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Inverse spectral problems for Dirac operators on a finite interval

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ABSTRACT

We consider the direct and inverse spectral problems for Dirac operators that are generated by the differential expressions

$$\mathfrak{t}_q := \frac{1}{i}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\frac{d}{dx} + \begin{pmatrix} 0 & q \\ q^* & 0 \end{pmatrix}$$

and some separated boundary conditions. Here q is an $r \times r$ matrix-valued function with entries belonging to $L_2((0,1),\mathbb{C})$ and I is the identity $r \times r$ matrix. We give a complete description of the spectral data (eigenvalues and suitably introduced norming matrices) for the operators under consideration and suggest an algorithm of reconstructing the potential q from the corresponding spectral data.

1. Introduction

Direct and inverse spectral problems for Dirac and Sturm–Liouville operators are the objects of interest in plenty of papers. In 1966, M. Gasymov and B. Levitan solved the inverse spectral problem for Dirac operators on a half-line by using the spectral function [1] and the scattering phase [2]. Their investigations were continued and further developed in many directions.

By now, the direct and inverse spectral problems for Dirac operators with potentials from different classes have been solved. For instance, the Dirac operators on a finite interval with continuous potentials were considered in [3,4] (reconstructing from two spectra), the ones on a half-line were treated in [5] (complete description of the spectral measures and the reconstruction procedure). The case of potentials belonging to $L_p(0,1)$, $p \geqslant 1$, was considered in [6] (reconstructing from two spectra and from one spectrum and the norming constants based on the Krein equation).

The Weyl–Titchmarsh m-functions were used in [7,8] to recover the Dirac operators acting in $L_2(\mathbb{R}_+, \mathbb{C}^{2r})$. More general canonical systems on $\mathbb{R}$ were considered in [9,10]. The matrix-valued Weyl–Titchmarsh functions were recently used in [11] for the characterization of vector-valued Sturm–Liouville operators on the unit interval.

There are many other interesting papers concerning the direct and inverse spectral problems for Dirac and Sturm–Liouville operators besides those mentioned here. We refer the reader to the extensive bibliography cited in [4–12] for further results on that subject.

The aim of the present paper is to extend the results of the recent paper [12] by Ya. Mykytyuk and N. Trush concerning the inverse spectral problems for Sturm–Liouville operators with matrix-valued potentials to the case of Dirac operators on a finite interval with square-summable potentials.

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1.1. Setting of the problem

Let M_r denote the Banach algebra of $r \times r$ matrices with complex entries, which we identify with the Banach algebra of linear operators $\mathbb{C}^r \to \mathbb{C}^r$ endowed with the standard norm. We write $I = I_r$ for the unit element of M_r and M_r^+ for the set of all matrices $A \in M_r$ such that $A = A^* \geqslant 0$. Also we use the notations

$$\mathbb{H} := L_2\big((0,1), \mathbb{C}^r\big) \times L_2\big((0,1), \mathbb{C}^r\big), \qquad \mathfrak{Q} := L_2\big((0,1), M_r\big).$$

Let $q \in \mathfrak{Q}$. Denote

$$\vartheta := \frac{1}{i}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \mathbf{q} := \begin{pmatrix} 0 & q \\ q^* & 0 \end{pmatrix} \tag{1.1}$$

and consider the differential expression

$$\mathfrak{t}_q := \vartheta \frac{d}{dx} + \mathbf{q} \tag{1.2}$$

on the domain $D(\mathfrak{t}_q) = \{y = (y_1, y_2)^\top \mid y_1, y_2 \in W_2^1((0,1), \mathbb{C}^r)\}$, where W_2^1 is the Sobolev space. The object of our investigation is a self-adjoint Dirac operator T_q that is generated by the differential expression (1.2) and the separated boundary conditions $y_1(0) = y_2(0)$, $y_1(1) = y_2(1)$:

$$T_q y = \mathfrak{t}_q(y), \qquad D(T_q) = \big\{y \in D(\mathfrak{t}_q) \;\big|\; y_1(0) = y_2(0),\ y_1(1) = y_2(1)\big\}.$$

The function $q \in \mathfrak{Q}$ will be conventionally called the *potential* of T_q.

The spectrum $\sigma(T_q)$ of the operator T_q consists of countably many isolated real eigenvalues of finite multiplicity, accumulating only at $+\infty$ and $-\infty$. We denote by $\lambda_j(q)$, $j \in \mathbb{Z}$, the pairwise distinct eigenvalues of the operator T_q labeled in increasing order so that $\lambda_0(q) \leqslant 0 < \lambda_1(q)$:

$$\sigma(T_q) = \big\{\lambda_j(q)\big\}_{j \in \mathbb{Z}}.$$

Denote by m_q the Weyl–Titchmarsh function of the operator T_q that is defined as in [7]. The function m_q is a matrix-valued meromorphic Herglotz function (i.e. $\operatorname{Im} m_q(\lambda) \geqslant 0$ whenever $\operatorname{Im}\lambda > 0$), and $\{\lambda_j(q)\}_{j\in\mathbb{Z}}$ is the set of its poles. We put

$$\alpha_j(q) := -\operatorname*{res}_{\lambda = \lambda_j(q)} m_q(\lambda), \quad j \in \mathbb{Z},$$

and call $\alpha_j(q)$ the *norming matrix* of the operator T_q corresponding to the eigenvalue $\lambda_j(q)$. Note that the multiplicity of the eigenvalue $\lambda_j(q)$ of T_q equals $\operatorname{rank}\alpha_j(q)$ and that $\alpha_j(q) \geqslant 0$ for all $j \in \mathbb{Z}$.

We call the collection $\mathfrak{a}_q := ((\lambda_j(q), \alpha_j(q)))_{j\in\mathbb{Z}}$ the *spectral data* of the operator T_q, and the matrix-valued measure

$$\mu_q := \sum_{j=-\infty}^{\infty} \alpha_j(q)\delta_{\lambda_j(q)}$$

is called its *spectral measure*. Here δ_λ is the Dirac delta-measure centered at the point λ. In particular, if $q \equiv 0$ then

$$\mu_0 = \sum_{n=-\infty}^{\infty} I\delta_{\pi n}.$$

The aim is to give a complete description of the class $\mathfrak{A} := \{\mathfrak{a}_q \mid q \in \mathfrak{Q}\}$ of spectral data for Dirac operators under consideration, which is equivalent to the description of the class $\mathfrak{M} := \{\mu_q \mid q \in \mathfrak{Q}\}$ of spectral measures, and to suggest an efficient method of reconstructing the potential q from the corresponding spectral data $\mathfrak{a}_q$.

1.2. Main results

We start from the description of spectral data for operators under consideration. In what follows $\mathfrak{a}$ will stand for an arbitrary sequence $((\lambda_j, \alpha_j))_{j\in\mathbb{Z}}$, in which $(\lambda_j)_{j\in\mathbb{Z}}$ is a strictly increasing sequence of real numbers such that $\lambda_0 \leqslant 0 < \lambda_1$ and α_j are non-zero matrices in M_r^+. By $\mu^{\mathfrak{a}}$ we denote the matrix-valued measure given by

$$\mu^{\mathfrak{a}} := \sum_{j=-\infty}^{\infty} \alpha_j \delta_{\lambda_j}. \tag{1.3}$$

We partition the real axis into pairwise disjoint intervals Δ_n, $n \in \mathbb{Z}$:

$$\Delta_n := \left(\pi n - \frac{\pi}{2}, \pi n + \frac{\pi}{2}\right], \quad n \in \mathbb{Z}.$$

A complete description of the class $\mathfrak{A}$ is given by the following theorem.

Theorem 1.1. *In order that a sequence $\mathfrak{a} = ((\lambda_j, \alpha_j))_{j\in\mathbb{Z}}$ should belong to $\mathfrak{A}$ it is necessary and sufficient that the following conditions are satisfied*:

(A_1) $\sup_{n\in\mathbb{Z}} \sum_{\lambda_j\in\Delta_n} 1 < \infty$, $\sum_{n\in\mathbb{Z}} \sum_{\lambda_j\in\Delta_n} |\lambda_j - \pi n|^2 < \infty$, $\sum_{n\in\mathbb{Z}} \|I - \sum_{\lambda_k\in\Delta_n} \alpha_k\|^2 < \infty$;

(A_2) $\exists N_0 \in \mathbb{N}\ \forall N \in \mathbb{N}$: $(N \geqslant N_0) \Rightarrow \sum_{n=-N}^{N} \sum_{\lambda_j\in\Delta_n} \operatorname{rank} \alpha_j = (2N+1)r$;

(A_3) *the system of functions* $\{de^{i\lambda_j t} \mid j \in \mathbb{Z},\ d \in \operatorname{Ran} \alpha_j\}$ *is complete in* $L_2((-1,1), \mathbb{C}^r)$.

By definition, every $\mathfrak{a} \in \mathfrak{A}$ forms the spectral data for Dirac operator T_q with some $q \in \mathfrak{Q}$. It turns out that this spectral data determine the potential q uniquely:

Theorem 1.2. *The mapping $\mathfrak{Q} \ni q \mapsto \mathfrak{a} = \mathfrak{a}_q \in \mathfrak{A}$ is bijective.*

We base our algorithm of reconstructing the potential q from the corresponding spectral data $\mathfrak{a}_q$ on Krein's accelerant method.

Definition 1.1. We say that a function $H \in L_2((-1,1), M_r)$ is an accelerant if for every $a \in [0,1]$ the integral equation

$$f(x) + \int_0^a H(x-t) f(t)\, dt = 0$$

has only trivial solution in $L_2((0,1), \mathbb{C}^r)$. We denote the set of accelerants by $\mathfrak{H}_2$ and endow it with the metric of the space $L_2((-1,1), M_r)$.

We set $\mathfrak{H}_2^s := \{H \in \mathfrak{H}_2 \mid H(x)^* = H(-x) \text{ a.e. for } x \in (-1,1)\}$.

The spectral data of the operator T_q generate Krein's accelerant as explained in the following theorem.

