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Non-homeomorphic Nilmanifolds with Identical Unitary Spectrum

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We show in every three-step nilpotent Lie group G with all coadjoint orbits flat the existence of a pair of discrete cocompact subgroups Γ_1 and Γ_2 such that $\Gamma_1\backslash G$ and $\Gamma_2\backslash G$ have the same unitary spectrum but Γ_1 is not isomorphic to Γ_2. This result generalizes an example of Gornet. We mention without giving a proof a result that would enunciate that for a large category of subgroups of the four dimensional three-step chain group with non-flat coadjoint orbits, this phenomenon of non-isomorphic representation equivalence cannot occur. We also prove some short structural results for three-step nilpotent Lie groups with one-dimensional center.

1. INTRODUCTION

Let Γ be a discrete cocompact subgroup of a nilpotent Lie group G. The quasi-regular representation U_Γ of G on $L^2(\Gamma\backslash G)$ acts on G by right translations; that is for $f\in L^2(\Gamma\backslash G)$, $x\in G$, $y\in \Gamma\backslash G$, $(U_\Gamma(x)\,f)(y)=f(yx)$. Two discrete cocompact subgroups Γ_1 and Γ_2 are said to be *representation equivalent* if the corresponding quasi-regular representations U_{Γ_1} and U_{Γ_2} are unitarily equivalent.

This paper arose from efforts to investigate an example which appeared in [G] of a phenomenon which has been considered to be rare: namely, the existence of two discrete cocompact subgroups Γ_1 and Γ_2 in a Lie group G such that $\Gamma_1\backslash G$ and $\Gamma_2\backslash G$ are representation equivalent but Γ_1 is not isomorphic to Γ_2.

In [G] the first known example of this phenomenon in the class of solvable Lie groups was given. In this example G was a specific three-step nilpotent Lie group and two discrete cocompact subgroups Γ_1 and Γ_2 of G such that Γ_1 is representation equivalent to Γ_2 with Γ_1 non-isomorphic to Γ_2 were presented.

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In the present paper we have been able to generalize this example and prove the following result:

> *Let G be any three-step nilpotent Lie algebra with rational structure constants and all coadjoint orbits flat. Then there exist discrete cocompact subgroups Γ_1 and Γ_2 in G such that Γ_1 is representation equivalent to Γ_2 but Γ_1 is not isomorphic to Γ_2.*

This theorem contains the example in [G] as a special case and demonstrates that this phenomenon occurs surprisingly often. Section 4 contains the proof of this new result.

The author investigated the role of flatness of orbits in this phenomenon of non-isomorphic representation equivalence by considering the lowest dimensional example of a nilpotent Lie group with non-flat coadjoint orbits. The author has been able to show that for a large category of discrete cocompact subgroups of this group this phenomenon of non-isomorphic representation equivalence cannot occur. Section 5 contains the statement of this result.

The author has also proven some short but apparently new structural results for three-step nilpotent Lie groups with one-dimensional center. Section 6 contains these.

Section 2 contains the background material for the work undertaken in Sections 4, 5 and 6. Section 3 contains some remarks about notation and terminology.

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2. BACKGROUND

The book by Corwin and Greenleaf [CG] provides a comprehensive introduction to nilpotent Lie groups and will serve as a reference to this Section. We list here some of the basic facts about the structure of nilpotent Lie groups and their discrete cocompact subgroups and some important representation theoretic results from this text, that we use in our results.

A Lie algebra $\mathfrak{g}$ is called *nilpotent* if its descending central series is finite, where

$$\mathfrak{g}^{(1)} = \mathfrak{g}.$$
$$\mathfrak{g}^{(k+1)} = [\mathfrak{g}, \mathfrak{g}^{(k)}] = R\text{-span}\{[X, Y]: X \in \mathfrak{g},\ Y \in \mathfrak{g}^{(k)}\}$$

defines that series inductively. The function $\exp: \mathfrak{g} \to G$ maps the algebra to the associated connected, simply connected Lie group, which is also nilpotent. In this case, the map exp (called the exponential map) is an analytic diffeomorphism. If log denotes the inverse of the exponential map and one defines an operation $*$ in $\mathfrak{g}$ by

$$X * Y = \log(\exp X \cdot \exp Y)$$

the Campbell–Baker–Hausdorff (C–B–H) formula expresses $X * Y$ in terms of a power series involving brackets in X and Y:

$$X * Y = X + Y + \tfrac{1}{2}[X, Y] + \tfrac{1}{12}[X, [X, Y]] - \tfrac{1}{12}[Y, [X, Y]] + \cdots,$$

where the higher order brackets are denoted by the dots. We note here that the coefficients of all the bracket terms in the above series are rational. Since $\mathfrak{g}$ is nilpotent, this series is always finite and gives global polynomial laws for the group multiplication, viewed in logarithmic co-ordinates.

If $\mathfrak{g}$ is a nilpotent Lie algebra and $\mathfrak{h}_1, ..., \mathfrak{h}_k$ are subalgebras of $\mathfrak{g}$ such that

$$\mathfrak{g} \supseteq \mathfrak{h}_k \supseteq \cdots \supseteq \mathfrak{h}_1$$

with $\dim \mathfrak{h}_j = m_j$ and $\dim \mathfrak{g} = n$, we say that $\{X_1, ..., X_n\}$ is a weak Malcev basis of $\mathfrak{g}$ *passing through* $\mathfrak{h}_1, ..., \mathfrak{h}_k$, if for each m, R-span$\{X_m, ..., X_n\}$ is a subalgebra of $\mathfrak{g}$ and for $1 \leqslant j \leqslant k$, $\mathfrak{h}_j = R$-span$\{X_{n-m_j}, ..., X_n\}$. If the subalgebras $\mathfrak{h}_1, ..., \mathfrak{h}_k$ are all ideals then we say that $\{X_1, ..., X_n\}$ is a strong Malcev basis of $\mathfrak{g}$ *passing through* $\mathfrak{h}_1, ..., \mathfrak{h}_k$, if for each m, R-span$\{X_m, ..., X_n\}$ is an ideal of $\mathfrak{g}$ and for $1 \leqslant j \leqslant k$, $\mathfrak{h}_j = R$-span$\{X_{n-m_j}, ..., X_n\}$.

If $\mathfrak{g}^*$ denotes the linear dual of $\mathfrak{g}$ as a vector space, then G acts on $\mathfrak{g}^*$ by the *co-adjoint map* Ad^*:

$$(\mathrm{Ad}^*(x))\, l(Y) = l((\mathrm{Ad}\, x^{-1})\, Y), \qquad \text{for all} \quad Y \in \mathfrak{g}, \quad l \in \mathfrak{g}^*, \text{ and } x \in G.$$

The group action given above is called the *coadjoint action*. The orbits of $\mathfrak{g}^*$ under the coadjoint map are called *coadjoint orbits*. The coadjoint orbit of an element $l \in \mathfrak{g}^*$ is denoted by $\mathcal{O}_l$. Thus $\mathcal{O}_l = (\mathrm{Ad}^*\, G)\, l$. We remark here that we will sometimes denote the coadjoint orbit

$$\mathcal{O}_l = (\mathrm{Ad}^*\, G)\, l$$

of $l \in \mathfrak{g}^*$, by $\mathcal{O}_\pi$ where π is the irreducible unitary representation induced by l.

