Journal of Combinatorial Theory, Series B 99 (2009) 843–868

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series B

www.elsevier.com/locate/jctb

# Tight spans of distances and the dual fractionality of undirected multiflow problems

Hiroshi Hirai

*Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan*

ARTICLE INFO

*Article history:*
Received 31 August 2007
Available online 26 March 2009

*Keywords:*
Tight spans
Metrics
Multicommodity flows
Polyhedral combinatorics

ABSTRACT

In this paper, we give a complete characterization of the class of weighted maximum multiflow problems whose *dual* polyhedra have bounded fractionality. This is a common generalization of two fundamental results of Karzanov. The first one is a characterization of commodity graphs $H$ for which the dual of maximum multiflow problem with respect to $H$ has bounded fractionality, and the second one is a characterization of metrics $d$ on terminals for which the dual of metric-weighed maximum multiflow problem has bounded fractionality. A key ingredient of the present paper is a *nonmetric* generalization of the *tight span*, which was originally introduced for metrics by Isbell and Dress. A theory of nonmetric tight spans provides a unified duality framework to the weighted maximum multiflow problems, and gives a unified interpretation of combinatorial dual solutions of several known min–max theorems in the multiflow theory.



## 1. Introduction and main results

Let $G = (V, E, c)$ be an undirected graph with a nonnegative edge capacity $c : E \to \mathbf{R}_+$, and let $S \subseteq V$ be a set of terminals and $\mu$ a nonnegative weight function on the set of pairs of elements in $S$. A path $P \subseteq E$ is called an *S-path* if its endpoints are distinct vertices in $S$. A *multiflow* (multicommodity flow) is a set $\mathcal{P}$ of $S$-paths in $G$ together with a nonnegative flow-value function $\lambda : \mathcal{P} \to \mathbf{R}_+$ satisfying the capacity constraint $\sum_{P \in \mathcal{P}:\, e \in P} \lambda(P) \leqslant c(e)$ for each $e \in E$. The *weighted maximum multiflow problem* with respect to $G$ and $(S, \mu)$, denoted by $M(G; S, \mu)$, is formulated as:

$$M(G; S, \mu) \quad \text{Maximize} \sum_{P \in \mathcal{P}} \mu(s_P, t_P)\lambda(P) \text{ over all multiflows } (\mathcal{P}, \lambda) \text{ in } G,$$

*E-mail address:* hirai@kurims.kyoto-u.ac.jp.

0095-8956/$ – see front matter 
doi:10.1016/j.jctb.2009.03.001

where $s_P, t_P \in S$ are the endpoints of $P$. One of the intriguing issues in the multiflow theory is the fractionality of optimal multiflows; see [19], [26, Part VII]. The *fractionality* of $(S, \mu)$ is the least positive integer $k$ such that $M(G; S, \mu)$ has a $1/k$-integral optimal flow for *any* integer-capacitated graph $G = (V, E, c)$ with $S \subseteq V$. If such a $k$ does not exist, the fractionality of $(S, \mu)$ is defined to be infinity. The question is:

(F) What is a necessary and sufficient condition for $(S, \mu)$ to have bounded fractionality?

The 0–1 weight case is of a particular combinatorial interest. In this case, the 0–1 weight $\mu$ can be regarded as a *commodity graph*, and $M(G; S, \mu)$ is the problem of maximizing the total sum of multiflows connecting pairs of terminals $s$ and $t$ specified by $\mu(s,t) = 1$. For example, when $S$ is a 2-set $\{s, t\}$ with $\mu(s,t) = 1$, which corresponds to the single-commodity flow problem, the famous maxflow–mincut theorem due to Ford and Fulkerson [10] states that there exists an integral optimal flow. The two-commodity flow problem corresponds to the case where $S$ is a 4-set $\{s, t, s', t'\}$ and $\mu$ is defined as $\mu(s,t) = \mu(s',t') = 1$ and the other weights are zero. Hu's biflow–mincut theorem [15] says that there exists a half-integral optimal flow. Lovász [24] and Cherkassky [5] have shown the existence of half-integral optimal flows in the case where $\mu(s,t) = 1$ for all distinct $s, t \in S$ (the *maximum free multiflow problem*). These results for 0–1 weights are further generalized by Karzanov and Lomonosov [22] to a certain class of commodity graphs. In cases of non 0–1 weights $\mu$, the so-called *multiflow locking theorem* by Karzanov and Lomonosov [22] states the existence of half-integral optimal flows for a class of cut-decomposable metrics $\mu$. All of these results give sufficient conditions, but a complete answer to (F) is still unknown (even for the 0–1 weight cases).

Since $M(G; S, \mu)$ is a linear program, we may think of its dual problem $M^*(G; S, \mu)$, which is given as

$$\begin{aligned} M^*(G; S, \mu) \quad & \text{Minimize } \sum_{e \in E} c(e) l(e) \\ & \text{subject to } \sum_{e \in P} l(e) \geqslant \mu(s_P, t_P) \text{ for all } S\text{-paths } P, \\ & \qquad l(e) \geqslant 0 \ (e \in E). \end{aligned}$$

Corresponding to the (primal) fractionality mentioned above, the *dual fractionality* of $(S, \mu)$ with integral $\mu$ is the least positive integer $k$ such that $M^*(G; S, \mu)$ has a $1/k$-integral optimal solution for *any* capacitated graph $G = (V, E, c)$ with $S \subseteq V$. Then the dual fractionality problem is described as follows.

(F∗) What is a necessary and sufficient condition for $(S, \mu)$ with integral $\mu$ to have bounded dual fractionality?

As was observed in [18], a necessary condition for bounded dual fractionality is also necessary for bounded primal fractionality. Namely, for a fixed $(S, \mu)$, if $M(G; S, \mu)$ has a $1/k$-integral optimal flow for any integer-capacitated graph $G$ with $S \subseteq V$, then $M^*(G; S, \mu)$ also has a $1/k$-integral optimal solution for any capacitated graph $G$. The converse is not true in general. More precisely, the primal fractionality is greater than or equal to the dual fractionality.

The main result of this paper is a complete answer to problem (F∗). To describe our result, we need some notation. We regard a nonnegative weight $\mu$ on $S$ as a distance on $S$. Here $\mu$ is called a *distance* on $S$ if $\mu(s,t) = \mu(t,s) \geqslant 0$, and $\mu(u,u) = 0$ for $s, t, u \in S$. In addition, if distance $\mu$ satisfies the triangle inequality $\mu(s,t) \leqslant \mu(s,u) + \mu(u,t)$ for all $s, t, u \in S$, then we call $\mu$ a *metric* on $S$. For a distance $\mu$, a polyhedral set $T_\mu \subseteq \mathbf{R}^S$, called the *tight span* of $\mu$, is defined to be the set of minimal elements of the polyhedron

$$P_\mu = \{ p \in \mathbf{R}^S \mid p(s) + p(t) \geqslant \mu(s,t) \ (s, t \in S) \}.$$

Note that $P_\mu$ is contained in the nonnegative orthant $\mathbf{R}^S_+$; see Fig. 1 for 2- and 3-dimensional examples.

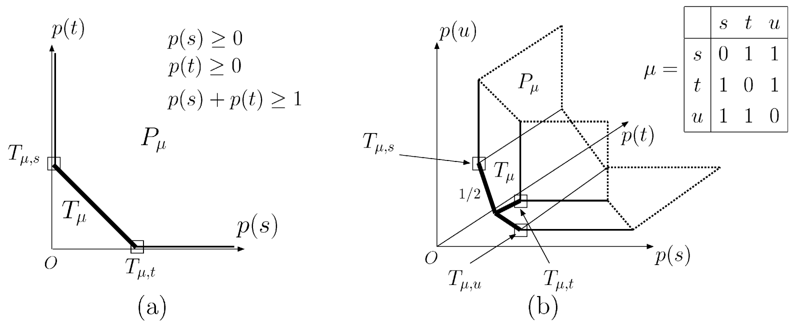



**Fig. 1.** (a) $T_\mu$ of a 2-point distance and (b) $T_\mu$ of all-one 3-point distance.

The tight span has been introduced independently by Isbell [17] and Dress [9] for a metric, and recently considered in [11] for a (nonmetric) distance; also see [6] for an appearance of tight spans in the context of online algorithms. Our main theorem provides a necessary and sufficient condition for bounded dual fractionality in terms of the dimension of the polyhedral space $T_\mu$, where the dimension $\dim T_\mu$ is defined to be the largest dimension of faces of $T_\mu$. We state our main result in a sharper form. A distance $\mu$ is called *cyclically even* if $\mu$ is integral and $\mu(s,t)+\mu(t,u)+\mu(u,s)$ is an even integer for all $s,t,u \in S$. Since $2\mu$ is always cyclically even for any integral distance $\mu$, we may consider (F$*$) for cyclically even distances without loss of generality.

**Theorem 1.1.** *For a cyclically even distance $\mu$ on $S$, the following two statements hold.*

(1) *If $\dim T_\mu \leqslant 2$, then there exists a half-integral optimal solution to $M^*(G;S,\mu)$ for any graph $G=(V,E,c)$ with $S \subseteq V$.*
(2) *If $\dim T_\mu > 2$, then there exists no integer $k$ such that $M^*(G;S,\mu)$ has a $1/k$-integral optimal solution for any graph $G=(V,E,c)$ with $S \subseteq V$.*

In particular, for an integral distance $\mu$ with $\dim T_\mu \leqslant 2$, $M^*(G;S,\mu)$ has a 1/4-integral optimal solution. This result unifies two fundamental results by Karzanov for metric-weights and 0–1 weights below.

**Theorem 1.2.** *(See [21].) For a cyclically even metric $\mu$ on $S$, the following two statements hold.*

(1) *If $\dim T_\mu \leqslant 2$, then there exists a half-integral optimal solution to $M^*(G;S,\mu)$ for any graph $G=(V,E,c)$ with $S \subseteq V$.*
(2) *If $\dim T_\mu > 2$, then there exists no integer $k$ such that $M^*(G;S,\mu)$ has a $1/k$-integral optimal solution for any graph $G=(V,E,c)$ with $S \subseteq V$.*

Although (2) in this theorem is not explicit in [21], it is a consequence of his characterization of primitively finite metrics.

For a 0–1 distance $\mu$ on $S$, the *commodity graph* $H_\mu=(S,F_\mu)$ is defined by $F_\mu=\{st \mid s,t \in S, \mu(s,t)=1\}$. Consider the following condition.

(P) For any three pairwise intersecting maximal stable sets $A,B,C$ of $H_\mu$, we have $A\cap B = B\cap C = C\cap A$.

**Theorem 1.3.** *(See [18].) For a 0–1 distance $\mu$ on $S$ whose commodity graph $H_\mu$ has no isolated vertices, the following two statements hold.*

(1) *If $H_\mu$ satisfies condition* (P), *then there exists a* $1/4$*-integral optimal solution to* $M^*(G; S, \mu)$ *for any graph* $G = (V, E, c)$ *with* $S \subseteq V$.
(2) *If $H_\mu$ violates condition* (P), *then there exists no integer k such that* $M^*(G; S, \mu)$ *has a* $1/k$*-integral optimal solution for any graph* $G = (V, E, c)$ *with* $S \subseteq V$.

It is not so obvious that condition (P) in Theorem 1.3 is equivalent to the 2-dimensionality of $T_\mu$ for a 0–1 distance $\mu$. We give a direct proof of this fact in Section 7.

Our result suggests that we cannot expect a combinatorial min–max theorem in $M(G; S, \mu)$ for a fixed $(S, \mu)$ with $\dim T_\mu \geqslant 3$ and any graph $G$, although we do not know whether the 2-dimensionality of $T_\mu$ is sufficient for bounded primal fractionality. Karzanov [19] conjectured that condition (P) is also sufficient for bounded (primal) fractionality in 0–1 problems. Therefore, it seems reasonable to extend it to a conjecture that the 2-dimensionality of $T_\mu$ is sufficient for bounded fractionality in $\mu$-weighted problems. This research direction will be further pursued by the author's subsequent papers.

*Overview*

The proof of Theorem 1.1 is based on a novel relationship between multiflows and the tight span $T_\mu$ as generalized for nonmetric distance $\mu$. This is the central topic in this paper. A certain duality relationship between multiflows and metrics was explored by Onaga and Kakusho [25] and Iri [16] in the 1970s, and further developed by Lomonosov and Karzanov [23,18]. Indeed, the LP-dual of $M(G; S, \mu)$ can also be represented as

$$\begin{aligned} &\text{Minimize } \sum_{xy \in E} c(xy) d(x, y) \\ &\text{subject to } d : \text{metric on } V, \\ &\qquad\qquad d(s,t) \geqslant \mu(s,t) \ (s,t \in S). \end{aligned} \tag{1.1}$$

This can be easily seen from the fact that we can replace $l$ in $M^*(G; S, \mu)$ by the path metric induced by $l$; see [23]. In the mid-1990s, a more sharper duality by using tight spans was found by Bandelt, Chepoi, and Karzanov [2,3,20,21] (in the metric case). Our approach to Theorem 1.1 also lies on this line of research developments.

