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Sums with Curves

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Let $C \subseteq \mathbb{R}^2$ be a fixed compact set and let $E \subseteq \mathbb{R}^2$ be a Borel set. We form the sum $E + C = \{e + c: e \in E, c \in C\}$, which is a measurable set, and inquire as to the behavior of the two-dimensional Lebesgue measure $m(E + C)$ as $m(E) \to 0$. If C is "thin" then we expect that $m(E + C)$ can be made small by choosing a suitable E. It is a consequence of the theorem below that if C is a curve (continuous image of $[0, 1]$) which is not a line segment, then $m(E + C)$ cannot, however, be small when compared to $m(E)$.

THEOREM. If C is a curve and E is a Borel set, then

$$m(E + C) \geqslant [m(C - C) \cdot m(E)]^{1/2}/13\sqrt{2}.$$

(If $E_1, E_2 \subseteq \mathbb{R}^2$, then $E_1 - E_2$ will denote the algebraic difference $\{e_1 - e_2 : e_1 \in E_1, e_2 \in E_2\}$ while $E_1 \sim E_2$ will denote the set-theoretic difference $\{e \in E_1 : e \notin E_2\}$.) It is an amusing exercise to show rigorously that if C is not a line segment, then $m(C - C) > 0$. On the other hand, it is clear that if C is a line segment then there exist sets E with $m(E)$ arbitrarily small and with $m(E + C)$ nearly a constant multiple of $m(E)$. The number $13\sqrt{2}$ in the statement of the theorem could be improved by a more careful analysis.

The proof of our theorem is given in Sections 1–3 below. The reader who would see the idea of the proof but wishes to avoid a few technical details can omit Sections 2 and 3.

1.

Fix a positive integer N and consider the lattice of all points $(j/N, k/N)$ where j and k are integers. Suppose that C_N is a curve formed by connecting

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lattice points with horizontal or vertical line segments. We will prove that

$$m(E+C_N) \geqslant [m(C_N-C_N)\cdot m(E)]^{1/2}/\sqrt{2} \tag{1}$$

for any Borel set E.

The curve C_N is the union of a finite number of horizontal or vertical line segments each having length $1/N$. Let S_h (resp. S_v) be the union of the corresponding collection of open horizontal (resp. vertical) line segments. Let $a=[m(S_h-S_v)]^{1/2}$. Then

$$a \geqslant [m(C_N-C_N)]^{1/2}/\sqrt{2}.$$

Let $I_1,..., I_K$ be the intervals comprising S_h, and put

$$A_1=I_1-S_v; \qquad A_k=(I_k-S_v)\sim \bigcup_{j=1}^{k-1} A_j, \qquad k=2,..., K.$$

Then each A_k is of the form $I_k-S(k)$ where $S(k)\subseteq S_v$ is the union of a certain collection of open vertical line segments each having length $1/N$. Let n_k be the cardinality of this collection. Then

$$m(A_k)=n_k/N^2, \qquad \sum_{k=1}^{K} n_k/N^2=a^2.$$

Let λ be the measure on S_h which on the interval I_k is n_k/N times arclength. The total variation $\|\lambda\|$ of λ is a^2. Now if $m(E+S_h)\geqslant a[m(E)]^{1/2}$, then (1) holds. So assume that $m(E+S_h)\leqslant a[m(E)]^{1/2}$. The convolution $\chi_E^*\lambda(x)=\int \chi_E(x-y)\,d\lambda(y)$ of the characteristic function of E with the measure λ is supported on $E+S_h$. Thus , by Fubini's theorem,

$$a[m(E)]^{1/2} \leqslant \frac{\|\lambda\|\cdot m(E)}{m(E+S_h)}=\frac{1}{m(E+S_h)}\int_{\mathbb{R}^2}\chi_E^*\lambda\,dm$$

$$=\frac{1}{m(E+S_h)}\int_{E+S_h}\chi_E^*\lambda\,dm.$$

From this it follows that there is some x such that

$$a[m(E)]^{1/2}\leqslant \chi_E^*\lambda(x)=\lambda(x-E).$$

Writing $m_1(J)$ for the linear measure of a subset J of an interval I_k, we see from the inequality above that

