A supplement to precise asymptotics in the law of the iterated logarithm

Deli Li \(^a,\,*\), Bao-Em Nguyen \(^a\), Andrew Rosalsky \(^b\)

\(^a\) Department of Mathematical Sciences, Lakehead University, Thunder Bay, ON, Canada P7B 5E1
\(^b\) Department of Statistics, University of Florida, Gainesville, FL 32611, USA

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Abstract

Let \(\{X, X_n; n \geq 1\}\) be a sequence of real-valued i.i.d. random variables with \(E(X) = 0\) and \(E(X^2) = 1\), and set \(S_n = \sum_{i=1}^{n} X_i, n \geq 1\). This paper studies the precise asymptotics in the law of the iterated logarithm. For example, using a result on convergence rates for probabilities of moderate deviations for \(\{S_n; n \geq 1\}\) obtained by Li et al. [Internat. J. Math. Math. Sci. 15 (1992) 481–497], we prove that, for every \(b \in (-1/2, 1]\),

\[
\lim_{n \to \infty} \frac{\log \log n}{n} - \frac{\log \log n}{n} \sum_{n \geq 3} \frac{(\log \log n)^b |S_n| \geq \sigma_n \sqrt{(2 + \epsilon) n \log \log n + a_n}}{n} = e^{-\sqrt{2} \gamma_b \sqrt{2/\pi} \Gamma'(b + (1/2))},
\]

whenever \(\lim_{n \to \infty} \left(\frac{\log \log n}{n}\right)^{1/2} a_n = \gamma \in [-\infty, \infty]\), where \(\Gamma'(s) = \int_0^{\infty} t^{s-1} e^{-t} dt, s > 0\), \(\sigma^2(t) = E(X^2 I(|X| < \sqrt{t})) - (E(X I(|X| < \sqrt{t})))^2, t \geq 0\), and \(\sigma_n^2 = \sigma^2(n \log \log n), n \geq 3\). This result generalizes and improves Theorem 2.8 of Li et al. [Internat. J. Math. Math. Sci. 15 (1992) 481–497] and Theorem 1 of Gut and Spătaru [Ann. Probab. 28 (2000) 1870–1883].

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* Corresponding author. Tel.: +1-807-343-8231; fax: +1-807-343-8821.
E-mail address: lideli@giant.lakeheadu.ca (D. Li).

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1. Introduction

Throughout this paper, let \( \{X, X_n; \ n \geq 1\} \) be a sequence of real-valued independent and identically distributed (i.i.d.) random variables and, as usual, let \( S_n = \sum_{i=1}^{n} X_i, \ n \geq 1 \), denote their partial sums. One of the most striking and fundamental results of probability theory is the renowned law of the iterated logarithm due to Hartman and Wintner [10] for a sequence of i.i.d. random variables. This result, which describes the almost sure (a.s.) asymptotic fluctuation behavior of the partial sums, is stated as follows.

**Theorem A.** Let \( S_n = \sum_{i=1}^{n} X_i, \ n \geq 1 \), where \( \{X, X_n; \ n \geq 1\} \) is a sequence of i.i.d. random variables. If

\[
E(X) = 0 \quad \text{and} \quad E(X^2) = 1,
\]

then

\[
\limsup_{n \to \infty} \frac{S_n}{\sqrt{2n \log \log n}} = 1 \quad \text{a.s.} \quad \text{and} \quad \liminf_{n \to \infty} \frac{S_n}{\sqrt{2n \log \log n}} = -1 \quad \text{a.s.}
\] (1.2)

Alternative proofs of the Hartman–Wintner [10] law of the iterated logarithm were discovered by Strassen [19], Heyde [12], Egorov [4], Teicher [21], Csörgő and Révész [2, p. 119], and de Acosta [1]. Strassen [20] proved the converse: \((1.2) \Rightarrow (1.1)\). Substantially simpler proofs of Strassen’s converse were obtained by Feller [5], Heyde [11], and Steiger and Zaremba [18]. Martikainen [15], Rosalsky [17], and Pruitt [16] simultaneously and independently obtained a “one-sided” converse to the Hartman–Wintner [10] law of the iterated logarithm. Specifically, they proved that each half of \((1.2)\) individually implies \((1.1)\).

