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ON RINGEISEN'S ISOLATION GAME

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We develop the theory of the Isolation Game on a graph G, in which two players alternately "switch" at successive vertices v not previously switched. The switching operation deletes all edges incident with v, and creates new edges between v and those vertices not previously adjacent to it. The game is won when a vertex is first isolated. Among other results, we show that n-vertex forced wins exist for all n, and that length-p forced wins exist for all p. We give generic examples of forced wins which (against best defense) can be won only very late in the game. We also prove several large classes of graphs to be unwinnable, and give a complexity result for a problem closely related to the identification of drawing strategies in $I_n(G)$.

1. Introduction

We shall deal exclusively with finite undirected graphs $G=(V, E)$ which are *simple* (no loops, no multiple edges), and set $n=|V|>1$ throughout. The neighborhood set of a vertex $v \in V$ will be denoted by $N(v)=\{x \in V: (v, x) \in E\}$ ($N(v, G)$ if the underlying graph needs specification); its cardinality, $d_G(v)$, is the *degree* of vertex v.

The operation of *switching* G at $v \in V$ (briefly: "switching v"), apparently introduced by van Lint and Seidel [8], replaces G by the graph obtained from G by deleting the edges $\{(v, x): x \in N(v)\}$ and adjoining new edges $\{(v, y): y \notin N(v)\}$. This switching operation and its induced equivalence relation have been studied, e.g. by Colbourn and Corneil [3], Mallows and Sloane [9], Taylor [13], and Goldman [6].

In 1974 Ringeisen [10] introduced the Isolation Game $I_n(G)$, describable for our purposes as follows: play begins with the n-vertex graph G. Players 1 and 2, denoted P1 and P2, switch alternately, each time at a vertex not previously used for switching. Play ends as soon as one player succeeds in isolating a vertex; otherwise the game is drawn after move n. For which graphs G is $I_n(G)$ a win for P1 (assuming best play), or a win for P2, or a draw? If a win, how long can the

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loser postpone defeat? For example [10], for $G = K_n$ (complete), any switch is an immediate win for P1, while for $G = C_n$ (*n*-*cycle*, $n > 3$) P2 can quickly win. In [10] it is also shown that for $G = K_{q,n-q}$ (complete bipartite, $2 \leq q \leq n-2$, $n > 4$), neither player has a forced win. Surprisingly, it appears that no further analyses of $I_n(G)$ have been published (Ringeisen, personal communications). The present paper and sequelae (based on [11]) redress this neglect, which may reflect the difficulty of "tracking" the more-than-local changes in G produced by switching operations.

The difficulty is overcome by our basic tool (Theorem 2.1), a result which allows reasoning about the progress of the game using only "static" knowledge of the initial graph. This result, exhibiting $I_n(G)$ as closely related to (although no subsumed under) the "positional games" of Berge [1] and the "achievement and avoidance" games of Harary [7], is then applied to the study of *forced p-wins*: graphs for which one of the players can force a win (for herself) in no more than p moves, but neither player can force a win in fewer than p moves. For example, G is a forced 1-win iff G contains a vertex of degree 1 or $n-1$. We show (Theorem 2.5) that there exist connected forced p-wins for any choice of p (for $p \neq 3$, even if the graphs are required to be bipartite), and also (Theorems 2.3, 2.4) characterize all forced p-wins for $p \leq 3$.

Section 3 shifts attention from forced wins to nonwinnability. Ringeisen's result [10] for bipartite graphs is generalized to the multipartite case (Theorem 3.1), and some classes of graphs admitting similar analyses are identified. Drawing strategies for one player in a large class of graphs are formulated and justified (Theorem 3.2). In addition, a problem closely related to nonwinnability of $I_n(G)$ is shown to be NP-complete (Theorem 3.3).

Ringeisen's definition [10] of $I_n(G)$ actually required G to be connected. This condition proved inessential in our work, and so while "preferring" connected graph in our constructions and examples, we shall require only (to avoid trivial "pre-won" cases) that the graph G be free of isolated vertices.

2. Forced wins

Here is the result, extending Theorem 4 of [6], that permits "static analysis" of $I_n(G)$:

Theorem 2.1. *A play of $I_n(G)$ ends, with v as isolated vertex and S the set of switched vertices, iff S is $N(v)$ or its complement $N(v)^c$.*

Proof. (Only if) If $v \notin S$ then certainly $N(v) \subset S$, else some $y \in N(v)$ remains adjacent to v. If further $w \in N(v)^c$ belongs to S, then v is now adjacent to w, contradicting its isolation. So $S = N(v)$. If $v \in S$, then $N(v)^c \subset S$ else some

$y \in N(v)^c$ is now adjacent to v. If further $w \in N(v) \cap S$, then v is adjacent to w in the final graph. So $S = N(v)^c$.

(If) If $S = N(v)$, implying $v \in S^c$, then v is now adjacent to none of $N(v)$, and is still nonadjacent to all members of $N(v)^c$. Hence v is isolated. If $S = N(v)^c$, implying $v \in S$, then all pairs $\{(v, y): y \in N(v)^c\}$ have been switched and so v remains nonadjacent to all $y \in N(v)^c$, but it has become nonadjacent to $N(v)$ since v has been switched and none of $N(v)$ have been switched. So again v is isolated. $\square$

Note that the identity of the winning player is determined by the parity of $|S|$, the number of moves in the win, which by the Theorem must be $d_G(v)$ or $n - d_G(v)$. For example, if all vertex-degrees in G are odd then (since n must be even) P2 cannot win $I_n(G)$, while if all degrees are even and n is even, P1 cannot win.