Our proof is based on a special duality relation that the dual of $M(G; S, \mu)$ is also represented as a continuous location problem on the tight span $T_\mu$ as follows. Recall the definitions of $P_\mu$ and $T_\mu$, and define a subset $T_{\mu,s} \subseteq T_\mu$ for $s \in S$ as

$$P_\mu = \{ p \in \mathbf{R}^S \mid p(s) + p(t) \geqslant \mu(s,t) \ (s,t \in S) \}, \tag{1.2}$$

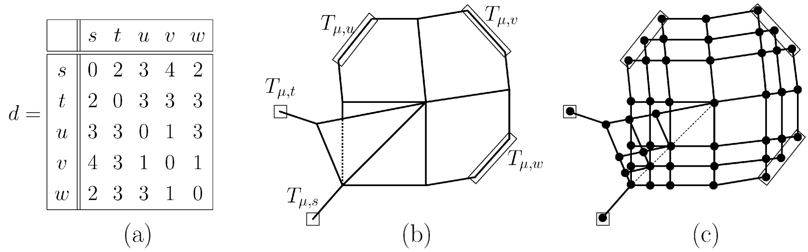
$$T_\mu = \text{the set of minimal elements of } P_\mu, \tag{1.3}$$

$$T_{\mu,s} = \{ p \in T_\mu \mid p(s) = 0 \} \ (s \in S). \tag{1.4}$$

Fig. 2(b) illustrates the tight span $T_\mu$ together with $T_{\mu,s}$ $(s \in S)$ of a 5-point (nonmetric) distance $\mu$. Then $T_\mu$ is a 2-dimensional (nonconvex) polyhedral set in 5-dimensional space, which is obtained by gluing three pentagons and three triangles. We consider a continuous location problem in $T_\mu$ as follows.

$$\begin{aligned} \text{(TSD)} \quad &\text{Minimize } \sum_{xy \in E} c(xy) \|\rho(x) - \rho(y)\|_\infty \\ &\text{subject to } \rho : V \to T_\mu, \\ &\qquad\qquad \rho(s) \in T_{\mu,s} \ (s \in S). \end{aligned}$$

We call it the *tight-span dual* to the weighted maximum multiflow problem. The tight-span dual is a problem of optimizing a *location* $\{\rho(x)\}_{x \in V}$ in the $l_\infty$-space $T_\mu$. A location problem of this type is called a *p-facility minisum problem with mutual communication* or a *multifacility location problem* in the location theory [27]. In fact, the dual of $M(G; S, \mu)$ is further reduced to (TSD) as follows.




| | s | t | u | v | w |
|---|---|---|---|---|---|
| s | 0 | 2 | 3 | 4 | 2 |
| t | 2 | 0 | 3 | 3 | 3 |
| u | 3 | 3 | 0 | 1 | 3 |
| v | 4 | 3 | 1 | 0 | 1 |
| w | 2 | 3 | 3 | 1 | 0 |

**Fig. 2.** (a) Distance $\mu$, (b) tight span $T_\mu$, and (c) $T_\mu \cap Z$.

**Theorem 1.4.** *The optimal value of $M(G; S, \mu)$ is equal to the minimum value of the tight-span dual* (TSD)*.*

This duality relation has already been recognized in the case of metrics by Karzanov [20,21]. Our contribution is to extend it to a nonmetric version. In analogy to the network flow theory, $\rho(x)$ is a *potential* at $x \in V$, and $\|\rho(x) - \rho(y)\|_\infty$ is a *potential difference*. In a single-commodity case, $S$ is a 2-set, $T_\mu$ is a segment (Fig. 1(a)), and therefore $\rho(x)$ can be regarded as an ordinary scalar potential.

For a finite set $Z$ of points in $T_\mu$, we consider the following *discrete* location problem:

$$\begin{aligned} (\mathrm{TSD}(Z)) \quad & \text{Minimize} \sum_{xy \in E} c(xy) \big\| \rho(x) - \rho(y) \big\|_\infty \\ & \text{subject to } \rho : V \to T_\mu \cap Z, \\ & \qquad \rho(s) \in T_{\mu,s} \cap Z \ \ (s \in S). \end{aligned}$$

Clearly, the minimum value of (TSD($Z$)) is greater than or equal to that of (TSD). Theorem 1.1(1) follows from the following characterization when the continuous location problem (TSD) can be reduced to the discrete one (TSD($Z$)) for some finite set $Z \subseteq T_\mu$.

**Theorem 1.5.** *For a rational distance $\mu$ on a finite set $S$, the following two statements hold.*

(1) *If $\dim T_\mu \leqslant 2$, then there exists a finite set $Z$ of points in $T_\mu$ such that for any graph $G = (V, E, c)$ with $S \subseteq V$, the optimal value of $M(G; S, \mu)$ is equal to the minimum value of* (TSD($Z$))*, i.e., we can always take an optimal solution $\rho$ of* (TSD) *satisfying $\rho(V) \subseteq Z$.*
(2) *In addition, if $\mu$ is cyclically even, then we can take $Z$ such that the $l_\infty$-distances on $Z$ are half-integral.*

We give some comments on our results. Theorem 1.5 can be regarded as a multiflow analogue of *discreteness of potential* in network flow theory. So the set $Z$ of points can also be regarded as *integer points* in $T_\mu$, although $Z$ is not a subset of the ordinary integer points $\mathbf{Z}^S$ in general. Fig. 2(c) illustrates $Z$ as the black dot points; also see Fig. 13 for further examples. Moreover, the constraints in (TSD($Z$)) imply that it is an optimization problem over certain partitions of $V$. Therefore, solutions of (TSD($Z$)) have a *combinatorial* meaning. This leads us to a unified interpretation of the combinatorial dual of several known min–max theorems in the multiflow theory mentioned above. For example, consider a distance of a 2-set, which corresponds to a single-commodity case. Then its tight span is a line segment (Fig. 1(a)), and $Z$ can be taken to be its endpoints, and hence (TSD($Z$)) is the problem of finding a minimum cut. Consider the case of all-one distance $\mu$ of a 3-set, which corresponds to a maximum free multiflow problem of three terminals. Then $T_\mu$ is a star with three edges of length 1/2 (Fig. 1(b)), and $Z$ can be taken to be its vertices, and (TSD($Z$)) immediately gives the Lovász–Cherkassky duality relation; see [21, p. 241] for a related argument.

An intuitive reason why the 2-dimensionality of $T_\mu$ implies bounded dual fractionality is the following well-known property of the $l_\infty$-space; see [8, p. 31].

$$\big(\mathbf{R}^2, l_\infty\big) \text{ is isomorphic to } \big(\mathbf{R}^2, l_1\big) \text{ by the map } (x_1, x_2) \mapsto \left( \frac{x_1 + x_2}{2}, \frac{x_1 - x_2}{2} \right).$$

In fact, it will turn out that $(T_\mu, l_\infty)$ can be obtained by gluing certain $l_\infty$-spaces (Proposition 3.2). If $\dim T_\mu \leqslant 2$, then $T_\mu$ is a 2-complex of $l_1$-spaces. Recall that every finite submetric in an $l_1$-space is cut-decomposable [8]. Therefore the metric space $(T_\mu, l_\infty)$ with $\dim T_\mu \leqslant 2$ shares nice decomposability properties similar to $l_1$-spaces.

Karzanov's proof of Theorem 1.2 is based on his elegant characterization of *minimizable graphs* [20], and a number of properties of modular closures and least generating graphs (LG-graphs) of metrics [21]. Such a graph metric approach does not seem to extend to the case of nonmetric distances. In particular, we do not know an analogue of LG-graphs and modular closures of nonmetric distances. Instead, our proof of Theorem 1.1 relies mainly on Theorem 1.4 and the geometry of the tight span $T_\mu$.

This paper is organized as follows. In Section 2, we prove Theorem 1.4. In Section 3, we study geometric properties of $T_\mu$ which are the basis for the subsequent arguments. In Section 4, we give a construction of $Z$ in Theorem 1.5 by drawing a global $l_1$-coordinate system on the tight span, and prove (1) in Theorem 1.5. In Section 5, we prove the half-integrality assertion (Theorem 1.1(1) and Theorem 1.5(2)). In Section 6, we prove the unbounded fractionality assertion (Theorem 1.1(2)). In Section 7, we verify that condition (P) in Theorem 1.3 is indeed equivalent to the 2-dimensionality of the tight span of a 0–1 distance, and also give an explicit combinatorial construction of tight spans for 2-dimensional 0–1 distances. Finally, Section 8 gives some remarks.

*Notation*

We use the following notation. Let $\mathbf{R}$ and $\mathbf{R}_+$ be the sets of reals and nonnegative reals, respectively. Let $\mathbf{Z}$ be the set of integers. The set of functions from a set $V$ to $\mathbf{R}$ is denoted by $\mathbf{R}^V$. For $p, q \in \mathbf{R}^V$, $p \leqslant q$ means $p(x) \leqslant q(x)$ for all $x \in V$. For $p \in \mathbf{R}^V$ and $S \subseteq V$, the restriction of $p$ to $S$ is denoted by $p|_S$. Similarly, for a distance $d$ on $V$ and $S \subseteq V$, the restriction of $d$ to $S$ is denoted by $d|_S$. The $l_\infty$-distance between two points $p, q \in \mathbf{R}^S$ is simply denoted by $\|p, q\|$, i.e.,

$$\|p, q\| := \|p - q\|_\infty = \sup_{s \in S} \bigl|p(s) - q(s)\bigr|. \tag{1.5}$$

We define the $l_\infty$-distance between two *subsets* $P, Q \subseteq \mathbf{R}^S$ by

$$\|P, Q\| := \inf\bigl\{\|p - q\|_\infty \bigm| p \in P,\ q \in Q\bigr\}. \tag{1.6}$$

We simply denote $\|\{p\}, Q\|$ by $\|p, Q\|$. The *characteristic vector* $\chi_S \in \mathbf{R}^V$ of $S \subseteq V$ is defined as $\chi_S(s) = 1$ for $s \in S$ and $\chi_S(s) = 0$ for $s \notin S$. We simply denote $\chi_{\{s\}}$ by $\chi_s$, which is the $s$th unit base vector. For an undirected graph $G = (V, E)$, the edge between $x, y \in V$ is denoted by $xy$ or $yx$. $xx$ means a loop. $E_V$ is the set of edges of the complete graph on vertices $V$. A *stable set* $A$ of $G$ is a subset of vertices such that there is no edge both of whose endpoints belong to $A$. A *partition* of $G$ is a partition of vertices such that each part is a stable set. In particular, if there is a bipartition, $G$ is called *bipartite*. $G$ is called a *complete multipartite graph* if $G$ has a partition such that each pair of vertices in different parts is connected by an edge. We often regard distance $d$ on $V$ as $d \in \mathbf{R}_+^{E_V}$. We often identify a distance space $(S, \mu)$ with distance $\mu$. We use the standard terminology of polytope theory such as *faces*, *extreme points*, *polyhedral complex* or *subdivision*, and so on; see [28].

## 2. The tight-span dual to the weighted maximum multiflow problem

In this section, we prove Theorem 1.4 saying that the maximum value of $M(G; S, \mu)$ is equal to the minimum value of the tight-span dual:

$$\begin{aligned} &\text{Minimize} \sum_{xy \in E} c(xy) \bigl\|\rho(x), \rho(y)\bigr\| \\ &\text{subject to } \rho : V \to T_\mu, \\ &\qquad\qquad \rho(s) \in T_{\mu,s}\ (s \in S). \end{aligned}$$

Recall the definitions of $P_\mu$, $T_\mu$, and $T_{\mu,s}$ in (1.2)–(1.4) and the notation $\|\cdot,\cdot\|$ in (1.5). The proof consists of two lemmas. The first lemma states that the dual of $M(G;S,\mu)$ is reduced to the location problem on $P_\mu$ as follows.

**Lemma 2.1.** *The optimal value of $M(G;S,\mu)$ is equal to the minimum value in the following problem*:

$$\begin{aligned}&\text{Minimize } \sum_{xy\in E} c(xy)\big\|\rho(x),\rho(y)\big\|\\ &\text{subject to } \rho: V\to P_\mu,\\ &\qquad\qquad \rho(s)\in P_{\mu,s}\ (s\in S),\end{aligned}\tag{2.1}$$

*where the subset $P_{\mu,s}\subseteq P_\mu$ for $s\in S$ is defined by*

$$P_{\mu,s}=\big\{p\in P_\mu \mid p(s)=0\big\}.$$

**Proof.** We use problem (1.1) instead of $M^*(G;S,\mu)$. For $\rho: V\to P_\mu$ with $\rho(s)\in P_{\mu,s}$ $(s\in S)$, define a metric $d^\rho$ on $V$ by

$$d^\rho(x,y):=\big\|\rho(x),\rho(y)\big\|\quad (x,y\in V).$$

Then for $s,t\in S$ we have

$$\begin{aligned}d^\rho(s,t)&=\big\|\rho(s),\rho(t)\big\|\geqslant \big(\rho(s)\big)(t)-\big(\rho(t)\big)(t)\\ &=\big(\rho(s)\big)(t)+\big(\rho(s)\big)(s)\geqslant \mu(s,t)\quad (s,t\in S),\end{aligned}$$

where we use $(\rho(s))(s)=(\rho(t))(t)=0$ and $\rho(s)\in P_\mu$. Therefore, $d^\rho$ is feasible to (1.1).

Conversely, take a metric $d$ feasible to (1.1). Define a map $\rho^d: V\to \mathbf{R}^S$ by

$$\big(\rho^d(x)\big)(s):=d(s,x)\quad (s\in S,\ x\in V).$$

By the definition of $\rho^d(x)$ and the triangle inequality, we have

$$\rho^d(x)(s)+\rho^d(x)(t)=d(x,s)+d(x,t)\geqslant d(s,t)\geqslant \mu(s,t).$$

This implies $\rho^d(x)\in P_\mu$. Moreover, $\rho^d(s)(s)=d(s,s)=0$ implies $\rho^d(s)\in P_{\mu,s}$. Therefore $\rho^d$ is feasible to (2.1). Furthermore, the triangle inequality $d(x,y)\geqslant |d(x,s)-d(s,y)|$ implies $d(x,y)\geqslant \|\rho(x),\rho(y)\|$. The nonnegativity of $c$ implies

$$\sum_{xy\in E} c(xy)d(x,y)\geqslant \sum_{xy\in E} c(xy)\big\|\rho(x),\rho(y)\big\|.$$

Hence we can always take an optimal solution of (1.1) as $d^\rho$ for some $\rho$ feasible to (2.1). □

The second lemma, due to Dress, states the existence of a nonexpansive retraction from $P_\mu$ to $T_\mu$. Although he stated this lemma for metrics, his proof in [9, Remark, p. 332] does not use the triangle inequality. Therefore it is applicable to nonmetric distances.

**Lemma 2.2.** *(See [9, (1.9), p. 331].) There is a map $\phi: P_\mu\to T_\mu$ such that*

(1) *$\|\phi(p),\phi(q)\|\leqslant \|p,q\|$ for $p,q\in P_\mu$, and*
(2) *$\phi(p)\leqslant p$ for $p\in P_\mu$, and thus $\phi$ is identical on $T_\mu$.*

Since $c$ is nonnegative, by Lemma 2.2, we can always take an optimal solution of (2.1) from the set of maps $\rho : V \to T_\mu$ with $\rho(s) \in T_{\mu,s} (s \in S)$. Thus we obtain Theorem 1.4.

In the rest of this section, we briefly discuss a relationship among the following three sets.

$$\mathcal{P}_{\mu,V} = \{d\colon \text{ metric on } V \mid d|_S \geqslant \mu\} + \mathbf{R}_+^{E_V},$$

$$\mathcal{T}_{\mu,V} = \text{the set of minimal elements of } \mathcal{P}_{\mu,V},$$

$$\Pi_{\mu,V} = \bigl\{\rho\colon\ V \to T_\mu \bigm| \rho(s) \in T_{\mu,s}\ (s \in S)\bigr\}.$$

Recall that (1.1) is a linear optimization over $\mathcal{P}_{\mu,V}$, its optimal solution can be taken from $\mathcal{T}_{\mu,V}$ by nonnegativity of $c$, and the tight-span dual is an optimization over $\Pi_{\mu,V}$. Note that each element of $\mathcal{T}_{\mu,V}$ is necessarily a metric.

