqs. (22) and (26), one can obtain yet another constraint equation, viz.,

$$\frac{d}{dt}(a\phi)=-\frac{1+F2a2\phi 2}{2Fb}-\frac{3\lambda\phi 2b}{F} \tag{28}$$

which can be used instead of Eq. (26). Differentiating equation (28) with respect to time and using the same equation once again in it, one obtains,

$$\frac{\ddot{a}}{a}+\frac{\ddot{\phi}}{\phi}=-2\frac{\dot{a}\dot{\phi}}{a\phi}+\frac{1+F2a2\phi 2}{2Fab2\phi}\dot{b}-\frac{3\lambda\phi}{Fa}\dot{b}-\frac{6\lambda b}{Fa}\dot{\phi} +\frac{1+F2a2\phi 2}{2b2}+3\lambda\phi 2. \tag{29}$$

In view of Eqs. (27) and (29) one can now easily observe that Eqs. (24) and (25) are trivially satisfied. Hence at this stage we are left with a pair of Eqs. (22) and (28) with three degrees of freedom viz., $a, b$ and $\phi$, leading to an underdetermined situation. This is the outcome of a degenerate Lagrangian. In order to obtain solution, one is now free to impose 'one' condition that would lead to physically acceptable solution. We are presenting here one such solution under the assumption,

$$a\phi=kb, \tag{30}$$

where $k$ is a constant and let it be positive definite ($k>0$). In view of Eqs. (30), (22) can immediately be integrated to yield,

$$b\phi=n\exp(Fkt), \tag{31}$$

where $n$ is a constant of integration and considered to be positive definite ($n>0$) too. Further for $\lambda=0$, Eq. (28) can also be integrated in view of Eq. (30). The result is

$$b=-\frac{1}{Fk}\big[m\exp(-Fkt)-1\big]^{1/2}, \tag{32}$$

where the overall negative sign has been chosen to reveal physically acceptable solution and $m$ is yet another constant of integration which is considered to be greater than one ($m>1$) for the same reason. Hence $\phi$ and $a$ can also be obtained via Eqs. (30), (31) and (32) as,

$$\phi=-nFk\frac{\exp(Fkt)}{[m\exp(-Fkt)-1]^{1/2}},$$
$$a=\frac{\exp(-Fkt)}{nkF2}[m\exp(-Fkt)-1]. \tag{33}$$

Now if one chooses $F=-c2$, then the solutions (32) and (33) take the following form,

$$a=\frac{\exp(c2kt)}{nkc4}\big[m\exp(c2kt)-1\big],$$
$$b=\frac{1}{kc2}\big[m\exp(c2kt)-1\big]^{1/2}, \tag{34}$$

$$ab2=\frac{\exp(c2kt)}{nk3c8}\big[m\exp(c2kt)-1\big]^2,$$
$$\phi=nkc2\frac{\exp(-c2kt)}{\big[m\exp(c2kt)-1\big]^{1/2}}. \tag{35}$$

The above solution reveals that the universe admits inflation starting from a finite proper volume, under the choice of the constants already made viz., $k>0$,

$n > 0$ and $m > 1$. The scalar field at the initial epoch is finite and it falls off exponentially as the universe expands. The solution is singularity free although there is no question of graceful exit from inflation. The big-bang singularity is pushed back to the infinite past.

## 4. Some other symmetries in Kantowski–Sachs model

In this section we shall first show that the whole laborious job that has been carried out in the preceding section (i.e., to find the set of Eqs. (12) to (18) and to solve them by the method of the separation of variables to obtain conditions under which the Lagrangian (3) admits Noether symmetry) is not at all required. Rather we can construct a pair of equations from the set of field equations (4) to (7). One of this pair can immediately extract the conditions for which the Lagrangian admits dynamical symmetry and the other can find the corresponding conserved current. We shall further show that one of the dynamical symmetries obtained in the process is of Noether class. Finally, we shall extract dynamical symmetries of some other type that we did not obtain by applying Noether's theorem in the preceding section.