The above theorem is useful because it allows reasoning about the progress of the game to be carried out in terms only of the initial graph: its underlying neighborhood sets and their complements. A *winning set* of vertices will refer to any collection of vertices whose switching (without switching of any other vertices) will result in an isolated vertex. By Theorem 2.1 these are precisely $N(v)$ and $N(v)^c$ for each $v \in V$. We now give different and simpler proofs of two results in [6].

Theorem 2.2. *$I_n(G)$ is won immediately by every move of P1 iff G is either K_n or a perfect matching of V.*

Proof. Sufficiency is clear by Theorem 2.1, since either choice of G makes every singleton a winning set, so assume $I_n(G)$ winnable on every first move by P1. The theorem being trivial for $n = 2$, assume $n > 2$. For $w \in V$, write $w \in V_1$ if $\{w\} = N(v_1)$ for some v_1 (this implies $v_1 \neq w$ and $v_1 \in N(w)$), and write $w \in V_2$ if $\{w\} = N(v_2)^c$ for some $v_2 \in V$ (which implies $v_2 = w$). By Theorem 2.1, $V = V_1 \cup V_2$.

If V_1 is all of V, then each $w \in V$ is the unique neighbor of some vertex v, which in turn is the unique neighbor of some vertex that can only be w. It follows readily that, as desired, G is a perfect matching on V. Therefore assume some $w \in V_2 - V_1$ (implying $d_G(w) = n - 1$); then each other vertex v has w as non-unique neighbor, implying $d_G(v) > 1$ and therefore $d_G(v) = n - 1$. Since every vertex has degree $n - 1$, we have $G = K_n$. $\square$

Theorem 2.3. *G is a forced 2-win iff no vertex has degree 1 or $n - 1$, and every vertex w either has degree $n - 2$, or is the unique non-neighbor of a vertex $v \neq w$ of degree $n - 2$, or is a neighbor of a vertex of degree 2.*

Proof. Suppose G is a forced 2-win. Since G does not admit a forced 1-win, it must give rise to no winning set of cardinality 1, i.e. it possesses no vertex of degree 1 or $n-1$. Since any $w \in V$ must be completable into a winning set of size 2, every w belongs to a neighborhood or complement set of size 2. This is precisely the statement made by the theorem.

Suppose G satisfies the stipulated conditions. Then every v belongs to a winning set of cardinality 2, and no vertex belongs to a singleton winning set. Thus G is a forced 2-win. □

The n-cycle C_n $(n>3)$ is an example [10] of a connected 2-win; Theorem 2.3 was used in [6] to concoct quite different-looking examples. Here we will give (in Corollary 2.3B) a structural characterization of a large class of forced 2-wins G which includes the latter examples; it will be more convenient to work with the complementary graphs G^c (this usage should cause no confusion with our notation $S^c = V - S$ where $S \subset V$). Theorem 2.3 immediately yields

Corollary 2.3A. *Graph H is the complement of a forced 2-win iff no vertex has degree 0 or $n-2$, and every vertex w is either (a) of degree 1, or (b) the unique neighbor of some other vertex, or (c) a non-neighbor of some other vertex of degree $n-3$.*

It will be useful first to characterize those graphs whose vertices all satisfy condition (a) or (b) of the corollary; the complements of these graphs are forced 2-wins. For any graph $H=(V_H, E_H)$ without isolated vertices, let V_H^+ denote the subset of vertices of degree >1, and $H(V_H^+)$ the subgraph induced by V_H^+. Note that if $H_1, \ldots, H_p$ are the components of H which are not single edges, and if $V_i = V(H_i)$, then $H(V_1^+), \ldots, H(V_p^+)$ are the corresponding components of $H(V_H^+)$. We call H *spiked* if each $v \in V_H^+$ is the (unique) neighbor in H of at least one vertex of degree 1; the spiked graphs are precisely those whose vertices all satisfy (a) or (b) of the corollary. There is a generic construction for such graphs; beginning with any graph $J=(V_J, E_J)$, one associates to the vertices $v \in V_J$ disjoint non-empty sets V_v of "new" vertices, and adjoins to J the edges $\{(v,x): v \in V_J, x \in V_v\}$. Single-edge components of such a "spiked extension" of J can arise only from isolated vertices of J.

Corollary 2.3B. *A disconnected graph H is the complement of a forced 2-win iff H is spiked.*

Proof. First assume H is spiked. Then it has no isolated vertex, hence each component contains 2 or more vertices, hence no vertex has degree $>n-3$. By spikiness, each vertex satisfies condition (a) or (b) of Corollary 2.3A, so H^c must be a forced 2-win.

Conversely, assume H^c a forced 2-win. By Corollary 2.3A no vertex of H can be isolated, so each component contains 2 or more vertices. If H has 3 or more components, then it contains no vertex of degree $n-3$ and so condition (c) of

Corollary 2.3A cannot apply; hence every vertex satisfies (a) or (b), so H is spiked. If H has 2 components but is not spiked, then $n > 4$ and by Corollary 2.3A H has at least one vertex of degree $n-3$, say $N(x, H) = \{u_1, \ldots, u_p, v_1, \ldots, v_q\}$ where each $d_H(v_i) = 1$, each $d_H(v_j) > 1$, and $p + q = n - 3$. The $n-2$ vertices of $\{x\} \cup N(x, H)$ form the vertex-set of one component H_1 of H; the other component is a single edge (y, z). Since H is not spiked, $q > 0$ must hold; furthermore each v_i cannot satisfy (a) or (b) and must therefore be a non-neighbor of some other vertex, of degree $n-3$, which can only be a v_j. But the 3-set $N(v_j, H)^c$ includes $\{v_j, y, z\}$, which leaves no room for v_i, a contradiction. □

It was thought for a time that the lengths of forced wins might be bounded; however the next few theorems show that forced p-wins exist for arbitrary p.

