JOURNAL OF APPROXIMATION THEORY **50**, 18–24 (1987)

Lagrange and Hermite Interpolation Processes on the Positive Real Line

KATHERINE BALÁZS

Karl Marx University, Institute of Mathematics, Budapest 5, Pf. 489, H-1828, Hungary

Communicated by Paul G. Nevai

Received November 19, 1984; revised June 11, 1985

DEDICATED TO THE MEMORY OF GÉZA FREUD

1. INTRODUCTION

In this paper we consider interpolation based on the Laguerre roots and the point 0 as nodes. First we show that this interpolation generates a convergent approximation process on $[0, \infty)$ for a wide class of functions. Moreover, we prove the following interesting fact: In order to have uniform convergence of the derivatives of the interpolating polynomials in every interval $[0, A]$, it is sufficient to prescribe the derivatives at 0 *only*, in addition to the function values at the above-mentioned nodes.

Interpolating polynomials of degree $2n-1$ based on the roots of nth Laguerre polynomials and the point 0 were introduced first by Egerváry and Turán [4] as the "most economical" stable interpolation on $[0, \infty)$. A convergence theorem was proved by Balázs and Turán [1] and later this process was investigated by Joó [7–10].

Lagrange interpolation for the Laguerre abscissas and its convergence were treated by Freud [5] and Nevai [11–13]. Let

$$L_n^{(\alpha)}(x) = \frac{e^x x^{-\alpha}}{n!} \{e^{-x} x^{n+\alpha}\}^{(n)}, \qquad n = 1, 2, \ldots,$$

be the Laguerre polynomial of degree n for $\alpha > -1$, with the usual normalization

$$L_n^{(\alpha)}(0) = \binom{n+\alpha}{n}.$$

0021-9045/87 $3.00

These polynomials are orthogonal on $[0, \infty)$ with respect to the weight function $e^{-x}x^{\alpha}$. The zeros of $L_n^{(\alpha)}(x)$ are

$$(0<)\, x_{1n}^{(\alpha)} < x_{2n}^{(\alpha)} < \cdots < x_{nn}^{(\alpha)}.$$

If there is no danger of misunderstanding we will write briefly x_{kn} or x_k, $k=1, 2,..., n$.

In what follows we will always suppose that α is *integer*. Let f be an α-times differentiable function on $[0, \infty)$. Let us denote by $Q_{n,\alpha}(f; x)$ its Hermite interpolating polynomial of degree $n+\alpha$ with nodes $x_{kn}^{(\alpha)}$, $k=1, 2,..., n$, and 0, the latter with multiplicity $\alpha+1$. That is,

$$Q_{n,\alpha}(f; x) = \sum_{k=1}^{n} f(x_k)\left(\frac{x}{x_k}\right)^{\alpha+1} l_k(x) + \sum_{i=0}^{\alpha} f^{(i)}(0)\, r_i(x) \tag{1.1}$$

where $l_k(x)$ are the fundamental polynomials of Lagrange interpolation based on the roots of $L_n^{(\alpha)}(x)$:

$$l_k(x) = l_{kn\alpha}(x) = \frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)\prime}(x_k)(x-x_k)}, \qquad k=1, 2,..., n,$$

and the polynomials $r_i(x) = r_{in\alpha}(x)$ are such that

$$r_i^{(s)}(0) = \begin{cases} 1, & \text{if } s=i, \\ 0, & \text{if } 0 \leqslant s < i, \end{cases}$$

and

$$r_i(x_k) = 0, \qquad \text{for} \quad k=1, 2,..., n;\ i=1, 2,..., \alpha,$$

so that, explicitly,

$$r_i(x) = \frac{x^i L_n^{(\alpha)}(x)}{i!\binom{n+\alpha}{n}}, \qquad i=0, 1,..., \alpha.$$

In the case $\alpha=0$ we have *Lagrange interpolation*:

$$Q_{n,0}(f; x) = \sum_{k=1}^{n} f(x_k)\frac{x}{x_k} l_k(x) + f(0)\, L_n^{(0)}(x). \tag{1.2}$$

Convergence theorems and estimates concerning $Q_{n,0}(f)$ were announced without proof by the author at the Varna Conference on Constructive Theory of Functions in 1984, [2].