Two special subalgebras of $\mathfrak{g}$ play a central role in the representation theory of nilpotent Lie groups. The first is called the *radical* of l, denoted by $\mathfrak{r}_l$ and is defined for all $l \in \mathfrak{g}^*$ by:

$$\mathfrak{r}_l = \{X \in \mathfrak{g} \colon l([X, Y]) = 0 \quad \forall Y \in \mathfrak{g}\}.$$

The second, called a *polarizer* for l, is a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ which has maximal dimension subject to the condition that it is isotropic:

$$l([X, Y])=0 \qquad \forall X, Y \in \mathfrak{h}.$$

Hence a polarizer for l is an *isotropic* subalgebra of maximal dimension.

If B_l denotes the skew-symmetric bilinear form $B_l(X, Y)=l([X, Y])$, then $\mathfrak{r}_l$ is the radical of B_l. If $\mathfrak{m}$ denotes a polarizer of l then the character $\chi(\exp \mathfrak{m})=e^{2\pi i l(\mathfrak{m})}$ defines a one-dimensional representation of the subgroup $M=\exp(\mathfrak{m})$, since $l([\mathfrak{m}, \mathfrak{m}])=0$. The machinery of induced representation theory of Mackey is then used to induce from this one-dimensional representation, a representation of the whole group G.

A nilpotent Lie algebra $\mathfrak{g}$ is said to have a *rational structure* if it has a basis $\{X_1, ..., X_n\}$ with respect to which the structure constants are rational:

$$[X_i, X_j]=\sum_{k=1}^{n} c_{jk} X_k, \qquad c_{jk} \in Q.$$

If $\mathfrak{g}_Q=Q\text{-span}\{X_1, ..., X_n\}$ then $\mathfrak{g}_Q$ is a rational nilpotent Lie algebra. Since a rational structure depends on a choice of basis it is important to note that $\mathfrak{g}$ could have more than one rational structure. Hence rationality is basis dependent.

Let $\mathfrak{g}$ have a fixed rational structure given by $\mathfrak{g}_Q$. A subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is said to be *rational* if $\mathfrak{h}=R\text{-span}(\mathfrak{h} \cap \mathfrak{g}_Q)$. A functional $l \in \mathfrak{g}^*$ is said to be *rational* if $l\colon \mathfrak{g}_Q \to Q$. Since certain algebras play an important role in our main results it is important to note the following.

PROPOSITION 1. *Let $\mathfrak{g}$ be a nilpotent Lie algebra with rational structure and G the corresponding nilpotent Lie group. Then*

(i) *All the algebras in the ascending and the descending series in G are rational.*

(ii) *If $l \in \mathfrak{g}^*$ is rational, then its radical $\mathfrak{r}_l$ is rational and l has a rational polarizer.*

We are now ready to introduce discrete cocompact subgroups.

Let G be a locally compact group and Γ a subgroup. We say that Γ is a *discrete cocompact subgroup* if Γ is discrete and $\Gamma \backslash G$ is compact. We shall sometimes call a discrete cocompact subgroup a *uniform subgroup* in accordance with [CG].

We shall be interested only in the case when G is a nilpotent Lie group. When G is nilpotent $\Gamma \backslash G$ is called a *nilmanifold.* Γ turns out to be the *fundamental group* of the compact manifold $\Gamma \backslash G$ whenever G is a locally compact topological group and $\Gamma \backslash G$ is compact [P].

Let Γ be a discrete subgroup of a nilpotent Lie group G. A strong or weak Malcev basis $\{X_1, ..., X_n\}$ of $\mathfrak{g}$ is said to be *strongly based* on Γ if

$$\Gamma = \{\exp(m_1 X_1) \cdots \exp(m_n X_n) : m_j \in Z, 1 \leqslant j \leqslant n\}$$

and to be *weakly based* on Γ if

$$\exp(X_j) \in \Gamma \qquad \text{for} \quad 1 \leqslant j \leqslant n.$$

The following theorem is fundamental to the study of discrete cocompact subgroups.

THEOREM 1. *The following statements are equivalent for an arbitrary discrete subgroup Γ of a nilpotent Lie group G*:

(a) *G has a strong Malcev basis strongly based on Γ.*

(b) *G has a strong Malcev basis weakly based on Γ.*

(c) *G has a weak Malcev basis strongly based on Γ.*

(d) *G has a weak Malcev basis weakly based on Γ.*

(e) *Γ is uniform in G.*

The following theorem gives the relationship between rationality and the existence of a uniform subgroup.

THEOREM 2. *Let G be a nilpotent Lie group with Lie algebra $\mathfrak{g}$.*

(a) *If G has a uniform subgroup Γ, then $\mathfrak{g}$ (hence G) has a rational structure such that $\mathfrak{g}_Q = Q$-span$(\log \Gamma)$.*

(b) *Conversely, if $\mathfrak{g}$ has a rational structure from some Q-algebra $\mathfrak{g}_Q \subseteq \mathfrak{g}$, then G has a uniform subgroup Γ such that $\log \Gamma \subseteq \mathfrak{g}_Q$.*

The following Proposition will be very useful in choosing a nice basis for $\mathfrak{g}$ on which Γ is strongly based. We will use this result to classify discrete cocompact subgroups of the three step chain group in Section 3.

PROPOSITION 2. *Let Γ be uniform in a nilpotent Lie group G, and let $G = H_k \supsetneq \cdots \supsetneq H_1$ be rational Lie subgroups of G. Let $\mathfrak{h}_1, ..., \mathfrak{h}_{k-1}, \mathfrak{h}_k = \mathfrak{g}$ be the corresponding Lie algebras. Then there exists a weak Malcev basis $\{X_1, ..., X_n\}$ for $\mathfrak{g}$ strongly based on Γ and passing through $\mathfrak{h}_1, ..., \mathfrak{h}_{k-1}$. If the H_j are all normal, the basis can be chosen to be a strong Malcev basis.*

We cite here a fundamental result that is known for a long time and is proven in [GGP].