The following three statements, related to the Hartman–Wintner [10] law of the iterated logarithm, are known to be equivalent:

\[
E(X) = 0 \quad \text{and} \quad E(X^2) = 1, \quad \text{(1.3)}
\]

\[
\sum_{n \geq 3} \frac{1}{n} P\left( |S_n| \geq \sqrt{(2 + \varepsilon)n \log \log n} \right) < \infty, \quad \text{if } \varepsilon > 0,
\]

\[
= \infty, \quad \text{if } -2 < \varepsilon < 0,
\] (1.4)

\[
\sum_{n \geq 3} \frac{1}{n} \log \log n P\left( |S_n| \geq \sqrt{(2 + \varepsilon)n \log \log n} \right) < \infty, \quad \text{if } \varepsilon > 0,
\]

\[
= \infty, \quad \text{if } -2 < \varepsilon < 0.
\] (1.5)

The implication “\((1.3) \Rightarrow (1.4)\)” should be due to Davis [3, Theorem 4] which was remedied by Li et al. [14, Corollary 2.3]. The equivalence “\((1.3) \Leftrightarrow (1.5)\)” was established by Li [13, Corollary 2.2]. For the implication “\((1.4) \Rightarrow (1.3)\),” see Gut [8, Theorem 6.2]. Necessary and sufficient conditions for \((1.4)\) and \((1.5)\), respectively, in a Banach space setting were obtained by Li [13].
In view of the fact that the expressions in (1.4) and (1.5) typically tend to infinity as \( \varepsilon \downarrow 0 \), it is of interest to find the rate; that is, one would be interested in finding appropriate normalizations in terms of functions of \( \varepsilon \) that yield nontrivial limits. Such results are referred to as \textit{precise asymptotics in the law of the iterated logarithm}. Under the conditions (1.3) and \( E(X^2 \log(1 + |X|)) < \infty \), Gafurov [7] studied the precise asymptotics in the law of the iterated logarithm by proving that

\[
\lim_{\varepsilon \downarrow 0} \varepsilon^{3/2} \sum_{n \geq 3} \frac{\log \log n}{n} P(|S_n| \geq \sqrt{(2 + \varepsilon)n \log \log n}) = \sqrt{2}.
\] (1.6)

Li et al. [14, Theorem 2.8] improved this result by establishing the following result. (Note that (1.6) is the special case \( b = 1 \) in (1.8).)

**Theorem B.** Suppose that (1.3) holds and that

\[
E\left(X^2 I(|X| \geq t)\right) = o\left(\frac{1}{\log \log t}\right) \quad \text{as } t \to \infty.
\] (1.7)

Then for every \( b \in [0, 1] \),

\[
\lim_{\varepsilon \downarrow 0} \varepsilon^{(2b+1)/2} \sum_{n \geq 3} \frac{\left(\log \log n\right)^b}{n} P\left(|S_n| \geq \sqrt{(2 + \varepsilon)n \log \log n}\right)
\]

\[= 2^b \sqrt{2/\pi} \Gamma(b + (1/2)), \]

where \( \Gamma(s) = \int_0^\infty t^{s-1} e^{-t} \, dt, \, s > 0 \).

Independently, the following result was proved in Gut and Spătaru [9, Theorem 1].

**Theorem C.** Suppose that (1.3) holds, that

\[
E\left(X^2 \left(\log \log (3 + |X|)\right)^{1+\delta}\right) < \infty \quad \text{for some } \delta > 0,
\] (1.9)

and that

\[a_n = O\left(\sqrt{n}\left(\log \log n\right)^{-h}\right) \quad \text{for some } h > \frac{1}{2}.
\] (1.10)

Then

\[
\lim_{\varepsilon \downarrow 0} \varepsilon^{1/2} \sum_{n \geq 3} \frac{1}{n} P\left(|S_n| \geq \sqrt{(2 + \varepsilon)n \log \log n + a_n}\right) = \sqrt{2}.
\]

Clearly, in the case \( a_n = 0, \, n \geq 1 \), Theorem C follows from Theorem B (by taking \( b = 0 \) in Theorem B) in view of the fact that (1.9) implies

\[
E\left(X^2 \log (3 + |X|)\right) < \infty
\]

which, in turn, implies (1.7). Also, as was pointed out by Li et al. [14], the condition (1.7) is the best possible for the validity of (1.8) which is not true if “\( o \)” is replaced by “\( O \)” in (1.7).

The main purpose of the present paper is to complete Theorems B and C under the condition (1.3) only. The main result of this paper, Theorem 1, appears in Section 2 and its
proof is provided in Section 3. The proof is based on convergence rates for probabilities of moderate deviations for partial sums \( \{S_n; n \geq 1\} \) obtained by Li et al. [14, Lemma 3.3]. In Section 4, some further results are provided and an open problem is posed.