We remark that convergence problems of Hermite interpolation of type $Q_{n,\alpha}$ based on the point 0 and Laguerre roots for non-integral α can be considered also, but these investigations require other means and will be treated in a forthcoming paper.

2. Results

We give weighted estimates which imply the convergence of interpolating polynomials $Q_{n,\alpha}(f)$ and their derivatives $Q_n^{(i)}(f)$ to f and $f^{(i)}$, respectively in $[0, \infty)$.

In what follows $O(1)$ is always independent from x and n. Our first theorem concerns Lagrange interpolation based on the roots of $L_n^{(0)}(x)$ and the origin (see (1.2)).

Theorem 1. *Let $f \in \operatorname{Lip} \gamma$, $\frac{1}{2} < \gamma \leqslant 1$, in $[0, \infty)$. Then*

$$|f(x) - Q_{n,0}(f; x)| = O(1)\, x^{1/2}\, e^{x/2} n^{-\gamma/2 + 1/4},$$

for $0 \leqslant x \leqslant x_{nn}^{(0)}$.

Note the important fact $x_{nn}^{(\alpha)} \sim n$ for the greatest zero of $L_n^{(\alpha)}(x)$, which follows from Lemma 3. We use the symbol $\sim$ in the sense of Szegő [14, p. 1]: if two sequences z_n and w_n of numbers have the property that $w_n \neq 0$ and the sequence $|z_n|/|w_n|$ has finite positive limits of indetermination, we write $z_n \sim w_n$.

Theorem 2. *Let $f^{(\alpha)} \in \operatorname{Lip} \gamma$, $0 < \gamma \leqslant 1$, in $[0, \infty)$ for some $\alpha > 0$ integer. Then*

$$|f(x) - Q_{n,\alpha}(f; x)| = O(1)\, x^{(\alpha+1)/2}\, e^{x/2}\, n^{-(\alpha+\gamma)/2 + 1/4}$$

for $0 \leqslant x \leqslant x_{nn}$.

If $f^{(r)}$ exists for some $r > \alpha$, then we may have better estimates:

Theorem 3. *Let $f^{(r)} \in \operatorname{Lip} \gamma$, $0 < \gamma \leqslant 1$, in $[0, \infty)$ for some $r > \alpha$, where $\alpha \geqslant 0$ and integer. Then*

$$|f(x) - Q_{n,\alpha}(f; x)| = O(1)\, x^{(\alpha+1)/2}\, e^{x/2}\, n^{-(r+\gamma)/2 + 1/4}$$

for $0 \leqslant x \leqslant x_{nn}^{(\alpha)}$.

Corollary. *The convergence of $Q_{n,\alpha}(f)$ to f is uniform in every finite subinterval of $[0, \infty)$ under the assumptions of the above theorems.*

THEOREM 4. *Suppose that $f^{(r)}$ exists in $[0, \infty)$ for some $r \geqslant \alpha$, where $\alpha \geqslant 0$ and integer. Let $f^{(r)} \in \operatorname{Lip} \gamma$, $\frac{1}{2} < \gamma \leqslant 1$ if r is even or $f^{(r)} \in \operatorname{Lip} \gamma$, $0 < \gamma \leqslant 1$ if r is odd. Then*

$$|f^{(i)}(x) - Q_{n,\alpha}^{(i)}(f; x)| = O(1)\, x^{(\alpha+1)/2 - i}\, e^x\, n^{-(r+\gamma)/2 + i + 1/4}$$

for $1 \leqslant i \leqslant [r/2]$ and $0 \leqslant x \leqslant x_{nn}^{(\alpha)}/2$.