The first one of this pair is the continuity equation. This equation is obtained by eliminating $\ddot{a}$ and $\ddot{b}$ from the field equations (4) to (6) and then comparing it with the Hamiltonian constraint equation (7). The equation thus formed is,

$$2(3f'^2 + f)\left(\ddot{\phi} + \frac{\dot{a}}{a}\dot{\phi} + 2\frac{\dot{b}}{b}\dot{\phi}\right) + f'(6f'' + 1)\dot{\phi}^2 + 2(fV' - 2Vf') = 0. \tag{36}$$

All dynamical symmetries are hidden in this equation. To find Noether symmetry one has to choose $f$ and $V$ in such a way that Eq. (36) is satisfied identically. The choice is quite trivial viz., the coefficients of the derivatives of $\phi$, $a$ and $b$ should vanish separately. This implies $3f'^2 + f = 0$ i.e., $f = -\frac{1}{12}\phi 2$ and as such $6f'' + 1 = 0$ too. $f2$ and hence is in the form $V = \lambda\phi 4$. These results are already obtained in Eqs. (19) and (20) of the preceding section. To obtain the conserved current we construct yet another equation and that is done simply by eliminating terms in the field equations which are free from time derivatives viz., $\frac{1}{b2}$, $V(\phi)$ and $V'(\phi)$ in the present context. This is done by taking the difference of Eqs. (1) and (4), which yields,

$$2\frac{\ddot{b}}{b} + \frac{f'}{f}\ddot{\phi} + \left(\frac{2f'' + 1}{2f}\right)\dot{\phi}^2 - \frac{f'\dot{a}}{fa}\dot{\phi} - 2\frac{\dot{a}\dot{b}}{ab} = 0. \tag{37}$$

This equation in view of the solution of $f$ obtained from Eq. (36), reads

$$\frac{\ddot{b}}{b} + \frac{\ddot{\phi}}{\phi} - 2\frac{\dot{\phi}^2}{\phi 2} - \frac{\dot{a}\dot{\phi}}{a\phi} - \frac{\dot{a}\dot{b}}{ab} = 0, \tag{38}$$

whose first integral yields the conserved current obtained in Eq. (22). Thus, we have shown that the Noether symmetry can even be obtained in view of the continuity Eq. (36) and the corresponding conserved current from Eq. (37), without invoking Eqs. (11) and (21).

Let us now proceed to find some other type of dynamical symmetry that we did not find in the preceding section, for which we choose $f$ in the form

$$f = \epsilon\phi 2. \tag{39}$$

Further we choose the potential in the form

$$V = \lambda\phi 4 \tag{40}$$

then Eq. (36) can be written as

$$(12\epsilon + 1)\left(\ddot{\phi} + \frac{\dot{a}}{a}\dot{\phi} + 2\frac{\dot{b}}{b}\dot{\phi} + \frac{\dot{\phi}^2}{\phi}\right) = 0. \tag{41}$$

For $\epsilon = -1/12$, we regain Noether symmetry. However, for any arbitrary $\epsilon$ other than zero or $(-1/12)$, Eq. (41) leads to

$$\frac{\ddot{\phi}}{\phi} + \frac{\dot{a}\dot{\phi}}{a\phi} + 2\frac{\dot{b}\dot{\phi}}{b\phi} + \frac{\dot{\phi}^2}{\phi 2} = 0 \tag{42}$$

whose first integral is

$$ab2\phi\dot{\phi} = \text{constant}. \tag{43}$$

Thus we obtain yet another dynamical symmetry of the system for arbitrary value of $\epsilon$, for which the conserved current is given by Eq (43). It is to be noted that the existence of this dynamical symmetry does not make the Lagrangian degenerate, since the Hessian determinant given by Eq. (8) does not turn out to be zero. Since this symmetry has not been obtained by the application of Noether's theorem, therefore, we conclude that not all the dynamical symmetries of a Lagrangian are of Noether class. The dynamical symmetry thus obtained here is of the same form that we have already seen in connection with the Robertson–Walker metric [13]. Let us now consider an action deliberately for nonminimally coupled theory in the form

$$A = \int d^4X\,\sqrt{-g}$$
$$\times\left[\left(1-\xi f(\phi)\right)R - \tfrac{1}{2}\phi_{,\mu}\phi^{,\mu} - V(\phi)\right] \qquad (44)$$

instead of (1). Proceeding as before, it is found to yield an equation,

$$\left(\frac{1}{\xi f'} + \frac{3\xi f'}{1-\xi f}\right)\left(\ddot{\phi} + \frac{\dot{a}\dot{\phi}}{a} + \frac{2\dot{b}\dot{\phi}}{b}\right)$$
$$+\left(\frac{3\xi f'' - 1/2}{1-\xi f}\right)\dot{\phi}^2 + \frac{V'}{\xi f'} + \frac{2V}{1-\xi f} = 0 \qquad (45)$$

instead of Eq. (36). It is not difficult to see that Noether symmetry in this case would again lead to the same results viz., $V(\phi) = \lambda\phi 4$ and $(1-\xi f) = -\phi 2/12$. This again confirms that Noether symmetry does not admit nonminimal coupling in the theory. This further implies that the attempt to find the form of the coupling $f(\phi)$ and that of the potential $V(\phi)$ using Noether's theorem, leads to negative Newton's gravitational constant. In addition the theory of nonminimal coupling along with its all pleasant features is found not to admit Noether symmetry. On the other hand the symmetry, other than Noether symmetry, that we have obtained for induced theory of gravity is good enough in the sense that Newton's gravitational constant remains positive. Finally, we observe that nonminimally coupled theory does not admit symmetry in any form.