Theorem 2.4. *G is a forced 3-win iff G is not a forced 1 or 2-win and there exists a collection of winning sets W_i each of cardinality 3 such that $\emptyset \neq \bigcap W_i \subset \bigcup W_i = V$.*

Proof. Suppose G is a forced 3-win. Let v be the first switch in such a win by P1. Let the winning sets of cardinality 3 containing v be named and indexed W_i. Every switch $w \in V$ available to P2 must belong to some W_i, else P2 can avoid a loss on the third move. Hence $V \subset \bigcup W_i$.

If G gives rise to winning sets with the above properties, let P1 switch first on any vertex $v \in \bigcap W_i$. For any response w by P2, $\{v, w\} \subset W_i$ for some i and so P1 can force completion of a winning set of cardinality 3. □

An example of a connected forced 3-win is given by the graph in Fig. 1. We can take, for instance, $W_1 = \{1, 3, 4\}$, $W_2 = \{1, 2, 5\}$.

Ringeisen [10] shows that the graph $K_{q,n-q}$, $2 \leq q \leq n-2$, is nonwinnable. While "most" bipartite graphs are nonwinnable, in fact there exist nontrivial instances of connected bipartite forced p-wins for arbitrarily large p. We exhibit such forced wins next, returning to the issue of nonwinnability (generalized to *multi*-partite graphs) in the later Theorem 3.1. In the following proof, for all $p \geq 7$ the desired graphs are constructed according to the residue of $|V|$ modulo 4. This is because 4-paths and 4-cycles turn out to be suitable

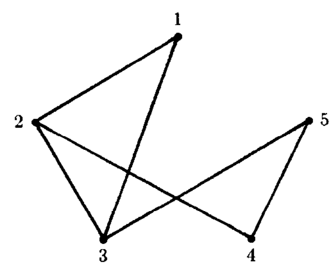

Fig. 1. A forced 3-win.

building blocks for designing winning-set structures easily shown to have desirable properties for our purpose: they give rise to a natural move-countermove pairing of most of the vertices, allowing forcing of a winning set on the pth move but no earlier, and permitting the winning player to control the game should the opponent switch a vertex "off" one of the building blocks.

Here and later we denote the set of switched-at vertices at move r by S_r, call a winning set w *completable* from S_r if $S_r \subset w$, and shall call it *killed* if it is no longer completable, i.e. if $S \cap w^c \neq \emptyset$.

Theorem 2.5. *For any p, there exists a connected graph on $n = p + 2$ vertices which is a forced p-win. G can be chosen bipartite iff $p \neq 3$.*

Proof. Consider some bipartite G. If any vertex had degree 1 then G would be a forced 1-win. If not, and either part of the partition $V_1 \cup V_2$ of V had only 2 vertices in it, then $G = K_{2,n-2}$, and by Theorem 5 of [10] would be nonwinnable. So we may assume $|V_1| \geqslant |V_2| \geqslant 3$, implying $N^c(v) \geqslant 3$ for all vertices v.

P2 should switch to ensure that S_2 contains one vertex from V_1 and one from V_2. This cannot be a winning set, since it neither lies in any neighborhood nor (by the last paragraph) is contained in any neighborhood-complement. Similarly, P1 cannot complete a winning set on the next move. Therefore G is not a forced 3-win.

Now for the existence claims. For $p = 1$, take G to be a path on 3 vertices. For $p = 2$ take G to be C_4. For $p = 3$, let G be the graph of Fig. 1.

For $p = 4$, take G to be the complete bipartite graph $K_{3,3}$ "between" $\{1, 3, 5\}$ and $\{2, 4, 6\}$, with one arc, say the one between vertices $(1, 2)$, removed. P2 can force S_2 to be either a winning set or $\{1, 2\}$. In the latter case, all possible third moves by P1 are covered by the two winning sets $N(1)^c, N(2)^c = \{1, 2, 3, 5\}, \{1, 2, 4, 6\}$.

If $p = 5$, form G from the graph $K_{3,4}$ with one part containing vertices $\{1, 3, 5\}$ and the other $\{2, 4, 6, 7\}$. Arcs $(1, 6)$, $(1, 7)$, $(3, 4)$ and $(5, 2)$ are removed.

Let $S_1 = \{7\}$. P2 cannot reply at any of the vertices 2, 4, 6, without giving P1 the opportunity to complete the winning sets $\{2, 6, 7\} = N(3)$ or $\{4, 6, 7\} = N(5)$. Thus P1 can force $S_3 = \{1, 6, 7\}$, $\{3, 6, 7\}$ or $\{5, 6, 7\}$. The winning sets $N(1)^c$, $N(2)^c$, $N(4)^c$ and $N(6)^c$ cover the possibilities for remaining play.

For $p = 6$, take G to be the graph given in Fig. 2.