THEOREM 3. *Let Γ be cocompact and discrete in a locally compact topological group G. The quasi-regular representation U_Γ splits into a discrete direct sum of a countable number of irreducible unitary representations, each of finite multiplicity.*

Now recall that two discrete cocompact subgroups Γ_1 and Γ_2 are representation equivalent if the corresponding quasi-regular representations U_{Γ_1} and U_{Γ_2} are unitary equivalent. It follows from the above theorem that Γ_1 and Γ_2 are representation equivalent if and only if every irreducible unitary representation π that occurs in the decomposition of U_{Γ_1} occurs in the decomposition of U_{Γ_2} with the same multiplicity and vice versa.

The fundamental papers of [R] and [H] give a condition for an irreducible unitary representation π to occur in the decomposition of U_Γ and also a formula that calculates its multiplicity.

Before stating the *occurrence condition* and the *multiplicity formula* from these papers we need some preliminaries. Given $\lambda \in \mathfrak{g}^*$ and a polarizer $\mathfrak{h}$ for λ we consider the family F of all pairs $(\bar{\lambda}, H)$, where $H = \exp \mathfrak{h}$ and $\bar{\lambda}$ is the character defined on H by $\bar{\lambda}(x) = \exp(2\pi i \lambda(\log(x)))$. We call $(\bar{\lambda}, H)$, a *rational point* if λ is rational and $\mathfrak{h}$ is a rational polarizer for λ. A rational point $(\bar{\lambda}, H)$ is said to be an *integral point* if $\bar{\lambda}$: $\log(\Gamma \cap H) \to 1$ (equivalently $\lambda(\log \Gamma \cap \mathfrak{h}) \subseteq Z$). G acts on the family F of all pairs $(\bar{\lambda}, H)$ in the following way.

$$x \circ (\bar{\lambda}, H) = (\overline{\lambda \circ Ad(x)}, x^{-1} H x).$$

THEOREM 4. *Let Γ be a discrete cocompact subgroup of a nilpotent Lie group G. For π to occur in U_Γ it is necessary and sufficient that there exists an integral point in $\mathcal{O}_\pi$ (i.e., there exists a rational function $\lambda \in \mathcal{O}_\pi$ and a rational polarizer $\mathfrak{h}$ for λ such that, $\lambda(\log \Gamma \cap \mathfrak{h}) \subseteq Z$). If π occurs in U_Γ then the multiplicity of π is equal to the number of Γ-orbits on the set of integral points in the G-orbit of $(\bar{\lambda}, H)$.*

THEOREM 5. *Let $\mathfrak{g}$ be a nilpotent Lie algebra and $\lambda \in \mathfrak{g}^*$. Then the following statements are equivalent.*

(i) $\mathcal{O}_\lambda = \lambda + \mathfrak{r}_\lambda^\perp$ *where* $\mathfrak{r}_\lambda^\perp = \{\lambda \in \mathfrak{g}^* : \lambda(X) = 0\ \forall X \in \mathfrak{r}_\lambda\}$.

(ii) *there is a subspace V of $\mathfrak{g}^*$ with $\mathcal{O}_\lambda = \lambda + V$ ('flat orbit' condition).*

(iii) $\mathfrak{r}_\lambda$ *is an ideal of* $\mathfrak{g}$.

An irreducible representation $\pi \in \hat{G}$ of a nilpotent Lie group G is said to be *square integrable mod the center* ($\pi \in SI/Z$), where Z denotes the center of G, if

$$\int_{G/Z} |\langle \pi(x)\, v, w \rangle|^2 \, d\dot{x} < \infty \qquad \forall v, w \in \mathcal{H}_\pi.$$

An irreducible representation $\pi \in \hat{G}$ of a nilpotent Lie group G is said to be *square integrable mod the kernel* ($\pi \in SI/K$), where K denotes the kernel of π, if

$$\int_{G/K} |\langle \pi(x)\, v, w \rangle|^2 \, d\dot{x} < \infty \qquad \forall v, w \in \mathscr{H}_\pi.$$

The following result (see [CG]) gives a necessary and sufficient condition for a representation to be square integrable mod the kernel.

THEOREM 6. *Let G be a nilpotent Lie group and Z its center. Let $\pi \in \hat{G}$, $K = \operatorname{Ker} \pi$ and $K_0 =$ connected component of the identity in K. Then $\pi \in SI/K$ if and only if the corresponding representation $\bar{\pi}$ on $G/K_0 = \bar{G}$ is in SI/Z.*

If $\lambda \in \mathfrak{g}^*$ and $\{X_1, ..., X_{2k}\}$ is a weak Malcev basis for $\mathfrak{r}_\lambda \backslash \mathfrak{g}$, then the *Pfaffian* $\mathrm{Pf}(\lambda)$ (see [CG]) is defined up to sign, by

$$\mathrm{Pf}(\lambda)^2 = \det B, \qquad B_{ij} = B_\lambda(X_i, X_j)$$

where B_λ is the skew-symmetric bilinear form defined earlier.

The occurrence condition and multiplicity formula in Theorem 10 become very simple for $\pi \in SI/Z$. The following result is found in [MW].

THEOREM 7. *Let G be a nilpotent Lie group and Γ a discrete cocompact subgroup of G. If $\pi \in \hat{G}$ is in SI/Z, then for π to occur in the decomposition of U_Γ it is both necessary and sufficient that there exists $\lambda \in \mathcal{O}_\pi$ such that $\lambda(\log(\Gamma \cap Z)) \subseteq \mathbb{Z}$. If π occurs then the multiplicity of π is $|Pf(\lambda)|$.*

The first necessary and sufficient condition for two subgroups Γ_1 and Γ_2 of a two-step nilpotent Lie group G was obtained in [G]. The author of this paper introduces a property called *Γ-equivalence*, which we state here.

Let Φ be an automorphism of a nilpotent Lie group G. Let Γ be a discrete cocompact subgroup of G. Φ is said to be a *Γ-equivalence* if for all γ in Γ there exists a_γ in G and γ'_γ in $\Gamma \cap G^{(2)}$ such that $\Phi(\gamma) = a_\gamma \gamma a_\gamma^{-1} \gamma'_\gamma$. Then Gornet proves the following result.

THEOREM 8. *Let G be a two-step nilpotent Lie group. Let Γ_1 and Γ_2 be two discrete cocompact subgroups of G. Γ_1 will be representation equivalent to Γ_2 if and only if there exists Φ, a Γ_1-equivalence of G, such that $\Phi(\Gamma_1) = \Gamma_2$.*

Finally, we would like to say a few words about the notation. We will use the symbol $\cong_R$ to denote representation equivalence. For instance,

$\Gamma_1 \cong_R \Gamma_2$ would mean that Γ_1 is representation equivalent to Γ_2. We will use the symbol $\cong$ to denote either isomorphism of groups or unitary equivalence of representations: the context will make the meaning clear. We will denote the centralizer of subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ by *cent*($\mathfrak{h}$).