2. Main results

We specify some notation first. Write

\[
\sigma^2(t) = E(X^2 I(|X| < \sqrt{t})) - (E(X I(|X| < \sqrt{t})))^2, \quad t \geq 0,
\]

\[
\sigma^2_n = \sigma^2(n \log \log n), \quad n \geq 3,
\]

\[
\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} \, dt, \quad s > 0.
\]

**Theorem 1.** Let \( \{X, X_n; n \geq 1\} \) be a sequence of real-valued i.i.d. random variables with \( E(X) = 0 \) and \( E(X^2) = 1 \). Let \( \{a_n; n \geq 1\} \) be a sequence of real numbers such that

\[
\lim_{n \to \infty} \left( \frac{\log \log n}{n} \right)^{1/2} a_n = \gamma \in [-\infty, \infty]. \tag{2.1}
\]

Then the following conclusions obtain:

(i) If \(-1/2 < b \leq 1\), then

\[
\lim_{\varepsilon \downarrow 0} \varepsilon^{(2b+1)/2} \sum_{n \geq 3} \frac{(\log \log n)^b}{n} P(\{|S_n| \geq \sigma_n \sqrt{(2 + \varepsilon)n \log \log n + a_n}\}) = e^{-\varepsilon \sqrt{2\gamma} \sqrt{2\pi} \Gamma(b + 1/2)}. \tag{2.2}
\]

(ii) If \( b = -1/2 \), then

\[
\lim_{\varepsilon \downarrow 0} \frac{1}{\log(\frac{1}{\varepsilon})} \sum_{n \geq 3} \frac{(\log \log n)^b}{n} P(\{|S_n| \geq \sigma_n \sqrt{(2 + \varepsilon)n \log \log n + a_n}\}) = e^{-\sqrt{2\gamma} \sqrt{2\pi}}. \tag{2.3}
\]

(iii) If \( b < -1/2 \) and \( \gamma > -\infty \), then

\[
\lim_{\varepsilon \downarrow 0} \sum_{n \geq 3} \frac{(\log \log n)^b}{n} P(\{|S_n| \geq \sigma_n \sqrt{(2 + \varepsilon)n \log \log n + a_n}\}) = \sum_{n \geq 3} \frac{(\log \log n)^b}{n} P(\{|S_n| \geq \sigma_n \sqrt{2n \log \log n + a_n}\}) < \infty. \tag{2.4}
\]
Theorem 1 is a very general result which includes (1.4), (1.5), and Theorems B and C as special cases. From Theorem 1, one can understand why (1.7) is the best possible for the validity of (1.8) and we see immediately that the condition (1.10) of Theorem C implies (2.1) with γ = 0; i.e.,

$$\lim_{n \to \infty} \left( \frac{\log \log n}{n} \right)^{1/2} a_n = 0.$$ 

Moreover, under the conditions (2.1) of Theorem 1 and (1.7) of Theorem B, we easily obtain a more precise result. To see this, write

$$\alpha_n = (1 - \sigma_n) \sqrt{n \log \log n + a_n}, \quad \beta_n = 2(1 - \sigma_n) \sqrt{n \log \log n + a_n}, \quad n \geq 3.$$ 

Under (1.3), 0 ≤ σ_n ≤ 1, n ≥ 3, we then have

$$\sigma_n \sqrt{(2 + \varepsilon)n \log \log n + \alpha_n} \leq \sqrt{(2 + \varepsilon)n \log \log n + \alpha_n}$$

$$\leq \sigma_n \sqrt{(2 + \varepsilon)n \log \log n + \beta_n}, \quad n \geq 3.$$ 

Consequently, for n ≥ 3,

$$P(|S_n| \geq \sigma_n \sqrt{(2 + \varepsilon)n \log \log n + \alpha_n}) \geq P(|S_n| \geq \sqrt{(2 + \varepsilon)n \log \log n + \alpha_n})$$

$$\geq P(|S_n| \geq \sigma_n \sqrt{(2 + \varepsilon)n \log \log n + \beta_n}).$$ 

Note that (1.7) is equivalent to

$$1 - \sigma(t) = \frac{1 - \sigma^2(t)}{1 + \sigma(t)} = o \left( \frac{1}{\log \log t} \right) \text{ as } t \to \infty.$$  (2.5)

So, under (1.7) and (2.1), it is easy to see that

$$\lim_{n \to \infty} \left( \frac{\log \log n}{n} \right)^{1/2} a_n = \lim_{n \to \infty} \left( \frac{\log \log n}{n} \right)^{1/2} \beta_n = \lim_{n \to \infty} \left( \frac{\log \log n}{n} \right)^{1/2} a_n = \gamma.$$ 

Thus, under the additional assumption that (1.7) holds, which is equivalent to (2.5), we have the following more precise result.