As in the proof of Lemma 2.1, for a map $\rho \in \Pi_{\mu,V}$ we define a metric $d^\rho$ on $V$ by

$$d^\rho(x, y) := \bigl\|\rho(x), \rho(y)\bigr\| \quad (x, y \in V), \tag{2.2}$$

and for a metric $d \in \mathcal{T}_{\mu,V}$ we define a map $\rho^d : V \to P_\mu$ by

$$\rho^d(x)(s) := d(s, x) \quad (s \in S,\ x \in V).$$

The relationship among $\mathcal{P}_{\mu,V}$, $\mathcal{T}_{\mu,V}$, and $\Pi_{\mu,V}$ is summarized as follows.

**Proposition 2.3.** *We have the following.*

(1) *For a metric $d \in \mathcal{T}_{\mu,V}$, we have $\rho^d \in \Pi_{\mu,V}$ and $d^{\rho^d} = d$.*
(2) *For a map $\rho \in \Pi_{\mu,V}$, we have $d^\rho \in \mathcal{P}_{\mu,V}$ and $\rho^{d^\rho} = \rho$.*
(3) *Suppose that $\mu$ is a metric. Then we have $d^\rho \in \mathcal{T}_{\mu,V}$. In particular, $\mathcal{T}_{\mu,V}$ and $\Pi_{\mu,V}$ are in one-to-one correspondence.*

We easily see the properties (1) and (2) by a similar argument as in the proof of Lemma 2.1. Consider (3). Suppose that $\mu$ is a metric. Then it is easy to see that $d|_S = \mu$ holds for any $d \in \mathcal{T}_{\mu,V}$. Therefore, $\mathcal{T}_{\mu,V}$ is exactly the set of all tight extensions of metric $\mu$. Here, a metric $d$ on $V$ $(\supseteq S)$ is called a *tight extension* of $\mu$ if $d|_S = \mu$ and there is no metric $d' \neq d$ on $V$ such that $d'|_S = \mu$ and $d' \leqslant d$. Then the bijection in (3) has already been established by Dress [9, Theorem 3].

**Remark 2.4.** By extending the notion of tight extension to general nonmetric distances, one can see that the following two sets are in one-to-one correspondence.

(i) The set of all maps $\rho : V \to T_\mu$.
(ii) The set of minimal elements of the polyhedron

$$\bigl\{d\colon \text{ distance on } V \bigm| d|_S = \mu,\ d(s,u) + d(u,t) \geqslant d(s,t)\ (u \in V \setminus S,\ s,t \in V)\bigr\} + \mathbf{R}_+^{E_V}.$$

See the preprint version of this paper [12] for details, in which a distance in (ii) is called a *tight extension* of $(S, \mu)$.

**Remark 2.5.** If $\mu$ is a metric, then it is known [11, Lemma 2.2] that $T_{\mu,s}$ is a single point $\mu_s \in \mathbf{R}^S$ defined by

$$\mu_s(t) := \mu(t, s) \quad (t \in S).$$

Namely, $\mu_s$ is the $s$th column vector of the distance matrix $\mu$. In this case, $\rho(s)$ is fixed to the point $\mu_s$ for $s \in S$ in (TSD).

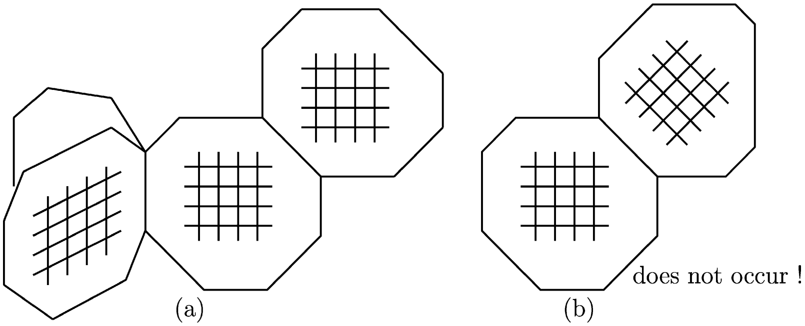



**Fig. 3.** Gluing $l_1$-octagons.

## 3. Geometry of $T_\mu$

The main aim of this section is to reveal several geometric properties of 2-dimensional tight spans $T_\mu$ which are the basis for the subsequent arguments. Among them, the following two propositions are particularly important for us; in fact, they (and Proposition 3.3) are sufficient to prove Theorem 1.5(1) in the next section. The first proposition concerns the shape of a 2-dimensional face. Here, we simply call a 2-dimensional face a 2-*face*.

**Proposition 3.1.** *Let $F$ be a 2-face of $T_\mu$. Then the metric space $(F, l_\infty)$ is isomorphic to the polygon $Q$ in the $l_\infty$-plane represented as*

$$Q = \left\{ (x_1, x_2) \in \mathbf{R}^2 \;\middle|\; \begin{array}{l} a_1 \leqslant x_1 \leqslant a_1',\ b \leqslant x_1 + x_2 \leqslant b' \\ a_2 \leqslant x_2 \leqslant a_2',\ c \leqslant x_1 - x_2 \leqslant c' \end{array} \right\} \tag{3.1}$$

*for some $a_1, a_1', a_2, a_2', b, b', c, c' \in \mathbf{R}$. Moreover, the isometry is given by the projection $\mathbf{R}^S \to \mathbf{R}^{\{s,t\}}$ for some $s, t \in S$.*

A polygon represented as (3.1) is exactly a convex polygon each of whose edges is parallel to one of the four vectors $(1,0), (0,1), (1,1), (1,-1)$. We call such a polygon in the $l_\infty$-plane an $l_\infty$-*octagon* (though it can be a $k$-gon with $3 \leqslant k \leqslant 8$). Recall that the $l_\infty$-plane is isomorphic to the $l_1$-plane. By the map $(x_1, x_2) \mapsto ((x_1 + x_2)/2, (x_1 - x_2)/2)$, we again obtain a convex polygon in the $l_1$-plane each of whose edges is parallel to one of the four vectors $(1,0), (0,1), (1,1), (1,-1)$. We call such a polygon in the $l_1$-plane an $l_1$-*octagon*. If we draw the $l_1/l_\infty$-coordinate on a 2-face $F$, then we observe that there are two types of edges of $F$: edges parallel to an $l_1$-axis and edges parallel to an $l_\infty$-axis. Here an $l_1$-*axis* means a vector $(1,1)$ or $(1,-1)$, and an $l_\infty$-*axis* means a vector $(1,0)$ or $(0,1)$ by the isometric projection to $(\mathbf{R}^2, l_\infty)$ in Proposition 3.1.

The second proposition says that if $\dim T_\mu \leqslant 2$, the metric space $(T_\mu, l_\infty)$ is constructed by gluing $l_1$-octagons *along the same type of edges*; see Fig. 3(a).

**Proposition 3.2.** *Suppose* $\dim T_\mu \leqslant 2$. *Let $F, F'$ be 2-faces of $T_\mu$ sharing an edge $e$. The edge $e$ is parallel to an $l_1$-axis on $F$ if and only if $e$ is parallel to an $l_1$-axis on $F'$.*

This property enables us to draw a global $l_1$-coordinate system on a 2-dimensional tight span, which gives a construction of $Z$ in Theorem 1.5 and will be discussed in the next section. The proofs of two propositions above will be given in Sections 3.3 and 3.4.

### *3.1. $T_\mu$ is geodesic*

Firstly, we verify that $(T_\mu, l_\infty)$ is *geodesic*. This means that for $p, q \in T_\mu$ there exists a path in $T_\mu$ connecting $p$ and $q$ with its length $\|p, q\|$, where $\| \cdot , \cdot \|$ denotes the $l_\infty$-distance; see (1.5). To avoid the measure-theoretic argument, a *path* $P$ in $T_\mu$ means a polygonal curve in $T_\mu$ and its length is defined to be the sum of the $l_\infty$-length of the segments in $P$.

**Proposition 3.3.** *The metric space $(T_\mu, l_\infty)$ is geodesic.*

**Proof.** For $p, q \in T_\mu$, consider the image of the segment $[p, q] \subseteq P_\mu$ by a nonexpansive retraction in Lemma 2.2. Since $T_\mu$ is a polyhedral set, we can modify it to a polygonal curve of length $\|p, q\|$. □

*3.2. The graph $K(p)$ and the moving process on $T_\mu$*

Secondly, we introduce an important technical tool to investigate $T_\mu$. For a point $p \in P_\mu$, we define an undirected graph $K_\mu(p) = K(p) = (S, E(p))$ by

$$st \in E(p) \quad \overset{\text{def}}{\Longleftrightarrow} \quad p(s) + p(t) = \mu(s,t) \quad (s, t \in S).$$

Note that a loop appears at $s \in S$ exactly when $p(s) = 0$. The graph $K(p)$ expresses the information of facets of $P_\mu$ which contain $p$.

Let $F(p)$ denote the minimal face of $P_\mu$ that $p$ belongs to. Then one can easily see the following characterization of elements of $T_\mu$; see also [9,11].

**Lemma 3.4.** *For $p \in P_\mu$, the following conditions are equivalent.*

(a) *$p$ belongs to $T_\mu$.*
(b) *For any $s \in S$, there is $t \in S$ such that $p(s) + p(t) = \mu(s,t)$.*
(c) *$K(p)$ has no isolated vertices.*
(d) *$F(p)$ is bounded.*

Note that in (b) the case $t = s$ is allowed and in this case $s$ has a loop. Also note that a vertex $s$ with $p(s) = 0$ is never isolated. In several places, the following observation is useful.

$$F(p) \subseteq F(q) \text{ if and only if } K(q) \text{ is a subgraph of } K(p). \tag{3.2}$$

Next we present a useful way of moving a point $p \in T_\mu$ to another point in $T_\mu$ using a stable set of $K(p)$. For a set $A$ of vertices of $K(p)$, the *neighborhood* $N(A)$ of $A$ is the set of vertices which are incident to $A$ in $K(p)$ and are not in $A$. For a stable set $A$ of $K(p)$ and a sufficiently small $\epsilon > 0$, one can easily see that the point

$$p^{A,\epsilon} := p + \epsilon(-\chi_A + \chi_{N(A)})$$

belongs to $P_\mu$. In particular, we observe that

$$K\bigl(p^{A,\epsilon}\bigr) \text{ is equal to } K(p) \text{ minus all edges joining } N(A) \text{ and } S \setminus A. \tag{3.3}$$

The following lemma gives a condition for $p^{A,\epsilon} \in T_\mu$, which immediately follows from (3.3) and (a) ⇔ (c) in Lemma 3.4.

**Lemma 3.5.** *For $p \in T_\mu$, let $A$ be a stable set in $K(p)$. If $A$ is maximal stable in $K(p)$ or in some connected component of $K(p)$, then for a sufficiently small $\epsilon > 0$, the point $p^{A,\epsilon}$ belongs to $T_\mu$.*

As an application of this lemma, we have the following geodesic properties of $T_\mu$ which will be used for the proof of (2) in Theorem 1.1. Recall the definition (1.6) of the $l_\infty$-distances among subsets.

**Lemma 3.6.** *The following two statements hold.*

(1) *$\mu(s,t) = \|T_{\mu,s}, T_{\mu,t}\|$ for $s, t \in S$.*
(2) *$p(s) = \|p, T_{\mu,s}\|$ for $p \in T_\mu$, $s \in S$.*

**Proof.** (1) For $p \in T_{\mu,s}$ and $q \in T_{\mu,t}$, we have $\|p,q\| \geqslant p(t) - q(t) = p(t) + p(s) \geqslant \mu(s,t)$ by $q(t) = p(s) = 0$. We show the reverse inequality. It is easy to see that there is $p \in T_{\mu,s}$ with $st \in E(p)$; take a minimal $p \in P_\mu$ with $p(s) = 0$ and $p(t) = \mu(s,t)$. We may assume $\mu(s,t) > 0$ since $\mu(s,t) = 0$ implies $p \in T_{\mu,s} \cap T_{\mu,t}$ and thus $\mu(s,t) = 0 = \|T_{\mu,s}, T_{\mu,t}\|$. We can take a maximal stable set $A$ containing $t$. Move $p \to p^{A,\epsilon}$ as much as $p^{A,\epsilon} \in T_\mu$. Then we have $\|p, p^{A,\epsilon}\| = \epsilon$. Reset $p \leftarrow p^{A,\epsilon}$, and repeat this process until $p(t) = 0$. This procedure terminates by the polyhedrality of $T_\mu$. In this procedure, the vertex $t$ is always in $N(A)$. Therefore, the resulting path from $T_{\mu,s}$ to $T_{\mu,t}$ has the length $\mu(s,t)$.

(2) Since each $q \in T_{\mu,s}$ satisfies $q(s) = 0$ by definition, we have $\|p, T_{\mu,s}\| \geqslant \inf_{q \in T_{\mu,s}} \{p(s) - q(s)\} = p(s)$. We show the reverse inequality by constructing a path from $p$ to $T_{\mu,s}$ with the length equal to $p(s)$. We may assume $p(s) > 0$ since $p(s) = 0$ implies $p \in T_{\mu,s}$ and thus $p(s) = \|p, T_{\mu,s}\| = 0$. We can take a maximal stable set $A$ containing $s$. Then move $p \to p^{A,\epsilon}$ as much as $p^{A,\epsilon} \in T_\mu$. Set $p \leftarrow p^{A,\epsilon}$. Repeat this process until $p(s) = 0$. Then we obtain a desired path of length $p(s)$. □

The first property (1) in Lemma 3.6 means that the distance $\mu$ is isometrically embedded into $T_\mu$ as the $l_\infty$-distance among subsets $\{T_{\mu,s}\}_{s \in S}$, which was shown in [11, Theorem 2.4]. The second property (2), which is an extension of [9, Theorem 3(ii)], gives an interpretation of $p$ as a *multiflow-potential*. Recall a relation between distances and potentials in the network flow theory. Since $\{T_{\mu,s}\}_{s \in S}$ corresponds to terminals, $p$ is regarded as a vector of distances from terminals.

### 3.3. The dimension and the local structure of faces of $T_\mu$

Thirdly, we study the dimension and the local structure of a face $F$ in terms of the graph $K(\cdot)$. Take $p^*$ in the relative interior of a face $F$. Suppose that $K(p^*)$ has $m$ bipartite components with bipartitions $\{A_1, B_1\}, \{A_2, B_2\}, \ldots, \{A_m, B_m\}$. Then it is easy to see that the set of vectors $\{\chi_{A_i} - \chi_{B_i}\}_{i=1}^m$ is a basis of the vector space $\{p \in \mathbf{R}^S \mid p(s) + p(t) = 0 \ (st \in E(p^*))\}$. Then every point $p$ in $F$ is uniquely represented as

$$p = p^* + \sum_{i=1}^{m} x_i (\chi_{A_i} - \chi_{B_i}) \tag{3.4}$$

for $x_1, x_2, \ldots, x_m \in \mathbf{R}$. Therefore we have the following.