For the sake of convenience of we will denote the set Γ, defined by

$$\Gamma = \{\exp(m_1 X_1) \cdots \exp(m_n X_n) \colon m_j \in Z, 1 \leqslant j \leqslant n\}$$

where $\{X_1, ..., X_n\}$ is a strong Malcev basis passing through $\mathfrak{g}^{(n)}, ..., \mathfrak{g}^{(2)}, \mathfrak{g}$ by listing the basis elements as follows:

$$\log \Gamma \colon X_1, ..., X_n.$$

When Γ actually turns out to be a group, we would call $\exp X_i$, $i = 1, ..., n$ *generators* of the corresponding uniform subgroup Γ. Sometimes we may write the set Γ given above as

$$\Gamma = \{\exp(m_1 X_1) \cdots \exp(m_n X_n) \colon m_j \in Z\}$$

for convenience.

3. THE FLAT ORBIT PHENOMENON

Remark 1. The definition of a strong/weak Malcev basis strongly based on Γ given in Section 2 needs some explanation. It should be noted that if $\{X_1, ..., X_n\}$ is a strong/weak Malcev basis for $\mathfrak{g}$ with rational structure constants, then the set Γ defined by $\Gamma = \{\exp(\alpha_1 X_1) \cdots \exp(\alpha_n X_n) \colon \alpha_j \in Z, 1 \leqslant j \leqslant n\}$ need not even be a group. To see this, one has to go only as far as the Heisenberg group N_3, with Lie algebra basis $\{X, Y, Z\}$ and Lie brackets given by $[X, Y] = Z$. Let

$$\Gamma = \{\exp(\alpha_1 X) \exp(\alpha_2 Y) \exp(\tfrac{3}{5}\alpha_3 Z) \colon \alpha_i \in Z\}.$$

Γ can be written as,

$$\Gamma = \{(\alpha_1, 0, 0) \cdot (0, \alpha_2, 0) \cdot (0, 0, \alpha_3) \colon \alpha_i \in Z\}$$

with the understanding that

$$(\alpha_1, \beta_1, \gamma_1) \cdot (\alpha_2, \beta_2, \gamma_2) = (\alpha_1 + \alpha_2, \beta_1 + \beta_2, \gamma_1 + \gamma_2 + \alpha_1 \beta_2).$$

Now,

$$\begin{aligned}\Gamma &= \left\{(\alpha_1, \alpha_2, \alpha_1\alpha_2) \cdot \left(0, 0, \frac{3}{5}\alpha_3\right) : \alpha_i \in Z\right\}\\ &= \left\{\left(\alpha_1, \alpha_2, \alpha_1\alpha_2 + \frac{3}{5}\alpha_3\right) : \alpha_i \in Z\right\}\\ &= \left\{\left(m, n, \frac{5mn+3t}{5}\right) : m, n, t \in Z\right\}.\end{aligned}$$

This set is not a group. This is because

$$\left(1, 1, \frac{8}{5}\right), \left(1, -1, \frac{-8}{5}\right) \in \Gamma$$

whereas,

$$\left(1, 1, \frac{8}{5}\right) \cdot \left(1, -1, \frac{-8}{5}\right) = (2, 0, -1) \notin \Gamma$$

since

$$n = 0 \qquad \text{and} \qquad \frac{3t}{5} = -1$$

doesn't have an integral solution. The following result gives a necessary and sufficient condition for such a set Γ to be a group.

PROPOSITION 3. *Let $\mathfrak{g}$ be any N-step nilpotent Lie algebra with rational structure constants with respect to the rational strong Malcev basis $\{X_1, ..., X_n\}$ passing through $\mathfrak{g}^{(N)}, ..., \mathfrak{g}^{(2)}, \mathfrak{g}$. Then*

$$\Gamma = \{\exp(\alpha_1 X_1) \cdots \exp(\alpha_n X_n) : \alpha_j \in Z, 1 \leqslant j \leqslant n\}$$
$$\textit{is a group (hence uniform)} \Leftrightarrow xyx^{-1}y^{-1} \in \Gamma \qquad \forall x, y \in \Gamma.$$

Proof. For ($\Leftarrow$) we first note that the canonical product (product of the exp's of the X_i's with integer coefficients) for every element of the set Γ is unique since $\{X_1, ..., X_n\}$ is a strong Malcev basis.

Define,

$$\Gamma_1 = \{\exp(\alpha_{i_1} X_{i_1}) \cdots \exp(\alpha_n X_n) : \alpha_i \in Z\}$$

where i_1 is the largest i such that the basis element $X_i \in \mathfrak{g}^{(N-1)} \sim \mathfrak{g}^{(N)}$. As $\mathfrak{g}^{(N)}$ is central it is obvious that Γ_1 is a group.

Proceeding inductively, let

$$\Gamma_k = \{\exp(\alpha_{i_1-k+1} X_{i_1-k+1}) \exp(\alpha_{i_1-k+2} X_{i_1-k+2}) \cdots \exp(\alpha_n X_n) : \alpha_i \in Z\}$$

be a group.

Γ_{k+1} is a group.

Let $x, y \in \Gamma_{k+1}$.

$$\begin{aligned} xy^{-1} &= (x_{i_1-k} \underbrace{x_{i_1-k+1} \cdots x_n)(y_n^{-1} \cdots y_{i_1-k+1}^{-1}}_{\gamma_k,\ \text{where } \gamma_k \in \Gamma_k} y_{i_1-k}^{-1}) \\ &= x_{i_1-k} \gamma_k y_{i_1-k}^{-1} \\ &= x_{i_1-k} y_{i_1-k}^{-1} \gamma_k u_k \end{aligned}$$

where $u_k = (\gamma_k^{-1}, y_{i_1-k}) = \gamma_k^{-1} y_{i_1-k} \gamma_k y_{i_1-k}^{-1}$.

Since u_k cannot have an $\exp(X_{i_1-k})$ term in its canonical product (y_{i_1-k} and $y_{i_1-k}^{-1}$ are both in the product for u_k) and u_k is in Γ it follows by the uniqueness of the canonical product and the definition of Γ_k that $u_k \in \Gamma_k$. Hence $\gamma_k u_k \in \Gamma_k$ and therefore $\gamma_k u_k$ can be expressed as a canonical product of the exponentials of $X_{i_1-k+1}, ..., X_n$ respectively. Hence $xy^{-1} \in \Gamma_{k+1}$ and consequently Γ_{k+1} is a group. The proof is now complete.

The following result is a minor variation of a result in [Pa] and we omit the proof since it is a standard result.

PROPOSITION 4. *Suppose $\mathfrak{g}$ is any three-step nilpotent Lie algebra with rational structure constants and $\mathfrak{g}_Q = Q$-span$\{X_1, ..., X_n\}$ provides a rational structure for $\mathfrak{g}$. Then $\mathfrak{g}_Q$ has a chain generator i.e., $\mathfrak{g}_Q$ contains an element X such that $ad^2\, X(Y) \neq 0$ for some $Y \neq 0$ in $\mathfrak{g}_Q$.*

We now define a chain basis and prove its existence.