P1 must switch vertex 1, 2 or 3 on the first move or lose on the second move. Whichever of these P1 switches, the completable winning sets are:

Description	*Winning set*
$N(1\text{–}2\text{–}3)^c$	$\{1, 2, 3, 4, 5, 8\}$
$N(4)^c$	$\{1, 2, 3, 4, 5, 7\}$
$N(5)^c$	$\{1, 2, 3, 4, 5, 6\}$
$N(6)$	$\{1, 2, 3, 4\}$
$N(7)$	$\{1, 2, 3, 5\}$
$N(8)^c$	$\{1, 2, 3, 6, 7, 8\}$

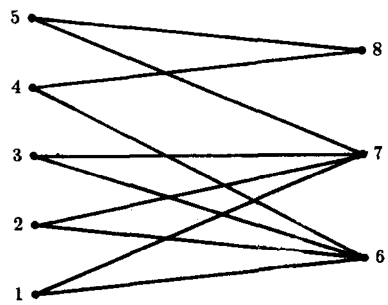

Fig. 2. A bipartite forced 6-win.

On the second move, P2 should reply with any other vertex in $\{1, 2, 3\}$, killing no more winning sets. P1, because of the presence of the winning sets $\{1, 2, 3, 4\}$ and $\{1, 2, 3, 5\}$ must switch on one of $\{6, 7, 8\}$. Whichever of these is switched, P2 should switch the remaining member of $\{1, 2, 3\}$ on the fourth move. It is not difficult to see that P2 can match a winning set on the sixth move.

For $p \geqslant 7$, the graphs are constructed as described below, according to the residue class of p modulo 4; their vertices are numbered $1, 2, \ldots, n$. At the referees' suggestion, we accompany the construction with a verifying analysis only for the case $p \equiv 3 \pmod 4$; the interested reader can readily adapt this to the other cases, or consult [11].

If $p = 4m - 1$, $m \geqslant 2$, the construction is on $n = 4m + 1$ vertices. G is obtained from a $4m$-cycle by joining every fourth vertex $v \equiv 0$ (note that these divide the $4m$-cycle into m 4-paths) to a special vertex indexed $4m + 1$. "Vertex 0" is a synonym for vertex $4m$. P1 must switch first on vertex $4m + 1$ to avoid completion by P2 of some $N(4j + 1) = \{4j, 4j + 2\}$ or $N(4j + 2) = \{4j + 1, 4j + 3\}$ on the second move. The winning sets completable from $4m + 1$ are listed below for $j = 0, \ldots, m - 1$:

Description	*Winning set*	*Size*
$N(4j+1)^c$	$V - \{4j, 4j+2\}$	$4m-1$
$N(4j+2)^c$	$V - \{4j+1, 4j+3\}$	$4m-1$
$N(4j+3)^c$	$V - \{4j+2, 4j+4 \pmod{4m}\}$	$4m-1$
$N(4j+4)$	$\{4m+1, 4j+3, 4j+5 \pmod{4m}\}$	3
$N(4m+1)^c$	$V - \{v : v \equiv 0\}$	$3m+1$

P1 will follow a "symmetric" strategy, ensuring that after each exchange either

$$S_{2r+1} = S_{2r-1} \cup \{4j, 4j + 2\}$$

or

$$S_{2r+1} = S_{2r-1} \cup \{4j + 1, 4j + 3\}$$

for some j. The symmetric strategy, furthermore, completes no winning sets prior to move $4m - 1$ ($N(4m + 1)^c$ cannot be completed for if a $v \equiv 2 \pmod 4$ is switched then a $v \equiv 0 \pmod 4$ must have been switched also). Thus $S_{4m-1} = S_p$ is forced to be a winning set of type $N(4j+1)^c$ or $N(4j+2)^c$.

G is not a forced 3-win, as P2 can avoid both $N(4j+4)$ and $N(4m+1)$ by a switch on move 2 at some $v \equiv 2 \pmod 4$. For $m > 2$, a move 3 or 4 at some $v \equiv 1 \pmod 4$ avoids a loss at move $3m+1 < 4m-1$.

If $p = 4m-2$, $m > 2$, then the construction is given on $n = 4m$ vertices. The basic building block B is a 4-cycle; m copies of B are indexed so that the vertices of odd index in each are adjacent to those with even index. The ith copy of B contains vertices $4i-3$, $4i-2$, $4i-1$, $4i$. In addition, vertices 2 and 4 are joined to all vertices $4j+1$.

If $p = 4m$, $m > 1$, then the construction is given on $n = 4m+2$ vertices. G consists of m copies of a four-cycle (indexed as before) together with two special vertices $4m+1$ and $4m+2$. All vertices $v \equiv 3 \pmod 4$ are connected to vertex $4m+2$. All vertices $v \equiv 0 \pmod 4$ are connected to vertex $4m+1$.

If $p = 4m+1$, $m > 1$, the construction is given on $n = 4m+3$ vertices. The graph G includes m copies of a four cycle (indexed as before). In addition there are three "special" vertices $4m+1$, $4m+2$ and $4m+3$. All vertices $v \equiv 3 \pmod 4$ are connected to vertices $4m+2$ and $4m+3$. Vertex $4m+1$ is connected to vertices $4m+2$ and $4m+3$.

The theorem is now proved. □

Remark. For all p, there also exist connected *non-bipartite* graphs which are forced p-wins on $p+2$ vertices (cf. [11]).