Theorem 1.3. *Take a sequence $\mathfrak{a} = ((\lambda_j, \alpha_j))_{j\in\mathbb{Z}}$ satisfying the asymptotics (A_1), and set $\mu := \mu^{\mathfrak{a}}$. Then the limit*

$$H_\mu(x) = \lim_{n\to\infty} \int_{-\pi(n-\frac{1}{2})}^{\pi(n+\frac{1}{2})} e^{2i\lambda x}\, d(\mu - \mu_0)(\lambda) \tag{1.4}$$

exists in the topology of the space $L_2((-1,1), M_r)$. If, in addition, (A_3) holds, then the function H_μ is an accelerant and belongs to $\mathfrak{H}_2^s$.

By virtue of Theorem 1.1, any $\mathfrak{a} \in \mathfrak{A}$ satisfies the conditions (A_1)–(A_3). In addition, if $q \in \mathfrak{Q}$ and $\mathfrak{a} = \mathfrak{a}_q$, then $\mu^{\mathfrak{a}} = \mu_q$. Therefore according to Theorem 1.3 we can define the mapping $q \mapsto \Upsilon(q) := H_{\mu_q}$ acting from $\mathfrak{Q}$ to $\mathfrak{H}_2^s$, and in order to solve the inverse spectral problem for the operator T_q we have to find the inverse mapping Υ^{-1}. As in [12], it can be done using the Krein equation.

It is known that for all $H \in \mathfrak{H}_2$ the Krein equation

$$R(x,t) + H(x-t) + \int_0^x R(x,s) H(s-t)\, ds = 0, \quad (x,t) \in \Omega^+, \tag{1.5}$$

where $\Omega^+ := \{(x,t) \mid 0 \leqslant t \leqslant x \leqslant 1\}$, has a unique solution R_H in the class $L_2(\Omega^+, M_r)$. Moreover, if we extend R_H by zero to the triangle $\Omega^- := \{(x,t) \mid 0 \leqslant x < t \leqslant 1\}$, we obtain that $R_H \in G_2(M_r)$ (see Appendix A).

Thus we can define the mapping $\Theta : \mathfrak{H}_2^s \to \mathfrak{Q}$ given by

$$\Theta(H) := i R_H(\cdot, 0). \tag{1.6}$$

The following theorem explains how to solve the inverse spectral problem for the operator T_q.

Theorem 1.4. *$\Upsilon^{-1} = \Theta$. In particular, if $q \in \mathfrak{Q}$, $\mathfrak{a} = \mathfrak{a}_q$, $\mu = \mu^{\mathfrak{a}}$, then*

$$q = \Theta(H_\mu). \tag{1.7}$$

According to this theorem the reconstruction algorithm can proceed as follows. Given $\mathfrak{a} \in \mathfrak{A}$ we construct the matrix-valued measure $\mu := \mu^{\mathfrak{a}}$ via (1.3), which generates the accelerant $H := H_\mu$ via (1.4). Solving the Krein equation (1.5) we find the function R_H, which gives us q via the formulas (1.7) and (1.6). That q is the function looked for follows from the fact that the Dirac operator T_q has the spectral data $\mathfrak{a}$ we have started with.

We visualize the reconstruction algorithm by means of the following diagram:

$$\mathfrak{a} \xrightarrow[s_1]{(1.3)} \mu^{\mathfrak{a}} =: \mu \xrightarrow[s_2]{(1.4)} H_\mu =: H \xrightarrow[s_3]{(1.5)} R_H \xrightarrow[s_4]{(1.6)} \Theta(H) = q.$$

Here s_j denotes the step number j. Steps s_1, s_2, s_4 are trivial. The basic and non-trivial step is s_3.

Remark 1.1. One can also consider the case of more general separated self-adjoint boundary conditions. Denote by $T_{q,a,b}$ the operator generated by the differential expression (1.2) and the boundary conditions

$$ay(0) = 0, \qquad by(1) = 0,$$

where a and b are $r \times 2r$ matrices with complex entries such that (see [7])

$$aa^* = bb^* = I, \qquad a\vartheta a^* = b\vartheta b^* = 0.$$

For the operator $T_{q,a,b}$, the analogues of Theorems 1.1–1.4 can be proved, but their formulations are more complicated since the spectrum of the non-perturbed operator $T_{0,a,b}$ has a more involved structure. Namely, it consists of $2r$ eigenvalue sequences of the form $(\lambda_j^0 + 2\pi k)_{k\in\mathbb{Z}}$, $j = 1, \dots, 2r$, counting multiplicities. The authors plan to consider the case of general (not necessarily separated) boundary conditions in a forthcoming paper.

2. Direct spectral analysis

In this section we study the properties of the spectral data for operators under consideration.

2.1. Basic properties of the operator T_q

Here we prove self-adjointness of T_q, construct its resolvent and the resolution of identity.

Let $\lambda \in \mathbb{C}$. For an arbitrary $q \in \mathfrak{Q}$ denote by $u_q = u_q(\cdot, \lambda) \in W_2^1((0,1), M_{2r})$ a solution of the Cauchy problem

$$\vartheta \frac{d}{dx} u + \mathbf{q} u = \lambda u, \qquad u(0,\lambda) = I_{2r}, \tag{2.1}$$

where ϑ and $\mathbf{q}$ are defined via (1.1). Note that if $q \equiv 0$ then

$$u_0(x,\lambda) = \begin{pmatrix} e^{i\lambda x} I & 0 \\ 0 & e^{-i\lambda x} I \end{pmatrix}. \tag{2.2}$$

Denote

$$\varphi_q(\cdot,\lambda) := u_q(\cdot,\lambda)\vartheta a^*, \qquad \psi_q(\cdot,\lambda) := u_q(\cdot,\lambda) a^*, \tag{2.3}$$

where

$$a := \frac{1}{\sqrt{2}} (\, I, \quad -I\,),$$

and set $s(\lambda, q) := a\varphi_q(1,\lambda)$, $c(\lambda, q) := a\psi_q(1,\lambda)$, $m_q(\lambda) := -s(\lambda,q)^{-1} c(\lambda,q)$. We call m_q *the Weyl–Titchmarsh* function of the operator T_q.

Some basic properties of the objects just introduced are described in the following lemma.

Lemma 2.1.

(i) *For every $q \in \mathfrak{Q}$ there exists a unique matrix-valued function $K_q \in G_2^+(M_{2r})$ such that for any $\lambda \in \mathbb{C}$ and $x \in [0,1]$,*

$$\varphi_q(x,\lambda) = \varphi_0(x,\lambda) + \int_0^x K_q(x,s)\varphi_0(s,\lambda)\, ds; \tag{2.4}$$

(ii) *the mapping* $\mathfrak{Q} \ni q \mapsto K_q \in G_2^+(M_{2r})$ *is continuous*;
(iii) *the matrix-valued functions* $\lambda \mapsto s(\lambda, q)$ *and* $\lambda \mapsto c(\lambda, q)$ *are entire and allow the representations*

$$s(\lambda, q) = (\sin\lambda)I + \int_{-1}^{1} e^{i\lambda t} g_1(t)\, dt, \qquad c(\lambda, q) = (\cos\lambda)I + \int_{-1}^{1} e^{i\lambda t} g_2(t)\, dt, \tag{2.5}$$

where g_1 *and* g_2 *are some (depending on* q*) functions from the space* $L_2((-1,1), M_r)$;
(iv) *for every* $q \in \mathfrak{Q}$ *the following relation holds*:

$$-\psi_q(x,\lambda)\varphi_q(x,\bar{\lambda})^* + \varphi_q(x,\lambda)\psi_q(x,\bar{\lambda})^* \equiv \vartheta. \tag{2.6}$$

Proof. Let us fix $q \in \mathfrak{Q}$ and set $q_1 := -\operatorname{Im} q = -\frac{1}{2}(q - q^*)$, $q_2 := \operatorname{Re} q = \frac{1}{2}(q + q^*)$. Consider the Cauchy problem

$$B\frac{d}{dx}v + Qv = \lambda v, \qquad v(0,\lambda) = I_{2r},$$

where

$$B := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \qquad Q := \begin{pmatrix} q_1 & q_2 \\ q_2 & -q_1 \end{pmatrix}.$$

It follows from [13] that this problem has a unique solution $v_q = v_q(\cdot,\lambda)$ in $W_2^1((0,1), M_{2r})$ and that $v_q(\cdot,\lambda)$ can be represented in the form

$$v_q(x,\lambda) = e^{-\lambda xB} + \int_0^x P^+(x,s)e^{-\lambda(x-2s)B}\, ds + \int_0^x P^-(x,s)e^{\lambda(x-2s)B}\, ds, \tag{2.7}$$

where $e^{xB} = (\cos x)I_{2r} + (\sin x)B$.

Note that $\vartheta = W^{-1}BW$ and $\mathbf{q} = W^{-1}QW$, where W is the unitary matrix

$$W = \frac{1}{\sqrt{2}}\begin{pmatrix} I & -iI \\ -iI & I \end{pmatrix}.$$

Therefore the function $u_q(\cdot,\lambda) = W^{-1}v_q(\cdot,\lambda)W$ solves the Cauchy problem (2.1).

Note that

$$e^{xB}J = Je^{-xB}, \quad J = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

Using now (2.7) and performing some calculations we easily obtain that

$$\varphi_q(x,\lambda) = \varphi_0(x,\lambda) + \int_0^x K_q(x,s)\varphi_0(s,\lambda)\, ds,$$

where $K_q(x,t) = W^{-1}P_Q(x,t)W$ and

$$P_Q(x,t) = \frac{1}{2}\left\{P^+\left(x, \frac{x-t}{2}\right) + P^+\left(x, \frac{x+t}{2}\right)J + P^-\left(x, \frac{x-t}{2}\right)J + P^-\left(x, \frac{x+t}{2}\right)\right\}.$$

It follows from [13] that the function P_Q belongs to $G_2^+(M_{2r})$ and that the mapping $L_2((0,1), M_{2r}) \ni Q \mapsto P_Q \in G_2^+(M_{2r})$ is continuous. Therefore the first two statements of the present lemma will be proved if we prove the uniqueness of the representation (2.4), but this can be easily done repeating the proof given in [13].