PROPOSITION 5. *Let $\mathfrak{g}$ be a three-step nilpotent Lie algebra with rational structure constants. Then there exists a basis of rational vectors*

$$\{X_1, ..., X_r, \underbrace{Z_1, ..., Z_k, \overbrace{W_1, ..., W_n}^{\text{span } \mathfrak{g}^{(3)}}}_{\text{span } \mathfrak{g}^{(2)}}\}$$

for $\mathfrak{g}$ such that

(i) $[X_1, X_r] = Z_1$; $[X_1, Z_1] = W_1$

(ii) $[X_r, Z_1] \in Z$-*span*$\{4W_2, ..., 4W_n\}$

(iii) $[X_i, X_j]$ *and* $[X_i, Z_t]$ *are both in* Z*-span*$\{Z_1, ..., Z_k, W_1, ..., W_n\}$ $\forall i, j, t$.

A basis with these properties is called a chain basis.

Proof. Such a basis exists because X'_1 can be chosen to be a chain generator. If $[X'_r, Z'_1]$ has a W'_1-term say $\alpha W'_1$ consider $X'_r - \alpha X'_1$ in place of X'_r. Choose a sufficiently large even integer K such that $\{KX'_1, ..., KX'_r, Z'_1, ..., Z'_k, W'_1, ..., W'_n\}$ has Z-constants.

We note the following facts:

(i) $\{KX'_1, K^3X'_2, ..., K^3X'_{r-1}, KX'_r, K^2Z'_1, Z'_2, ..., Z'_k, W'_1, ..., W'_n\}$ has Z-constants as well. Also,

$$\begin{aligned}[KX'_r, K^2Z'_1] &= K^2[KX'_r, Z'_1] \\ &= K^2 \cdot \text{integral linear combinations of } W'_2, ..., W'_n \\ &\in Z\text{-span}\{4W'_2, ..., 4W'_n\}.\end{aligned}$$

Label the above basis as $\{X''_1, X''_2, ..., X''_r, Z''_1, Z'_2, ..., Z'_k, W'_1, ..., W'_n\}$. Note that this basis has Z-constants.

(ii) Note that $[X''_1, X''_r] = Z''_1$, $[X''_r, Z''_1] \in Z\text{-span}\{4W'_2, ..., 4W'_n\}$ and $[X''_1, Z''_1] = \gamma W'_1$ for some $\gamma \in Z$. Now, consider the basis

$$\{X''_1, \gamma X''_2, ..., \gamma X''_{r-1}, X''_r, Z''_1, \gamma Z'_2, ..., \gamma Z'_k, \gamma W'_1, W'_2, ..., W'_n\}.$$

Label this basis as

$$\{X_1, X_2, ..., X_{r-1}, X_r, Z_1, Z_2, ..., Z_k, W_1, W_2, ..., W_n\}.$$

Then,

$$[X_1, X_r] = Z_1; \qquad [X_1, Z_1] = W_1$$
$$[X_r, Z_1] \in Z\text{-span}\{4W_2, ..., 4W_n\}$$
$$[X_i, X_j], [X_i, Z_t] \in Z\text{-span}\{Z_1, ..., Z_k, W_1, ..., W_n\}.$$

Thus we have a basis satisfying all the properties stated in the proposition.

We modify the occurrence condition given in Theorem 4 for three-step algebras with flat coadjoint orbits.

PROPOSITION 6. *Let Γ be a uniform subgroup of G, with Lie algebra $\mathfrak{g}$ (hence with rational structure constants) that is three-step nilpotent and has all its coadjoint orbits flat. Then the irreducible unitary representation π occurs in $L^2(\Gamma \backslash G) \Leftrightarrow$ there exists $\lambda \in \mathcal{O}_\pi$ such that*

(i) *λ is rational on* $\mathfrak{g}_Q^{(2)}$

(ii) $\lambda(\log \Gamma \cap \mathfrak{r}_\lambda) \subseteq Z$.

Proof. Conversely suppose (i) and (ii) hold. Then there exists a rational polarizer $\mathfrak{h}$ for λ such that $\mathfrak{h} \supseteq \mathfrak{g}^{(2)}$. To see this consider a rational basis $\{X_1, ..., X_n\}$ for $\mathfrak{g}$. Define $\lambda_j(X) = \lambda([X, X_j])$. Since λ is rational on $\mathfrak{g}_Q^{(2)}$,

$$\mathfrak{r}_\lambda = (R\text{-span}\{\lambda_1, ..., \lambda_n\})^\perp$$

is rational by (5.1.2, [CG]). Now appeal to Vergne's construction (1.3.5, [CG]), choosing $\mathfrak{g}_j$'s to be rational and taking $\mathfrak{g}_t = \mathfrak{g}^{(2)}$ for a suitable t.

Let $\lambda_j = \lambda \mid \mathfrak{g}_j$. Then for each j, $\mathfrak{r}_{\lambda_j}$ is rational by the argument given in the preceding paragraph.

$\mathfrak{g}$ being three-step, $\mathfrak{r}_{\lambda_t} = \mathfrak{g}^{(2)}$. Let $\mathfrak{h} = \sum_{j=1}^n \mathfrak{r}_{\lambda_j}$. Then $\mathfrak{h}$ is rational for λ, polarizes λ and contains $\mathfrak{g}^{(2)}$.

Consider $\Gamma \cap H$, where $H = \exp \mathfrak{h}$. $\Gamma \cap H$ being uniform in H has a strong Malcev basis $\{X_1, ..., X_r, \underbrace{W_1, ..., W_k}_{\text{span } \mathfrak{r}_\lambda}\}$, with $X_i, W_j \in \log \Gamma$ on which it is strongly based i.e.,

$$\Gamma \cap H = \{\exp(\alpha_1 X_1) \cdots \exp(\alpha_r X_r) \exp(\beta_1 W_1) \cdots \exp(\beta_k W_k) \colon \alpha_i, \beta_i \in Z\}.$$

Define $\mu \in \mathfrak{g}^*$ by

$$\mu \equiv \lambda(W_1)\, W_1^* + \cdots + \lambda(W_k)\, W_k^*.$$

Then $\mu \in \mathcal{O}_\lambda$, since orbits are flat and $\mathfrak{h}$ (being an ideal) polarizes μ. Let

$$x = \exp(\alpha_1 X_1) \cdots \exp(\alpha_r X_r) \exp(\beta_1 W_1) \cdots \exp(\beta_k W_k) \in \Gamma \cap H.$$

If $X = \log x$, then

$$\begin{aligned}\mu(X) &= (\lambda(W_1)\, W_1^* + \cdots + \lambda(W_k)\, W_k^*)(\beta_1 W_1 + \cdots + \beta_k W_k) \\ &= \beta_1 \lambda(W_1) + \cdots + \beta_k \lambda(W_k) \in Z\end{aligned}$$

since $\mu \equiv 0$ on brackets ($\mathfrak{h}$ being an ideal) and $\lambda(W_i) \in Z$. This completes the proof of Proposition 6.