The relationship $n = p+2$ which appeared in Theorem 2.5 and the preceding remark is in fact extremal (i.e. n is minimal). This is an immediate consequence of the following "Isolation Game Theorem", Theorem 5.2 of [11]:

The Isolation Game Theorem. *If the isolation game $I_n(G)$ with $n > 2$ is winnable by either player, then the graph G is one of the following*:

(i) *A forced p-win with $p \leqslant 5$ or $p = n-2$.*
(ii) *A forced* 6*-win on* 10, 12 *vertices.*
(iii) *A forced* 7*-win on* 10, 11, 13 *vertices.*
(iv) *A forced* 8*-win on* 12 *vertices.*

Although the present paper's results are independent of this theorem, several remarks about it seem appropriate here. *First,* the existential status of its "main case", (i), is affirmatively settled for all p by Theorems 2.2–2.5, even under the further requirement of connectivity. *Second,* the existential status of the "exceptional cases" (ii)–(iv) is at the moment unsettled, but in any event the main content of the theorem is that with at most a few stipulated exceptions, forced wins in the Isolation Game either occur very *early* (i.e. in the first five moves) or else can be delayed by the opponent until very *late* (i.e. just two moves before the end of play). *Third,* our current proof of the Isolation Game Theorem unfortunately requires too much machinery for inclusion here (separate paper in preparation); the reason is that the class of isolation games lacks an obvious proof-facilitating recursive structure (e.g. the result of a partial play of $I_n(G)$ does not seem to correspond in general to any $I_m(H)$), so that our method of analysis required embedding in a larger class of games.

The Isolation Game Theorem implies that for all $p > 5$, the number of vertices in a forced p-win is at most $p + 6$ (at most $p + 2$ if cases (ii)–(iv) can be ruled out). The next result shows that in this regard, 5 is indeed the critical value of p:

Theorem 2.6. *For any $p \leqslant 5$ there exist connected graphs with $|V|$ arbitrarily large which are forced p-wins.*

Proof. K_n, C_v are forced 1 and 2 wins respectively for any $n > 3$.

For $p = 3$, we give a construction with $n = 4r + 1$, where r is any positive integer. Two adjacent vertices from each of r 4-cycles are joined to a special vertex, which we call vertex 1. Since no vertices are of degree 1 or $n - 1$, G is not a forced 1-win. Every vertex on a 4-cycle belongs to a winning 2-set, so to avoid immediate loss, P1 must switch first on vertex 1; then there are no completable winning sets of cardinality 2, so G is not a forced 2-win. The vertex P2 switches on the second move must belong to one of the r 4-cycles. By design, every such vertex belongs to a winning 3-set with vertex 1. P1 can thus ensure a win on the third move.

For $p = 4$, we given a similar construction with $n = 4r + 2$. Two adjacent vertices on each of r 4-cycles are adjoined to two special vertices, indexed 1 and 2. Since there are no vertices of degree 1 or $n - 1$, G is not a forced 1-win. Every vertex on a 4-cycle belongs to a winning 2-set, so to avoid immediate loss, P1 must first switch vertex 1 or 2. Then P2 can complete no winning set on the next move, so G is not a forced 2-win. Assume P2 augments S, the set of switched-at vertices, to $\{1, 2\}$. Since there are no completable winning sets of odd cardinality, P1 cannot win on the third move. But it is possible to force a win on the fourth move, for every vertex is a member of a neighborhood winning set of cardinality 4 which contains $\{1, 2\}$. So G is indeed a forced 4-win.

For $p = 5$, we give a construction for *odd* $n \geqslant 7$. With $n = 2m + 1$, we suppose the vertices to be indexed from 1 to $2m + 1$. Here is the construction: Each *odd* vertex $2j + 1$, $j \geqslant 2$ is joined to all vertices except $\{1, 2, 3, 2j, 2j + 1\}$. Each *even* vertex $2j$, $j \geqslant 2$ is joined to all vertices except for $\{1, 2j, 2j + 1\}$. Vertex 1 is joined to vertices 2 and 3.

G is not a forced 1-win because it has no winning sets of cardinality 1.

Suppose P1 begins by switching vertex 1. The completable winning sets are then (O, E denote sets of odd and even vertices respectively) precisely those listed below.

Description	*Winning set*	*Size*
$N(1)^c$	$V-\{2,3\}$	$2m-1$
$N(2)=N(3)$	$\{1\}\cup E-\{2\}$	m
$N(2k+1)^c, k\geq 2$	$\{1,2,3,2k,2k+1\}$	5
$N(2k)^c, k\geq 2$	$\{1,2k,2k+1\}$	3

The last listed class of these winning sets enables P1 after three moves to force (via "pairing") either a win or that the switched-at set $S=\{1,2,3\}$. Since every other $v\in V$ belongs to a winning set of cardinality 5 containing $\{1,2,3\}$, but none of cardinality 4, P1 can force via pairing a win by five moves. In particular G cannot be a forced win for P2, e.g. a forced 2-win or 4-win.

To see why G is not a forced 3-win, observe that P2 can immediately switch at 2 to avoid all sets $\{1,2k,2k+1\}$. There is another winning set of cardinality 3 only if $m=3$, and then the unique such set is $\{1\}\cup E-\{2\}=\{1,4,6\}$, also avoidable by P2's switching at 2.

Remark. The construction we gave for $p=5$ involved an odd number of vertices; we conjecture that no forced 5-wins exist on graphs with an even number of vertices.

3. Nonwinnability

The first results in this section lead to a generalization (Theorem 3.1) of Ringeisen's result [10] on the nonwinnability of most complete bipartite graphs.