**Proposition 3.7.** *(See [9].) For $p \in T_\mu$, we have*

$$\dim F(p) = \text{the number of bipartite components of } K(p),$$

*where loops are regarded as odd cycles.*

In the expression (3.4), the map $p \mapsto (x_1, x_2 \ldots, x_m)$ is an injective isometry from $(F, l_\infty)$ to $(\mathbf{R}^m, l_\infty)$ since each $\chi_{A_i} - \chi_{B_i}$ is a 0–1 vector. From this fact, we easily obtain Proposition 3.1. Indeed, consider the case $m = 2$. Then (3.4) is

$$p = p^* + x_1(\chi_{A_1} - \chi_{B_1}) + x_2(\chi_{A_2} - \chi_{B_2}). \tag{3.5}$$

By substituting this equation to linear inequalities $p(s) + p(t) \geqslant \mu(s,t)$ $(s, t \in S)$, we obtain the linear inequality representation (3.1). Furthermore, the isometry is given by the projection $\mathbf{R}^S \to \mathbf{R}^{\{s,t\}}$ for $s \in A_1 \cup B_1$, $t \in A_2 \cup B_2$.

### 3.4. Classification of faces of $T_\mu$

Fourthly, we classify faces of $T_\mu$ in terms of graph $K(p)$. Note that $K(p)$ may have a connected component each of whose vertices has a loop. Such a component is called a *loop-component*. In this case, $p(s) = p(t) = 0$ and $\mu(s,t) = 0$ hold for vertices $s, t$ of the loop-component. In particular, the loop-component is a complete graph with all loops, and is unique if it exists. A connected component of $K(p)$ that is not a loop-component is said to be *proper*. The next lemma summarizes the classification of faces of $T_\mu$ in terms of $K(\cdot)$.

**Lemma 3.8.** *Suppose that* $\dim T_\mu \leqslant 2$. *For* $p \in T_\mu$, *we have the following.*

(1) $F(p)$ *is an extreme point if and only if*
  (1-a) *the proper components of* $K(p)$ *consist of one nonbipartite component, or*
  (1-b) *the proper components of* $K(p)$ *consist of two nonbipartite components.*
(2) $F(p)$ *is an edge if and only if*
  (2-a) *the proper components of* $K(p)$ *consist of one bipartite component, or*
  (2-b) *the proper components of* $K(p)$ *consist of one bipartite component and one nonbipartite component.*
(3) $F(p)$ *is a* 2*-face if and only if the proper components of* $K(p)$ *consist of two bipartite components.*
(4) $F(p)$ *is a maximal face if and only if the proper components of* $K(p)$ *consist of complete bipartite components.*

**Proof.** We show that $K(p)$ has at most two proper components. Indeed, suppose that $K(p)$ has at least three proper components. Take a maximal stable set $A$ in $K(p)$ and small $\epsilon > 0$. Then we have $p^{A,\epsilon} \in T_\mu$ by Lemma 3.5. By (3.3) and maximality of $A$, the proper components of $K(p^{A,\epsilon})$ consist of edges in $K(p)$ joining $A$ and $N(A)$, and $A$ meets all proper components. In particular, all proper components in $K(p^{A,\epsilon})$ are bipartite. Therefore $K(p^{A,\epsilon})$ has at least three bipartite components since $K(p^{A,\epsilon})$ is a (bipartite) subgraph of $K(p)$. This is a contradiction to $\dim T_\mu \leqslant 2$ by Proposition 3.7. From this fact and Proposition 3.7, we have (1-3). Suppose that $F(p)$ is a maximal face. By the same argument above, $K(p)$ has no proper nonbipartite components. Suppose that $K(p)$ has a bipartite component $K$ of bipartition $\{A, B\}$ that is not complete. Then there is a maximal stable set $A'$ in $K$ intersecting both $A$ and $B$. Therefore, for small $\epsilon > 0$ we have $p^{A',\epsilon} \in T_\mu$ by Lemma 3.5, and $K(p^{A',\epsilon})$ is a proper subgraph of $K(p)$, which implies $F(p^{A',\epsilon}) \supset F(p)$ by (3.2). This is a contradiction to the maximality. Then we have the only-if-part of (4). The proof of the if-part is omitted since it is not difficult and is not used in the subsequent arguments. □

In particular, there are two types of edges in $T_\mu$: (2-a) and (2-b) in Lemma 3.8. An edge $e$ of $T_\mu$ is called an $l_1$*-edge* if the type of $K_e$ is (2-a), and is called an $l_\infty$*-edge* if the type of $K_e$ is (2-b), where $K_e := K(p)$ for a relative interior point $p$ in $e$. An edge that is a maximal face is necessarily an $l_1$-edge by Lemma 3.8(4). The names "$l_1/l_\infty$-edge" are justified by the following lemma.

**Lemma 3.9.** *Let* $F$ *be a* 2*-face and* $e$ *an edge of* $F$. *Then* $e$ *is parallel to an* $l_1$*-axis in* $F$ *if and only if* $e$ *is an* $l_1$*-edge.*

**Proof.** Let $F$ be a 2-face, and let $K_F$ be the graph corresponding to $F$, i.e., $K_F := K(p)$ for a relative interior point $p$ in $F$. By Lemma 3.8, the graph $K_F$ has exactly two complete bipartite components $K_1$ and $K_2$ with bipartitions $\{A_1, B_1\}$ and $\{A_2, B_2\}$, respectively. By (3.5), the directions of $l_\infty$-axes in $F$ are $\chi_{A_1} - \chi_{B_1}$ and $\chi_{A_2} - \chi_{B_2}$, and the directions of $l_1$-axes in $F$ are $\chi_{A_1 \cup A_2} - \chi_{B_1 \cup B_2}$ and $\chi_{A_1 \cup B_2} - \chi_{B_1 \cup A_2}$. Let $e$ be an edge of $F$, and let $K_e$ be the graph corresponding to $e$. Then $K_F$ is a subgraph of $K_e$ by (3.2). By Lemma 3.8, the type of $K_e$ is (2-a) or (2-b). If the type of $K_e$ is (2-b), then $K_e$ has exactly one of $K_1$ and $K_2$ as a (proper) component, and thus $e$ is parallel to $\chi_{A_1} - \chi_{B_1}$ or $\chi_{A_2} - \chi_{B_2}$ by (3.4). If the type of $K_e$ is (2-a), then both $K_1$ and $K_2$ are subgraphs of one bipartite component of $K_e$ whose bipartition is $\{A_1 \cup A_2, B_1 \cup B_2\}$ or $\{A_1 \cup B_2, B_1 \cup A_2\}$. Therefore, $e$ is parallel to an $l_1$-axis in $F$. Thus we are done. □

Since the property (2-a) or (2-b) is independent on the choice of $F$, we obtain Proposition 3.2.

## 4. $l_1$-Grids

In this section, we introduce a global $l_1$-coordinate system on a 2-dimensional tight span $T_\mu$, called an $l_1$*-grid*, and show that the finite set $Z$ in Theorem 1.5 can be taken as the set of the *grid-points* of an $l_1$-grid satisfying a certain orientability condition. The idea of drawing the $l_1$-coordinate was used

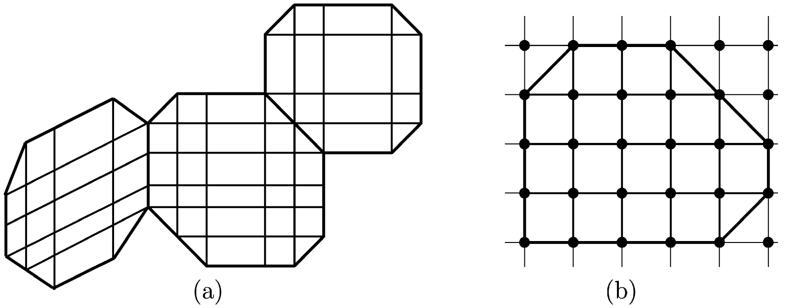



**Fig. 4.** (a) An $l_1$-grid and (b) decomposing an integral $l_1$-octagon by $\mathbf{Z}^2$.

in [3] for tight spans of 5-point metrics. The argument here extends it to general 2-dimensional tight spans.

Now suppose that $\dim T_\mu \leqslant 2$. Recall that, by Propositions 3.1 and 3.2, $T_\mu$ can be constructed by gluing $l_1$-octagons. An $l_1$*-grid* $\Delta$ of $T_\mu$ is a 2-dimensional polyhedral subdivision such that each 2-face $C$ of $\Delta$ is

(r) a rectangle with edge parallel to $l_1$-axes of $F$, or
(t) an isosceles triangle such that its two equal edges are parallel to $l_1$-axes of $F$ and the remaining edge is parallel to an $l_\infty$-axis of $F$,

where $F$ is the unique 2-face of $T_\mu$ containing $C$. In particular, by the projection to $\mathbf{R}^2$ in Proposition 3.1, a triangle in $\Delta$ is an isosceles *right* triangle (regarding $\mathbf{R}^2$ as the Euclidean plane) such that its equal edges are parallel to $(1,1)$ or $(1,-1)$ and its longer edge (the *hypotenuse*) is parallel to $(0,1)$ or $(1,0)$. See Figs. 4(a) and 2(c) in the introduction. A vertex (a zero-dimensional face) of an $l_1$-grid is called a *grid-point*. The longer edge of a triangle is called an $l_\infty$*-edge*, and other edges are called $l_1$*-edges*.

If $\mu$ is rational, then an $l_1$-grid always exists. In this case, we obtain an $l_1$-grid all of whose $l_1$-edges have the same length by the following construction. By rationality, we may assume that the polyhedron $P_\mu$ is $2/k$-integral for some integer $k \geqslant 2$. For an edge $e$ that is a maximal face, we can subdivide it to segments of the $l_\infty$-length $1/k$. For a 2-face $F$, we can subdivide it to triangles and squares of size $1/k$ by the following way, where the *size* of a triangle or a square is defined to be the $l_\infty$-length of its $l_1$-edge. $F$ is regarded as a $2/k$-integral $l_\infty$-octagon by the projection to $\mathbf{R}^2$ in Proposition 3.1. By the transformation $(x_1,x_2) \mapsto ((x_1+x_2)/2, (x_1-x_2)/2)$, the resulting $l_1$-octagon $Q$ is $1/k$-integral in $\mathbf{R}^2$. Then the $1/k$-integer grid naturally decomposes $Q$ into triangles and squares of size $1/k$, which are the closure of the connected components obtained by deleting the coordinate lines $(i/k)(1,0)+\mathbf{R}(0,1)$, $\mathbf{R}(1,0)+(j/k)(0,1)$ $(i,j \in \mathbf{Z})$ from $Q$; see Fig. 4(b). From this construction, we obtain a subdivision of $T_\mu$ consisting of squares and triangles satisfying (r) and (t). By the gluing property (Proposition 3.2), it is indeed a polyhedral subdivision of $T_\mu$ and thus is an $l_1$-grid. This $l_1$-grid is called the $1/k$*-uniform* $l_1$*-grid*.

**Remark 4.1.** If $\mu$ is irrational, then an $l_1$-grid may not exist. For example, consider the distance $\mu$ on 4-set $\{s,s',t,t'\}$ defined as $\mu(s,s')=1$, $\mu(t,t')=\alpha$ for irrational positive $\alpha$, and the other distances are zero. Then $T_\mu$ is a rectangle of four $l_\infty$-edges with the edge length ratio $(1:\alpha)$. Clearly $T_\mu$ has no $l_1$-grids.

The graph of $l_1$-edges behaves nicely as follows.

**Proposition 4.2.** *Let $\Delta$ be an $l_1$-grid of $T_\mu$. For two grid-points $p, q$ in $\Delta$, there is a geodesic between $p$ and $q$ consisting of $l_1$-edges of $\Delta$.*

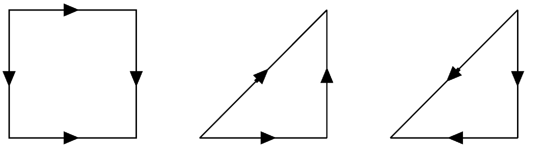


**Fig. 5.** Orientations of a rectangle and a triangle.

**Proof.** Let $L \subseteq T_\mu$ be a geodesic from $p$ to $q$. Suppose that $L$ does not lie on the union of $l_1$-edges of $\Delta$. Then there is a member $F$ in $\Delta$ such that $L$ meets a point not in $l_1$-edges of $F$. Let $F$ be the first (along $L$) among such members of $\Delta$. Let $p', q'$ be the endpoint of $L \cap F$. We may assume that $p'$ is a grid-point of $\Delta$ and $q'$ is in the boundary of $F$. Suppose that $F$ is a rectangle. Then we modify $L$ so that $p'$ and $q'$ are connected by a geodesic boundary path in $F$. Then the resulting path is also geodesic. Suppose that $F$ is a triangle. If $q'$ lies on an $l_1$-edge (a shorter edge) of $F$, then we modify $P$ as above. If $q'$ lies on the longer edge of $F$, then there is a triangle $F'$ in $\Delta$ such that $F'$ and $F$ share the longer edge by Proposition 3.2. Let $q''$ $(\neq q')$ be the endpoint of $P \cap F'$. Then $q''$ lies on an $l_1$-edge of $F'$. Then we modify $L$ so that $p'$ and $q''$ are connected by a geodesic boundary path in $F \cup F'$. The modified path is also a geodesic between $p$ and $q$. Repeating this process, we eventually obtain a desired geodesic consisting of $l_1$-edges of $\Delta$. □

**Remark 4.3.** Chepoi [4] studied 2-dimensional complexes constructed by gluing rectangles and isosceles right triangles, and explored some of interesting geodesic and graph-theoretic properties. By using his arguments in [4, Section 7], one can show that the graph of $l_1$-edges of an $l_1$-grid of a 2-dimensional tight span is a *hereditary modular graph without induced* $K_{3,3}$ *and* $K_{3,3}^-$. A hereditary modular graph is just a bipartite graph without isometric cycles of length $k \geqslant 6$ [1].

We will show that the finite set $Z$ in Theorem 1.5 can be taken as the set of the grid-points of an $l_1$-grid satisfying a certain orientability condition. So we introduce the definition of orientability of $l_1$-grids and related concepts. Such a notion was originally introduced by Karzanov [20] for hereditary modular graphs in a purely graph-theoretical sense. In particular, we will explain a simple modification of Karzanov's *orbit splitting method* [21]. The essential distinction is that we need to deal with $l_\infty$-edges explicitly.

