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New Generalizations of Browder's Variational Inequalities and the Ky Fan Minimax Inequality

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Abstract—In the present paper, some new generalized versions of Browder's theorems for variational inequalities and the Ky Fan minimax inequality are obtained.

Keywords—H-space, H-quasiconvex function, H-convex set, Transfer open valued.

1. INTRODUCTION AND PRELIMINARIES

In 1968, Browder [1] proved the following theorem.

THEOREM A. *Let E be a locally convex Hausdorff topological vector space, K a compact convex subset of E, and T an upper semicontinuous mapping of K into E^* (the conjugate space of E), such that for each $x \in K$, $T(x)$ is a nonempty compact convex subset of E^*. Then there exists an element $u_0 \in K$ and $w_0 \in T(u_0)$ such that*

$$\langle w_0, u_0 - u \rangle \geq 0,$$

for all $u \in K$.

In 1972, Ky Fan [2] proved the following famous minimax inequality.

THEOREM B. *Let E be a Hausdorff topological vector space, X a nonempty compact convex subset of E. If a functional $\varphi : X \times X \to R$ such that*

(i) *for each $y \in X$, $\varphi(x, y)$ is lower semicontinuous in x,*
(ii) *for each $x \in X$, $\varphi(x, y)$ is quasiconcave in y,*

then there exists a point $x_0 \in X$ such that

$$\sup_{y \in X} \varphi(x_0, y) \leq \sup_{x \in X} \varphi(x, x).$$

In 1981, Yen [3] generalized Theorem B as the following.

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THEOREM C. *Let E be a Hausdorff topological vector space, X a nonempty compact convex subset of E. If two functions $f, g : X \times X \to R$ such that*

(i) *for each $y \in X$, $g(x, y)$ is lower semicontinuous in x,*
(ii) *for each $x \in X$, $f(x, y)$ is quasiconcave in y,*
(iii) *$g(x, y) \le f(x, y)$, for all $(x, y) \in X \times X$,*

then there exists a point $x_0 \in X$ such that

$$\sup_{y \in X} g(x_0, y) \le \sup_{x \in X} f(x, x).$$

The main objects of this paper are to generalize Theorem A, Theorem B, and Theorem C. Before proceeding to the main results, we consider some elementary concepts.

DEFINITION 1.1. *(See [4].) Let X be a topological space and $\mathcal{F}(X)$ be a family of all nonempty finite subsets of X. Let $\{F_A\}$ be a family of nonempty contractible subsets of X indexed by $A \in \mathcal{F}(X)$ such that $F_A \subset F_{A'}$, whenever $A \subset A'$. The pair $(X, \{F_A\})$ is called an H-space. Given an H-space $(X, \{F_A\})$, a nonempty set D is called H-convex if $F_A \subset D$ for each nonempty finite subset A of D.*

Let $(X, \{F_A\})$ be an H-space and Y be a topological space. Let C be a nonempty subset of X, and $\varphi : C \times Y \to R \cup \{+\infty\}$ a functional. For each $y \in Y$, $\varphi(x, y)$ is called to be H-quasiconvex (quasiconcave) in x, iff for each finite subset $\{x_1, \dots, x_n\} \subset C$ and each $x \in F_{\{x_1,\dots,x_n\}}$, the following holds:

$$\varphi(x, y) \le \max_{1 \le i \le n} \varphi(x_i, y)$$

($\varphi(x, y) \ge \min_{1 \le i \le n} \varphi(x_i, y)$, respectively).

DEFINITION 1.2. *(See [5].) Let X and Y be two topological spaces. A multivalued mapping $T : X \to 2^Y$ is called to be transfer open valued, iff for each $x \in X$ and each $y \in T(x)$, there exists $x' \in X$ such that $y \in \operatorname{int}[T(x')]$, where $\operatorname{int}[T(x')]$ denotes interior of $T(x')$.*

2. GENERALIZATIONS OF THE BROWDER VARIATIONAL INEQUALITIES

In this section, we shall prove some generalized versions of the Browder variational inequalities. The following lemma is equivalent to Corollary 2.3 in [6].