Now let us prove (iii). By virtue of the definition of $s(\lambda, q)$ and the representation (2.4) we obtain that

$$s(\lambda, q) = (\sin\lambda)I + \int_0^1 aK_q(1,s)\varphi_0(s,\lambda)\, ds$$

and simple calculations yield the formula for $s(\lambda)$ in (2.5) with some $g_1 \in L_2((-1,1), M_r)$. Having noted that $\psi_q(x,\lambda) = u_q(x,\lambda)a^* = W^{-1}v_q(x,\lambda)Wa^*$ and taking into consideration (2.7) we can analogously obtain the formula for $c(\lambda)$.

It remains to prove (iv). A direct verification shows that

$$\frac{d}{dx}\left\{u_q(x,\bar{\lambda})^*\vartheta u_q(x,\lambda)\right\}\equiv 0,$$

and therefore we obtain the relation $u_q(x,\bar{\lambda})^*\vartheta u_q(x,\lambda)\equiv\vartheta$. From this equality we obtain that $\vartheta u_q(x,\lambda)\vartheta u_q(x,\bar{\lambda})^*\equiv -I_{2r}$, and thus $u_q(x,\lambda)\vartheta u_q(x,\bar{\lambda})^*\equiv\vartheta$. Having noted that $\vartheta = a^*a\vartheta+\vartheta a^*a$ we conclude that

$$u_q(x,\lambda)a^*a\vartheta u_q(x,\bar{\lambda})^*+u_q(x,\lambda)\vartheta a^*a u_q(x,\bar{\lambda})^*\equiv\vartheta,$$

which proves the relation (2.6). □

For $\lambda\in\mathbb{C}$ denote by $\Phi_q(\lambda)$ the operator acting from $\mathbb{C}^r$ to $\mathbb{H}$ by the formula

$$\left[\Phi_q(\lambda)c\right](x):=\varphi_q(x,\lambda)c. \tag{2.8}$$

Taking into consideration (2.4) we obtain that

$$\Phi_q(\lambda)=(\mathscr{I}+\mathscr{K}_q)\Phi_0(\lambda),\quad \lambda\in\mathbb{C}, \tag{2.9}$$

where $\mathscr{K}_q$ is an integral operator with kernel K_q and $\mathscr{I}$ is the identity operator in $\mathscr{B}(\mathbb{H})$, which is the algebra of bounded linear operators acting in $\mathbb{H}$. Note that since K_q belongs to $G_2^+(M_{2r})$, the operator $\mathscr{K}_q$ belongs to $\mathscr{G}_2^+(M_{2r})$ (see Appendix A), and hence it is a Volterra operator (see [14]).

Some properties of the operators $\Phi_q(\lambda)$ and the Weyl–Titchmarsh function $m_q(\lambda)$ are formulated in the following lemma.

Lemma 2.2. *Let $q\in\mathfrak{Q}$. Then the following statements hold*:

(i) *the operator function $\lambda\mapsto\Phi_q(\lambda)$ is analytic in $\mathbb{C}$; moreover, for $\lambda\in\mathbb{C}$,*

$$\ker\Phi_q(\lambda)=\{0\},\qquad \operatorname{Ran}\Phi_q(\lambda)^*=\mathbb{C}^r, \tag{2.10}$$

$$\ker(T_q-\lambda\mathscr{I})=\Phi_q(\lambda)\ker s(\lambda,q). \tag{2.11}$$

(ii) *the operator functions $\lambda\mapsto s(\lambda,q)^{-1}$ and*

$$\lambda\mapsto m_q(\lambda)=-s(\lambda,q)^{-1}c(\lambda,q)$$

are meromorphic in $\mathbb{C}$; moreover, $m_0(\lambda)=-\cot\lambda I$ and

$$\left\|m_q(\lambda)+\cot\lambda I\right\|=o(1) \tag{2.12}$$

as $\lambda\to\infty$ within the domain $\mathcal{O}=\{z\in\mathbb{C}\mid\forall n\in\mathbb{Z}\ |z-\pi n|>1\}$.

Proof. The proof of this lemma is analogous to the proof of Lemma 2.3 in [12]. □

Finally, basic properties of the operator T_q are described in the following theorem.

Theorem 2.1. *Let $q\in\mathfrak{Q}$. Then the following statements hold*:

(i) *the operator T_q is self-adjoint*;
(ii) *the spectrum $\sigma(T_q)$ of T_q consists of isolated real eigenvalues and*

$$\sigma(T_q)=\left\{\lambda\ \middle|\ \ker s(\lambda,q)\neq\{0\}\right\};$$

(iii) *let $\lambda_j=\lambda_j(q)$ and let $P_{j,q}$ be the orthogonal projector on $\ker(T_q-\lambda\mathscr{I})$, then*

$$\sum_{j=-\infty}^{\infty}P_{j,q}=\mathscr{I};$$

(iv) *the norming matrices $\alpha_j=\alpha_j(q)$ satisfy the relations $\alpha_j\geqslant 0$, $j\in\mathbb{Z}$; moreover, for all $j\in\mathbb{Z}$ we have*

$$P_{j,q}=\Phi_q(\lambda_j)\alpha_j\Phi_q^*(\lambda_j),$$

where $\Phi_q^*(\lambda):=[\Phi_q(\lambda)]^*$.

Proof. A direct verification shows that the operator T_q is symmetric. Take an arbitrary λ such that the matrix $s(\lambda, q)$ is non-singular, and let $f \in \mathbb{H}$. Then the function

$$g(x) = \left[\mathscr{T}(\lambda) f\right](x) := \psi_q(x, \lambda) \int_0^x \varphi_q(t, \bar{\lambda})^* f(t)\, dt + \varphi_q(x, \lambda) \int_x^1 \psi_q(t, \bar{\lambda})^* f(t)\, dt$$

belongs to the domain of differential expression $\mathfrak{t}_q$ and solves the Cauchy problem

$$\mathfrak{t}_q(g) = \lambda g + f, \qquad ag(0) = 0,$$

as can be directly verified using (2.6). A generic solution of this problem takes the form $h = \varphi_q(\cdot, \lambda)c + \mathscr{T}(\lambda) f$, $c \in \mathbb{C}^r$. The choice

$$c = m_q(\lambda) \int_0^1 \varphi_q(t, \bar{\lambda})^* f(t)\, dt$$

gives that $ah(1) = 0$, i.e. the boundary conditions $h_1(0) = h_2(0)$, $h_1(1) = h_2(1)$ are satisfied. This implies that λ is a resolvent point of the operator T_q, and the resolvent of T_q is given by

$$(T_q - \lambda \mathscr{I})^{-1} = \Phi_q(\lambda) m_q(\lambda) \Phi_q^*(\bar{\lambda}) + \mathscr{T}(\lambda).$$

Since $\mathscr{T}(\lambda)$ is a Hilbert–Schmidt operator, the operator T_q has a compact resolvent, and therefore the statements (i)–(iii) are proved.

Recall that $-\alpha_j(q)$ is a residue of the Weyl–Titchmarsh function at the point $\lambda_j = \lambda_j(q)$, $j \in \mathbb{Z}$. Taking $\varepsilon > 0$ small enough we obtain that

$$P_{j,q} = -\frac{1}{2\pi i} \oint_{|\lambda - \lambda_j| = \varepsilon} (T_q - \lambda \mathscr{I})^{-1}\, d\zeta = \Phi_q(\lambda_j) \alpha_j(q) \Phi_q^*(\lambda_j)$$

for every $j \in \mathbb{Z}$.

By virtue of (2.10) we obtain that $\alpha_j(q) \geqslant 0$ for all $j \in \mathbb{Z}$, and the statement (iv) is also proved. $\square$

2.2. Description of the spectral data: the necessity part

Here we show that if $q \in \mathfrak{Q}$, then the spectral data $\mathfrak{a}_q$ satisfy the conditions (A_1)–(A_3), which is the necessity part of Theorem 1.1.

2.2.1. The condition (A_1)

In the sequel we shall use the following notations. If $(\lambda_j)_{j \in \mathbb{Z}}$ is a strictly increasing sequence of non-negative real numbers and $(\alpha_j)_{j \in \mathbb{Z}}$ is a sequence in M_r^+, then

$$\beta_n := I - \sum_{\lambda_k \in \Delta_n} \alpha_k, \qquad \widetilde{\lambda}_j := \lambda_j - \pi n, \quad \lambda_j \in \Delta_n,\ n \in \mathbb{Z}, \tag{2.13}$$

with Δ_n being defined in Subsection 1.2.

We start from the condition (A_1), which describes the asymptotics of spectral data.

Theorem 2.2. *Let $q \in \mathfrak{Q}$. Then for the sequence $\mathfrak{a} = \mathfrak{a}_q$ the condition (A_1) holds.*

Sketch of the proof. The proof of this theorem is analogous to the proof in [12], and therefore we give here only its sketch. Let $q \in \mathfrak{Q}$ and $\lambda_j = \lambda_j(q)$, $\alpha_j = \alpha_j(q)$ for $j \in \mathbb{Z}$. The eigenvalues λ_j are zeros of the sine-type function $\lambda \mapsto s(\lambda)$ (see (2.5)) that belongs to the following class of functions $\mathbb{C} \to M_r$:

$$\mathcal{F}_f(\lambda) := \sin\lambda I + \int_{-1}^1 f(t) e^{i\lambda t}\, dt, \quad \lambda \in \mathbb{C},$$

where $f \in L_2((-1,1), M_r)$. It is shown in [15] that the set of zeros of a function $\det \mathcal{F}_f$, with $\mathcal{F}_f$ as above, can be indexed (counting multiplicities) by the set $\mathbb{Z}$ so that the corresponding sequence $(\omega_n)_{n \in \mathbb{Z}}$ of its zeros has the asymptotics

$$\omega_{kr+j} = \pi k + \hat{\omega}_{j,k}, \quad k \in \mathbb{Z},\ j = 0, \dots, r-1,$$

where the sequences $(\hat{\omega}_{j,k})_{k\in\mathbb{Z}}$ belong to $\ell_2(\mathbb{Z})$. Therefore,

$$\sup_{n\in\mathbb{Z}} \sum_{\lambda_j\in\Delta_n} 1 < \infty, \qquad \sum_{n\in\mathbb{Z}} \sum_{\lambda_j\in\Delta_n} |\widetilde{\lambda}_j|^2 < \infty, \tag{2.14}$$

and thus it is left to prove only that (see (2.13))

$$\sum_{n=-\infty}^{\infty} \|\beta_n\|^2 < \infty.$$

It can be done in exactly the same way as in [12]. □

2.2.2. The condition (A_2)

We start from proving the following lemma, which is an analogue of Lemma 2.12 in [12].