Two edges $e$ and $e'$ of an $l_1$-grid $\Delta$ are said to be *projective* if there is a sequence of edges $e = e_0, e_1, \ldots, e_m = e'$ such that for $0 \leqslant i \leqslant m-1$ there is a triangle in $\Delta$ containing $e_i$ and $e_{i+1}$, or a rectangle in $\Delta$ containing $e_i$ and $e_{i+1}$ as its nonadjacent edges. The projectivity is an equivalence relation on the set of edges of an $l_1$-grid. An equivalence class is called an *orbit*. An $l_1$-grid is said to be *orientable* if we can orient its edges in such a way that in each rectangle nonadjacent edges have the same direction with respect to the coordinate axes, and in each triangle an acute angle is a source or a sink; see Fig. 5. We call such an orientation *admissible*. It is easy to see that an $l_1$-grid is nonorientable if and only if there is an orbit containing a sequence of edges $p_0q_0, p_1q_1, \ldots, p_mq_m$ with $p_m = q_0$, $q_m = p_0$ such that for $0 \leqslant i \leqslant m-1$ there is a rectangle of edges $\{p_iq_i, p_{i+1}q_{i+1}, p_ip_{i+1}, q_iq_{i+1}\}$ or a triangle of vertices $\{p_i, q_i = q_{i+1}, p_{i+1}\}$ with an acute angle $q_i$ or $\{q_i, p_i = p_{i+1}, q_{i+1}\}$ with an acute angle $p_i$. Such an orbit is called a *nonorientable orbit*. Fig. 6 illustrates the 1/2-uniform $l_1$-grid for the tight span given in Fig. 2(b) in the introduction. This $l_1$-grid has one nonorientable orbit.

By subdividing some of faces meeting a (possibly nonorientable) orbit $o$, we can make $o$ orientable as follows. For a triangle all of whose edge belonging to $o$, subdivide it to two triangles and one square of the half-size as in Fig. 7(a). For a rectangle with exactly two edges belonging to $o$, split it into two rectangles by cutting it along the segment joining the midpoints of two nonadjacent edges belonging to $o$ as in Fig. 7(b). For a square with all edges belonging to $o$, subdivide it into four squares of the half-size as in Fig. 7(c). For the (exceptional) case that $o$ consists of a single edge $e$, subdivide $e$ into two edges of the half-size. This operation is called the *orbit splitting* (with respect to $o$). The edges of this subdivided orbit can be oriented so that the original vertices are sources as in Fig. 7. In particular, if $o$ is nonorientable, then $o$ is transformed into one orientable orbit of the double size

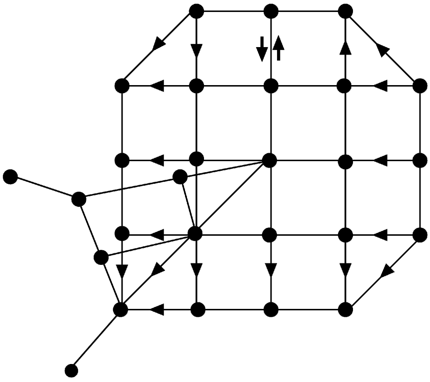


**Fig. 6.** Nonorientable 1/2-uniform $l_1$-grid.

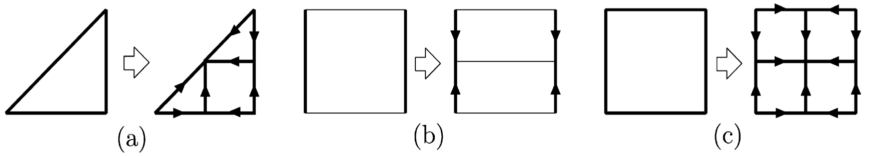


**Fig. 7.** Splitting and orienting a triangle and rectangles.

that turns around the original orbit *twice*. If $o$ is orientable, then $o$ is split into two orientable orbits of the same size that turn around the original orbit only once. The orbit splitting to $o$ does not affect the orientability of other orbits. Applying the orbit splitting to each nonorientable orbit, we have an orientable $l_1$-grid. Fig. 2(c) in the introduction is the result of an orbit splitting for Fig. 6.

**Remark 4.4.** If an $l_1$-grid exists, then there is a unique "minimal" $l_1$-grid $\Delta$ with the property that every $l_1$-grid is a refinement of $\Delta$. By applying the orbit splitting to each nonorientable orbit of $\Delta$, we obtain a unique minimal orientable $l_1$-grid $\Delta^*$. For more details of this unique minimal orientable $l_1$-grid, see the preprint version of this paper [12].

Related to the orbit splitting operation, we introduce the subdivision operation as follows. Let $k$ be a positive integer. For each rectangle $R$ in $\Delta$, divide it equally into $k^2$ rectangles congruent to $(1/k)R$. For each triangle $T$ of size $l$ in $\Delta$, divide it into $k$ triangles of size $l/k$ and $(k^2-k)/2$ squares of size $l/k$, where the size of a triangle is defined to be the length of its $l_1$-edge. Similarly, divide each edge that is maximal in $\Delta$ equally into $k$ edges. The resulting $l_1$-grid, denoted by $\Delta^k$, is called the *k-subdivision* of $\Delta$; see Fig. 8(b). Note that the 2-subdivision is always orientable.

*Proof of (1) in Theorem 1.5*

Assume that $\mu$ is rational. We are ready to prove Theorem 1.5(1).

**Proposition 4.5.** *Let $Z$ be the set of the grid-points of an orientable $l_1$-grid $\Delta$ of $T_\mu$. Then for every graph $G=(V,E,c)$ with $S\subseteq V$ there exists an optimal solution $\rho$ of* (TSD) *with $\rho(V)\subseteq Z$.*

Take an optimal solution $\rho: V\to T_\mu$ of (TSD). Since $\mu$ is rational, we may assume that the image of $V$ by $\rho$ are rational(-valued). Then there is an integer $k$ such that the image of $V$ by $\rho$ lies on the set $Z^k$ of the grid-points on $\Delta^k$.

Fix an admissible orientation of $\Delta$. Each edge $e$ of $\Delta$ is subdivided into $k$ edges $e_1,e_2,\dots,e_k$ in $\Delta^k$. We number their indices by the orientation as follows. If $e$ has ends $p$ and $q$, is oriented as $\overrightarrow{pq}$, and

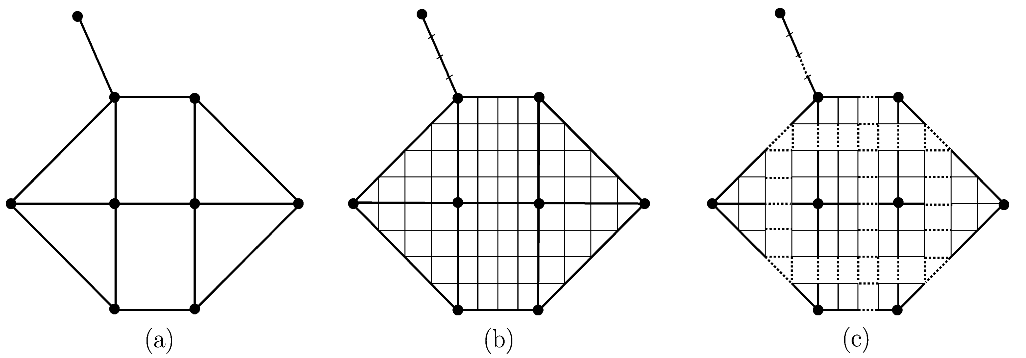


**Fig. 8.** (a) $l_1$-grid $\Delta$, (b) 4-subdivision $\Delta^4$, and (c) edge set $U$.

is subdivided into $p_0p_1, p_1p_2, \ldots, p_{k-1}p_k$ for $p_0 = p$ and $p_k = q$, then let $e_i := p_{i-1}p_i$. Take arbitrary $i \in \{1, 2, \ldots, k\}$. Let $U$ be the set of edges that are projective to the $i$th subdivided edge $e_i$ of some edge $e$ in $\Delta$. Then $U$ is the union of several orbits, and does not meet any $j$th subdivided edge $e_j$ for $j \neq i$; one can verify this fact by considering $U$ in each subdivided face. See Fig. 8(c), where the broken lines represent the edge set $U$. Consider the 1-skeleton graph of $\Delta^k$. Contract all edges in $U$ and delete multiple edges appeared. Then resulting graph coincides with the 1-skeleton graph of $\Delta^{k-1}$ (as graph); see Fig. 8(c). Therefore we obtain a map $\phi : Z^k \to Z^{k-1}$ by defining $\phi(p)$ to be the point in $Z^{k-1}$ corresponding to the contracted point of $p$ in the 1-skeleton graph of $\Delta^{k-1}$. Also contract all edges not in $U$ and delete multiple edges appeared. Then resulting graph coincides with the 1-skeleton graph of $\Delta$. Similarly we obtain a map $\psi : Z^k \to Z$ by defining $\psi(p)$ to be the contracted point. By construction, if $\rho(x)$ belongs to some face $C \in \Delta$, then both $\phi \circ \rho(x)$ and $\psi \circ \rho(x)$ belong to $C$. This implies that both compositions $\phi \circ \rho$ and $\psi \circ \rho$ are feasible to (TSD).

Therefore it suffices to show the following.

$$d^\rho \geqslant \frac{k-1}{k} d^{\phi\circ\rho} + \frac{1}{k} d^{\psi\circ\rho}. \tag{4.1}$$

(In fact, the equality holds.) Recall that $d^\rho$ is defined as $d^\rho(x, y) := \|\rho(x), \rho(y)\|$; see (2.2). If (4.1) holds, then at least one of $\phi \circ \rho$ and $\psi \circ \rho$ is an optimal solution by nonnegativity of $c$. If $\psi \circ \rho$ is optimal, then the image of $\psi \circ \rho$ lies on $Z$, and we are done. If $\phi \circ \rho$ is optimal, then the image of $\psi \circ \rho$ lies on the grid-points of $\Delta^{k-1}$, and we can repeat the same process to $\phi \circ \rho$.

By Proposition 4.2, there is a geodesic $L$ between $p$ and $q$ consisting of $l_1$-edges of $\Delta^k$. We regard $L$ as a set of $l_1$-edges of $\Delta^k$. By applying $\phi$ to (vertices in) $L$, we obtain a path connecting $\phi(p)$ and $\phi(q)$ whose length is $k/(k-1)$ times as longer as the sum of the length of all edges in $L \setminus U$. Also by applying $\psi$ to $L$, we obtain a path connecting $\psi(p)$ and $\psi(q)$ whose length is $k$ times as longer as the sum of the length of all edges in $L \cap U$. Therefore, we have

$$\|p, q\| \geqslant \frac{k-1}{k} \big\|\phi(p), \phi(q)\big\| + \frac{1}{k} \big\|\psi(p), \psi(q)\big\|.$$

Consequently, we have (4.1).

## 5. Proof of the half-integrality

In this section, we prove (2) in Theorem 1.5 that immediately implies (1) in Theorem 1.1 by the correspondence $\rho \mapsto d^\rho$ in (2.2). We begin with the fundamental lemma.

**Lemma 5.1.** *If $\mu$ is a cyclically even distance, then the polyhedron $P_\mu$ is integral.*

**Proof.** Let $p$ be an extreme point of $T_\mu$. Then $K(p)$ has no bipartite components. Take a nonbipartite component $K$. Then there is an odd cycle $C$ in $K$. We order vertices in $C$ cyclically as $(s_0, s_1, \ldots, s_{k-1})$. Then $p(s_0)$ is given as

$$p(s_0) = \frac{1}{2}\sum_{j=0}^{k-1}(-1)^j \mu(s_j, s_{j+1}), \tag{5.1}$$

where the indices are taken modulo $k$. By the cyclically evenness, $p(s_0)$ is integral, and thus $p(s_j)$ is integral. Let $s'$ be an arbitrary vertex of $K$. There is a path in $K(p)$ connecting $s'$ to $C$. Then $p(s')$ is determined by substituting $p(s) + p(s') = \mu(s, s')$ along this path. Consequently $p$ is integral. □

Now to show the 1/4-integrality is easy. Indeed, by the previous lemma, we can take the 1/2-uniform $l_1$-grid $\Delta$ of $T_\mu$. $\Delta$ may be nonorientable. By applying the orbit splitting to each orbit, we obtain the 1/4-uniform $l_1$-grid that is orientable. By Propositions 4.2 and 4.5, the $l_\infty$-distances among the grid-points of the 1/4-uniform $l_1$-grid are quarter-integral. Consequently, we can take a quarter-integral optimal solution in (1.1) and in $M^*(G; S, \mu)$. In fact, surprisingly, this 1/2-uniform $l_1$-grid $\Delta$ is orientable. The rest of this section is devoted to proving this fact.

**Theorem 5.2.** *Suppose that $\mu$ is a cyclically even distance with $\dim T_\mu \leqslant 2$. The 1/2-uniform $l_1$-grid for $T_\mu$ is orientable.*

The proof is relatively complicated. A key is the following observation.

($*$1) If an $l_\infty$-octagon is *integral* in the lattice $\{(x_1, x_2) \in \mathbf{Z}^2 \mid x_1 + x_2 \in 2\mathbf{Z}\}$, then by the map $(x_1, x_2) \mapsto ((x_1 + x_2)/2, (x_1 - x_2)/2)$, the resulting $l_1$-octagon is integral in $\mathbf{Z}^2$.

Therefore, if all 2-faces of $T_\mu$ have such a property, then $T_\mu$ has the integral uniform $l_1$-grid and consequently the 1/2-uniform $l_1$-grid is orientable by the orbit splitting.

Motivated by ($*$1), for $U \subseteq S$, we define a lattice $L_U$ in $\mathbf{Z}^S$ by

$$L_U = \bigl\{p \in \mathbf{Z}^S \bigm| p(s) = 0 \ (s \in U),\ \ p(t) + p(u) \in 2\mathbf{Z}\ (t, u \in S \setminus U)\bigr\},$$

and define a subset $T_{\mu,U} \subseteq T_\mu$ by

$$T_{\mu,U} = \text{the union of maximal faces } F \text{ of } T_\mu$$
$$\text{whose } K_F \text{ has the loop-component of vertex set } U,$$

where $K_F := K(p)$ for a relative interior point $p$ in $F$, and $U = \emptyset$ means that $K_F$ has no loop-component. Recall that the loop-component is a connected component all of whose vertices have a loop. A loop-component is unique if it exists. Other connected components are said to be proper. Then $T_{\mu,U}$ and $L_U$ have the following property.

($*$2) For a 2-face $F \subseteq T_{\mu,U}$, the isometric projection of $F \cap L_U$ to $\mathbf{R}^2$ in Proposition 3.1 coincides with the intersection of an $l_\infty$-octagon and the lattice $\{(x_1, x_2) \in \mathbf{Z}^2 \mid x_1 + x_2 \in 2\mathbf{Z}\}$.

This immediately follows from the local coordinate (3.5) in a 2-face. In the sequel, we try to make each 2-face $F \subseteq T_{\mu,U}$ *integral* in the affine lattice of some translation of $L_U$.