COROLLARY. *The convergence of $Q_{n,\alpha}^{(i)}(f)$ to $f^{(i)}$ is uniform in every finite subinterval of $[0, \infty)$ if $1 \leqslant i \leqslant [\alpha/2]$.*

3. LEMMAS AND PROOFS

LEMMA 1. *If $f^{(r)}$ exists and is continuous in $[0, \infty)$, $r \geqslant 0$, then there exists a polynomial G_n of degree $n \geqslant 4r + 5$ at most, that*

$$|f^{(i)}(x) - G_n^{(i)}(f; x)| = O(1)\, \omega\left(f^{(r)}; \frac{\sqrt{x(x_n - x)}}{n}\right)\left(\frac{\sqrt{x(x_n - x)}}{n}\right)^{r-i}$$

$$0 \leqslant x \leqslant x_n, \quad i = 0, 1, \ldots, r,$$

where $\omega(f^{(r)}; \cdot)$ denotes the modulus of continuity of $f^{(r)}$ on $[0, x_n]$.

The lemma shows that $G_n^{(i)}(f; 0) = f^{(i)}(0)$, $i = 0, 1, 2, \ldots, r$.

Proof. The lemma is an easy consequence of Gopengauz's theorem [6].

LEMMA 2 (Joó [10, inequality (11)]).

$$\frac{e^x}{x^{\alpha+1}} - \sum_{k=1}^{n} \frac{e^{x_k}}{x_k^{\alpha+1}} \left(\frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)\prime}(x_k)(x - x_k)}\right)^2 \geqslant 0, \qquad x > 0, \alpha > -1.$$

LEMMA 3. *Let $\alpha > -1$. Then the following asymptotic relation holds for the zeros $x_k = x_{kn}^{(\alpha)}$ of $L_n^{(\alpha)}(x)$:*

$$x_{kn}^{(\alpha)} \sim \frac{k^2}{n}, \qquad k = 1, 2, \ldots, n; n = 1, 2, \ldots.$$

Proof. Lemma 3 follows from Theorem 6.31.3 of Szegő [14], e.g.,

LEMMA 4. *Let $\alpha > -1$ and $\beta > \alpha/2 + \frac{1}{4}$. Then for the zeros of $L_n^{(\alpha)}(x)$ the estimate*

$$\sum_{k=1}^{n} x_k^{\beta - \alpha - 1}(x_n - x_k)^{\beta}\, x^{\alpha+1}\, |l_k(x)| = O(1)\, n^{\beta + 1/4} x^{(\alpha+1)/2}\, e^{x/2}$$

holds for $x \geqslant 0$.

Proof. By Lemma 3 our sum is equal to

$$S_n = x_n^\beta \sum_{k=1}^{n} x_k^{\beta-\alpha-1} \left(1 - \frac{x_k}{x_n}\right)^\beta x^{\alpha+1} |l_k(x)|$$
$$= O(1)\, n^\beta x^{(\alpha+1)/2} \sum_{k=1}^{n} x_k^{\beta-(\alpha+1)/2} e^{-x_k/2} e^{x_k/2} \left(\frac{x}{x_k}\right)^{(\alpha+1)/2} |l_k(x)|.$$

Using Cauchy's inequality and Lemma 2 we obtain

$$S_n = O(1)\, n^\beta x^{(\alpha+1)/2} \left\{ \sum_{k=1}^{n} x_k^{2\beta-(\alpha+1)} e^{-x_k} \right\}^{1/2} e^{x/2}. \tag{3.1}$$

Let $-\frac{1}{2} < 2\beta - (\alpha+1) \leqslant 0$. Then denoting the sum under square root by T_n, we have by Lemma 3

$$T_n = \sum_{k=1}^{n} x_k^{2\beta-(\alpha+1)} e^{-x_k} = O(1) \sum_{k=1}^{n} \left(\frac{k^2}{n}\right)^{2\beta-(\alpha+1)} e^{-ck^2/n}$$
$$= O(1) \int_0^\infty \left(\frac{x^2}{n}\right)^{2\beta-(\alpha+1)} e^{-cx^2/n}\, dx = O(1)\, n^{1/2}, \tag{3.2}$$

where c is a positive constant.