Lemma 2.3. *Assume that $q \in \mathfrak{Q}$, and let $\mathfrak{a}$ be a collection satisfying the asymptotics (A_1). For $j \in \mathbb{Z}$ set $\hat{P}_j := \Phi_q(\lambda_j)\alpha_j\Phi_q^*(\lambda_j)$. Then the series $\sum_{j\in\mathbb{Z}} \hat{P}_j$ converges in the strong operator topology and*

$$\sum_{n=-\infty}^{\infty} \left\| P_{n,0} - \sum_{\lambda_j\in\Delta_n} \hat{P}_j \right\|^2 < \infty. \tag{2.15}$$

Sketch of the proof. Let the assumptions of the present lemma hold, and let $\mathfrak{a} = ((\lambda_j, \alpha_j))_{j\in\mathbb{Z}}$. Using (2.9) and the fact that $\mathscr{K}_q$ is a Hilbert–Schmidt operator, it can be observed that

$$\sum_{n\in\mathbb{Z}} \sum_{\lambda_j\in\Delta_n} \left\| \Phi_q(\pi n) - \Phi_0(\pi n) \right\|^2 < \infty.$$

Since $\|\Phi_q(\lambda_j) - \Phi_q(\pi n)\| \leqslant C|\widetilde{\lambda}_j|$ ($\lambda_j \in \Delta_n$, $n \in \mathbb{Z}$) for some $C > 0$,

$$\sum_{n\in\mathbb{Z}} \sum_{\lambda_j\in\Delta_n} \left\| \Phi_q(\lambda_j) - \Phi_0(\pi n) \right\|^2 < \infty. \tag{2.16}$$

From (2.16) we easily obtain that

$$\sum_{j\in\mathbb{Z}} \left\| \Phi_q^*(\lambda_j) f \right\|^2 < \infty \tag{2.17}$$

for all $f \in \mathbb{H}$. Indeed, it is enough to note that $\sum_{n\in\mathbb{Z}} \|\Phi_0(\pi n)^* f\|^2 = \|f\|^2$, $f \in \mathbb{H}$, and that $\sup_{n\in\mathbb{Z}} \sum_{\lambda_j\in\Delta_n} 1 < \infty$.

Taking into account that the sequence $(\alpha_j)_{j\in\mathbb{Z}}$ is bounded we conclude that $\sum_{j\in\mathbb{Z}} \|\alpha_j \Phi^*(\lambda_j) f\|^2 < \infty$, $f \in \mathbb{H}$. Moreover, it can also be shown that for every sequence $c \in l_2(\mathbb{Z}, \mathbb{C}^r)$ the series $\sum_{j\in\mathbb{Z}} \Phi_q(\lambda_j) c_j$ is convergent, which justifies the convergence of $\sum_{j\in\mathbb{Z}} \hat{P}_j$.

Now let us prove (2.15). Recall that $P_{n,0} = \Phi_0(\pi n)\Phi_0^*(\pi n)$. By virtue of the definition of β_n we obtain that

$$P_{n,0} = \Phi_0(\pi n)\beta_n\Phi_0^*(\pi n) + \sum_{\lambda_j\in\Delta_n} \Phi_0(\pi n)\alpha_j\Phi_0^*(\pi n),$$

and thus we can write

$$P_{n,0} - \sum_{\lambda_j\in\Delta_n} \hat{P}_j = \Phi_0(\pi n)\beta_n\Phi_0^*(\pi n) + \sum_{\lambda_j\in\Delta_n} \left[\Phi_0(\pi n)\alpha_j\Phi_0^*(\pi n) - \Phi_q(\lambda_j)\alpha_j\Phi_q^*(\lambda_j)\right].$$

Thus, since the sequences $(\Phi_q(\lambda_j))$ and (α_j) are bounded, we obtain that

$$\left\| P_{n,0} - \sum_{\lambda_j\in\Delta_n} \hat{P}_j \right\|^2 \leqslant C_1\|\beta_n\|^2 + C_2 \sum_{\lambda_j\in\Delta_n} \left\| \Phi_q(\lambda_j) - \Phi_0(\pi n) \right\|^2,$$

where C_1 and C_2 are non-negative constants independent of n. Taking now into consideration (2.16) and (A_1), we obtain (2.15). □

The following lemma is proved in [12] (Lemma B.1).

Lemma 2.4. *Suppose that H is a Hilbert space. Let $(P_n)_{n=1}^\infty$ and $(G_n)_{n=1}^\infty$ be sequences of pairwise orthogonal projectors of finite rank in H such that $\sum_{n=1}^\infty P_n = \sum_{n=1}^\infty G_n = \mathscr{I}_H$, where $\mathscr{I}_H$ is the identity operator in H, and let $\sum_{n=1}^\infty \|P_n - G_n\|^2 < \infty$. Then there exists $N_0 \in \mathbb{N}$ such that for all $N \geqslant N_0$,*

$$\sum_{n=1}^N \operatorname{rank} P_n = \sum_{n=1}^N \operatorname{rank} G_n.$$

We use Lemmas 2.3 and 2.4 to prove (A_2). If $\mathfrak{a} = \mathfrak{a}_q$, then the operators $\hat{P}_j$, $j \in \mathbb{Z}$, from Lemma 2.3 coincide with the orthogonal projectors $P_{j,q}$ corresponding to the eigenvalues λ_j (see Theorem 2.1). Since $\{P_{j,q}\}$, $j \in \mathbb{Z}$, forms a complete system of orthogonal projectors, by virtue of Lemmas 2.3 and 2.4 we justify that

$$\sum_{n=-N}^N \operatorname{rank} P_{n,0} = \sum_{n=-N}^N \sum_{\lambda_j \in \Delta_n} \operatorname{rank} P_{j,q}$$

for $N \geqslant N_0$. Taking into consideration (2.10), we obtain that $\operatorname{rank} P_{j,q} = \operatorname{rank} \alpha_j$ and $\operatorname{rank} P_{n,0} = r$ for all $j, n \in \mathbb{Z}$, and thus we justify that the condition (A_2) is satisfied.

2.2.3. The operators $\mathscr{U}_{\mathfrak{a},q}$

Before proving the condition (A_3) we have to introduce some operators that play an important role below.

Let $q \in \mathfrak{Q}$, and let $\mathfrak{a} = ((\lambda_j, \alpha_j))_{j\in\mathbb{Z}}$ be any collection satisfying the asymptotics (A_1). Construct the operator $\mathscr{U}_{\mathfrak{a},q} : \mathbb{H} \to \mathbb{H}$ by the formula

$$\mathscr{U}_{\mathfrak{a},q} := \sum_{j\in\mathbb{Z}} \Phi_q(\lambda_j)\alpha_j \Phi_q^*(\lambda_j). \tag{2.18}$$

By virtue of Lemma 2.3 the operator $\mathscr{U}_{\mathfrak{a},q}$ is continuous, and, since $\alpha_j \geqslant 0$ for all $j \in \mathbb{Z}$ (see Theorem 2.1), it is also non-negative.

In particular,

$$\mathscr{U}_{\mathfrak{a}_q,q} = \mathscr{I}, \tag{2.19}$$

as follows from Theorem 2.1.

Now we are going to show that the operator $\mathscr{U}_{\mathfrak{a},q}$ is the sum of the identity one and a compact one. We start from proving the following lemma.

Lemma 2.5. *Let $\mathfrak{a}$ be any collection satisfying the condition (A_1). Then the limit (1.4) exists in the topology of the space $L_2((-1,1), M_r)$, and the following relation holds:*

$$H_\mu(x)^* = H_\mu(-x). \tag{2.20}$$

Proof. Taking into consideration the definitions of measures μ and μ_0 it is easy to observe that the function $H := H_\mu$ can be rewritten as

$$H(x) = \sum_{n\in\mathbb{Z}} \left\{ \left(\sum_{\lambda_j\in\Delta_n} e^{2i\lambda_j x}\alpha_j \right) - e^{2i\pi n x} I \right\}, \tag{2.21}$$

and thus we have to show that the series (2.21) is convergent in $L_2((-1,1), M_r)$.

Note that

$$\left(\sum_{\lambda_j\in\Delta_n} e^{2i\lambda_j x}\alpha_j \right) - e^{2i\pi n x} I = e^{2i\pi n x}\gamma_n(x) + x e^{2i\pi n x}\eta_n - e^{2i\pi n x}\beta_n, \tag{2.22}$$

where

$$\gamma_n(x) := \sum_{\lambda_j\in\Delta_n} \big(e^{2i\widetilde{\lambda}_j x} - 1 - 2i\widetilde{\lambda}_j x\big)\alpha_j, \qquad \eta_n := \sum_{\lambda_j\in\Delta_n} 2i\widetilde{\lambda}_j\alpha_j,$$

β_n and $\widetilde{\lambda}_j$ are given by (2.13). Since the sequence $(\alpha_j)_{j\in\mathbb{Z}}$ is bounded and $|e^z - 1 - z| \leqslant |z|^2 e^{|z|}$, $z \in \mathbb{C}$, in view of the condition (A_1) we obtain that

$$\sum_{n\in\mathbb{Z}} \sup_{x\in[0,1]} \|\gamma_n(x)\| < \infty, \qquad \sum_{n\in\mathbb{Z}} \|\eta_n\|^2 < \infty, \qquad \sum_{n\in\mathbb{Z}} \|\beta_n\|^2 < \infty.$$

Therefore, taking into consideration (2.22) it is easy to observe that the series (2.21) is convergent in the topology of the space $L_2((-1,1), M_r)$.

The relation (2.20) follows directly from the formula (2.21). □

For $H \in L_2((-1,1), M_r)$ denote

$$F_H(x,t) := \frac{1}{2}\begin{pmatrix} H(\frac{x-t}{2}) & H(\frac{x+t}{2}) \\ H^\sharp(\frac{x+t}{2}) & H^\sharp(\frac{x-t}{2}) \end{pmatrix}, \tag{2.23}$$

where $H^\sharp(x) := H(-x)$. Note that $F_H \in G_2(M_{2r})$.