$$a[m(E)]^{1/2}\leqslant \sum_{k=1}^{K} m_1((x-E)\cap I_k)\cdot\frac{n_k}{N}.$$

Now if $J \subseteq I_k$, then $m(J - S(k)) = m_1(J) \cdot n_k/N$. Since the sets $I_k - S(k)$ are pairwise disjoint, it follows that

$$a[m(E)]^{1/2} \leqslant m(x - E - S_v).$$

Thus (1) holds.

2.

Let C be a fixed curve and let C_N be a polygonal curve as in Section 1 which approximates C so well that $|C(t) - C_N(t)| < 3/N$ for every $t \in [0, 1]$. (Here we identify a curve with a function which defines it.) Such a C_N is easily constructed. We will show that

$$m(C_N - C_N) \geqslant m(C - C)/169. \tag{2}$$

Fix $c_1, c_2 \in C$ and let $d_1, d_2 \in C_N$ be such that if $e_i = c_i - d_i (i = 1, 2)$ then $|e_i| < 3/N$. Now $C_N - C_N$ is the union of a certain set of closed squares whose sides are parallel to the coordinate axes in $\mathbb{R}^2$ and have length $1/N$. Let S be one of these squares which contains $d_1 - d_2$. Since

$$c_1 - c_2 = (d_1 - d_2) + (e_1 - e_2), \qquad |e_1 - e_2| < 6/N,$$

it follows that

$$c_1 - c_2 \in \bigcup_{-6 \leqslant i, j \leqslant 6} \left(S + \left(\frac{i}{N}, \frac{j}{N}\right)\right).$$

Now (2) follows from

$$C - C \subseteq \bigcup_{-6 \leqslant i, j \leqslant 6} \left(C_N - C_N + \left(\frac{i}{N}, \frac{j}{N}\right)\right).$$

3.

Let E be a Borel set. We will show that

$$m(E + C) \geqslant [m(C - C) \cdot m(E)]^{1/2}/13\sqrt{2}. \tag{3}$$

We begin with some reductions. First, by approximating a Borel set from inside with compact sets, it is enough to prove (3) when E is compact. Now if E is compact and if (3) is true with E replaced by each set $E_j = \{x\colon \mathrm{dist}(x, E) < 1/j\} (j = 1, 2, \ldots)$, then (3) is true for E. Since each E_j is open, it is enough to establish (3) for open E. Finally, (3) is true for every open set E if it is true whenever E is the union of a finite number of disjoint and congruent open squares. So it is enough to establish (3) for such E.

Write $E = \bigcup_1^M x_i + S$, where S is an open square and the translates $x_i + S$ are pairwise disjoint. Let $D(0, \delta)$ denote the open disk with center at the origin and radius $\delta > 0$. Fix a small number $\varepsilon > 0$ and let S' be any Borel subset of S such that $m(S \sim S') < \varepsilon/M$ and $S' + D(0, \delta) \subseteq S$ for some $\delta > 0$. Let N be so large that $3/N < \delta$, and let C_N be as in Section 2. Then for $t \in [0, 1]$,

$$\begin{aligned} C_N(t) + S' &= C(t) + S' + C_N(t) - C(t) \\ &\subseteq C(t) + S' + D(0, \delta) \subseteq C(t) + S. \end{aligned}$$

Write E' for $\bigcup_1^M x_i + S'$. Then $C_N + E' \subseteq C + E$. From this and Section 1 it follows that

$$m(E + C) \geqslant [m(C_N - C_N) \cdot m(E')]^{1/2}/\sqrt{2}.$$

By Section 2 and the fact that $m(S \sim S') < \varepsilon/M$ we have

$$m(E + C) \geqslant [m(C - C) \cdot (m(E) - \varepsilon)]^{1/2}/13\sqrt{2}.$$

Since ε was arbitrary, (3) follows.