Lemma 3.1. *If $U\subset V$ is such that $N(v)\cap U\neq\emptyset$ and $N(v)^c\cap U\neq\emptyset$, then v cannot be isolated in any play of $I_n(G)$ continuing from $S=U$.*

Proof. Suppose that play continues from $S=U$ and an isolation of v occurs when $S=U^*\supseteq U$. Then by Theorem 2.1, $U^*=N(v)$ or $V^*=N(v)^c$. But the latter is incompatible with $\emptyset\neq N(v)\cap U\subset N(v)\cap U^*$, the former with $\emptyset\neq N(v)^c\cap U\subset N(v)^c\cap U^*$. $\square$

For reasons apparent from the next result, we call $U\subset V$ *a stopping set* if every $v\in V$ satisfies the hypothesis of Lemma 3.1 with respect to U.

Corollary 3.1A. *If U is a stopping set, then no continuation of $I_n(G)$ from $S=U$ yields a win for either player.* $\square$

Theorem 3.1. *Let $G=K_{k(1),\ldots,k(p)}$ (complete p-partite) with $p\geq 2$, $\max k(i)>2$, and $2\leq k(i)\leq n-2(p-1)$ for $i=1,\ldots,p$. Then neither player of $I_n(G)$ has a forced win.*

Proof. (a) Let $V = \bigcup V_i$ be a partition with $|V_i| = k(i)$. The winning sets are thus $\{V_i, V - V_i: i = 1, \ldots, p\}$. Under the given hypotheses the graph is neither a forced 1-win nor (cf. Theorem 2.3) a forced 2-win. Any p vertices, one from each V_i, form a minimal stopping set. By a *diversifying* strategy for a player, we mean one in which that player switches at a vertex in some V_i none of whose vertices have previously been switched, so long as such a choice is possible.

(b) To show P1 has no forced win, assume P2 adopts a diversifying strategy. If uninterrupted by a win, it will produce a stopping set on or before move $2p - 2$ of the game. By the corollary, therefore, a win for P1 requires isolation of some vertex $v \in V_j$, say, with S, the set of switched vertices, such that $|S| < 2p - 2$. By P2's first diversifying move, S cannot be a V_i so by Theorem 2.1 the vertex v must be isolated by $V - V_j = S$. then $|S| = n - k(j) \geq n - (n - 2(p-1)) = 2(p-1)$ which contradicts $|S| < 2p - 2$.

(c) To show that P2 has no forced win, assume P1 adopts a diversifying strategy, first switching a vertex w in some V_r with $k(r) > 2$. This would produce a stopping set on or before move $2p - 1$ of the game. So P2 can win only by isolating some vertex v, lying in some V_j, with a set of switched-at vertices S such that $|S| < 2p - 1$. If $v \in S$, then $S = V_j$ which is impossible after P1's second move under the diversifying strategy. So $S = V - V_j = \bigcup \{V_i : i \neq j\}$. Since all $k(i) \geq 2$, this implies that $|S| \geq 2(p-1)$, which with $|S| < 2p - 1$ yields $|S| = 2p - 2$, and thus $k(i) = 2$ for all $i \neq j$. Since $\max k(i) > 2$, we have $j = r$, but then $w \in S \cap V_j$ contradicts $S = V - V_j$. $\square$

Many other classes of graphs possess enough special structure to permit a relatively simple specification (as in the last proof) of a loss-avoiding move-countermove strategy. This structure could involve limits on the possible cardinalities of winning sets, or a natural pairing of the vertices into move-countermove pairs. Examples of such analyses (for Generalized Petersen graphs, permutation graphs, and cycle permutation graphs, cf. [12]) are given in [11].

Turning from specific families of graphs, we now formulate a more general sufficient condition for nonwinnability by P1. The analysis is "constructive" in the sense that specific loss-avoiding strategies for P2 will be exhibted. We denote the minimum distance from u to v (with unit arc lengths) by $d(u, v)$. A connected graph G will be called a *web* if for each edge (x, y), there exist at least two vertices z for which x is not adjacent to z but y is; the intuitive idea is that G is "spread out". For example, if all vertices of G have degree ≥ 3, then the absence of triangles is a sufficient condition for G to be a web, and if $\mathrm{rad}(G) > 2$ also holds then the following theorem applies for *any* first move by P1. This remark covers, for instance, the familiar graphs (Tutte, Grinberg, Meredith, (4, 6)-cage) on pp. 161, 162, 238, 239 of [2]; preliminary results on $I_n(G)$ for *regular* graphs imply that these four graphs are also nonwinnable by P2.

In the following analysis, we say as in Section 2 that a winning set w is *killed* when the growth of S makes $S \cap w^c \neq \emptyset$. We shall also use δ to denote the minimum vertex degree in G.

Theorem 3.2. *If G is a web, with $\delta \geqslant 3$, and* P1 *switches first at a vertex u for which there exists a v satisfying $d(v, v) \geqslant 3$, then* P1 *cannot force a win.*

Proof. On her first move (i.e. the second move of the game), P2 should switch at the vertex v guaranteed by the hypotheses of the theorem. Since $d(u, v) \geqslant 3$ implies that no neighborhood contains $\{u, v\}$, this choice kills all neighborhood winning sets: an isolation, if such occurs, must be by completion of a neighborhood-complement set.

We first show that the game cannot be won in fewer than 5 moves by P1. As $\{u, v\} \cup N(u) \subset N(v)^c$ and $\{u, v\} \cup N(v) \subset N(u)^c$, it follows from $\delta \geqslant 3$ that neither u nor v can be isolated on or before the third move. If $N(x)^c = \{u, v, x\}$ say, then $d(x, u) = d(x, v) = 2$ else $N(u) \subset N(x)^c$ or $N(v) \subset N(x)^c$. Furthermore, u, v are the *only* vertices distance $\geqslant 2$ away from x, since $N(x)^c = \{x, u, v\}$. Since G is a web, therefore, each $y \in N(x)$ must be adjacent to vertices u and v. But this in turn implies that $d(u, v) = 2$, a contradiction.