**Theorem 2.** Let \{X, X_n: n ≥ 1\} be a sequence of real-valued i.i.d. random variables and suppose that E(X) = 0, E(X^2) = 1, and (1.7) holds. Let \{a_n; n ≥ 1\} be a sequence of real numbers such that (2.1) holds. Then the following conclusions obtain:

(i) If −1/2 < b ≤ 1, then

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{(2b+1)/2} \sum_{n \geq 3}^{(\log \log n)^b} \frac{\log \log n}{n} P(|S_n| \geq \sqrt{(2 + \varepsilon)n \log \log n + a_n})$$

$$= e^{-\sqrt{2} \gamma (2b+1/2)} \Gamma(b + (1/2)).$$
(ii) If \( b = -1/2 \), then
\[
\lim_{\epsilon \downarrow 0} \frac{1}{\log(\frac{1}{\epsilon})} \sum_{n \geq 3} \frac{(\log \log n)^b}{n} P(|S_n| \geq \sqrt{(2 + \epsilon)n \log \log n + a_n})
\]
\[
= \lim_{\epsilon \downarrow 0} \frac{1}{\log(\frac{1}{\epsilon})} \sum_{n \geq 3} \frac{1}{n \sqrt{\log \log n}} P\left(\frac{|S_n|}{\sqrt{\log \log n}} \geq \sqrt{(2 + \epsilon)n \log \log n + a_n}\right)
\]
\[
= e^{-\sqrt{2}\gamma} \sqrt{\pi}.
\]
(iii) If \( b < -1/2 \) and \( \gamma > -\infty \), then
\[
\lim_{\epsilon \downarrow 0} \sum_{n \geq 3} \frac{(\log \log n)^b}{n} P\left(\frac{|S_n|}{\sqrt{\log \log n}} \geq \sqrt{(2 + \epsilon)n \log \log n + a_n}\right)
\]
\[
= \sum_{n \geq 3} \frac{(\log \log n)^b}{n} P\left(\frac{|S_n|}{\sqrt{2n \log \log n + a_n}} \geq \infty\right).
\]

3. Proof of Theorem 1

For convenience, we need some additional notation. Let
\[
F_n(x) = P\left(\frac{S_n}{\sqrt{n}} \leq x\right), \quad -\infty < x < \infty, \ n \geq 1.
\]
Let \( \Phi(\cdot) \) and \( \phi(\cdot) \) denote the distribution function and the density function, respectively, of the standard normal distribution.

For the proof of Theorem 1 we need the following three lemmas. The first lemma, which follows directly from Lemma 3.3 of Li et al. [14], plays an important role in the derivation of Theorem 1.

Lemma 1. Let \( \{X, X_n; n \geq 1\} \) be a sequence of real-valued i.i.d. random variables with \( E(X) = 0 \) and \( E(X^2) = 1 \). Then, for every \( b \in (-\infty, 1] \), we have
\[
\sum_{n \geq 3} \frac{(\log \log n)^b}{n} \sup_{|x| \geq \sqrt{\log \log n}} \left| F_n(x) - \Phi\left(\frac{x}{\sigma_n}\right)\right| < \infty.
\]

Proof. Let \( \varphi(t) = \sqrt{\log \log t}, \ t \geq 3 \). Applying Lemma 3.3 of Li et al. [14], we get
\[
\sum_{n \geq 3} \frac{\varphi^2(n)}{n} \sup_{|x| \geq \varphi(n)} \left| F_n(x) - \Phi\left(\frac{x}{\sigma_n}\right)\right|
\]
\[
= \sum_{n \geq 3} \frac{(\log \log n)^b}{n} \sup_{|x| \geq \sqrt{\log \log n}} \left| F_n(x) - \Phi\left(\frac{x}{\sigma_n}\right)\right| < \infty.
\]
Clearly, (3.2) implies (3.1). \( \Box \)
Lemma 2. Let \( \{ f_n(\cdot); \ n \geq 1 \} \) be a sequence of nonnegative functions defined on \([0, \infty)\) such that
\[
\sum_{n \geq 1} f_n(\epsilon) < \infty \quad \text{for every fixed } \epsilon > 0 \quad \text{and} \quad \lim_{\epsilon \downarrow 0} \sum_{n \geq 1} f_n(\epsilon) = \infty.
\]
Let \( g(\cdot) \) be a nonnegative function defined on \([0, \infty)\) such that
\[
\lim_{\epsilon \downarrow 0} g(\epsilon) = 0 \quad \text{and} \quad \lim_{\epsilon \downarrow 0} g(\epsilon) \sum_{n \geq 1} f_n(\epsilon) = \lambda \in [0, \infty].
\]
Let \( \{ h_n(\cdot); \ n \geq 1 \} \) be a sequence of nonnegative functions defined on \([0, \infty)\) such that
\[
\lim_{n \to \infty} \sup_{0 \leq \epsilon \leq \delta} |h_n(\epsilon) - 1| = 0 \quad \text{for some } \delta > 0.
\]
Then
\[
\lim_{\epsilon \downarrow 0} g(\epsilon) \sum_{n \geq 1} h_n(\epsilon) f_n(\epsilon) = \lambda.
\]