Recall Lemma 3.8. There are two types of extreme points in $T_\mu$: (1-a) and (1-b) in Lemma 3.8. An extreme point of type (1-a) is said to be *normal*. An extreme point $p$ of type (1-b) is called a *core*.

**Lemma 5.3.** *For $U \subseteq S$, let $p, q \in T_{\mu,U}$ be normal extreme points of $T_\mu$. Then we have*

$$p - q \in L_U.$$

**Proof.** Since $p$ is normal, $K(p)$ has exactly one proper component $K$ by definition. Then both $s, t \in S \setminus U$ belong to $K$. By a simple calculation from (5.1), $p(s) + p(t)$ is given by $\sum_{e \in P} \pm\mu(e)$ for some (possibly nonsimple) path $P$ connecting $s$ and $t$ in $K$. Also $q(s) + q(t)$ is given by the sum of $\pm\mu(e)$ along a path $P'$ connecting $s$ and $t$ in $K$. Therefore $(p - q)(s) + (p - q)(t)$ is given by the sum of $\pm\mu(e)$

along some (possibly nonsimple) cycle $P \cup P'$. Therefore, $(p-q)(s)+(p-q)(t)$ is even by the cyclically evenness of $\mu$. □

**Lemma 5.4.** *If $T_{\mu,U} \neq \emptyset$, then there exists a normal extreme point in $T_{\mu,U}$.*

The proof will be given in the end of this section. For $U \subseteq S$ with $T_{\mu,U} \neq \emptyset$, we can define an affine lattice $A_{\mu,U}$ by

$$A_{\mu,U} = p + L_U,$$

where $p$ is any normal extreme point in $T_{\mu,U}$. The affine lattices $\{A_{\mu,U}\}_{U\subseteq S}$ together with $\{T_{\mu,U}\}_{U\subseteq S}$ have the following gluing property.

**Lemma 5.5.** *For $U, U' \subseteq S$ with $T_{\mu,U} \cap T_{\mu,U'} \neq \emptyset$, the following holds.*

$$A_{\mu,U} \cap T_{\mu,U} \cap T_{\mu,U'} = A_{\mu,U'} \cap T_{\mu,U} \cap T_{\mu,U'}. \tag{5.2}$$

**Proof.** Take $q \in A_{\mu,U} \cap T_{\mu,U} \cap T_{\mu,U'}$. Let $p$ and $p'$ be normal extreme points in $T_{\mu,U}$ and $T_{\mu,U'}$, respectively. Then $p - q \in L_U$. It suffices to show $p' - q \in L_{U'}$. By the same argument as in the proof of Lemma 5.3, for $s,t \in S \setminus U$, $p(s)+p(t)$ is the sum of $\pm\mu(e)$ along some $s$–$t$ path, and for $s,t \in S \setminus U'$, $p'(s)+p'(t)$ is the sum of $\pm\mu(e)$ along some $s$–$t$ path. By $p - q \in L_U$, for $s,t \in S \setminus U$, $q(s)+q(t)$ is equal to $p(s)+p(t)$ modulo 2.

It suffices to show that for $s,t \in S \setminus U'$, $q(s)+q(t)$ is equal to the sum of $\pm\mu(e)$ along some $s$–$t$ path modulo 2.

**Case 1.** $s,t \in U \setminus U'$. Then we have $q(s)+q(t) = 0 = \mu(s,t)$ since $q(u) = 0$ for any $u \in U \cup U'$.

**Case 2.** $s,t \in S \setminus (U \cup U')$. We have $q(s)+q(t) = (q-p)(s)+(q-p)(t)+p(s)+p(t) \equiv p(s)+p(t) \pmod 2$ by $q - p \in L_U$. Then $p(s)+p(t)$ is the sum of $\pm\mu(e)$ along some $s$–$t$ path, and so is $q(s)+q(t)$ modulo 2.

**Case 3.** $s \in U \setminus U'$, $t \in S \setminus (U \cup U')$. We may assume that $K(q)$ has no loop-component of vertex set $U'' = U \cup U'$. Indeed, if $K(q)$ has such a loop-component, then every maximal face containing $q$ belongs to $T_{\mu,U''}$, and this implies $U = U' = U''$ (the statement (5.2) is trivial). Therefore there are $s' \in U \cup U'$ and $t' \in S \setminus (U \cup U')$ with $s't' \in E(q)$. Then we have $q(s)+q(t) = (q(s)+q(s')) + (q(s')+q(t')) + (q(t)-q(t')) \equiv \mu(s,s') + \mu(s',t') + (q(t)-q(t')) \pmod 2$, where we use $q(s) = q(s') = 0 = \mu(s,s')$. By Case 2 above, $q(t)-q(t')$ is equal to the sum of $\pm\mu(e)$ along $t$–$t'$ path modulo 2. Then we are done. □

By this gluing property, if all extreme points of $T_\mu$ lie on the finite set $Z' := \bigcup_{U\subseteq S} T_{\mu,U} \cap A_{\mu,U}$, then each 2-face satisfies the property $(*1)$ and thus there exists the integral uniform $l_1$-grid. Although all normal extreme points lie on $Z'$ by Lemma 5.4 and the definition of $A_{\mu,U}$, some of cores may *not* lie on $Z'$. Next we study the local property of a core $p$. By definition of a core (an extreme point of type (1-b) in Lemma 3.8), $K(p)$ consists of two proper nonbipartite components and the (possibly empty) loop-component. A more detailed description of $K(p)$ is given as follows.

**Lemma 5.6.** *Let $p$ be a core. There is a partition $\{A_1,\dots A_m, B_1,\dots,B_n, C\}$ of $S$ having the following properties.*

(1) *$C$ is the set of vertices having a loop ($C$ may be empty).*
(2) *The subgraph of $K(p)$ induced by $S \setminus C$ consists of two complete multipartite components with partitions $\{A_1,\dots,A_m\}$ and $\{B_1,\dots,B_n\}$.*
(3) *If some vertex of $A_i$ (respectively $B_j$) is joined to $t \in C$, then all vertices of $A_i$ (respectively $B_j$) are joined to $t$.*

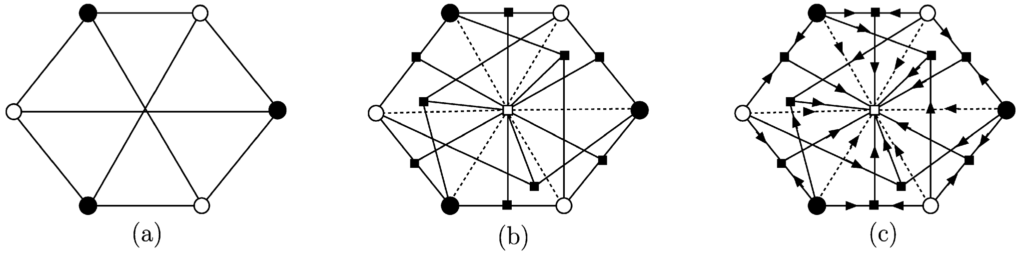



**Fig. 9.** (a) $K_{3,3}$, (b) the complex $\Delta^p$, and (c) an orientation of $\Delta^p$.

**Proof.** Let $K_1$ and $K_2$ be proper nonbipartite components of $K(p)$. Let $A_1$ and $A_2$ be maximal stable sets of $K_1$ and $K_2$, respectively. Then $A := A_1 \cup A_2$ is a maximal stable set of $K(p)$. By Lemma 3.5, $p' := p + \epsilon(-\chi_A + \chi_{N(A)})$ belongs to $T_\mu$ for small $\epsilon > 0$. In particular $K(p')$ has exactly two *complete* bipartite components by (3.3) and Lemma 3.8(3)–(4). From this, we easily see the existence of the partition above. □

The subpartition $(A_1, \ldots A_m; B_1, \ldots, B_n)$ is called the *type of p*. The (proper) component containing $\{A_i\}$ is called the *A-component*, and the (proper) component containing $\{B_j\}$ is called the *B-component*. By (3.2) and Lemma 3.8, all edges adjacent to $p$ are $l_\infty$-edges. Such an $l_\infty$-edge is given explicitly as follows. Since each $A_i$ is maximal stable in the $A$-component, by Lemma 3.5, a point $p' := p + \epsilon(-\chi_{A_i} + \chi_{N(A_i)})$ belongs to $T_\mu$ for small $\epsilon > 0$. Then $K(p')$ consists of the $B$-component of $K(p)$, one complete bipartite component with bipartition $\{A_i, N(A_i)\}$, and the (possibly empty) loop-component. Therefore $p'$ lies on an $l_\infty$-edge adjacent to $p$. Conversely, any edge adjacent to $p$ is given in this way. Motivated by this fact, we denote the edges adjacent to $p$ with directions $-\chi_{A_i} + \chi_{N(A_i)}$ and $-\chi_{B_i} + \chi_{N(B_i)}$ by $e(p, A_i)$ and $e(p, B_j)$, respectively. Moreover, we easily see, by perturbing $p$ as above, that $e(p, A_i)$ and $e(p, B_j)$ belong to a common 2-face, and that $e(p, A_i)$ and $e(p, A_j)$ do not belong to a common 2-face if $i \neq j$. Therefore, the local structure around a core $p$ is given as follows.

**Corollary 5.7.** *Let $p$ be a core of type $(A_1, \ldots, A_m; B_1, \ldots, B_n)$. Then we have the following.*

(1) *$e$ is an edge adjacent to $p$ if and only if $e$ is $e(p, A_i)$ or $e(p, B_j)$ for some $i, j$.*
(2) *Two edges $e', e''$ adjacent to $p$ belong to the common 2-face if and only if $\{e', e''\}$ coincides with $\{e(p, A_i), e(p, B_j)\}$ for some $i, j$.*

Let $\Delta$ be the 1/2-uniform $l_1$-grid. For a core $p$, $\Delta^p$ denotes the subcomplex consisting of members of $\Delta$ containing $p$ and their faces, i.e., $\Delta^p$ is the *star* at $p$ of $\Delta$. By the previous corollary, we obtain a combinatorial description of $\Delta^p$ as follows.

**Corollary 5.8.** *Let $p$ be a core of type $(A_1, \ldots, A_m; B_1, \ldots, B_n)$. Then $\Delta^p$ is isomorphic to the join of one point and the subdivision of the complete bipartite graph $K_{n,m}$.*

See Fig. 9 for (a) the complete bipartite graph $K_{3,3}$ and (b) the complex $\Delta^p$ obtained by taking the join of one point and the subdivision of $K_{3,3}$, where the broken lines represent $l_\infty$-edges.

A core $p$ is called *odd* if $p$ is not in $\bigcup_{U \subseteq S} T_{\mu,U} \cap A_{\mu,U}$. Let $\{p_i\}_{i \in I}$ be the set of odd cores. The proof of Theorem 5.2 is completed by showing that the set of odd cores $\{p_i\}_{i \in I}$ has the following property.

(∗3) For a 2-face $F$ in $T_{\mu,U}$, (the closure of) the set $F \setminus \bigcup_{i \in I} |\Delta^{p_i}|$ is also an $l_\infty$-octagon (by the projection to $\mathbf{R}^2$) and is *integral* in the affine lattice $A_{\mu,U}$, where $|\Delta^{p_i}|$ is the union of faces of $\Delta^{p_i}$.

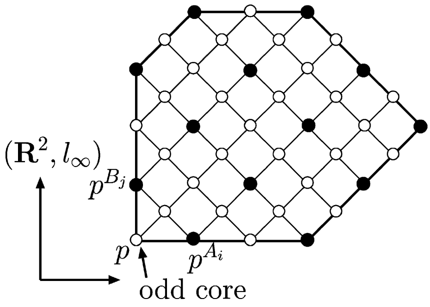



**Fig. 10.** A 2-face with an odd core.

Namely we can remove $|\Delta^{p_i}|$ from $T_\mu$ to make the resulting polyhedral set, which is also a complex of $l_1$-octagons, have the integral uniform $l_1$-grid $\Delta^*$. Apply the orbit splitting to each orbit of $\Delta^*$ and orient it as in Fig. 5. Moreover, $\Delta^{p_i}$ itself is orientable, and can be oriented as in Fig. 9(c), i.e., orient the graph of $\Delta^{p_i}$ so that $p_i$ is the unique sink and vertices adjacent to $p_i$ by $l_\infty$-edges are sources. Restore each $\Delta^{p_i}$ to the original position. Then we obtain the original 1/2-uniform $l_1$-grid $\Delta$ together with an admissible orientation. Thus we can conclude that the 1/2-uniform $l_1$-grid $\Delta$ is orientable. See Fig. 10, where the black and white points are grid-points of the 1/2-uniform $l_1$-grid, and the black points are elements of $A_{\mu,U}$. The property (∗3) can be immediately seen from the following lemma.

**Lemma 5.9.** *Let $p$ be an odd core of type $(A_1, \ldots, A_m; B_1, \ldots, B_n)$, let $F \subseteq T_{\mu,U}$ be the unique 2-face containing $e(p, A_i)$ and $e(p, B_j)$, and let $p^{A_i}$ and $p^{B_j}$ be the grid-points in $\Delta$ adjacent to $p$ by $e(p, A_i)$ and $e(p, B_j)$, respectively. Then both $p^{A_i}$ and $p^{B_j}$ belong to $A_{\mu,U} \cap T_{\mu,U}$.*

**Proof.** Note that $p^{A_i}$ and $p^{B_j}$ are given as

$$p^{A_i} = p + (-\chi_{A_i} + \chi_{N(A_i)}), \qquad p^{B_j} = p + (-\chi_{B_j} + \chi_{N(B_j)}).$$

Let $A$ and $B$ be the sets of vertices of the $A$-component and the $B$-component of $K(p)$, respectively. Let $q \in T_{\mu,U}$ be a normal extreme point. Then, by the same argument as in the proof of Lemma 5.3, $(p-q)(s) + (p-q)(t) \in 2\mathbf{Z}$ holds for $s, t \in A \setminus U$ or $s, t \in B \setminus U$. By Lemma 5.5 and the assumption that $p$ is odd, we have $p \notin A_{\mu,U}$ and therefore $(p-q)(s) + (p-q)(t) \in 1 + 2\mathbf{Z}$ holds for $s \in A \setminus U$ and $t \in B \setminus U$. From this fact, $A_i \cup N(A_i) = A \setminus U$, and $B_j \cup N(B_j) = B \setminus U$, we can conclude $p^{A_i}, p^{B_j} \in A_{\mu,U} \cap T_{\mu,U}$. □

Finally we verify Lemma 5.4 and complete the proof of Theorem 5.2.