In the case $2\beta - (\alpha+1) > 0$ the function $y(x) = (x^2/n)^{2\beta-(\alpha+1)} e^{-cx^2/n}$ $(x > 0)$ attains its maximum at $x_0 = \sqrt{n(2\beta-(\alpha+1))/c}$, and y decreases monotonically, if $x > x_0$. Let $N = [x_0] + 1$, $N = O(1) n^{1/2}$ evidently. We get by repeated applications of Lemma 3,

$$T_n = \sum_{k=1}^{N} x_k^{2\beta-(\alpha+1)} e^{-x_k} + O(1) \sum_{k=N+1}^{n} \left(\frac{k^2}{n}\right)^{2\beta-(\alpha+1)} e^{-ck^2/n}$$
$$= O(1)\, N x_N^{2\beta-(\alpha+1)} + O(1) \int_N^\infty \left(\frac{x^2}{n}\right)^{2\beta-(\alpha+1)} e^{-cx^2/n}\, dx$$
$$= O(1)\, n^{1/2}. \tag{3.3}$$

The lemma follows from (3.1)–(3.3).

LEMMA 5 (Bernstein [3]). *Let* $M = \max_{0 \leqslant x \leqslant A} |P_n(x)|$, *where* $P_n(x)$ *is a polynomial of degree* n, *then*

$$|P_n^{(k)}(x)| \leqslant \left(\frac{k}{x(A-x)}\right)^{k/2} n^k M, \qquad k = 1, 2, \dots, n;\ 0 \leqslant x \leqslant A.$$

Proofs of Theorems 1, 2, *and* 3. Only the proof of Theorem 3 $(r>\alpha)$ will be detailed, since the proofs of Theorems 2 and 1 can be treated as analog cases where $r=\alpha>0$ and $r=\alpha=0$, respectively.

Let $G_{n+\alpha}(f)$ be the polynomial defined in Lemma 1. Then we may write by Lemma 1,

$$\begin{aligned}|f(x)-Q_{n,\alpha}(f;x)| &\leqslant |f(x)-G_{n+\alpha}(f;x)|+|G_{n+\alpha}(f;x)-Q_{n,\alpha}(f;x)|\\ &=O(1)\,\omega\left(f^{(r)};\frac{\sqrt{x(x_n-x)}}{n}\right)\left(\frac{\sqrt{x(x_n-x)}}{n}\right)^r\\ &\quad+|Q_{n,\alpha}(G_{n+\alpha}f-f;x)|\end{aligned}$$

where $\omega(f^{(r)};\cdot)$ denotes the modulus of continuity of $f^{(r)}$ in $[0,\infty)$.

Using Lemma 3 and Lemma 1 again we get

$$\begin{aligned}&|f(x)-Q_{n,\alpha}(f;x)|\\ &\quad=O(1)\,x^{(r+\gamma)/2}n^{-(r+\gamma)/2}\\ &\quad\quad+O(1)\sum_{k=1}^{n}\omega\left(f^{(r)};\frac{\sqrt{x_k(x_n-x_k)}}{n}\right)\left(\frac{\sqrt{x_k(x_n-x_k)}}{n}\right)^r\\ &\quad\quad\times\left(\frac{x}{x_k}\right)^{\alpha+1}|l_k(x)|.\end{aligned}$$

Applying Lemma 4 $(\beta=(r+\gamma)/2)$ we obtain our theorem.

Proof of Theorem 4. Let $G_{n+\alpha}(f)$ be the polynomial defined in Lemma 1. Then we have by that lemma,

$$\begin{aligned}&|f^{(i)}(x)-Q_{n,\alpha}^{(i)}(f;x)|\\ &\quad\leqslant|f^{(i)}(x)-G_{n+\alpha}^{(i)}(f;x)|+|G_{n+\alpha}^{(i)}(f;x)-Q_{n,\alpha}^{(i)}(f)|\\ &\quad=O(1)\,\omega\left(f^{(r)};\frac{\sqrt{x(x_n-x)}}{n}\right)\left(\frac{\sqrt{x(x_n-x)}}{n}\right)^{r-i}+|Q_{n,\alpha}^{(i)}(G_{n+\alpha}f-f;x)|\end{aligned}$$

where $\omega(f^{(r)};\cdot)$ denotes the modulus of continuity of $f^{(r)}$ in $[0,\infty)$.