Proof. For arbitrary \( \tau > 0 \), there exists a positive integer \( N_\tau \) such that
\[
1 - \eta \leq h_n(\epsilon) \leq 1 + \eta \quad \text{whenever } n \geq N_\tau \text{ and } 0 \leq \epsilon \leq \delta.
\]
Since \( \lim_{\epsilon \downarrow 0} g(\epsilon) = 0 \), it is easy to see that
\[
\liminf_{\epsilon \downarrow 0} g(\epsilon) \sum_{n \geq 1} h_n(\epsilon) f_n(\epsilon) \geq \liminf_{\epsilon \downarrow 0} g(\epsilon) \sum_{n \geq N_\tau} (1 - \tau) f_n(\epsilon)
\]
\[
= \liminf_{\epsilon \downarrow 0} g(\epsilon) \sum_{n \geq 1} (1 - \tau) f_n(\epsilon) = (1 - \tau) \lambda.
\]
and
\[
\limsup_{\epsilon \downarrow 0} g(\epsilon) \sum_{n \geq 1} h_n(\epsilon) f_n(\epsilon) \leq \limsup_{\epsilon \downarrow 0} g(\epsilon) \sum_{n \geq N_\tau} (1 + \tau) f_n(\epsilon)
\]
\[
= \limsup_{\epsilon \downarrow 0} g(\epsilon) \sum_{n \geq 1} (1 + \tau) f_n(\epsilon) = (1 + \tau) \lambda.
\]
Letting \( \tau \downarrow 0 \), the conclusion of the lemma follows. \( \square \)

The following lemma is useful in the study of the precise asymptotics in the law of the iterated logarithm.

Lemma 3. Let \( \{ a_n; \ n \geq 1 \} \) be a sequence of real numbers such that (2.1) holds. Then

(i) For every \( b \in (-1/2, 1] \), we have
\[
\lim_{\epsilon \downarrow 0} \epsilon^{(2b+1)/2} \sum_{n \geq 3} \frac{(\log \log n)^b}{n} \Phi \left( -\sqrt{2 + \epsilon} \log \log n - \frac{a_n}{\sqrt{n}} \right)
\]
\[
= e^{-\sqrt{2} \gamma} 2^{b-1} \sqrt{2/\pi} \Gamma((b + 1/2)).
\]
(ii) For $b = -1/2$, we have
\[
\lim_{\epsilon \to 0} \frac{1}{\log \left( \frac{\epsilon}{13} \right)} \sum_{n \geq 3} \frac{(\log \log n)^{-\frac{1}{2}}}{n} \phi \left( -\sqrt{2 + \epsilon} \log \log n - \frac{a_n}{\sqrt{n}} \right) = \lim_{\epsilon \to 0} \frac{1}{\log \left( \frac{\epsilon}{13} \right)} \sum_{n \geq 3} \frac{1}{n \sqrt{\log \log n}} \phi \left( -\sqrt{2 + \epsilon} \log \log n - \frac{a_n}{\sqrt{n}} \right) = \frac{e^{-\sqrt{\pi}}}{2\sqrt{\pi}}. \tag{3.4}
\]

(iii) If $b < -1/2$ and $\gamma > -\infty$, we have
\[
\lim_{\epsilon \to 0} \frac{1}{\log \left( \frac{\epsilon}{13} \right)} \sum_{n \geq 3} \frac{(\log \log n)^{-b\epsilon}}{n} \phi \left( -\sqrt{2 + \epsilon} \log \log n - \frac{a_n}{\sqrt{n}} \right) = \sum_{n \geq 3} \frac{(\log \log n)^{-b\epsilon}}{n} \phi \left( -\sqrt{2 \log \log n - \frac{a_n}{\sqrt{n}}} \right) < \infty. \tag{3.5}
\]

**Proof.** We only give the proof of (3.3). The other two statements (3.4) and (3.5) can be proved in the same vein.

We first prove that (3.3) holds if $|\gamma| < \infty$. Since
\[
\left( 1 - \frac{1}{x^3} \right) \phi(x) < \Phi(-x) < \frac{1}{x} \phi(x), \quad x > 0
\]
(see, e.g., Feller [6, p. 175]) and $|\gamma| < \infty$, it follows that
\[
\lim_{n \to \infty} \sup_{0 \leq \epsilon \leq 1} \left| \frac{\Phi \left( -\sqrt{(2 + \epsilon) \log \log n - \frac{a_n}{\sqrt{n}}} \right)}{\Phi \left( \sqrt{(2 + \epsilon) \log \log n + \frac{a_n}{\sqrt{n}}} / \sqrt{(2 + \epsilon) \log \log n + \frac{a_n}{\sqrt{n}}} \right) - 1} - 1 \right| = 0. \tag{3.6}
\]