**Proof of Lemma 5.4.** Take an arbitrary $t \in S \setminus U$ ($S = U$ implies $\mu = 0$). Take an extreme point $p$ in $T_{\mu,U}$ with $p(t)$ minimum. If $p$ is normal, then we are done. Suppose that $p$ is a core of type $(A_1, \ldots, A_m; B_1, \ldots, B_n)$. Suppose that $K(p)$ has the loop-component of vertex set $U' \supseteq U$. Then every face containing $p$ must belong to $T_{\mu,U'}$, and thus $U' = U$. We may assume $t \in A_i$. The extreme point $p'$ incident to $p$ by edge $e(p, A_i)$ belongs to $T_{\mu,U}$ and $p'(t) < p(t)$. This is a contradiction to the choice of $p$.

Therefore we may assume that the $A$-component of $K(p)$ has vertex set $U$. Then there is a small perturbation vector $v \in \mathbf{R}^S$ with $v|_{\bigcup_j B_j} = 0$ and $p + v \in T_\mu$ such that $K(p+v)$ has the loop-component of vertex set $U$ (and thus $p + v \in T_{\mu,U}$). Take an arbitrary vertex $s$ in the $B$-component and take $B_j$ with $s \in B_j$. Consider the extreme point $p'$ incident to $p$ by edge $e(p, B_j)$. Since the $A$-component is invariant on the half-open segment $[p, p')$, perturbing a point $p'' \in [p, p')$ by $v$ yields the loop-component of vertex set $U$ in $K(p'' + v)$. Therefore $e(p, B_j)$ is in $T_{\mu,U}$ and thus so is $p'$. If $p'$ is normal, then we are done. Suppose that $p'$ is a core. Then $K(p')$ still has the $A$-component of $K(p)$. Set $p \leftarrow p'$ and repeat the same process. In this process, $p(s)$ strictly decreases. Suppose that

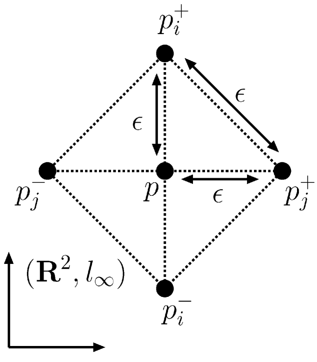



**Fig. 11.** The points $p_i^{\pm}$ in the proof of Lemma 6.1.

$p(s)$ becomes zero. Then $s$ is incident to $U$ in $K(p)$ since vertices having a loop are pairwise adjacent. This implies that $K(p)$ has exactly one proper component and thus $p$ is normal. Therefore, we can find a normal extreme point in $T_{\mu,U}$ in this procedure. $\square$

**Remark 5.10.** Suppose that $\mu$ is a metric. One can see that $T_{\mu,U} = \emptyset$ for all nonempty $U$. Then the argument in this section becomes considerably simpler. This idea is used in [13].

## 6. Proof of unbounded fractionality

The goal of this section is to prove (2) in Theorem 1.1. Recall that $M^*(G; S, \mu)$ or (1.1) is a linear optimization over the polyhedron

$$\mathcal{P}_{\mu,V} = \{d:\ \text{metric on } V \mid d|_S \geqslant \mu\} + \mathbf{R}_+^{E_V}.$$

So it suffices to show that if $\dim T_\mu \geqslant 3$, then there is no integer $k$ such that $\mathcal{P}_{\mu,V}$ is $1/k$-integral for every set $V$ containing $S$. Note that all extreme points of $\mathcal{P}_{\mu,V}$ lie on the set of minimal element of $\mathcal{P}_{\mu,V}$. Motivated by this fact, we call a metric $d$ on $S$ a *minimal dominant* of $\mu$ if $d$ is a minimal element in $\mathcal{P}_{\mu,S}$. First we show the following.

**Lemma 6.1.** *For a distance $\mu$ with* $\dim T_\mu \geqslant k$*, there exists a minimal dominant $\tilde{\mu}$ of $\mu$ such that* $\dim T_{\tilde{\mu}} \geqslant k$.

**Proof.** Let $F$ be a $k$-dimensional face of $T_\mu$ and $p$ a point of the relative interior of $F$. By Proposition 3.7, $K(p)$ has $k$ bipartite components with bipartitions $\{A_1, B_1\}, \{A_2, B_2\}, \ldots, \{A_k, B_k\}$. For small $\epsilon > 0$, points $p_i^{\pm} := p + \epsilon(\mp\chi_{A_i} \pm \chi_{B_i})$ $(i = 1, \ldots, k)$ are in $F$ by Lemma 3.5; see Fig. 11 that illustrates the configuration $p_i^{\pm}$ in the local coordinate (3.4) of $F$. We take an edge $u_i^+ u_i^- \in E(p)$ with $u_i^+ \in A_i$, $u_i^- \in B_i$ for each $i = 1, \ldots, k$. By construction of $p_i^{\pm}$ and Lemma 3.6(2), we have

$$\begin{aligned}\mu(u_i^+, u_i^-) &= p(u_i^+) + p(u_i^-) = p_i^+(u_i^+) + 2\epsilon + p_i^-(u_i^-) \\ &= \|T_{\mu,u_i^+}, p_i^+\| + \|p_i^+, p_i^-\| + \|p_i^-, T_{\mu,u_i^-}\|. \end{aligned} \tag{6.1}$$

We take $q_i^{\pm} \in T_{\mu,u_i^{\pm}}$ with $\|q_i^{\pm}, p_i^{\pm}\| = \|T_{\mu,u_i^{\pm}}, p_i^{\pm}\|$. Then, $\|q_i^+, q_i^-\| = \mu(u_i^+, u_i^-)$ must hold by (6.1) and Lemma 3.6(1). We define a metric $\mu'$ on $2k$-set $U := \{u_i^+, u_i^-\}_{i=1}^k$ by $\mu'(u_i^+, u_j^{\pm}) := \|q_i^+, q_j^{\pm}\|$ $(\geqslant \|T_{\mu,u_i^+}, T_{\mu,u_j^{\pm}}\| = \mu(u_i^+, u_j^{\pm}))$. Then $\mu' \geqslant \mu|_U$ with $\mu'(u_i^+, u_i^-) = \mu(u_i^+, u_i^-)$. Consider $T_{\mu'} \subseteq \mathbf{R}^U$. Then $p|_U$, the restriction of $p$ to $U$, has the following property.

(∗) $p|_U \in T_{\mu'}$ and the graph $K_{\mu'}(p|_U)$ is exactly $k$-matching $\{u_i^+ u_i^-\}_{i=1}^k$.

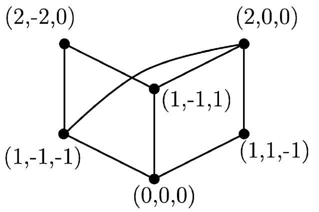



**Fig. 12.** $K_{3,3}^-$.

Indeed, $p(u_i^+)+p(u_i^-)=\mu'(u_i^+,u_i^-)$ is obvious by construction. We show $p(u_i^+)+p(u_j^\pm)>\mu'(u_i^+,u_j^\pm)$ if $i\neq j$. By constructions of $p_i^\pm$ and $q_j^\pm$, we have

$$\begin{aligned}\mu'(u_i^+,u_j^\pm) := \|q_i^+,q_j^\pm\| &\leqslant \|q_i^+,p_i^+\| + \|p_i^+,p_j^\pm\| + \|p_j^\pm,q_j^\pm\| \\ &= \|T_{\mu,u_i^+},p_i^+\| + \|p_i^+,p_j^\pm\| + \|p_j^\pm,T_{\mu,u_j^\pm}\| \\ &= \|T_{\mu,u_i^+},p_i^+\| + \epsilon + \|p_j^\pm,T_{\mu,u_j^\pm}\| \\ &< \|T_{\mu,u_i^+},p_i^+\| + 2\epsilon + \|p_j^\pm,T_{\mu,u_j^\pm}\| \\ &= p_i^+(u_i^+) + 2\epsilon + p_j^\pm(u_j^\pm) = p(u_i^+) + p(u_j^\pm).\end{aligned}$$

Therefore, $\dim T_{\mu'}\geqslant k$ by Proposition 3.7. Let $\mu''$ be a minimal dominant of $\mu|_U$ on $U$ with $\mu''\leqslant\mu'$. By $\mu(u_i^+,u_i^-)=\mu'(u_i^+,u_i^-)=\mu''(u_i^+,u_i^-)$, again $p|_U\in T_{\mu''}$, and $K_{\mu''}(p|_U)$ is still $k$-matching $\{u_i^+u_i^-\}_{i=1}^k$. Therefore, $\dim T_{\mu''}\geqslant k$. We can extend $\mu''$ to a minimal dominant $\tilde{\mu}$ of $\mu$ with $\tilde{\mu}|_U=\mu''$. Dress' dimension criterion (see Theorem 7.1 in the next section) implies $\dim T_{\tilde{\mu}}\geqslant k$. □

Second we recall the notion of extreme metrics and extreme extensions. A metric $d$ on a finite set $V$ is called *extreme* if $d$ lies on an extreme ray of the *metric cone*, which is a polyhedral cone in $\mathbf{R}_+^{E_V}$ defined by the triangle inequalities. A metric $(V,d)$ is called an *extension* of a metric $(S,\mu)$ if $S\subseteq V$ and $d|_S=\mu$. An extension $(V,d)$ of $(S,\mu)$ is called *extreme* if $d$ is an extreme point of the polyhedron

$$\{d\colon \text{ metric on } V \mid d|_S=\mu\}+\mathbf{R}_+^{E_V}. \tag{6.2}$$

Recall that a minimal element of (6.2) is called a *tight extension* of $\mu$; see Section 2. We use the following observations.

(∗1) If a metric $d$ on $V$ is a tight extension of a minimal dominant $\tilde{\mu}$ of $\mu$, then $d$ is minimal in $\mathcal{P}_{\mu,V}$.
(∗2) If a metric $d$ on $V$ is extreme in $\mathcal{P}_{\mu,V}$ and a metric $d'$ on $V'$ is an extreme extension of $d$, then $d'$ is extreme in $\mathcal{P}_{\mu,V'}$.

We are ready to prove (2) in Theorem 1.1. Suppose that $\dim T_\mu\geqslant 3$. Then, by the previous lemma, there is a minimal dominant $\tilde{\mu}$ such that $\dim T_{\tilde{\mu}}\geqslant 3$. Therefore $T_{\tilde{\mu}}$ has a 3-dimensional face $F$. Since $(F,l_\infty)$ is isomorphic to a 3-dimensional polytope in $(\mathbf{R}^3,l_\infty)$ by the argument in Section 3, we can take six points $Z$ from $F$ isometric to a dilation of the following configuration $Z_1$ in $(\mathbf{R}^3,l_\infty)$.

$$Z_1=\{(0,0,0),(1,1,-1),(1,-1,1),(1,-1,-1),(2,0,0),(2,-2,0)\}.$$

Then, $(Z_1,l_\infty)$ is extreme. Indeed, it is the graph metric $d_{K_{3,3}^-}$ of $K_{3,3}^-$ (the graph $K_{3,3}$ minus one edge); see Fig. 12. The graph metric $d_{K_{3,3}^-}$ is known to be extreme [21]. By Proposition 2.3(3), the set of points $Z$ corresponds to a tight extension of $\tilde{\mu}$. Therefore, there is a tight extension $(V,d)$ of $(S,\tilde{\mu})$ such that $d$ has $\alpha d_{K_{3,3}^-}$ as a submetric for $\alpha>0$. By (∗1), $d$ is minimal in $\mathcal{P}_{\mu,V}$. Then we can decompose $d$ into a convex combination of extreme points of $\mathcal{P}_{\mu,V}$. By extremality of $d_{K_{3,3}^-}$, there is a summand $d'$ in the convex combination such that $d'$ has $\alpha' d_{K_{3,3}^-}$ as a submetric for a positive $\alpha' > 0$. In [21, Section 3], Karzanov showed the following.

(∗3) If a metric $d'$ has $\alpha' d_{K_{3,3}^-}$ as a submetric for a positive $\alpha' > 0$, then there is an extreme extension $d''$ of $d'$ that has $\frac{1}{2}\alpha' d_{K_{3,3}^-}$ as a submetric.

Therefore, by (∗2) and (∗3), we obtain an infinite sequence of extreme points of $\{\mathcal{P}_{\mu,V}\}_{V \supseteq S}$ such that the fractionality strictly increases.

## 7. 0–1 Distances

In this section, we verify that condition (P) in Theorem 1.3 is indeed equivalent to the 2-dimensionality of the tight span of a 0–1 distance, and give an explicit combinatorial construction of the tight span of a 2-dimensional 0–1 distance. Here a distance $\mu$ is said to be *k-dimensional* if $\dim T_\mu \leqslant k$. First we present Dress' criterion [9, Theorem 9] of the dimension of tight spans. As is indicated by [9, Remark 5.4, p. 370], his criterion holds for nonmetric distances; also see [11, Appendix] for an elementary proof based on linear programming.

**Theorem 7.1.** *(See [9].) For a distance $\mu$ on a finite set $S$ and a positive integer $n$, the following two conditions are equivalent.*

(a) $\dim T_\mu \geqslant n$.
(b) *There exists a $2n$-element subset $\{s_1, s_{-1}, s_2, s_{-2}, \dots, s_n, s_{-n}\} \subseteq S$ such that*

$$\sum_{i \in \{\pm 1, \pm 2, \dots \pm n\}} \mu(s_i, s_{-i}) > \sum_{i \in \{\pm 1, \pm 2, \dots \pm n\}} \mu(s_i, s_{\sigma(i)})$$

*holds for any permutation $\sigma$ of $\{\pm 1, \pm 2, \dots \pm n\}$ with $\sigma(i) \neq -i$ for any $i \in \{\pm 1, \pm 2, \dots \pm n\}$.*

Specializing Theorem 7.1 to 0–1 distance $\mu$ and $n = 3$, we have the following. Recall the definition of commodity graph $H_\mu = (S, F_\mu)$ defined as $F_\mu = \{st \mid s, t \in S,\ \mu(s,t) = 1\}$.

**Proposition 7.2.** *For a 0–1 distance $\mu$ on $S$ whose $H_\mu$ has no isolated vertex, the following conditions are equivalent.*

(a) $\dim T_\mu \leqslant 2$.
(b) *There is no six-element subset $U \subseteq S$ such that the induced subgraph $H_\mu(U)$ of $H_\mu$ by $U$ has a unique perfect matching and has no vertex-disjoint two triangles.*
(P) *For any three distinct pairwise intersecting maximal stable sets $A, B, C$ of $H_\mu$, we have $A \cap B = B \cap C = C \cap A$.*

**Proof.** First note that the condition (b) in Theorem 7.1 is equivalent to the following condition.

(∗) There exist a $2n$-element subset $U \subseteq S$ and a perfect matching $M \subseteq E_U$ such that $M$ attains the unique maximum of

$$\max_{M', C_1, \dots, C_m} \sum_{e \in M'} \mu(e) + \frac{1}{2} \sum_{k=1}^{m} \sum_{e \in C_i} \mu(e),$$

where the maximum is taken over pairwise vertex-disjoint matchings $M'$ and odd cycles (possibly including loops) $C_1, \dots, C_m (m \geqslant 0)$.