We are now ready to prove the main theorem.

THEOREM 9. *Let $\mathfrak{g}$ be a three-step nilpotent Lie algebra, with rational structure constants and all coadjoint orbits flat. Then there exist uniform subgroups Γ_1 and Γ_2 in G such that $\Gamma_1 \cong_R \Gamma_2$ and $\Gamma_1 \not\cong \Gamma_2$.*

Proof. For the sake of clarity in exposition we shall call the vector X_r in the above basis Y_1. With this change in the labelling of the vector X_r in the chain basis, the new chain basis will be

$$\{X_1, ..., X_{r-1}, Y_1, \underbrace{Z_1, ..., Z_k, \overbrace{W_1, ..., W_n}^{\text{span } \mathfrak{g}^{(3)}}}_{\text{span } \mathfrak{g}^{(2)}}\}.$$

Define sets Γ_1 and Γ_2 as follows:

$$\log \Gamma_1 : 2X_1, 4X_2, ..., 4X_{r-1}, Y_1, \underbrace{Z_1, 2Z_2, ..., 2Z_k, \overbrace{W_1, W_2, ..., W_n}^{\text{span } \mathfrak{g}^{(3)}}}_{\text{span } \mathfrak{g}^{(2)}}$$

$$\log \Gamma_2 : 2X_1, 4X_2, ..., 4X_{r-1}, Y_1 + \tfrac{1}{2}Z_1, \underbrace{Z_1, 2Z_2, ..., 2Z_k, \overbrace{W_1, W_2, ..., W_n}^{\text{span } \mathfrak{g}^{(3)}}}_{\text{span } \mathfrak{g}^{(2)}}$$

At the outset, we would like to note that the only difference between the sets Γ_1 and Γ_2 is in the place where the vector Y_1 occurs. It is also crucial to note that this change occurs above $\mathfrak{g}^{(2)}$ since $Y_1 \in \mathfrak{g} \sim \mathfrak{g}^{(2)}$.

Claim 1. Γ_i, $i = 1, 2$ are groups (hence uniform).

Proof. We note the following facts:

(i) The following list exhausts all possible first order brackets capable of producing a non-central term in the C-B-H product for x:

$$\frac{[4X_i, 4X_j]}{2} = 8[X_i, X_j]; \qquad \frac{[2X_1, 4X_j]}{2} = 4[X_1, X_j];$$

$$\frac{[2X_1, Y_1]}{2} = Z_1; \qquad \frac{[4X_j, Y_1]}{2} = 2[X_j, Y_1].$$

(ii) The following list is the rest of the first order brackets that occur in the C-B-H product for x and by the construction of the chain basis all these brackets will be in Z-span$\{Z_1, 2Z_2, ..., 2Z_k, W_1, ..., W_n\}$:

$$\frac{[4X_i, 2Z_j]}{2} = 4[X_i, Z_j]; \qquad \frac{[4X_i, Z_1]}{2} = 2[X_i, Z_1];$$

$$\frac{[2X_1, 2Z_j]}{2} = 2[X_1, Z_j]; \qquad \frac{[2X_1, Z_1]}{2} = W_1;$$

$$\frac{[Y_1, 2Z_i]}{2} = [Y_1, Z_i]; \qquad \frac{[Y_1, Z_1]}{2} \in Z\text{-span}\{2W_2, ..., 2W_n\}.$$

Note that it follows from (i) that every element $X \in \Gamma_1$ can be written as $x = \exp(T_1 + Z)$ where T_1 is an integral linear combination of the logarithms of all the generators of Γ_1 above $\mathfrak{g}^{(3)}$ and Z is in the Q-span of the logarithms of the generators of Γ_1 in $\mathfrak{g}^{(3)}$. Note that Z is central.

If $y \in \Gamma_1$, say $y = \exp(T_2 + Z')$, consider

$$\begin{aligned}
xyx^{-1}y^{-1} &= \exp(T_1)\exp(T_2)\exp(-T_1)\exp(-T_2) \\
&= \exp\left(T_1 + T_2 + \frac{[T_1, T_2]}{2}\right)\exp\left(-T_1 - T_2 + \frac{[T_1, T_2]}{2}\right) \\
&\quad \text{(Bi-linearity of the Lie bracket applied to the three} \\
&\quad \text{bracket terms in } \exp(T_1)\exp(T_2) \text{ and} \\
&\quad \exp(-T_1)\exp(-T_2) \text{ respectively)} \\
&= \exp(T + U)\exp(-T + U) \qquad \text{where} \quad T = T_1 + T_2 \text{ and} \\
&\quad U = \frac{[T_1, T_2]}{2} \\
&= \exp\left(2U + \frac{[T, U]}{2} + \frac{[U, -T]}{2}\right) \\
&= \exp(2U + [T, U])
\end{aligned}$$

Note that from (i) and (ii) $U \in Z\text{-span}\{Z_1, 2Z_2, \ldots, 2Z_k, W_1, \ldots, W_n\}$. Hence $[T, U] \in Z\text{-span}\{W_1, \ldots, W_n\}$. Therefore,

$$\log(xyx^{-1}y^{-1}) \in Z\text{-span}\{Z_1, 2Z_2, \ldots, 2Z_k, W_1, \ldots, W_n\}.$$

This means $xyx^{-1}y^{-1} \in \Gamma_1$, since $\mathfrak{g}^{(2)}$ is abelian. By Proposition 1, Γ_1 is a group.

To show that Γ_2 is a group, first note that

$$\exp\left(Y_1 + \frac{Z_1}{2}\right) = \exp(Y_1)\exp\left(\frac{Z_1}{2}\right)\exp\left(-\frac{[Y_1, Z_1]}{2}\right),$$

with the last factor in Z-span$\{W_2, \ldots, W_n\}$. Hence every element $x \in \Gamma_2$ can be written as $x = x_1 \exp(\frac{1}{2}\alpha Z_1)$, where $x_1 \in \Gamma_1$ and $\alpha \in Z$, since $\exp(\frac{1}{2}\alpha Z_1)$ commutes with all factors from $\exp(\mathfrak{g}^{(2)})$.

Also note that if $\mathfrak{g}$ is any three-step algebra, $X \in \mathfrak{g}$ and $U \in \mathfrak{g}^{(2)}$, then

$$\exp(U)\exp(X) = \exp(X)\exp(U)\exp(-[X, U]).$$

Since we will be using this identity repeatedly in the following computation, we shall use underbraces to denote the places where it is used.