Clearly, if $|\gamma| < \infty$ then
\[
\lim_{n \to \infty} \sup_{0 \leq \epsilon \leq 1} \left| \frac{\sqrt{(2 + \epsilon) \log \log n + \frac{a_n}{\sqrt{n}}}}{\sqrt{(2 + \epsilon) \log \log n}} - 1 \right| = 0
\]
which together with (3.6) gives
\[
\lim_{n \to \infty} \sup_{0 \leq \epsilon \leq 1} \left| \frac{\Phi \left( -\sqrt{(2 + \epsilon) \log \log n - \frac{a_n}{\sqrt{n}}} \right)}{\Phi \left( \sqrt{(2 + \epsilon) \log \log n + \frac{a_n}{\sqrt{n}}} / \sqrt{(2 + \epsilon) \log \log n} \right) - 1} - 1 \right| = 0.
\]

For $n \geq 3$, write
\[
f_n(\epsilon) = \Phi \left( \frac{\sqrt{(2 + \epsilon) \log \log n + \frac{a_n}{\sqrt{n}}}}{\sqrt{(2 + \epsilon) \log \log n}} \right)
\]
and
\[
h_n(\epsilon) = \frac{\Phi \left( -\sqrt{(2 + \epsilon) \log \log n - \frac{a_n}{\sqrt{n}}} \right)}{f_n(\epsilon)}.
\]
Thus, by Lemma 2, (3.3) follows if we can show that

\[
\lim_{\varepsilon \to 0} \varepsilon^{(2b+1)/2} \sum_{n \geq 3} \frac{(\log \log n)^b}{n} \times \frac{\phi\left(\sqrt{(2 + \varepsilon) \log log n + \frac{a_n}{\sqrt{n}}}\right)}{\sqrt{(2 + \varepsilon) \log log n}} = \lim_{\varepsilon \to 0} \varepsilon^{(2b+1)/2} \sum_{n \geq 3} \frac{(\log \log n)^b}{n} \times \frac{\phi\left(\sqrt{(2 + \varepsilon) \log log n + \frac{a_n}{\sqrt{n}}}\right)}{\sqrt{(2 + \varepsilon) \log log n}} = e^{-\sqrt{2} \gamma} 2^{b-1} \sqrt{2/\pi} \Gamma(b + (1/2)).
\]

(3.7)

Since \(|\gamma| < \infty\) and, for \(n \geq 3,\)

\[
\phi\left(\sqrt{(2 + \varepsilon) \log \log n + \frac{a_n}{\sqrt{n}}}\right) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\left(1 + \frac{\varepsilon}{2}\right) \log \log n - \left(\frac{(2 + \varepsilon) \log \log n}{n}\right)^{1/2} a_n - \frac{a_n^2}{n}\right\},
\]

we get

\[
\lim_{n \to \infty} \sup_{0 \leq \varepsilon \leq 1} \left| \frac{\phi\left(\sqrt{(2 + \varepsilon) \log \log n + \frac{a_n}{\sqrt{n}}}\right)}{\sqrt{2\pi}} \exp\left\{-\left(1 + \frac{\varepsilon}{2}\right) \log \log n - \left(\frac{(2 + \varepsilon) \log \log n}{n}\right)^{1/2} a_n - \frac{a_n^2}{n}\right\} - 1 \right| = \lim_{n \to \infty} \sup_{0 \leq \varepsilon \leq 1} \left| \frac{\phi\left(\sqrt{(2 + \varepsilon) \log \log n + \frac{a_n}{\sqrt{n}}}\right)}{\sqrt{2\pi}} (2\pi)^{-1/2} e^{-\sqrt{2} \gamma} (\log n)^{1 - \varepsilon/2} - 1 \right| = 0.
\]

Again by Lemma 2, (3.7) holds if we can show that

\[
\lim_{\varepsilon \to 0} \varepsilon^{(2b+1)/2} \sum_{n \geq 3} \frac{(\log \log n)^b}{n} \frac{1}{\sqrt{(2 + \varepsilon) \log \log n}} = \lim_{\varepsilon \to 0} \varepsilon^{(2b+1)/2} \sum_{n \geq 3} \frac{(\log \log n)^b}{n} \frac{1}{\sqrt{(2 + \varepsilon) \log \log n}} = e^{-\sqrt{2} \gamma} 2^{b-1} \sqrt{2/\pi} \Gamma(b + (1/2)).
\]

(3.8)

To prove (3.8), note that

\[
\lim_{\varepsilon \to 0} \varepsilon^{(2b+1)/2} \sum_{n \geq 3} \frac{1}{n(\log n)^{1+\varepsilon/2}(\log \log n)^{1/2-b}} = \lim_{\varepsilon \to 0} \varepsilon^{(2b+1)/2} \int_{3}^{\infty} \frac{1}{x(\log x)^{1+\varepsilon/2}(\log \log x)^{1/2-b}} dx
\]

\[
= \lim_{\varepsilon \to 0} \varepsilon^{(2b+1)/2} \int_{\log \log 3}^{\infty} \frac{z^{-b-1/2} e^{-(\varepsilon/2)z}}{dz}
\]
Thus (3.8) follows. Hence (3.3) holds if \(|\gamma| < \infty\).