Proposition 2.1. *Let $\mathfrak{a}$ be any collection satisfying the asymptotics (A_1), and set $\mu := \mu^{\mathfrak{a}}$, $H := H_\mu$. Then*

$$\mathscr{U}_{\mathfrak{a},0} = \mathscr{I} + \mathscr{F}_H, \tag{2.24}$$

where $\mathscr{F}_H$ is a Hilbert–Schmidt operator with kernel F_H, i.e.

$$(\mathscr{F}_H f)(x) = \int_0^1 F_H(x,s) f(s)\,ds, \quad f \in L_2\big((0,1), \mathbb{C}^{2r}\big).$$

Proof. The proof can be obtained by direct verification. □

2.2.4. The condition (A_3)

Now let us prove that for all $q \in \mathfrak{Q}$ the spectral data $\mathfrak{a}_q$ satisfy the condition (A_3). In view of (2.19), this fact directly follows from the following lemma.

Lemma 2.6. *Let $q \in \mathfrak{Q}$, and let $\mathfrak{a} = ((\lambda_j, \alpha_j))_{j\in\mathbb{Z}}$ be any collection satisfying the asymptotics (A_1). Then*

$$(A_3) \quad \Leftrightarrow \quad \mathscr{U}_{\mathfrak{a},q} > 0. \tag{2.25}$$

Proof. Taking into consideration the relation (2.9), we obtain that

$$\mathscr{U}_{\mathfrak{a},q} = (\mathscr{I} + \mathscr{K}_q)\mathscr{U}_{\mathfrak{a},0}\big(\mathscr{I} + \mathscr{K}_q^*\big). \tag{2.26}$$

Since the operator $\mathscr{I} + \mathscr{K}_q$ is a homeomorphism of the space $\mathbb{H}$, it is enough to prove the equivalence (2.25) only for the case $q = 0$.

Since $\mathscr{U}_{\mathfrak{a},0} \geqslant 0$ and the operator $\mathscr{F}_H$ in (2.24) is compact, we obtain that $\mathscr{U}_{\mathfrak{a},0} > 0$ if and only if $\ker \mathscr{U}_{\mathfrak{a},0} = \{0\}$. Thus it is enough to prove the equivalence

$$(A_3) \quad \Leftrightarrow \quad \ker \mathscr{U}_{\mathfrak{a},0} = \{0\}. \tag{2.27}$$

Set $\mathcal{X} := \{e^{i\lambda_j t} d \mid j \in \mathbb{Z},\ d \in \operatorname{Ran}\alpha_j\} \subset L_2((-1,1), \mathbb{C}^r)$ and note that the condition (A_3) is equivalent to the equality $\mathcal{X}^\perp = \{0\}$. Consider the unitary transformation $U : L_2((-1,1), \mathbb{C}^r) \to \mathbb{H}$ given by

$$(Uf)(x) := \big(f(-x), f(x)\big) \in \mathbb{C}^{2r}, \quad x \in (0,1).$$

It follows from the definitions of $\mathscr{U}_{\mathfrak{a},0}$ and $\Phi_0(\lambda)$ that

$$\ker \mathscr{U}_{\mathfrak{a},0} = \bigcap_{j\in\mathbb{Z}} \ker \alpha_j \Phi_0^*(\lambda_j) = (U\mathcal{X})^\perp = U\mathcal{X}^\perp,$$

and therefore (2.27) is proved. □

3. Inverse spectral problem

In this section we solve the inverse spectral problem for the operator T_q. We show that if a collection $\mathfrak{a}$ satisfies the conditions (A_1)–(A_3), then $\mathfrak{a} = \mathfrak{a}_q$ for some $q \in \mathfrak{Q}$ and suggest a method of constructing such q.

3.1. The Krein accelerant: proof of Theorem 1.3

Here we prove Theorem 1.3, i.e. we show that any collection $\mathfrak{a}$ satisfying the conditions (A_1) and (A_3) generates the Krein accelerant belonging to $\mathfrak{H}_2^s$.

Since the convergence of (1.4) was already proved, it is left to prove only the following lemma.

Lemma 3.1. *Let $\mathfrak{a}$ satisfy the conditions (A_1) and (A_3), $\mu := \mu^{\mathfrak{a}}$, $H := H_\mu$. Then the function H is an accelerant and belongs to $\mathfrak{H}_2^s$.*

Proof. Let us prove that H belongs to $\mathfrak{H}_2$. It is enough to prove that the operator $\mathscr{I} + \mathscr{H}$ is positive in $L_2((0,1),\mathbb{C}^r)$, where $\mathscr{H}$ is given by

$$(\mathscr{H} f)(x) = \int_0^1 H(x-t) f(t)\,dt. \tag{3.1}$$

By virtue of (2.24) and Lemma 2.6, the condition (A_3) implies the positivity of the operator $\mathscr{I} + \mathscr{F}_H$ in the space $L_2((0,1),\mathbb{C}^{2r})$. Consider the unitary transformation $V : L_2((0,1),\mathbb{C}^{2r}) \to L_2((0,1),\mathbb{C}^r)$,

$$(Vf)(t) = \begin{cases} \sqrt{2} f_2(1-2t), & t \in (0,1/2], \\ \sqrt{2} f_1(2t-1), & t \in (1/2,1). \end{cases}$$

A direct verification shows that $\mathscr{I} + \mathscr{H} = V(\mathscr{I} + \mathscr{F}_H)V^{-1}$, and thus the operators $\mathscr{I} + \mathscr{H}$ and $\mathscr{I} + \mathscr{F}_H$ are unitary equivalent. Therefore $I + \mathscr{H} > 0$ in $L_2((0,1),\mathbb{C}^r)$. It is left to notice that by virtue of the relation (2.20) the function H belongs to $\mathfrak{H}_2^s$. $\square$

3.2. Factorization of $\mathscr{U}_{\mathfrak{a},0}$

Given a collection $\mathfrak{a}$ satisfying the asymptotics (A_1), put $\mu := \mu^{\mathfrak{a}}$, $H := H_\mu$, and construct the operator $\mathscr{U}_{\mathfrak{a},0}$ via (2.18). In this subsection we show that $\mathscr{U}_{\mathfrak{a},0}$ admits a factorization in $\mathscr{G}_2(M_r)$. Some statements concerning the theory of factorization can be found in Appendix B.

3.2.1. Basic properties of R_H

Recall that for $H \in \mathfrak{H}_2$ we denote by R_H the solution of the Krein equation (1.5). Here we prove some basic properties of R_H.

Lemma 3.2.

(i) *If $H \in \mathfrak{H}_2$, then $R_H \in G_2^+(M_r)$ and the mapping*

$$\mathfrak{H}_2 \ni H \mapsto R_H \in G_2^+(M_r)$$

is continuous;

(ii) *if $H \in \mathfrak{H}_2^s$, then $H^\sharp \in \mathfrak{H}_2^s$ and*

$$R_{H^\sharp}(\cdot,0) = \big[R_H(\cdot,0)\big]^*; \tag{3.2}$$

(iii) *the mapping $\Theta : \mathfrak{H}_2^s \to \mathfrak{Q}$ given by $\Theta(H) := iR_H(\cdot,0)$ is continuous*;

(iv) *if $H \in \mathfrak{H}_2 \cap C^1([-1,1],M_r)$, then $R_H \in C^1(\Omega^+,M_r)$.*

Proof. We start from proving (i). Suppose that $H \in \mathfrak{H}_2$. Denote by $\mathscr{H}$ the operator given by (3.1), and set $\mathscr{H}^a := \chi_a \mathscr{H} \chi_a$ (see Appendix B). Since $H \in \mathfrak{H}_2$, $\ker(\mathscr{I} + \mathscr{H}^a) = \{0\}$ for all $a \in [0,1]$, and the operator $\mathscr{I} + \mathscr{H}^a$ is invertible in the algebra $\mathscr{B}(L_2)$ of bounded linear operators acting in $L_2((0,1),\mathbb{C}^r)$. Since $\mathscr{H}^a$ depends continuously on $a \in [0,1]$, the mapping $[0,1] \ni a \mapsto (\mathscr{I} + \mathscr{H}^a)^{-1} \in \mathscr{B}(L_2)$ is continuous. Denote by $\Gamma_{a,H}$ the kernel of the integral operator $-\mathscr{H}^a(\mathscr{I} + \mathscr{H}^a)^{-1}$. Since $\mathscr{H}$ is a Hilbert–Schmidt operator, the mapping

$$[0,1] \times \mathfrak{H}_2 \ni (a,H) \mapsto \Gamma_{a,H} \in L_2\big((0,1)^2, M_r\big)$$

is also continuous.

For $(x,t) \in \Omega^+$ put

$$\hat{R}_H(x,t) := \int_0^x H(x-y)H(y-t)\,dy + \int_0^x \int_0^x H(x-u)\Gamma_{x,H}(u,v)H(v-t)\,dv\,du. \tag{3.3}$$

It is easily seen that the mapping $\mathfrak{H}_2 \ni H \mapsto \hat{R}_H \in C(\Omega^+, M_r)$ is continuous. A direct verification shows that the function

$$R_H(x,t) := \begin{cases} \hat{R}_H(x,t) - H(x-t), & (x,t) \in \Omega^+, \\ 0, & (x,t) \in \Omega^- \end{cases} \tag{3.4}$$

solves the Krein equation (1.5). Therefore R_H belongs to $G_2^+(M_r)$, and the mapping $\mathfrak{H}_2 \ni H \mapsto R_H \in G_2^+(M_r)$ is continuous.