LEMMA 2.1. *Let $(x, \{F_A\})$ be a compact H-space and $T : X \to 2^X$ be a multivalued mapping such that:*

(i) *for each $x \in X$, $T(x)$ is a nonempty H-convex subset,*
(ii) *$T^{-1} : X \to 2^X$ is transfer open valued, where $T^{-1}(y) = \{x \in X : y \in T(x)\}$ for each $y \in X$.*

Then there is a point $x_0 \in T(x_0)$.

THEOREM 2.2. *Let $(X, \{F_A\})$ be a compact H-space and Y a topological space and $\varphi : X \times Y \times X \to R$ a functional. Suppose the $T : X \to 2^Y$ is an upper semicontinuous multivalued mapping with nonempty compact values. If the following conditions hold:*

(i) *$\varphi(x, y, z)$ is H-quasiconvex in z,*
(ii) *$\varphi(x, y, z)$ is upper semicontinuous in (x, y),*
(iii) *for each $x \in X$, there exists an element $y \in T(x)$ such that $\varphi(x, y, x) \ge 0$,*

then there exists an element $x_0 \in X$ such that

$$\sup_{w \in Tx_0} \varphi(x_0, w, z) \ge 0,$$

for all $z \in X$.

PROOF. If the conclusion is false, then for each $u \in X$, there exists a point $z \in X$ such that

$$\sup_{w \in T(u)} \varphi(u, w, z) < 0.$$

Let $S(u) = \{v \in X : \sup_{w \in T(u)} \varphi(u, w, z) < 0\}$. Then $S : X \to 2^X$ is a multivalued mapping with nonempty values.

For each $u \in X$ and each finite subset $A = \{v_1, \dots, v_n\}$ of $S(u)$, we have

$$\sup_{w \in T(u)} \varphi(u, w, v_i) < 0, \qquad i = 1, 2, \dots, n.$$

Hence there is a real number $r \in R$ such that

$$\sup_{w \in T(u)} \varphi(u, w, v_i) < r < 0, \qquad i = 1, 2, \dots, n.$$

For each $v \in F_A$ and each $w \in T(u)$, by Condition (i),

$$\varphi(u, w, v) \le \max_{1 \le i \le n} \varphi(u, w, v_i) < r.$$

Hence

$$\sup_{w \in T(u)} \varphi(u, w, v) \le r < 0,$$

i.e., $v \in S(u)$ and so $F_A \subset S(u)$. It implies that $S(u)$ is H-convex.

Since $T : X \to 2^Y$ is an upper semicontinuous multivalued mapping with nonempty compact values and $\varphi(x, y, z)$ is upper semicontinuous in (x, y), by virtue of Proposition 21 in [7], we know that $\sup_{w \in T(u)} \varphi(u, w, v)$ is upper semicontinuous in u. Consequently, for each $v \in X$,

$$\begin{aligned} S^{-1}(v) &= \{u \in X : v \in S(u)\} \\ &= \left\{ u \in X : \sup_{w \in T(u)} \varphi(u, w, v) < 0 \right\} \end{aligned}$$

is open. By Lemma 2.1, there exists a point $\bar{u} \in X$ such that $\bar{u} \in S(\bar{u})$, i.e., $\sup_{w \in T(\bar{u})} \varphi(\bar{u}, w, \bar{u}) < 0$. It contradicts Condition (iii). Therefore, Theorem 2.2 is true.

COROLLARY 2.3. *Let X be a nonempty compact convex subset of a locally convex topological vector space E, E^* the conjugate space of E. Suppose that $T : X \to 2^{E^*}$ is an upper semicontinuous multivalued mapping with nonempty compact values. Then there exists an element $u_0 \in X$ such that*

$$\sup_{w \in T(u_0)} \langle w, u_0 - u \rangle \ge 0,$$

for all $u \in X$.

PROOF. It is sufficient to take $(X, \{F_A\}) = (X, \{\operatorname{co} A\})$, $Y = E^*$, and $\varphi(x, y, z) = \langle y, x - z \rangle$ in Theorem 2.2.

REMARK. Corollary 2.3 contains Theorem 2 of [1] as its special case.