This immediately follows from the fact that every permutation is decomposed into cyclic permutations.

Then it is easy to see that the condition (b) is equivalent to the negation of the condition $(*)$ for 0–1 distances and $n = 3$. Although the equivalence between (b) and (P) can be seen from [18, Statement 4.2], we show (b) $\Rightarrow$ (P) and (P) $\Rightarrow$ (a) for completeness.

(b) $\Rightarrow$ (P). Suppose that there are three distinct pairwise intersecting maximal stable sets $A, B, C$ of $H_\mu$ such that $(B \cap C) \setminus A$ is nonempty. Take $s \in (B \cap C) \setminus A$. Since $A$ is a *maximal* stable set, there is $s' \in A \setminus (B \cup C)$ with $ss' \in F_\mu$.

**Case 1.** Suppose that $A \cap B \cap C$ is empty. Then both $(A \cap C) \setminus B$ and $(A \cap B) \setminus C$ are nonempty. Take $t \in (A \cap C) \setminus B$ and $u \in (A \cap B) \setminus C$. There are $t' \in B \setminus (A \cup C)$, $u' \in C \setminus (A \cup B)$ with $tt', uu' \in F_\mu$. Let $U = \{s, s', t, t', u, u'\}$. Then the induced subgraph $H_\mu(U)$ consists of three edges $\{ss', tt', uu'\}$ and a subset of $\{s't', t'u', s'u'\}$. Thus $H_\mu(U)$ has a unique perfect matching $\{ss', tt', uu'\}$ and has no vertex-disjoint two triangles.

**Case 2.** Suppose that $A \cap B \cap C$ is not empty. Take $t \in B \setminus C$. Then there is $t' \in C \setminus B$ with $tt' \in F_\mu$. Take $u \in A \cap B \cap C$. By the condition that $H_\mu$ has no isolated vertex, there is $u' \in S \setminus (A \cup B \cup C)$ with $uu' \in F_\mu$. Let $U = \{s, s', t, t', u, u'\}$. Consider the induced subgraph $H_\mu(U)$; it has a perfect matching $\{ss', tt', uu'\}$. In $H_\mu(U)$, a vertex $u$ is covered by edge $uu'$ only. Therefore, $H_\mu(U)$ does not have vertex-disjoint two triangles. Moreover, any perfect matching must use edge $uu'$. A vertex $s$ is not adjacent to $t$ and $t'$. Therefore $\{ss', tt', uu'\}$ is a unique perfect matching of $H_\mu(U)$.

(P) $\Rightarrow$ (a). Suppose that $\dim T_\mu \geqslant 3$. Then there is $p \in T_\mu$ such that $K(p)$ has three bipartite components by Proposition 3.7. We can take three edges $s_1s'_1, s_2s'_2, s_3s'_3 \in E(p)$ from different bipartite components. Since $\mu$ is a 0–1 distance, we have $s_ks'_k \in F_\mu$ for $k = 1, 2, 3$. By $p(s_k) + p(s'_k) = 1$, we may assume that $p(s_k) \geqslant 1/2 \geqslant p(s'_k)$ and $p(s_1) \geqslant p(s_2) \geqslant p(s_3)$. Consequently we have $p(s'_1) \leqslant p(s'_2) \leqslant p(s'_3)$. Since $p(s) + p(t) \leqslant 1$ and $st \notin E(p)$ imply $st \notin F_\mu$, three sets $\{s'_1, s'_2, s'_3\}$, $\{s'_1, s'_2, s_3\}$, and $\{s'_1, s_2\}$ are pairwise intersecting stable sets of $H_\mu(U)$ violating condition (P). Then we can extend this triple to pairwise intersecting maximal stable sets of $H_\mu$ violating condition (P). $\square$

Finally, we give an explicit combinatorial construction of $T_\mu$ for a 2-dimensional 0–1 distance $\mu$. Let $\mathcal{A}_\mu$ be the set of maximal stable sets of $H_\mu$ and $\mathcal{K}_\mu$ the set of maximal cliques of the intersection graph of $\mathcal{A}_\mu$.

**Proposition 7.3.** *Let $\mu$ be a 2-dimensional 0–1 distance on $S$ whose $H_\mu$ has no isolated vertices. Let $\{p_A\}_{A \in \mathcal{A}_\mu}$, $\{p_K\}_{K \in \mathcal{K}_\mu}$, and $p_O$ be the points defined as*

$$p_A = \chi_{S \setminus A} \quad (A \in \mathcal{A}_\mu),$$
$$p_K = (1/2)\chi_{\bigcup_{A \in K} A \setminus \bigcap_{A \in K} A} + \chi_{S \setminus \bigcup_{A \in K} A} \quad (K \in \mathcal{K}_\mu),$$
$$p_O = (1/2)\chi_S.$$

*Then we have*

$$T_\mu = \bigcup \{\text{convex hull of } \{p_A, p_K, p_O\} \mid A \in K \in \mathcal{K}_\mu\}. \tag{7.1}$$

**Proof.** $(\supseteq)$ in (7.1) is straightforward. We show $(\subseteq)$. Take a generic point $p \in T_\mu$ in the relative interior of a maximal face of $T_\mu$. By the facts that $0 \leqslant p \leqslant 1$ and that $H_\mu$ has no isolated vertices, the graph $K(p)$ has no loop-component. By the maximality and Lemma 3.8, $K(p)$ is one complete bipartite graph or the (vertex-disjoint) sum of two complete bipartite graphs $K_1, K_2$.

For the first case, let $\{A, B\}$ be the bipartition of $K(p)$. Then we have $p(s) = \alpha$, $p(t) = \beta$ for $s \in A$, $t \in B$ and $\alpha, \beta$ with $\alpha + \beta = 1$ and $0 < \alpha < 1/2 < \beta < 1$ by genericity. Then $A$ is a maximal stable set of $H_\mu$. Therefore $p = (\beta - \alpha)p_A + 2\alpha p_O$.

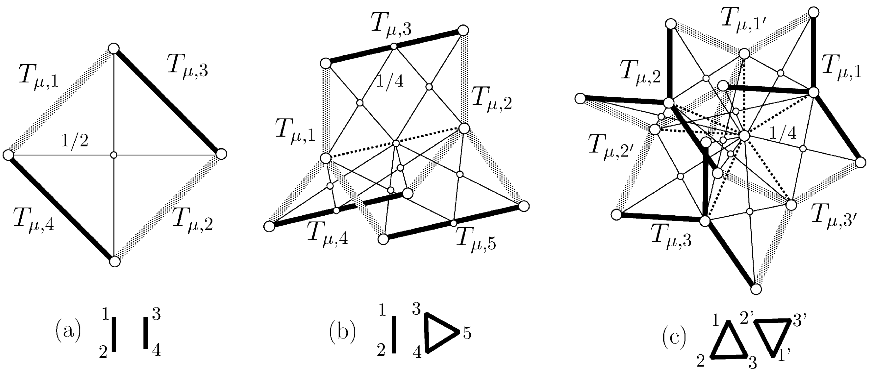



**Fig. 13.** Tight spans for 0–1 distances.

For the second case, let $\{A_i, B_i\}$ be the bipartition of $K_i$ for $i = 1, 2$. Similarly, $(p(s), p(t), p(u), p(v)) = (\alpha_1, \beta_1, \alpha_2, \beta_2)$ for $(s,t,u,v) \in A_1 \times B_1 \times A_2 \times B_2$ and $1 < \alpha_1 < \alpha_2 < 1/2 < \beta_2 < \beta_1 < 1$ with $\alpha_1 + \beta_1 = \alpha_2 + \beta_2 = 1$. Then $A_1 \cup A_2$ is a maximal stable set of $H_\mu$, and there is no edge in $H_\mu$ between $A_1$ and $B_2$. By condition (P), there is a maximal set $K$ of pairwise intersecting maximal stable sets such that $A := A_1 \cup A_2 \in K$, and the union and the intersection of members in $K$ are $S \setminus B_1$ and $A_1$, respectively. By calculation, we have $p = 2\alpha_1 p_O + (\alpha_1 + \beta_1 - 2\alpha_2)p_A + (2\alpha_2 - 2\alpha_1)p_K$. □

Namely, $T_\mu$ is the complex of the join of the point $p_O$ and the clique-vertex incidence graph of $\mathcal{A}_\mu$ and $\mathcal{K}_\mu$. Fig. 13 illustrates the tight spans with their minimal orientable $l_1$-grids for commodity graphs (a) $H_\mu = K_2 + K_2$, (b) $H_\mu = K_2 + K_3$, and (c) $H_\mu = K_3 + K_3$. Karzanov's original proof [18] of Theorem 1.3 is based on the concept of *frameworks* of graph $G = (V, E, c)$ and commodity graph $H_\mu$, which is a certain subpartition of $V$. He has shown that $M^*(G; S, \mu)$ is equivalent to a discrete optimization over all possible frameworks. In our setting, frameworks can be interpreted as feasible configurations to (TSD($Z$)) of the 1/4-uniform $l_1$-grid.

## 8. Concluding remarks

A natural question is: does there exist a duality relation similar to tight-span dual in weighted maximum *directed* multiflow problems? The forthcoming paper [14] answers this question. For a *not necessarily symmetric* distance $\gamma : S \times S \to \mathbf{R}_+$ on $S$, define two polyhedral sets $P_\gamma$ and $T_\gamma$ by

$$P_\gamma := \left\{(p,q) \in \mathbf{R}_+^{S \times S} \;\middle|\; p(s) + q(t) \geqslant \gamma(s,t) \;\; (s,t \in S)\right\},$$

$$T_\gamma := \text{the set of minimal elements of } P_\gamma.$$

Then $T_\gamma$ plays the same role as a tight span. Interestingly, this space $T_\gamma$ is closely related to the *tropical polytopes* introduced by Develin and Sturmfels [7].

Apart from the fractionality issues, the design of combinatorial or practical algorithms specialized to general multiflow problems is still a challenging problem. The tight-span dual problem and the geometry of $T_\mu$ explored in this paper might give a basis against this challenge.

## Acknowledgments

The author thanks Shungo Koichi for several critical remarks, Shuji Kijima for a discussion concerning Section 5, and Kazuo Murota, Satoru Fujishige, and Satoru Iwata for improving the presentation of this paper, and also thanks the referees for careful reading, numerous linguistic corrections, and helpful suggestions. This work is supported by a Grant-in-Aid for Scientific Research from the Ministry of Education, Culture, Sports, Science and Technology of Japan.

## References


[1] H.-J. Bandelt, Hereditary modular graphs, Combinatorica 8 (1988) 149–157.
[2] H.-J. Bandelt, V. Chepoi, A. Karzanov, A characterization of minimizable metrics in the multifacility location problem, European J. Combin. 21 (2000) 715–725.
[3] V. Chepoi, $T_X$-approach to some results on cuts and metrics, Adv. in Appl. Math. 19 (1997) 453–470.
[4] V. Chepoi, Graphs of some $CAT(0)$ complexes, Adv. in Appl. Math. 24 (2000) 125–179.
[5] B.V. Cherkassky, Solution of a problem of multiproduct flows in a network, Ekon. Mat. Metody 13 (1977) 143–151 (in Russian).
[6] M. Chrobak, L.L. Larmore, Generosity helps or an 11-competitive algorithm for three servers, J. Algorithms 16 (1994) 234–263.
[7] M. Develin, B. Sturmfels, Tropical convexity, Doc. Math. 9 (2004) 1–27.
[8] M.M. Deza, M. Laurent, Geometry of Cuts and Metrics, Springer-Verlag, Berlin, 1997.
[9] A.W.M. Dress, Trees, tight extensions of metric spaces, and the cohomological dimension of certain groups: A note on combinatorial properties of metric spaces, Adv. Math. 53 (1984) 321–402.
[10] L.R. Ford, D.R. Fulkerson, Flows in Networks, Princeton Univ. Press, Princeton, 1962.
[11] H. Hirai, Characterization of the distance between subtrees of a tree by the associated tight span, Ann. Comb. 10 (2006) 111–128.
[12] H. Hirai, Tight extensions of distance spaces and the dual fractionality of undirected multiflow problems, RIMS-preprint 1606, 2007.
[13] H. Hirai, Metric packing for $K_3 + K_3$, RIMS-preprint 1608, 2007.
[14] H. Hirai, S. Koichi, Tight spans for directed distances, tropical polytopes, and multicommodity flows in digraphs, in preparation.
[15] T.C. Hu, Multi-commodity network flows, Oper. Res. 11 (1963) 344–360.
[16] M. Iri, On an extension of the maximum-flow minimum-cut theorem to multicommodity flows, J. Oper. Res. Soc. Japan 13 (1970/1971) 129–135.
[17] J.R. Isbell, Six theorems about injective metric spaces, Comment. Math. Helv. 39 (1964) 65–76.
[18] A.V. Karzanov, Polyhedra related to undirected multicommodity flows, Linear Algebra Appl. 114/115 (1989) 293–328.
[19] A.V. Karzanov, Undirected multiflow problems and related topics—Some recent developments and results, in: Proceedings of the International Congress of Mathematician, vol. II, Kyoto, Japan, 1991, pp. 1561–1571.
[20] A.V. Karzanov, Minimum 0-extensions of graph metrics, European J. Combin. 19 (1998) 71–101.
[21] A.V. Karzanov, Metrics with finite sets of primitive extensions, Ann. Comb. 2 (1998) 211–241.
[22] A.V. Karzanov, M.V. Lomonosov, Systems of flows in undirected networks, in: O.I. Larichev (Ed.), Mathematical Programming, Institute for System Studies, Moscow, 1978, pp. 59–66 (in Russian).
[23] M.V. Lomonosov, Combinatorial approaches to multiflow problems, Discrete Appl. Math. 11 (1985), 93 pp.
[24] L. Lovász, On some connectivity properties of Eulerian graphs, Acta Math. Acad. Sci. Hungar. 28 (1976) 129–138.
[25] K. Onaga, O. Kakusho, On feasibility conditions of multicommodity flows in networks, IEEE Trans. Circuit Theory CT-18 (1971) 425–429.
[26] A. Schrijver, Combinatorial Optimization—Polyhedra and Efficiency, Springer-Verlag, Berlin, 2003.
[27] B.C. Tansel, R.L. Francis, T.J. Lowe, Location on networks: A survey. II. Exploiting tree network structure, Manag. Sci. 29 (1983) 498–511.
[28] G.M. Ziegler, Lectures on Polytopes, Springer-Verlag, Berlin, 1995.