Let us prove (ii). Assume that $H \in \mathfrak{H}_2^s$. First let us show that $H^\sharp \in \mathfrak{H}_2^s$. Construct the integral operator $\mathscr{H}^\sharp$ via the formula (3.1) with $H^\sharp$ instead of H. The operators $\mathscr{I} + \mathscr{H}^\sharp$ and $\mathscr{I} + \mathscr{H}$ are unitary equivalent under the unitary transformation $f(t) \mapsto f(1-t)$. Therefore $I + \mathscr{H} > 0$ if and only if $I + \mathscr{H}^\sharp > 0$, and thus $H^\sharp \in \mathfrak{H}_2^s$. The equality (3.2) can be easily verified having noted that $\Gamma_{a,H}(a-x, a-t) = \Gamma_{a,H^\sharp}(x,t)$ and $\Gamma_{a,H}(x,t) = [\Gamma_{a,H}(t,x)]^*$ for all $x, t \in [0,a]$ and for all $H \in \mathfrak{H}_2^s$.

The continuity of Θ easily follows from its definition and continuity of the mapping $H \mapsto R_H$, and thus the statement (iii) is proved.

It is left to prove (iv). It follows from [14, Chapter IV] that if $H \in \mathfrak{H}_2 \cap C^1([-1,1], M_r)$, then the function $a \mapsto \Gamma_{a,H}(u,v)$ is continuously differentiable for $a \geqslant \max\{u, v\}$. Therefore taking into consideration (3.4) and (3.3) we conclude that $R_H \in C^1(\Omega^+, M_r)$. □

3.2.2. The GLM equation

Here we establish structure of the solution of Gelfand–Levitan–Marchenko (GLM) equation.

Lemma 3.3. *Let $H \in L_2((-1,1), M_r)$. If $H \in \mathfrak{H}_2^s$, then the GLM equation*

$$L(x,t) + F_H(x,t) + \int_0^x L(x,s) F_H(s,t)\, ds = 0, \quad (x,t) \in \Omega^+ \tag{3.5}$$

has a unique solution in the class $L_2(\Omega^+, M_{2r})$; moreover, this solution belongs to $G_2^+(M_{2r})$ and takes the form

$$L_H(x,t) = \frac{1}{2} \begin{pmatrix} R_H(x, \frac{x+t}{2}) & R_H(x, \frac{x-t}{2}) \\ R_{H^\sharp}(x, \frac{x-t}{2}) & R_{H^\sharp}(x, \frac{x+t}{2}) \end{pmatrix}. \tag{3.6}$$

Proof. A direct verification shows that the function L_H given by (3.6) solves the GLM equation (3.5). Since $F_H \in G_2(M_{2r})$ and $L_H \in L_2(\Omega^+, M_{2r})$, the results of Appendix B yield that $L_H \in G_2^+(M_{2r})$. □

Remark 3.1. Since the mapping $H \mapsto R_H$ is continuous, it is easily seen that the mapping $H \mapsto L_H$ given by (3.6) is also continuous.

3.2.3. Theorem on factorization of $\mathscr{U}_{\mathfrak{a},0}$

Main result of the present subsection is the following theorem.

Theorem 3.1. *Let $\mathfrak{a} = ((\lambda_j, \alpha_j))_{j \in \mathbb{Z}}$ be a collection satisfying the conditions (A_1) and (A_3), $\mu := \mu^{\mathfrak{a}}$, $H := H_\mu$. Set $q := \Theta(H)$. Then*

$$\mathscr{U}_{\mathfrak{a},0} = (\mathscr{I} + \mathscr{K}_q)^{-1} \big(\mathscr{I} + \mathscr{K}_q^*\big)^{-1}, \tag{3.7}$$

where $\mathscr{K}_q$ is an integral operator with kernel K_q (see Lemma 2.1).

Proof. By virtue of Lemma 3.3, the function L_H given by (3.6) solves the GLM equation (3.5). Thus, as follows from Appendix B, the equality

$$\mathscr{U}_{\mathfrak{a},0} = (\mathscr{I} + \mathscr{L}_H)^{-1} \big(\mathscr{I} + \mathscr{L}_H^*\big)^{-1}$$

takes place, where $\mathscr{L}_H$ is an integral operator with kernel L_H. Therefore it is left to show that $\mathscr{L}_H = \mathscr{K}_q$, i.e. it suffices to show that

$$L_H = K_q, \quad q = \Theta(H). \tag{3.8}$$

Notice that it is enough to prove (3.8) only for the case $H \in \mathfrak{H}_2^s \cap C^1([-1,1], M_r)$. Indeed, the set $\mathfrak{H}_2^s \cap C^1([-1,1], M_r)$ is dense everywhere in $\mathfrak{H}_2^s$, and the mappings $q \mapsto K_q$, Θ and $H \mapsto L_H$ are continuous (see Lemma 2.1, Lemma 3.2 and Remark 3.1 respectively).

Let $H \in \mathfrak{H}_2^s \cap C^1([-1,1], M_r)$. Taking into consideration Lemma 2.1, it is easily seen that the equality (3.8) is equivalent to the fact that the function

$$\varphi(x,\lambda) := \varphi_0(x,\lambda) + \int_0^x L_H(x,s)\varphi_0(s,\lambda)\,ds \tag{3.9}$$

solves the Cauchy problem

$$\vartheta \frac{d}{dx}\varphi + \mathbf{q}\varphi = \lambda\varphi, \qquad \varphi(0,\lambda) = \vartheta a^*. \tag{3.10}$$

Thus it is left to prove (3.10). Let us introduce the auxiliary functions

$$\widetilde{H} := \begin{pmatrix} H & 0 \\ 0 & H^\sharp \end{pmatrix}, \qquad \widetilde{R}_H := \begin{pmatrix} R_H & 0 \\ 0 & R_{H^\sharp} \end{pmatrix}.$$

The definitions of R_H and $\widetilde{R}_H$ yield that the following relation holds:

$$\widetilde{R}_H(x,t) + \widetilde{H}(x-t) + \int_0^x \widetilde{R}_H(x,s)\widetilde{H}(s-t)\,ds = 0, \quad (x,t) \in \Omega^+. \tag{3.11}$$

Moreover, by virtue of Lemma 3.2 we obtain that $\widetilde{R}_H \in C^1(\Omega^+, M_{2r})$.

Noting that

$$L_H(x,t) = \frac{1}{2}\left\{\widetilde{R}_H\left(x, \frac{x+t}{2}\right) + \widetilde{R}_H\left(x, \frac{x-t}{2}\right)J\right\}$$

and

$$J\varphi_0(x,\lambda) = \varphi_0(-x,\lambda), \quad J = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

we can rewrite (3.9) as

$$\varphi(x,\lambda) = \varphi_0(x,\lambda) + \int_0^x \widetilde{R}_H(x,x-s)\varphi_0(x-2s,\lambda)\,ds.$$

From this equality, taking into consideration that $\vartheta \frac{d}{dx}\varphi_0(x,\lambda) - \lambda\varphi_0(x,\lambda) = 0$, we easily obtain that

$$\begin{aligned}\vartheta \frac{d}{dx}\varphi(x,\lambda) + \mathbf{q}(x)\varphi(x,\lambda) - \lambda\varphi(x,\lambda) &= \left\{\vartheta \widetilde{R}_H(x,0)J\varphi_0(x,\lambda) + \mathbf{q}(x)\varphi_0(x,\lambda)\right\} \\ &\quad + \int_0^x \left\{\vartheta \frac{\partial}{\partial x}\left[\widetilde{R}_H(x,x-s)\right] + \mathbf{q}(x)\widetilde{R}_H(x,x-s)\right\}\varphi_0(x-2s,\lambda)\,ds. \end{aligned} \tag{3.12}$$

Taking into consideration (3.2) we conclude that $\mathbf{q}(x) = -\vartheta \widetilde{R}_H(x,0)J$ and thus the relation (3.12) can be rewritten as

$$\vartheta \frac{d}{dx}\varphi(x,\lambda) + \mathbf{q}(x)\varphi(x,\lambda) - \lambda\varphi(x,\lambda) = \vartheta \int_0^x \left\{\frac{\partial}{\partial x}\left[\widetilde{R}_H(x,x-s)\right] - \widetilde{R}_H(x,0)J\widetilde{R}_H(x,s)J\right\}\varphi_0(x-2s,\lambda)\,ds.$$

If we show that

$$\frac{\partial}{\partial x}\left[\widetilde{R}_H(x,x-s)\right] - \widetilde{R}_H(x,0)J\widetilde{R}_H(x,s)J = 0 \tag{3.13}$$

for $(x,t) \in \Omega^+$, then (3.10) will be proved.

Let us show (3.13). From (3.11) we obtain that

$$\widetilde{R}_H(x,x-t) + \widetilde{H}(t) + \int_0^x \widetilde{R}_H(x,x-s)\widetilde{H}(t-s)\,ds = 0, \quad (x,t) \in \Omega^+,$$

and differentiating this expression with respect to x we can write

$$\frac{\partial}{\partial x}\widetilde{R}_H(x,x-t) + \widetilde{R}_H(x,0)\widetilde{H}(t-x) + \int_0^x \frac{\partial}{\partial x}\left[\widetilde{R}_H(x,x-s)\right]\widetilde{H}(t-s)\,ds = 0. \tag{3.14}$$

Now we multiply the relation (3.11) by $\widetilde{R}_H(x,0)J$ from the left and by J from the right, and write

$$\widetilde{R}_H(x,0)J\widetilde{R}_H(x,t)J+\widetilde{R}_H(x,0)J\widetilde{H}(x-t)J+\int_0^x \widetilde{R}_H(x,0)J\widetilde{R}_H(x,s)\widetilde{H}(s-t)J\,ds=0 \tag{3.15}$$

for $(x,t)\in\Omega^+$. Subtracting now (3.14) from (3.15) and taking into consideration that $\widetilde{H}(x)J=J\widetilde{H}(-x)$, we obtain that the function

$$X(x,t)=\frac{\partial}{\partial x}\bigl[\widetilde{R}_H(x,x-s)\bigr]-\widetilde{R}_H(x,0)J\widetilde{R}_H(x,s)J$$

solves the equation

$$X(x,t)+\int_0^x X(x,s)\widetilde{H}(s-t)\,ds=0,\quad (x,t)\in\Omega^+.$$

Since $\widetilde{R}_H\in C^1(\Omega^+,M_{2r})$, $X\in C(\Omega^+,M_{2r})$ and thus by virtue of Lemma B.1 and the relation (3.11) we conclude that $X(x,t)\equiv 0$. Therefore the relation (3.13) follows, and the proof is complete. □

Remark 3.2. Let $\mathfrak{a}=((\lambda_j,\alpha_j))_{j\in\mathbb{Z}}$ be a collection satisfying the conditions (A_1) and (A_3), $\mu:=\mu^{\mathfrak{a}}$, $H:=H_\mu$, $q:=\Theta(H)$. Then from the equalities (3.7) and (2.26) we obtain that

$$\mathscr{U}_{\mathfrak{a},q}=\mathscr{I}. \tag{3.16}$$

3.3. Description of the spectral data: the sufficiency part

In this subsection we show that if a collection $\mathfrak{a}$ satisfies the conditions (A_1)–(A_3), then it belongs to $\mathfrak{A}$, i.e. that $\mathfrak{a}=\mathfrak{a}_q$ for some $q\in\mathfrak{Q}$. This is the sufficiency part of Theorem 1.1.