THEOREM 2.4. *Let $(X, \{F_A\})$ be a compact H-space, E a locally convex Hausdorff topological vector space. Let Y be a nonempty convex subset of E and $T : X \to 2^Y$ an upper semicontinuous multivalued mapping with nonempty compact convex values. If an upper semicontinuous function $\varphi : X \times Y \times X \to R$ such that*

(i) *$\varphi(x, y, z)$ is quasiconcave in y and is H-quasiconvex in z, respectively,*
(ii) *for each $x \in X$, there exists $y \in T(x)$ such that $\varphi(x, y, x) \ge 0$,*

then there exist $\bar{x} \in X$ *and* $\bar{y} \in T(\bar{x})$ *such that*

$$\varphi(\bar{x}, \bar{y}, x) \geq 0,$$

for all $x \in X$.

PROOF. By virtue of Theorem 2.2, there exists $\bar{x} \in X$ such that

$$\sup_{y \in T(\bar{x})} \varphi(\bar{x}, y, x) \geq 0, \qquad \forall x \in X.$$

Since φ is upper semicontinuous and $T(\bar{x})$ is compact, for each $x \in X$, there exists $y(x) \in T(\bar{x})$ such that

$$\varphi(\bar{x}, y(x), x) \geq 0.$$

Let $S(x) = \{y \in T(\bar{x}) : \varphi(\bar{x}, y, x) \geq 0\}$, then $S : X \to 2^{T(\bar{x})}$ is a multivalued mapping with nonempty closed convex values.

If the conclusion of Theorem 2.4 is false, then for each $y \in T(\bar{x})$, there is $x_0 \in X$ such that

$$\varphi(\bar{x}, y, x_0) < 0.$$

Let $G(y) = \{x \in X : \varphi(\bar{x}, y, x) < 0\}$. $G : T(\bar{x}) \to 2^X$ is a multivalued mapping with nonempty H-convex values and for each $x \in X$,

$$\begin{aligned} G^{-1}(x) &= \{y \in T(\bar{x}) : x \in G(y)\} \\ &= \{y \in T(\bar{x}) : \varphi(\bar{x}, y, x) < 0\} \end{aligned}$$

is relatively open in $T(\bar{x})$. By virtue of Corollary 2.1 in [6], G has a continuous selection $f : T(\bar{x}) \to X$.

For each $y \in T(\bar{x})$, let $H(y) = S(f(y))$. Then $H : T(\bar{x}) \to 2^{T(\bar{x})}$ is a multivalued mapping with nonempty closed convex values. Assume that $\{(y_\alpha, z_\alpha)\}_{\alpha \in D}$ is a net in $\mathrm{Gr}(H)$ ($\mathrm{Gr}(H)$ denotes the graph of H) such that $(y_\alpha, z_\alpha) \to (u, v)$. Then $v \in T(\bar{x})$, $f(y_\alpha) \to f(u)$, and

$$\varphi(\bar{x}, z_\alpha, f(y_\alpha)) \geq 0, \qquad \forall \alpha \in D.$$

Consequently, by the upper semicontinuity of φ, we have

$$\varphi(\bar{x}, v, f(u)) \geq 0,$$

i.e., $v \in S(f(u)) = H(u)$. Hence, $\mathrm{Gr}(H)$ is closed and so $H : T(\bar{x}) \to 2^{T(\bar{x})}$ is upper semicontinuous as $T(\bar{x})$ is compact.

By virtue of Kakutani-Fan-Glicksberg's fixed point theorem (i.e., Theorem 5.5.2 in [8]), there exists a point $\bar{y} \in T(\bar{x})$ such that $\bar{y} \in H(\bar{y}) = S(f(\bar{y}))$, i.e.,

$$\varphi(\bar{x}, \bar{y}, f(\bar{y})) \geq 0.$$

On the other hand, as $f(\bar{y}) \in G(\bar{y})$, we have

$$\varphi(\bar{x}, \bar{y}, f(\bar{y})) < 0.$$

This is a contradiction. Therefore, Theorem 2.4 is true.