For $i \geqslant 3$, let y_i denote the vertex switched at move i. Assume P2 switches according to the following rule.

Rule 1. Choose y_{2k} adjacent to S_{2k-1}, and adjacent to y_{2k-1} unless y_{2k-1} was adjacent to S_{2k-2}.

That Rule 1 is feasible (so long as $2k < n$) follows from the connectivity of G. This rule assures that after move $2k$ is made, the winning sets $\{N(y_i)^c : 3 \leqslant i \leqslant 2k\}$ have all been killed, with all but possibly $N(y_{2k-1})^c$ being killed prior to move $2k$. Thus maintaining Rule 1 limits P2's concerns to the possibilities that some switch y_{2k-1} by P1 may

(a) complete $N(y_{2k-1})$ or

(b) complete $N(u)^c$ or $N(v)^c$.

The above arguments rule out possibility (a) for $2k - 1 = 3$. Possibility (b) motivates:

Rule 2. If $N(u) \cap S_{2k-1} = \emptyset$ and $|N(u)^c - S_{2k-1}| = 2$, or $N(v) \cap S_{2k-1} = \emptyset$ and $|N(v)^c - S_{2k-1}| = 2$, then choose y_{2k} in $N(u)$ or $N(v)$ respectively if consistent with Rule 1.

Note that since $\{u, v\} \subset S_{2k-2}$, Rule 2's directive is consistent with Rule 1 whenever y_{2k-1} is adjacent to S_{2k-2}. Observe also that both clauses of Rule 2 cannot apply simultaneously: if $N(u) \cap S_{2k-1} = \emptyset$ then $N(u)$ provides at least three members for $N(v)^c - S_{2k-1}$. We first show that Rule 2 is adequate for its purpose:

Claim 1. *In play subsequent to $S_2 = \{u, v\}$, neither u nor v is isolated by* P1.

Proof of Claim 1. Suppose without loss of generality that u is isolated by P1 on move $2k+1=q>4$, so that $N(u)^c = S_{q-2} \cup \{y_{q-1}, y_q\}$. Thus $N(u) \cap S_{q-2} = \emptyset$ and $|N(u)^c - S_{q-2}| = 2$. If y_{q-2} was adjacent to S_{q-3}, or nonadjacent to S_{q-3} but adjacent to $N(u)$, then Rule 2 would direct P2 to choose y_{q-1} in $N(u)$. Since such a choice is incompatible with $S_q = N(u)^c$, it follows that y_{q-2} is adjacent neither to $S_{q-3} = N(u)^c - \{y_{q-2}, y_{q-1}, y_q\}$ nor to $N(u)$. But this contradicts $d_G(y_{q-2}) \geqslant 3$. Hence Claim 1 is proved. □

We next augment P2's Rules 1–2 by the natural

Rule 3. If some choice y_{2k} compatible with Rules 1 and 2 would prevent loss on move $2k+1$, then make such a choice.

We can now rule out P2's last concern, that for some $2k+1=q>4$, P1's choice of y_q will complete $N(y_q)^c = S_q$. Suppose such an isolation occurs. Since $N(y_q)^c$ was not killed earlier, y_q has no neighbor in S_{q-1}, hence none in S_{q-2}.

Claim 2. *y_q is the only vertex in $V - \{u, v, y_{q-2}\}$ without a neighbor in S_{q-2}.*

Proof of Claim 2. By the remarks following Rule 1, the only possible violators of Claim 2 are vertices $y \in N(y_q)$. But since G is a web, for each y there exist at least two vertices z with $d(y_q, z) = 2$ and $d(y, z) = 1$. Since the set of vertices at distance >1 from y_q is $S_{q-2} \cup \{y_{q-1}\}$, y must have at least one neighbor z in S_{q-2}, and hence is not a violator.

Consequence of Claim 2. When P2 is about to choose $y_{q-1} \in N(y_q)^c$, the only unkilled winning sets are $N(y_q)^c$ and possibly $N(u)^c$, $N(v)^c$, $N(y_{q-2})$. We will show that Rule 3 dictates to P2 a choice that would avoid loss on move q, thus obtaining a contradiction to the actual choice of y_{q-1}. In the context of Rule 1, note the implication of Claim 2 that every $x \in N(y_q)$ is adjacent to S_{q-2}.

First suppose that neither $N(u)^c$ nor $N(v)^c$ is unkilled. Then P2's choice is not inhibited by Rule 2. If y_{q-2} is adjacent to S_{q-3}, so that $N(y_{q-2})^c$ has also been killed, then the "unless" clause of Rule 1 permits choosing any $x \in N(y_q)$, killing $N(y_q)^c$ and establishing the contradiction. And if y_{q-2} is *not* adjacent to S_{q-3}, then the fact $d_G(y_{q-2}) \geqslant 3$ implies the existence of at least two vertices $x \in N(y_q) \cap N(y_{q-2})$, again providing choices that leave all winning sets killed and thus establish the contradiction.