If $y \in \Gamma_2$, say $y = y_1 \exp(\frac{1}{2}\beta Z_1)$ with $y_1 \in \Gamma_1$ and $\beta \in Z$, consider

$$
\begin{aligned}
xyx^{-1}y^{-1} &= x_1 \underbrace{\exp\left(\frac{\alpha}{2} Z_1\right) y_1}\exp\left(\frac{\beta}{2} Z_1\right)\exp\left(-\frac{\alpha}{2} Z_1\right) x_1^{-1} \\
&\quad \exp\left(-\frac{\beta}{2} Z_1\right) y_1^{-1} \\
&= x_1 y_1 \exp\left(\frac{\alpha}{2} Z_1\right) u_1 \exp\left(\frac{\beta}{2} Z_1\right)\exp\left(-\frac{\alpha}{2} Z_1\right) x_1^{-1} \\
&\quad \exp\left(-\frac{\beta}{2} Z_1\right) y_1^{-1} \\
&\qquad \left(\text{where } u_1 = \exp\left(-\left[\log y_1, \frac{\alpha}{2} Z_1\right]\right)\right. \\
&\quad \left. \in Z\text{-span}\{W_1, ..., W_n\}\right) \\
&= x_1 y_1 \underbrace{\exp\left(\frac{\beta}{2} Z_1\right) x_1^{-1}}\exp\left(-\frac{\beta}{2} Z_1\right) y_1^{-1} u_1 \\
&= x_1 y_1 x_1^{-1} \exp\left(\frac{\beta}{2} Z_1\right) u_2 \exp\left(-\frac{\beta}{2} Z_1\right) y_1^{-1} u_1 \\
&\qquad \left(\text{where } u_2 = \exp\left(-\left[\log x_1^{-1}, \frac{\beta}{2} Z_1\right]\right)\right. \\
&\quad \left. \in Z\text{-span}\{W_1, ..., W_n\}\right) \\
&= x_1 y_1 x_1^{-1} y_1^{-1} u_1 u_2
\end{aligned}
$$

Since $\log(x_1 y_1 x_1^{-1} y_1^{-1}) \in Z\text{-span}\{Z_1, 2Z_2, ..., 2Z_k, W_1, ..., W_n\}$ from the first part of this proof and $u_1, u_2 \in Z\text{-span}\{W_1, ..., W_n\}$, it follows that $xyx^{-1}y^{-1} \in \Gamma_2$. Hence Γ_2 is a group by Proposition 3.

Claim 2. $\Gamma_1 \cong_R \Gamma_2$.

Subclaim 1. $\pi \in (\Gamma_1 \backslash G)^\wedge \Leftrightarrow \pi \in (\Gamma_2 \backslash G)^\wedge$.

For this let us first consider representations π induced by $\lambda \in \mathfrak{g}^*$ such that $\lambda \equiv 0$ on $\mathfrak{g}^{(3)}$. Such a λ can be identified with an element of $\bar{\mathfrak{g}}^*$, where $\bar{\mathfrak{g}} = \mathfrak{g}/\mathfrak{g}^{(3)}$. Let

$$\bar{\Gamma}_1 = \Gamma_1 / \Gamma_1 \cap G^{(3)} \qquad \text{and} \qquad \bar{\Gamma}_2 = \Gamma_2 / \Gamma_2 \cap G^{(3)}.$$

Note that $\bar{\Gamma}_1$ and $\bar{\Gamma}_2$ will be two-step nilpotent Lie groups. Further $\bar{\Gamma}_1$ and $\bar{\Gamma}_2$ will be described by

$$\log \bar{\Gamma}_1 : 2\bar{X}_1, 4\bar{X}_2, ..., 4\bar{X}_{r-1}, \bar{Y}_1, \bar{Z}_1, 2\bar{Z}_2, ..., 2\bar{Z}_k$$
$$\log \bar{\Gamma}_2 : 2\bar{X}_1, 4\bar{X}_2, ..., 4\bar{X}_{r-1}, \bar{Y}_1 + \tfrac{1}{2}\bar{Z}_1, \bar{Z}_1, 2\bar{Z}_2, ..., 2\bar{Z}_k$$

Define $\Phi_* : \bar{\mathfrak{g}} \to \bar{\mathfrak{g}}$ by

$$\Phi_*(\bar{X}_i) = \bar{X}_i; \qquad \Phi_*(\bar{Y}_1) = \bar{Y}_1 + \tfrac{1}{2}\bar{Z}_1; \qquad \Phi_*(\bar{Z}_j) = \bar{Z}_j.$$

Φ_* is obviously a Lie algebra automorphism.

Further Φ_* is a $\bar{\Gamma}_1$-equivalence at the Lie algebra level. For this consider, $\bar{X} \in \log \bar{\Gamma}_1$. Then

$$\bar{X} = \alpha_1(2\bar{X}_1) + \cdots + \alpha_{r-1}(4\bar{X}_{r-1}) + \alpha_r(\bar{Y}_1) + q_1(\bar{Z}_1)$$
$$+ q_2(2\bar{Z}_2) + \cdots + q_k(2\bar{Z}_k)$$

where $\alpha_i \in Z$, $q_j \in Q$. This means $\Phi_*(\bar{X}) = \bar{X} + \frac{1}{2}\alpha_r \bar{Z}_1$. Since

$$\left[\bar{X}, \frac{-1}{2}\bar{X}_1\right] = \frac{\alpha_r}{2}\bar{Z}_1 + 2\alpha_2[\bar{X}, \bar{X}_2] + \cdots + 2\alpha_{r-1}[\bar{X}_1, \bar{X}_{r-1}]$$

it follows that $\frac{1}{2}\alpha_r \bar{Z}_1 \in [\bar{X}, \bar{\mathfrak{g}}] + \log(\bar{\Gamma}_1 \cap \bar{\mathfrak{g}}^{(2)})$. Hence the corresponding map Φ on the group $\bar{G}$ is a $\bar{\Gamma}_1$-equivalence.

Therefore $\bar{\Gamma}_1 \cong_R \bar{\Gamma}_2$ by Theorem 8. This proves Subclaim 1, for π induced by λ such that $\lambda \equiv 0$ on $\mathfrak{g}^{(3)}$.

Now we shall show Subclaim 1 for π induced by $\lambda \in \mathfrak{g}^*$ such that $\lambda \not\equiv 0$ on $\mathfrak{g}^{(3)}$. We shall prove $(\Rightarrow)$ first.