REMARK. In Theorem 2.4, if X is a nonempty compact convex subset of a Hausdorff locally convex space E, $Y = E^*$ (the conjugate space of E), and $\varphi(x, y, z) = \langle y, x - z \rangle$, then we obtain Theorem 6 in [1] (i.e., Theorem A).

3. NEW GENERALIZATIONS OF THE KY FAN MINIMAX INEQUALITY

THEOREM 3.1. *Let $(X, \{F_A\})$ be a compact H-space and Y a topological space and $\varphi, \psi : X \times Y \to R$ two functions. Suppose that $T : X \to 2^Y$ is an upper semicontinuous multivalued mapping with nonempty compact values. If the following conditions hold:*

(i) *$\psi(x, y)$ is H-quasiconvex in x,*
(ii) *$\varphi(x, y)$ is upper semicontinuous in y,*
(iii) *for each $x \in X$, there exists an element $y \in T(x)$ such that $\psi(x, y) \geq c$ (c is a constant),*
(iv) *$\psi(x, y) \leq \varphi(x, y)$, for all $(x, y) \in X \times Y$,*

then there exists an element $x_0 \in X$ such that

$$\max_{y \in Tx_0} \varphi(x, y) \geq c,$$

for all $x \in X$.

PROOF. For each $x \in X$, let

$$S(x) = \left\{ z \in X : \max_{y \in T(x)} \varphi(z, y) < c \right\},$$

$$H(x) = \left\{ z \in X : \sup_{y \in T(x)} \psi(z, y) < c \right\}.$$

Then $S, H : X \to 2^X$ are two multivalued mappings such that $S(x) \subset H(x)$, for all $x \in X$. By (i) we know that

$$H(x) = \bigcap_{y \in T(x)} \{ z \in X : \psi(z, y) < c \}$$

is H-convex. Now we prove $f(z, x) := \max_{y \in T(x)} \varphi(z, y)$ is upper semicontinuous in x. In fact, for each fixed $z \in X$ and each $r \in R$, let

$$D = \{ x \in X : f(z, x) \geq r \}.$$

If $\{x_\alpha : \alpha \in I\}$ is a net in D such that $x_\alpha \to u$, then

$$f(z, x_\alpha) \geq r, \qquad \forall \alpha \in I,$$

i.e.,

$$\max_{y \in T(x_\alpha)} \varphi(z, y) \geq r, \qquad \forall \alpha \in I.$$

Consequently, for each $\alpha \in I$, there exists a point $y_\alpha \in T(x_\alpha)$ such that $\varphi(z, y_\alpha) \geq r$. By Proposition 1 in [9], there exists a point $v \in T(u)$ and a subnet $\{y_\beta\}$ of $\{y_\alpha\}_{\alpha \in I}$ such that $y_\beta \to v$. By (ii), $\varphi(z, v) \geq r$, hence $\max_{y \in T(u)} \varphi(z, y) \geq r$, i.e., $f(z, u) \geq r$, i.e., $u \in D$. Hence D is closed. Consequently, $f(z, x)$ is upper semicontinuous in x. Hence for each $z \in X$,

$$\begin{aligned} S^{-1}(z) &= \{ x \in X : z \in S(x) \} \\ &= \{ x \in X : f(z, x) < c \} \end{aligned}$$

is open.

If $S(x) \neq \emptyset$ for all $x \in X$, then $H(x) \neq \emptyset$ for all $x \in X$. For each $y \in X$ and each $x \in H^{-1}(y)$, since $S(x) \neq \emptyset$, there exists a point $y' \in S(x)$ so that $x \in S^{-1}(y') \subset H^{-1}(y')$, and hence $x \in \operatorname{int}[H^{-1}(y')]$ because $S^{-1}(y')$ is open. This shows that $H^{-1} : X \to 2^X$ is transfer open valued. By virtue of Lemma 2.1, there exists a point $\bar{x} \in X$ such that $\bar{x} \in H(\bar{x})$, i.e.,

$$\sup_{y \in T(\bar{x})} \psi(\bar{x}, y) < c.$$

This contradicts (iii). Hence there exists a point $x_0 \in X$ such that $S(x_0) = \emptyset$, i.e.,

$$\max_{y \in Tx_0} \varphi(x, y) \geq c,$$

for all $x \in X$.