We start from proving the following lemma.

Lemma 3.4. *Let $\mathfrak{a}=((\lambda_j,\alpha_j))_{j\in\mathbb{Z}}$ be a collection satisfying the conditions (A_1)–(A_3), $\mu:=\mu^{\mathfrak{a}}$, $H:=H_\mu$, $q:=\Theta(H)$. Set*

$$\hat{P}_j:=\Phi_q(\lambda_j)\alpha_j\Phi_q^*(\lambda_j).$$

Then the collection $\{\hat{P}_j\}_{j\in\mathbb{Z}}$ forms a complete system of pairwise orthogonal projectors.

In order to prove this lemma we need the following additional statements, that are proved in [12] (Lemmas B.2 and B.3 respectively).

Lemma 3.5. *Let H be a Hilbert space. Assume that $(A_j)_{j=1}^\infty$ is a sequence in $\mathscr{B}(H)$ and that $(G_j)_{j=1}^\infty$ is a sequence of pairwise orthogonal projectors such that the following statements hold*:

(i) *the series $\sum_{j=1}^\infty A_j$ converges in the strong operator topology to an operator A*;
(ii) *the orthogonal projector $G:=\mathscr{I}_H-\sum_{j=1}^\infty G_j$ is of finite rank*;
(iii) $\sum_{j=1}^\infty\|A_j-G_j\|^2<1$ *and* $\operatorname{rank}A_j\leqslant\operatorname{rank}G_j<\infty$ *for every* $j\in\mathbb{N}$.

Then $\operatorname{codim}\operatorname{Ran}A\geqslant\operatorname{rank}G$.

Lemma 3.6. *Let H be a Hilbert space, and let $\{A_j\}_{j=0}^n$ be a set of self-adjoint operators from the algebra $\mathscr{B}(H)$ that are of finite rank for $j\neq 0$. If*

$$\sum_{j=0}^n A_j=\mathscr{I}_H,\qquad \sum_{j=1}^n\operatorname{rank}A_j\leqslant\operatorname{codim}\operatorname{Ran}A_0,$$

then $\{A_j\}_{j=0}^n$ is the set of pairwise orthogonal projectors.

We use these statements to prove Lemma 3.4.

Proof of Lemma 3.4. It follows from Lemma 2.3 that the series $\sum_{j\in\mathbb{Z}}\hat{P}_j$ converges in the strong operator topology, and in view of (2.18) and (3.16) we obtain that

$$\sum_{j=-\infty}^{\infty} \hat{P}_j = \mathscr{I}.$$

Thus, it is enough to show that the operators $\hat{P}_j$, $j \in \mathbb{Z}$ are pairwise orthogonal projectors.

Denote

$$A_n := \sum_{\lambda_j \in \Delta_n} \hat{P}_j.$$

By virtue of Lemma 2.3 we obtain that $\sum_{n=-\infty}^{\infty} \|P_{n,0} - A_n\|^2 < \infty$, and therefore there exists an $N_0 \in \mathbb{N}$ such that $\sum_{|n|>N_0} \|P_{n,0} - A_n\|^2 < 1$. Moreover, due to the conditions (A_1) and (A_2) we conclude that N_0 can be taken so large that

$$\sum_{n=-N}^{N} \sum_{\lambda_j \in \Delta_n} \operatorname{rank} \alpha_j = (2N+1)r, \quad N \geqslant N_0, \tag{3.17}$$

$$\left\| \sum_{\lambda_j \in \Delta_n} \alpha_j - I \right\| < 1, \quad |n| \geqslant N_0. \tag{3.18}$$

First let us show that

$$\sum_{\lambda_j \in \Delta_n} \operatorname{rank} \alpha_j = r, \quad |n| \geqslant N_0. \tag{3.19}$$

Indeed, it follows from (3.18) that $\sum_{\lambda_j \in \Delta_n} \operatorname{rank} \alpha_j \geqslant r$ and from (3.17) that $\sum_{\lambda_j \in \Delta_n} \operatorname{rank} \alpha_j + \sum_{\lambda_j \in \Delta_{-n}} \operatorname{rank} \alpha_j = 2r$ for $|n| \geqslant N_0$, and thus we obtain (3.19).

Fix $N > N_0$ and set

$$P := \mathscr{I} - \sum_{|n|>N} P_{n,0}, \qquad A := \sum_{|n|>N} A_n.$$

Since $\operatorname{rank} \hat{P}_j = \operatorname{rank} \alpha_j$ for all $j \in \mathbb{Z}$ (which follows directly from the definition of $\hat{P}_j$ and (2.10)), taking into consideration (3.19) we conclude that

$$\operatorname{rank} A_n \leqslant \sum_{\lambda_j \in \Delta_n} \operatorname{rank} \hat{P}_j = \sum_{\lambda_j \in \Delta_n} \operatorname{rank} \alpha_j = r = \operatorname{rank} P_{n,0}, \quad |n| > N_0.$$

Recalling also that $\{P_{n,0}\}_{j \in \mathbb{Z}}$ forms a complete system of pairwise orthogonal projectors, by virtue of Lemma 3.5 we obtain that

$$\operatorname{codim} \operatorname{Ran} A \geqslant \operatorname{rank} P = (2N+1)r.$$

Moreover, $A + \sum_{n=-N}^{N} A_n = \sum_{j \in \mathbb{Z}} \hat{P}_j = \mathscr{I}$ and

$$\sum_{n=-N}^{N} \operatorname{rank} A_n \leqslant \sum_{n=-N}^{N} \sum_{\lambda_j \in \Delta_n} \operatorname{rank} \hat{P}_j = \sum_{n=-N}^{N} \sum_{\lambda_j \in \Delta_n} \operatorname{rank} \alpha_j = (2N+1)r \leqslant \operatorname{codim} \operatorname{Ran} A.$$

Therefore, since the operators A and A_n, $|n| \leqslant N$, are self-adjoint, by virtue of Lemma 3.6 we obtain that the set

$$\left\{ \hat{P}_j \,\middle|\, \lambda_j \in \bigcup_{n=-N}^{N} \Delta_n \right\}$$

is a set of pairwise orthogonal projectors. Since N is arbitrary, we conclude that projectors $\{\hat{P}_j\}_{j \in \mathbb{Z}}$ are orthogonal ones. □

In order to prove the sufficiency part of Theorem 1.1 it obviously suffices to find $q \in \mathfrak{Q}$ such that $\mathfrak{a} = \mathfrak{a}_q$.

Theorem 3.2. *Let $\mathfrak{a} = ((\lambda_j, \alpha_j))_{j \in \mathbb{Z}}$ be a collection satisfying the conditions (A_1)–(A_3), $\mu := \mu^{\mathfrak{a}}$, $H := H_\mu$, $q := \Theta(H)$. Then $\mathfrak{a} = \mathfrak{a}_q$.*

Proof. It is enough to prove the relation

$$\operatorname{Ran}\hat{P}_j \subset \ker(T_q - \lambda_j \mathscr{I}), \quad j \in \mathbb{Z}. \tag{3.20}$$

Indeed, taking into account the completeness of system $\{\hat{P}_j\}_{j\in\mathbb{Z}}$, from (3.20) we immediately conclude that $\lambda_j(q) = \lambda_j$ for all $j \in \mathbb{Z}$, where $\lambda_j(q)$ are the eigenvalues of T_q. From this equality and (3.20) we obtain the relation $P_{j,q} - \hat{P}_j \geqslant 0$, $j \in \mathbb{Z}$, where $P_{j,q}$ are corresponding orthogonal projectors of T_q. However, by virtue of completeness of the systems $\{\hat{P}_j\}_{j\in\mathbb{Z}}$ and $\{P_{j,q}\}_{j\in\mathbb{Z}}$ we conclude that $\sum_{j\in\mathbb{Z}}(P_{j,q} - \hat{P}_j) = 0$, and therefore $P_{j,q} - \hat{P}_j = 0$ for all $j \in \mathbb{Z}$. Therefore, taking into account Lemma 3.4 and the definition of $\hat{P}_j$, we conclude that

$$\Phi_q(\lambda_j)\big\{\alpha_j(q) - \alpha_j\big\}\Phi_q^*(\lambda_j) = 0, \quad j \in \mathbb{Z},$$

and by virtue of (2.10) we justify that $\alpha_j(q) = \alpha_j$, which, together with $\lambda_j(q) = \lambda_j$, means that $\mathfrak{a} = \mathfrak{a}_q$.