We now show that (3.3) holds if \(\gamma = -\infty\). For any given \(\Lambda < 0\), write
\[
\hat{a}_n = \max\left\{ a_n, \Lambda \sqrt{\frac{n}{\log \log n}} \right\}, \quad n \geq 3.
\]
It is easy to see that
\[
\lim_{n \to \infty} \left( \frac{\log \log n}{n} \right)^{1/2} \hat{a}_n = \Lambda
\]
and that
\[
\liminf_{\epsilon \downarrow 0} \epsilon^{(2b+1)/2} \sum_{n \geq 3} \frac{(\log \log n)^b}{n} \phi \left( -\sqrt{(2 + \epsilon)} \log \log n - \frac{\hat{a}_n}{\sqrt{n}} \right) \\
\geq \lim_{\epsilon \downarrow 0} \epsilon^{(2b+1)/2} \sum_{n \geq 3} \frac{(\log \log n)^b}{n} \phi \left( -\sqrt{(2 + \epsilon)} \log \log n - \frac{\hat{a}_n}{\sqrt{n}} \right) \\
= e^{-\sqrt{\pi}} \lambda^{b-\frac{1}{2}} \sqrt{\frac{2}{\pi}} \Gamma \left( b + \frac{1}{2} \right).
\]

Letting \(\Lambda \to -\infty\), it follows that
\[
\lim_{\epsilon \downarrow 0} \epsilon^{(2b+1)/2} \sum_{n \geq 3} \frac{(\log \log n)^b}{n} \phi \left( -\sqrt{(2 + \epsilon)} \log \log n - \frac{\hat{a}_n}{\sqrt{n}} \right) = \infty.
\]
Similarly, if \(\gamma = \infty\) then
\[
\lim_{\epsilon \downarrow 0} \epsilon^{(2b+1)/2} \sum_{n \geq 3} \frac{(\log \log n)^b}{n} \phi \left( -\sqrt{(2 + \epsilon)} \log \log n - \frac{\hat{a}_n}{\sqrt{n}} \right) = 0.
\]
This completes the proof of (3.3). \(\square\)

**Proof of Theorem 1.** Theorem 1(i) will be established by considering the following three cases.

**Case I.** If \(|\gamma| < \infty\), then
\[
\sigma_n \sqrt{2n \log \log n + a_n} \geq \sqrt{n \log \log n}
\]
for all sufficiently large \(n\). Thus, by Lemma 1, we have, for every \(b \in (-1/2, 1]\), that (2.2) is equivalent to
\[\lim_{\varepsilon \downarrow 0} \varepsilon^{(2h+1)/2} \sum_{n \geq 3} \frac{(\log \log n)^b n}{n} P(\sigma_n \geq \sigma_n \sqrt{(2 + \varepsilon)n \log \log n + a_n})\]

\[= \lim_{\varepsilon \downarrow 0} \varepsilon^{(2h+1)/2} \sum_{n \geq 3} \frac{(\log \log n)^b n}{n} 2\Phi \left(-\sqrt{(2 + \varepsilon)n \log \log n + \frac{a_n}{\sigma_n}}\right)\]

\[= e^{-\sqrt{2} \gamma} 2^b \sqrt{2/\pi} \Gamma(b + 1/2). \quad (3.9)\]

Note that

\[\lim_{n \to \infty} \left(\frac{\log \log n}{n}\right)^{1/2} (a_n/\sigma_n) = \gamma.\]

So, applying Lemma 3(i), (3.9) follows and thus (2.2) has been established if \(|\gamma| < \infty\).