## 5. Concluding remarks

In a recent communication [13] we have come across an important and wonderful result, while reviewing the works of Capozziello et al. [12] in connection with the Noether symmetry in the Robertson–Walker metric. The result is that, even if there exists certain forms of $f(\phi)$ and $V(\phi)$ and hence a vector field $X$ such that $\pounds_X L = 0$; the form of $V(\phi)$ might not satisfy the field equations. We have not come across such a result earlier and as such do not know the reason as yet. However, we have observed that only one, viz., the continuity equation suffices to check whether the field equations are satisfied or not. In the context of the induced theory of gravity, this equation turned out to be a very important one to explore all sorts of existing dynamical symmetries of a Lagrangian.

In the present Letter, we have shown that the Noether symmetry of the induced theory of gravity in the Kantowski–Sachs model can also be found from the continuity equation. Instead of using the Cartan's one form, the conserved current can simply be obtained from Eqs. (37) or (38), which is obtained from yet another combination of the field equations. The Noether symmetry makes the Lagrangian degenerate that introduces a constraint reducing the number of independent field equations to the number of true degrees of freedom by one, causing underdeterminacy. However, the field equations are found to admit inflationary solution under a suitable assumption. This result definitely proves that the conclusion made by Capozziello et al. [12], viz., degeneracy leads to trivial solutions, is wrong. However, Noether symmetry yields one very undesirable feature. It makes Newton's gravitational constant negative. Further it has been observed that nonminimal coupling does not admit Noether symmetry.

It has been further observed that all sorts of dynamical symmetries of a Lagrangian can be explored from the continuity equation only, at least in induced theory of gravity. In view of this equation one can obtain dynamical symmetries of a Lagrangian other than Noether symmetry. These symmetries are good enough in the sense that it does not make Newton's gravitational constant negative. This confirms that not all the dynamical symmetries of a system belong to the Noether class. It has also been noticed that nonminimally coupled theory does not have symmetry in any form.

## References


[1] S. Adler, Rev. Mod. Phys. 54 (1982) 729.
[2] M. Green, J. Schwarz, E. Witten, Superstring Theory, Cambridge Univ. Press, Cambridge, 1989.
[3] T. Futamase, K.I. Maeda, Phys. Rev. D 39 (1989) 399.
[4] A. Guth, Phys. Rev. D 23 (1981) 347;
A. Guth, Phys. Lett. B 108 (1982) 389;
D. La, P.J. Steinhardt, Phys. Rev. Lett. 62 (1989) 376;
A.D. Linde, Phys. Lett. B 238 (1990) 160;

F.S. Accetta, J.J. Trester, Phys. Rev. D 39 (1989) 2854.
[5] R. Fakir, W.G. Unruh, Phys. Rev. D 41 (1990) 1783;
R. Fakir, S. Habib, W.G. Unruh, Ap. J. 394 (1992) 396.
[6] R. Fakir, Phys. Rev. D 41 (1990) 3012.
[7] A. Vilenkin, Phys. Rev. D 27 (1983) 2848.
[8] J.B. Hartle, S.W. Hawking, Phys. Rev. D 28 (1983) 2960.
[9] D.H. Coule, Class. Quantum Grav. 9 (1992) 2353.
[10] A.K. Sanyal, Int. J. Mod. Phys. A 10 (1995) 2231.
[11] S. Kalara, N. Kaloper, A. Olive, Nucl. Phys. B 341 (1990) 252.
[12] S. Capozziello, R. De Ritis, Phys. Lett. A 177 (1993) 1;
S. Capozziello, R. De Ritis, P. Scudellaro, Nuovo Cimento B 109 (1994) 159;
S. Capozziello, R. De Ritis, P. Scudellaro, Phys. Lett. A 188 (1994) 130;
S. Capozziello, R. De Ritis, Class. Quantum Grav. 11 (1994) 107.
[13] A.K. Sanyal, B. Modak, Class. Quantum Grav. 18 (2001) 3767.
[14] K. Sundermeyer, Constrained Dynamics, Springer-Verlag, Berlin, 1982.