Applying Lemma 3, Lemma 5 for $Q_{n,\alpha}(f)$ if $A=2x$, and Lemma 1 again, we get

$$\begin{aligned}&|f^{(i)}(x)-Q_{n,\alpha}^{(i)}(f;x)|\\ &\quad=O(1)\,x^{(\gamma+r-i)/2}\,n^{-(\gamma+r-i)/2}\\ &\quad\quad+i^{i/2}x^{-i}n^{i}\max_{0\leqslant t\leqslant 2x}|Q_{n,\alpha}(G_{n+\alpha}f-f;t)|\end{aligned}$$

$$= O(1)\, x^{(\gamma+r-i)/2} n^{-(\gamma+r-i)/2}$$

$$+ O(1)\, x^{-i} n^{i} \max_{0 \leqslant t \leqslant 2x} \sum_{k=1}^{n} \omega\left(f^{(r)}; \frac{\sqrt{x_k(x_n - x_k)}}{n}\right)$$

$$\times \left(\frac{\sqrt{x_k(x_n - x_k)}}{n}\right)^{r} \left(\frac{t}{x_k}\right)^{\alpha+1} |l_k(t)|.$$

Using Lemma 4 ($\beta = (\gamma + r)/2$) we can estimate the maximum of the last sum by

$$O(1)\, n^{-(\gamma+r)/2+1/4} x^{(\alpha+1)/2} e^{x},$$

which proves the theorem.

References

1. J. Balázs and P. Turán, Notes on interpolation VII, *Acta Math. Hungar.* **9** (1959), 63–68.
2. K. Balázs, Lagrange interpolation for the completed Laguerre abscissas, *in* "Proceedings, Internat. Conf. on Constructive Theory of Functions," Varna, 1984, pp. 150–153.
3. S. N. Bernstein, "Collected Works," Vol. 1, Academy of Sciences of the USSR, Moscow, 1952. [Russian]
4. E. Egerváry and P. Turán, Notes on interpolation VI, *Acta Math. Hungar.* **10** (1959), 55–62.
5. G. Freud, Convergence of the Lagrange interpolation on the infinite interval, *Mat. Lapok* **18** (1967), 289–292.
6. I. E. Gopengauz, On Timan's theorem on approximation of functions by polynomials, *Mat. Zametki* **1** (1967), 163–172. [Russian]
7. I. Joó, Stable interpolation on an infinite interval, *Acta Math. Hungar.* **25** (1974), 147–157.
8. I. Joó, On positive linear interpolation operators, *Anal. Math.* **1** (1975), 273–281.
9. I. Joó, Interpolation on the roots of Laguerre polynomials, *Ann. Univ. Sci. Budapest Eötvös Sect. Math.* **17** (1974), 183–188.
10. I. Joó, An interpolation-theoretical characterization of the classical orthogonal polynomials, *Acta Math. Hungar.* **26** (1975), 163–169.
11. P. Nevai, On Lagrange interpolation based on Laguerre roots, *Mat. Lapok* **22** (1971), 149–164. [Hungarian]
12. P. Nevai, On the convergence of Lagrange interpolation based on Laguerre roots, *Publ. Math.* **20** (1973), 235–239. [Russian]
13. P. Nevai, Lagrange interpolation at zeros of orthogonal polynomials, "Approximation Theory" (G. G. Lorentz, Ed.), Academic Press, New York, pp. 163–201, 1976.
14. G. Szegő, "Orthogonal Polynomials," AMS Colloq. Publ. Vol. 23, Providence, R. I., 1978.