**Case II.** If \(\gamma = -\infty\), then

\[\lim_{\varepsilon \downarrow 0} \varepsilon^{(2h+1)/2} \sum_{n \geq 3} \frac{(\log \log n)^b n}{n} P(\sigma_n \geq \sigma_n \sqrt{(2 + \varepsilon)n \log \log n + \hat{a}_n})\]

\[\downarrow \lim_{\varepsilon \downarrow 0} \varepsilon^{(2h+1)/2} \sum_{n \geq 3} \frac{(\log \log n)^b n}{n} 2\Phi \left(-\sqrt{(2 + \varepsilon)n \log \log n + \frac{\hat{a}_n}{\sigma_n}}\right)\]

\[= e^{-\sqrt{2} \Delta} 2^b \sqrt{2/\pi} \Gamma(b + 1/2),\]

where \(\Delta\) is any given negative number and

\[\hat{a}_n = \max\left\{a_n, \Delta \sqrt{\frac{n}{\log \log n}}\right\}, \quad n \geq 3.\]

Letting \(\Delta \downarrow -\infty\), we get

\[\lim_{\varepsilon \downarrow 0} \varepsilon^{(2h+1)/2} \sum_{n \geq 3} \frac{(\log \log n)^b n}{n} P(\sigma_n \geq \sigma_n \sqrt{(2 + \varepsilon)n \log \log n + a_n}) = \infty.\]

**Case III.** If \(\gamma = \infty\), then

\[\limsup_{\varepsilon \downarrow 0} \varepsilon^{(2h+1)/2} \sum_{n \geq 3} \frac{(\log \log n)^b n}{n} P(\sigma_n \geq \sigma_n \sqrt{(2 + \varepsilon)n \log \log n + a_n})\]

\[\leq \lim_{\varepsilon \downarrow 0} \varepsilon^{(2h+1)/2} \sum_{n \geq 3} \frac{(\log \log n)^b n}{n} 2\Phi \left(-\sqrt{(2 + \varepsilon)n \log \log n + \hat{a}_n}\right)\]

\[= e^{-\sqrt{2} \Delta} 2^b \sqrt{2/\pi} \Gamma(b + 1/2),\]

where \(\Delta\) is any given positive number and

\[\hat{a}_n = \min\left\{a_n, \Delta \sqrt{\frac{n}{\log \log n}}\right\}, \quad n \geq 3.\]

Letting \(\Delta \uparrow \infty\), we get
\[
\lim_{\varepsilon \downarrow 0} \varepsilon^{(2b+1)/2} \sum_{n \geq 3} \frac{(\log \log n)^b}{n} P\left( |S_n| \geq \sigma_n \sqrt{(2 + \varepsilon)n \log \log n + a_n} \right) = 0.
\]

This completes the proof of Theorem 1(i).

Applying Lemma 1 and, respectively, Lemma 3(ii) and (iii), a similar approach can be used to establish Theorem 1(ii) and (iii). \qed

4. Further comments

In this section we provide some further comments. First, by following the proof of Theorem 1 with some routine modifications we obtain the following additional precise asymptotics in the law of the iterated logarithm result. We leave the proof to the reader.

**Theorem 3.** Let \( \{X, X_n; n \geq 1\} \) and \( \{\sigma_n; n \geq 3\} \) be as in Theorem 1. Then, for every \( b \in (-\infty, 1] \),

\[
\lim_{\varepsilon \downarrow 0} \varepsilon \sum_{n \geq 3} \frac{(\log \log n)^b}{n} P\left( \frac{|S_n|}{\sqrt{2n \log \log n}} \geq \sigma_n \left( 1 + (b + 0.5 + \varepsilon) \frac{\log \log n}{2 \log \log n} \right) \right) = \frac{1}{\sqrt{\pi}}.
\]

Moreover, under the additional assumption that

\[
E\left( X^2 I(|X| \geq t) \right) = o\left( \frac{\log \log t}{\log t} \right) \quad \text{as } t \to \infty,
\]

we have

\[
\lim_{\varepsilon \downarrow 0} \varepsilon \sum_{n \geq 3} \frac{(\log \log n)^b}{n} P\left( \frac{|S_n|}{\sqrt{2n \log \log n}} \geq 1 + (b + 0.5 + \varepsilon) \frac{\log \log n}{2 \log \log n} \right) = \frac{1}{\sqrt{\pi}}.
\]

Second, it is easy to see that Theorem 1 holds with \( |S_n| \) replaced by \( S_n, n \geq 1, \) in (2.2)–(2.4) and with the right-hand sides of (2.2) and (2.3) each multiplied by 1/2. The following open problem is suggested by the current work. The authors hope that this and other related problems will be further investigated.

**Problem 1.** In view of the equivalence between (1.1) and each half of (1.2) and the equivalence between (1.3), (1.4), and (1.5) as was discussed in Section 1, it is natural to pose the question as to whether (1.3), (1.4)', and (1.5)' are equivalent where (1.4)' and (1.5)' are (1.4) and (1.5), respectively, but with \( |S_n| \) replaced by \( S_n, n \geq 1. \) In other words, does a “one-sided” version of the equivalence between (1.3), (1.4), and (1.5) prevail?
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References