COROLLARY 3.2. *Let $(X, \{F_A\})$ be a compact H-space and Y a topological space and $f, g : X \times Y \to R$ two functions. Suppose that $T : X \to 2^Y$ is an upper semicontinuous multivalued mapping with nonempty compact values. If the following conditions hold:*

(i) *$f(x, y)$ is H-quasiconcave in x,*
(ii) *$g(x, y)$ is lower semicontinuous in y,*
(iii) *for each $x \in X$, there exists an element $y \in T(x)$ such that $f(x, y) \leq c$ (c is a constant),*
(iv) *$g(x, y) \leq f(x, y)$, for all $(x, y) \in X \times Y$,*

then there exists an element $x_0 \in X$ such that

$$\min_{y \in Tx_0} g(x, y) \leq c,$$

for all $x \in X$.

PROOF. For each $(x, y) \in X \times Y$, let $\varphi(x, y) = -g(x, y)$, $\psi(x, y) = -f(x, y)$. Then by virtue of Theorem 3.1, there exists an element $x_0 \in X$ such that

$$\max_{y \in T(x_0)} \varphi(x, y) \geq -c,$$

for all $x \in X$, i.e.,

$$\min_{y \in Tx_0} g(x, y) \leq c,$$

for all $x \in X$.

COROLLARY 3.3. *Let $(X, \{F_A\})$ be a compact H-space and $f, g : X \times X \to R$ two functions. If the following conditions hold:*

(i) *$f(x, y)$ is H-quasiconcave in x,*
(ii) *$g(x, y)$ is lower semicontinuous in y,*
(iii) *$g(x, y) \leq f(x, y)$, for all $(x, y) \in X \times X$,*

then there exists an element $x_0 \in X$ such that

$$g(x, x_0) \leq \sup_{z \in X} f(z, z),$$

for all $x \in X$.

PROOF. Let $c = \sup_{z \in X} f(z, z)$. If $c = +\infty$, then the conclusion holds, obviously. If $c < +\infty$, let $T(x) = x$ for each $x \in X$. Then the conclusion follows from Corollary 3.2.

REMARK. Corollary 3.3 generalizes Theorem C (consequently, Theorem B) to H-spaces without introducing additional assumptions. Hence Corollary 3.2 contains the Ky Fan inequality as a special case. Moreover, Theorem 3.1 generalizes Theorem 5.7.2 of [8] to H-space X and general topological space Y without introducing additional assumptions and we do not need $\varphi = \psi$.

By Theorem 3.1, we can prove the following existence theorem of solutions for quasi-variational inequalities similar to Theorem 2.4.

THEOREM 3.4. *Let $(X, \{F_A\})$ be a compact H-space and Y a nonempty convex subset of a Hausdorff locally convex topological vector space, and $\varphi : X \times Y \to R$ an upper semicontinuous functional. Suppose that $T : X \to 2^Y$ is an upper semicontinuous multivalued mapping with nonempty compact values. If the following conditions hold:*

(i) *$\varphi(x, y)$ is H-quasiconvex in x,*
(ii) *$\varphi(x, y)$ is H-quasiconcave in y,*
(iii) *for each $x \in X$, there exists an element $y \in T(x)$ such that $\varphi(x, y) \geq c$ (c is a constant),*

then there exists an element $\bar{x} \in X$ *and a point* $\bar{y} \in T(\bar{x})$ *such that*

$$\varphi(x, \bar{y}) \geq c,$$

for all $x \in X$.

THEOREM 3.5. *Let* $(X, \{F_A\})$ *be a compact* H*-space,* $(Y, \{F_B\})$ *an* H*-space, and* $\varphi : X \times Y \to R$ *a functional. Suppose that* $T : X \to 2^Y$ *is a multivalued mapping with nonempty* H*-convex values. If the following conditions hold:*

(i) $T^{-1} : Y \to 2^X$ *is transfer open valued,*
(ii) $\varphi(x, y)$ *is* H*-quasiconvex in* x*,*
(iii) $\varphi(x, y)$ *is upper semicontinuous in* y*,*
(iv) $\varphi(x, y) \geq c$ *(c is a constant) for each* $x \in X$ *and each* $y \in T(x)$*,*

then there exists an element $\bar{x} \in X$ *and a point* $\bar{y} \in T(\bar{x})$ *such that*

$$\varphi(x, \bar{y}) \geq c,$$

for all $x \in X$.