Next, suppose that $N(u)^c$ is unkilled (the argument is similar if $N(v)^c$ is assumed unkilled). Then $N(u) \subset N(y_q) \cup \{y_{q-1}\} \subset V - S_{q-2}$. Since $d_G(u) \geqslant 3$, it follows that there exist at least two vertices $x_u \in N(y_q, G) \cap N(u)$; also, there exist at least three vertices in $N(u) - S_{q-2}$ and thus in $N(v)^c - S_{q-2}$. The last clause shows that continuation from S_{q-2} cannot isolate v on move q, so that isolation of v need not figure via Rule 3 (or Rule 2) in P2's choice. If y_{q-2} is adjacent to S_{q-3} so that $N(y_{q-2})^c$ has been killed, then the choice of any x_u would be consistent with Rule 3 and would avoid loss on move q, establishing the contradiction. So assume y_{q-2} *not* adjacent to S_{q-3}. If any x_u is adjacent to y_{q-2}, then its choice again establishes the contradiction. So assume, further, that no x_u is adjacent to y_{q-2}. Then $N(y_{q-2})^c - S_{q-2}$ contains y_q and at least two x_u's, so that $N(y_{q-2})^c$ is not completable from S_{q-2} by move q: isolation of y_{q-2} does not figure via Rule 3 in P2's choice. As in the last paragraph, there exist at least two vertices $x_y \in N(y_q) \cap N(y_{q-2})$, and our most recent assumption is that no x_y is an x_u.

Consider the value K of $|N(u)^c - S_{q-2}|$. If $K = 0$, the game would already be over (with u isolated), while if $K = 1$ then P2 can and should win on move $q - 1$. If $K > 2$ then Rule 2 does not apply and u cannot be isolated on move q; thus Rule 3 dictates choosing a vertex x_y, again establishing the contradiction. The remaining case is $K = 2$; let $\{z\} = N(u)^c - S_{q-2} - \{y_q\} \subset N(y_q) \cup \{y_{q-1}\}$. Then all but at most one member (z) of $N(y_q)$ lies in $N(u)$, hence at most one x_y can fail to be an x_u, yielding a contradiction.

This completes the proof of the theorem. □

It does not follow from the fact that G is a web and $\delta \geqslant 3$, that for each $u \in V$ there exists a $v \in V$ with $d(u, v) \geqslant 3$. Indeed, this can fail for *all* $u \in V$. An example is provided by the complete bipartite graphs $K_{q,n-q}$ $(3 \leqslant q \leqslant n-3,\ n \geqslant 6)$, which shows that for establishing nonwinnability by P1, Theorem 3.1 is not subsumed by Theorem 3.2.

We conclude with a brief excursion into computational-complexity theory. It is suspected (work still in progress) that the complexity of the decision problem for $I_n(G)$—i.e. the problem of deciding if a graph is a forced win for either player—is polynomial of low order. By contrast, we show that the following problem, related to nonwinnability of $I_n(G)$ through the Corollary to Lemma 3.1, is NP-complete:

STOPPING SET

INSTANCE: A finite, simple graph $G = (V, E)$ and a positive ingeger $K \leqslant |V|$.
QUESTION: Is there a subset $V' \subset V$ with $|V'| \leqslant K$ such that $N(v) \cap V' \neq \emptyset$ and $N(v)^c \cap V' \neq \emptyset$ for all $v \in V$?

Theorem 3.3. *STOPPING SET is NP-complete.*

Proof. It is easy to see that STOPPING SET is in the class NP, for we need only guess V' and check in a polynomial number of steps that $N(v) \cap V' \neq \emptyset$, $N(v)^c \cap V' \neq \emptyset$ for all $v \in V$.

We transform the known NP-complete problem HITTING SET to STOPPING SET. For reference ([5] p. 222), the definition of HITTING SET is now given:

HITTING SET
INSTANCE: A collection C of subsets of a finite set D, positive integer $M \leq |D|$.
QUESTION: Is there a subset $D' \subset D$ with $|D'| \leq M$ such that D contains at least one element from each subset in C?

For each instance of HITTING SET we must construct (in polynomial time) a problem instance of STOPPING SET which has a stopping set of cardinality $\leq K$ if and only if HITTING SET has a hitting set of size $\leq M$.

The transformation is defined in the following way: Take any instance (D, C, M) of HITTING SET. Using four special vertices p, q, r, s, a graph $G = (V, E)$ is formed with $V = D \cup C \cup \{p, q, r, s\}$ and edges

$$(p, q), (p, r), (q, s), \{(p, d): d \in D\}, \{(d, c): d \in D, c \in C, d \in c\}.$$

It is easy to see that the construction of G may be accomplished in time that is polynomial in the input length of the data for HITTING SET. Let $K = M + 2$, givening an instance (G, K) of STOPPING SET.

Claim. *If H is a hitting set of cardinality $\leq M$, then $V' = H \cup \{p, q\}$ is a stopping set (of cardinality $\leq K$).*

Proof of claim. V' contains: non-neighbor p and some neighbor $h \in H$ of each vertex $c \in C$, neighbor p and non-neighbor q of each vertex $d \in D$, neighbor q and non-neighbor p of p and of s, and neighbor p and non-neighbor q of q and of r.

Conversely, suppose now we have a solution V' to the instance (G, K) of STOPPING SET. $\{p, q\} \subset V'$, for p, q are the unique neighbors of r, s respectively. Each vertex $c \in C$ must have an adjacent vertex in V': these can only be in D and must form a hitting set, H say. But then $|H| \leq |V' - \{p, q\}| \leq (M + 2) - 2 = M$. Thus we have a solution to the hitting set problem. Hence the theorem is proved. □

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