Thus it only remains to prove (3.20). Due to the definition of $\Phi_q(\lambda)$ and (2.10) we obtain that $\operatorname{Ran}\hat{P}_j = \{\varphi_q(\cdot,\lambda_j)\alpha_j c \mid c \in \mathbb{C}^r\}$. From the other side, by virtue of Lemma 2.2 we obtain that

$$\ker(T_q - \lambda_j\mathscr{I}) = \big\{\varphi_q(\cdot,\lambda_j)c \,\big|\, a\varphi_q(1,\lambda_j)c = 0\big\}.$$

Therefore we conclude that it is enough to show that

$$a\varphi_q(1,\lambda_j)\alpha_j = 0. \tag{3.21}$$

Let $j,k \in \mathbb{Z}$ and $c,d \in \mathbb{C}^r$. Then, taking into account that $\mathbf{q} = \mathbf{q}^*$, $\vartheta^* = -\vartheta$ and integrating by parts, we obtain that

$$\lambda_j\big(\Phi_q(\lambda_j)c \,\big|\, \Phi_q(\lambda_k)d\big) = \big(\vartheta\varphi_q(1,\lambda_j)c \,\big|\, \varphi_q(1,\lambda_k)d\big) + \lambda_k\big(\Phi_q(\lambda_j)c \,\big|\, \Phi_q(\lambda_k)d\big),$$
$$\lambda_j\Phi_q(\lambda_k)^*\Phi_q(\lambda_j) - \lambda_k\Phi_q(\lambda_k)^*\Phi_q(\lambda_j) = \varphi_q(1,\lambda_k)^*\vartheta\varphi_q(1,\lambda_j). \tag{3.22}$$

Since $\hat{P}_k\hat{P}_j = 0$ if $k \neq j$, we obtain that $\Phi_q(\lambda_k)\alpha_k\Phi_q^*(\lambda_k)\Phi_q(\lambda_j)\alpha_j\Phi_q^*(\lambda_j) = 0$, and by virtue of (2.10) we conclude that $\alpha_k\Phi_q^*(\lambda_k)\Phi_q(\lambda_j)\alpha_j = 0$. Multiplying now (3.22) by α_k from the left and by α_j from the right we obtain that

$$\alpha_k\varphi_q(1,\lambda_k)^*\vartheta\varphi_q(1,\lambda_j)\alpha_j = 0,$$

and therefore we can write

$$\Bigg\{\sum_{\lambda_k\in\Delta_n}(-1)^n\alpha_k\varphi_q(1,\lambda_k)^*\Bigg\}\vartheta\varphi_q(1,\lambda_j)\alpha_j = 0, \quad \lambda_j \notin \Delta_n. \tag{3.23}$$

Taking into account (2.4), it follows from the Riemann–Lebesgue lemma and the asymptotic behavior of the sequences (λ_k) and (α_k) that

$$\lim_{n\to\infty}\Bigg\{\sum_{\lambda_k\in\Delta_n}(-1)^n\varphi_q(1,\lambda_k)\alpha_k\Bigg\} = \vartheta a^*,$$

and passing to the limit in (3.23) we obtain the relation (3.21). □

3.4. *Potential reconstruction: proof of Theorems 1.2 and 1.4*

Finally, we prove Theorems 1.2 and 1.4.

Proof of Theorem 1.2. Suppose that $q_1, q_2 \in \mathfrak{Q}$, and let $\mathfrak{a}_{q_1} = \mathfrak{a}_{q_2}$. Let us show that $q_1 = q_2$. Write $\mathfrak{a}_{q_1} = \mathfrak{a}_{q_2} =: \mathfrak{a}$ for short, and set $\mu := \mu^{\mathfrak{a}}$, $H := H_\mu$. Then by virtue of Theorem 3.1 the operator $\mathscr{U}_{\mathfrak{a},0} = \mathscr{I} + \mathscr{F}_H$ admits a factorization, and we can write

$$\mathscr{U}_{\mathfrak{a},0} = (\mathscr{I} + \mathscr{K}_{q_1})^{-1}\big(\mathscr{I} + \mathscr{K}_{q_1}^*\big)^{-1} = (\mathscr{I} + \mathscr{K}_{q_2})^{-1}\big(\mathscr{I} + \mathscr{K}_{q_2}^*\big)^{-1}.$$

Since any operator may admit at most one factorization of the above form (see Appendix B), we conclude that

$$\mathscr{K}_{q_1} = \mathscr{K}_{q_2}.$$

It is left to notice that $\mathscr{K}_{q_1} = \mathscr{K}_{q_2} \Rightarrow q_1 = q_2$. Taking into account (2.4) we conclude that $\mathscr{K}_{q_1} = \mathscr{K}_{q_2} \Rightarrow \varphi_{q_1}(\cdot,0) = \varphi_{q_2}(\cdot,0) =: \varphi$, and therefore we obtain

$$\vartheta\varphi' + \mathbf{q}_1\varphi = \vartheta\varphi' + \mathbf{q}_2\varphi = 0,$$

and thus $\{\mathbf{q}_1 - \mathbf{q}_2\}\varphi = 0$.

Thus it is left to show that for all x the matrix $\varphi(x)$ is invertible. Assume the contrary. Then there exist $x_0 \in (0,1]$ and $c \in \mathbb{C}^r \setminus \{0\}$ such that $\varphi(x_0)c = 0$, and therefore the function $f = \varphi_{q_1}(\cdot,0)c$ is a non-zero solution of the Cauchy problem $\vartheta f' + \mathbf{q}_1 f = 0$, $f(x_0) = 0$. But this is in contradiction with the uniqueness theorem; thus $\varphi(x)$ is non-singular, and $\mathbf{q}_1 = \mathbf{q}_2$.

Besides this, by definition of the spectral data we obviously have $q_1 = q_2 \Rightarrow \mathfrak{a}_{q_1} = \mathfrak{a}_{q_2}$, and therefore we conclude that the mapping $\mathfrak{Q} \ni q \mapsto \mathfrak{a}_q \in \mathfrak{A}$ is bijective. $\square$

Proof of Theorem 1.4. Theorem 1.4 now directly follows from Theorems 1.2 and 3.2.

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Appendix A. Spaces

By $G_2(M_r)$ we denote the set of all measurable functions $K : [0,1]^2 \to M_r$, such that for all x and t in $[0,1]$ the functions $K(x,\cdot)$ and $K(\cdot,t)$ belong to $L_2((0,1),M_r)$ and, moreover, the mappings

$$[0,1] \ni x \mapsto K(x,\cdot) \in L_2\big((0,1),M_r\big), \qquad [0,1] \ni t \mapsto K(\cdot,t) \in L_2\big((0,1),M_r\big)$$

are continuous on the interval $[0,1]$. It can easily be seen that $G_2(M_r) \subset L_2([0,1]^2,M_r)$. The set $G_2(M_r)$ becomes a Banach space upon introducing the norm

$$\|K\|_{G_2(M_r)} = \max\Big\{\max_{x\in[0,1]}\big\|K(x,\cdot)\big\|_{L_2((0,1),M_r)}, \max_{t\in[0,1]}\big\|K(\cdot,t)\big\|_{L_2((0,1),M_r)}\Big\}.$$

By $\mathscr{G}_2(M_r)$ we denote the space of all integral operators with kernels $K \in G_2(M_r)$. It forms a subalgebra in the algebra $\mathscr{B}_\infty$ of compact operators in $L_2((0,1),\mathbb{C}^r)$.

We denote

$$\Omega^+ := \big\{(x,t)\ \big|\ 0 \leqslant t \leqslant x \leqslant 1\big\}, \qquad \Omega^- := \big\{(x,t)\ \big|\ 0 \leqslant x < t \leqslant 1\big\}.$$

We write $G_2^+(M_r)$ for the set of all functions $K \in G_2(M_r)$ such that $K(x,t) = 0$ a.e. in Ω^-, and $G_2^-(M_r)$ for set of all $K \in G_2(M_r)$ such that $K(x,t) = 0$ a.e. in Ω^+. By $\mathscr{G}_2^\pm(M_r)$ we denote the subalgebra of $\mathscr{G}_2(M_r)$ consisting of all operators with kernels $K \in G_2^\pm(M_r)$.

Appendix B. Factorization of operators

Here we state some well-known facts from the theory of factorization. In particular, these facts are mentioned in [12,6]. See also [14] for details.

We say that an operator $\mathscr{I} + \mathscr{F}$, $\mathscr{F} \in \mathscr{G}_2(M_r)$ admits a factorization (in $\mathscr{G}_2(M_r)$) if there exist $\mathscr{L}^+ \in \mathscr{G}_2^+(M_r)$ and $\mathscr{L}^- \in \mathscr{G}_2^-(M_r)$ such that

$$\mathscr{I} + \mathscr{F} = \big(\mathscr{I} + \mathscr{L}^+\big)^{-1}\big(\mathscr{I} + \mathscr{L}^-\big)^{-1}.$$

It is known that if $\mathscr{I} + \mathscr{F}$ admits a factorization, then the corresponding operators $\mathscr{L}^+$ and $\mathscr{L}^-$ are unique. Moreover, the set of operators $\mathscr{F} \in \mathscr{G}_2(M_r)$, such that $\mathscr{I} + \mathscr{F}$ admits a factorization, is open, and the mappings $\mathscr{F} \mapsto \mathscr{L}^\pm \in \mathscr{G}_2(M_r)$ are continuous.

An operator $\mathscr{I} + \mathscr{F}$, $\mathscr{F} \in \mathscr{G}_2(M_r)$ admits a factorization if and only if the operators $I + \chi_a \mathscr{F} \chi_a$ have trivial kernels for all $a \in [0,1]$. Here χ_a is an operator of multiplication by the indicator of the interval $(0,a]$, i.e.

$$(\chi_a f)(x) = \begin{cases} f(x), & x \in (0,a], \\ 0, & x \in (a,1). \end{cases}$$

If $\mathscr{F}$ is self-adjoint, then this condition is equivalent to the positivity of $\mathscr{I} + \mathscr{F}$.

From the other side, it is known that $\mathscr{I} + \mathscr{F}$ admits a factorization in $\mathscr{G}_2(M_r)$ if and only if the equation

$$X(x,t) + F(x,t) + \int_0^x X(x,s)F(s,t)\,ds = 0, \quad (x,t) \in \Omega^+, \tag{B.1}$$

where F is the kernel of $\mathscr{F}$, is solvable in $L_2(\Omega^+,M_r)$. In this case its solution is unique and belongs to $G_2^+(M_r)$. Eq. (B.1) is usually called *the Gelfand–Levitan–Marchenko (GLM) equation.*

Also we formulate the following lemma (see Lemma A.3 in [12]).

Lemma B.1. *Let $F \in L_2((0,1)^2, M_r)$. Then the GLM equation* (B.1) *has at most one solution*; *if* (B.1) *is solvable, then the equation*

$$X(x,t) + \int_0^x X(x,s)F(s,t)\,ds = 0, \quad (x,t) \in \Omega^+$$

has only trivial solution in $L_2(\Omega^+, M_r)$.

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