PROOF. Since Condition (i) is equivalent to Conditions (ii) and (iii) of Theorem 2.2 in [6], T satisfies all conditions of Theorem 2.2 in [6]. By virtue of Theorem 2.2 in [6], T has a continuous selection $f : X \to Y$.

If the conclusion of Theorem 3.5 is false, then for each $x \in X$, there exists a point $z_0 \in X$ such that $\varphi(z_0, f(x)) < c$. Let

$$G(x) = \{z \in X : \varphi(z, f(x)) < c\}.$$

Then $G : X \to 2^X$ has nonempty H-convex values and for each $z \in X$, by the upper semicontinuity of $\varphi(x, y)$ in y and the continuity of f, we know that

$$\begin{aligned} G^{-1}(z) &= \{x \in X : z \in G(x)\} \\ &= \{x \in X : \varphi(z, f(x)) < c\} \end{aligned}$$

is open in X. Consequently, by Lemma 2.1, there exists a point $\bar{x} \in X$ such that $\bar{x} \in G(\bar{x})$, i.e.,

$$\varphi(\bar{x}, f(\bar{x})) < c.$$

This contradicts (iv). Therefore Theorem 3.5 is true.

REFERENCES

1. F.E. Browder, The fixed point theory of multi-valued mapping in topological vector spaces, *Math. Ann.* **177**, 283–301 (1968).
2. K. Fan, A minimax inequality and applications, *Inequalities*, Volume III, (Edited by O. Shisha), pp. 103–113, Academic Press, (1972).
3. C.L. Yen, A minimax inequality and its applications to variational inequalities, *Pacific J. Math.* **97** (2) (1981).
4. C. Horvath, Some results on multivalued mappings and inequalities without convexity, In *Nonlinear and Convex Analysis*, (Edited by B.L. Lin and S. Simons), pp. 99–106, Marcel Dekker, (1987).
5. G.Q. Tian, Generalizations of FKKM theorem and the Ky Fan minimax inequality, with applications to maximal elements, price equilibrium, and complementarity, *J. Math. Anal. Appl.* **170**, 457–471 (1992).
6. E. Tarafdar, Fixed point theorems in H-spaces and equilibrium point of abstract economies, *J. Austral. Math. Soc.* **53**, 252–260 (1992).
7. J.P. Aubin and I. Ekeland, *Applied Nonlinear Analysis*, John Wiley and Sons, New York, (1984).
8. S.-S. Chang, *Variational Inequality and Complement Problem Theory with Applications*, Shanghai Scientific and Technological Literature, Shanghai, (1991).
9. T. Kacynski and V. Zeidan, An application of Ky Fan fixed point theorem to an optimization problem, *Nonlinear Anal. TMA* **13** (3), 259–261 (1989).

10. F.E. Browder, Nonlinear monotone operators and convex sets in Banach spaces, *Bull. Amer. Math. Soc.* **71**, 780–785 (1965).
11. F.E. Browder, A new generalization of the Schauder fixed point theorem, *Math. Ann.* **174**, 285–290 (1967).
12. S.-S. Chang and Y.-H. Ma, Generalized KKM theorem on H-space with applications, *J. Math. Anal. Appl.* **163**, 406–421 (1991).
13. S.-S. Chang and Y. Zhang, Generalized KKM theorem and variational inequalities, *J. Math. Anal. Appl.* **158**, 10–15 (1991).
14. P. Hartman and G. Stampacchia, On some non-linear elliptic differential-functional equations, *Acta. Math.* **115**, 271–310 (1966).
15. C.H. Su and V.M. Sehgal, Some fixed point theorems for condensing multifunctions in locally convex spaces, *Proc. Amer. Math. Soc.* **50**, 150–154 (1975).