Let $\pi \in (\Gamma_1 \backslash G)\,\hat{}$. By virtue of Proposition 6 we assume without loss in generality that λ is rational on $\mathfrak{g}_Q$ and $\lambda(\log \Gamma_1 \cap \mathfrak{r}_\lambda) \subseteq Z$. We also note that every $\gamma_2 \in \Gamma_2$ can be written as $\gamma_2 = \gamma_1 \exp(\frac{1}{2}\alpha Z_1)$ for some $\alpha \in Z$ and $\gamma_1 \in \Gamma_1$. This means

$$\log \gamma_2 = \log \gamma_1 + \frac{\alpha}{2} Z_1 + \frac{\alpha}{4}[\log \gamma_1, Z_1].$$

Case 1. $Z_1 \in \mathfrak{r}_\lambda$ and $Y_1 \notin \mathfrak{r}_\lambda$.

$$\pi \in (\Gamma_1 \backslash G)\,\hat{} \Rightarrow \exists \lambda_1 \in \mathcal{O}_\lambda \text{ such that } \lambda_1 : \log \Gamma_1 \to Q$$

and a rational polarizer $\mathfrak{h}$ for λ_1 containing $\mathfrak{g}^{(2)}$ such that $\lambda_1(\log \Gamma_1 \cap \mathfrak{h}) \subseteq Z$. Since $\mathfrak{g}$ has flat orbits, $\lambda_1 \equiv \lambda$ on $\mathfrak{r}_\lambda$ and $\mathfrak{r}_{\lambda_1} = \mathfrak{r}_\lambda$. Therefore

$$\lambda_1(\log \Gamma_1 \cap \mathfrak{r}_\lambda) \subseteq Z.$$

We shall now show that $\lambda_1(\log \Gamma_2 \cap \mathfrak{r}_\lambda) \subseteq Z$. Let $\gamma_2 \in \Gamma_2 \cap R_\lambda$, where $R_\lambda = \exp(\mathfrak{r}_\lambda)$. By the observation made before $\gamma_2 = \gamma_1 \exp(\frac{1}{2}\alpha Z_1)$.

$$Z_1 \in \mathfrak{r}_\lambda \Rightarrow \gamma_1 \in \Gamma_1 \cap R_\lambda.$$

Hence $\lambda_1(\log \gamma_1) \in Z$. By applying the C-B-H formula to the canonical product representing γ_1 we can write γ_1 as,

$$\gamma_1 = \exp(T + \alpha Y_1)$$

where $T = T_1 + W'$, $T_1 \in Z\text{-span}\{2X_1, 4X_2, \ldots, 4X_{r-1}, Z_1, 2Z_2, \ldots, 2Z_k\}$ and $W' \in \mathfrak{g}^{(3)}$. Since $T + \alpha Y_1 \in \mathfrak{r}_\lambda$, $\lambda_1([T + \alpha Y_1, X_1]) = 0$. This means, $\lambda_1([T, X_1]) = \alpha\lambda_1([X_1, Y_1]) = \alpha\lambda_1(Z_1)$. Also, $\lambda_1([T, X_1]) = \lambda_1([T_1 + W', X_1]) = \lambda_1([T_1, X_1])$. Therefore, $\lambda_1([T_1, X_1]) = \alpha\lambda_1(Z_1)$. This means that for some $\alpha_i, \beta_j \in Z$,

$$\begin{aligned}
\alpha\lambda_1(Z_1) &= \lambda_1([T_1, X_1]) \\
&= \lambda_1([\alpha_1 \cdot 2X_1 + \alpha_2 \cdot 4X_2 + \cdots + \alpha_{r-1} \cdot 4X_{r-1} \\
&\quad + \beta_1 \cdot Z_1 + \beta_2 \cdot 2Z_2 + \cdots + \beta_k \cdot 2Z_k, X_1]) \\
&= 2\alpha_2 \cdot \lambda_1([2X_2, X_1]) + \cdots + 2\alpha_{r-1} \cdot \lambda_1([2X_{r-1}, X_1]) \\
&\quad + \beta_1 \cdot \lambda_1([Z_1, X_1]) + 2\beta_2 \cdot \lambda_1([Z_2, X_1]) + \cdots \\
&\quad + 2\beta_k \cdot \lambda_1([Z_k, X_1]) \in 2Z,
\end{aligned}$$

by virtue of the following observations:

(1) Since $\mathfrak{g}^{(2)}$ is abelian, $[2X_i, X_1] \in \log \Gamma_1 \cap \mathfrak{g}^{(2)}$ and therefore

$$\lambda_1(\log \Gamma_1 \cap \mathfrak{g}^{(2)}) \subseteq Z \Rightarrow \lambda_1([2X_i, X_1]) \in Z \text{ for all } i.$$

(2) $\lambda_1([Z_1, X_1]) = 0$ since $Z_1 \in \mathfrak{r}_{\lambda_1}$.

(3) Since $[Z_j, X_1] \in \log \Gamma_1 \cap \mathfrak{g}^{(3)}$ and $\lambda_1(\log \Gamma_1 \cap \mathfrak{r}_\lambda) \subseteq Z$,

$$\lambda_1([Z_j, X_1]) \in Z \text{ for all } j.$$

Therefore,

$$\lambda_1(\log \gamma_2) = \lambda_1(\log \gamma_1) + \frac{\alpha}{2}\lambda_1(Z_1) + \frac{\alpha}{4}\lambda_1([\log \gamma_1, Z_1]) \in Z,$$

since $\alpha\lambda_1(Z_1)$ is even and $Z_1 \in \mathfrak{r}_\lambda (= \mathfrak{r}_{\lambda_1})$. Hence $\lambda_1 \in \mathcal{O}_\lambda$ is such that $\lambda_1(\log \Gamma_2 \cap \mathfrak{r}_{\lambda_1}) \subseteq Z$.

Further note that λ_1 is also rational on $\mathfrak{g}_Q^{(2)}$ with respect to the rational structure given by $\log \Gamma_2$ since $\lambda_1 : \log \Gamma_1 \to Q$ and $\log \Gamma_1 \cap \mathfrak{g}^{(2)} = \log \Gamma_2 \cap \mathfrak{g}^{(2)}$. Now applying Proposition 6 to $\lambda_1 \in \mathcal{O}_\pi$ and Γ_2 we have finally shown that

$$\pi \in (\Gamma_2 \backslash G)\,\hat{}\,.$$

This proves Subclaim 1 for Case 1.

Case 2. $Y_1 \in \mathfrak{r}_\lambda$ (This implies that $Z_1 \in \mathfrak{r}_\lambda$, since $[X_1, Y_1] = Z_1$ and $\mathfrak{r}_\lambda$ is an ideal). First note that λ is rational on $\mathfrak{g}_Q^{(2)}$ with respect to the rational structure given by $\log \Gamma_2$ since $\log \Gamma_1 \cap \mathfrak{g}^{(2)} = \log \Gamma_2 \cap \mathfrak{g}^{(2)}$. Also, $\lambda(Z_1) = \lambda([X_1, Y_1]) = 0$. Since $\mathfrak{r}_\lambda$ is an ideal (by flatness of orbits)

$$[X_1, Y_1] = Z_1 \in \mathfrak{r}_\lambda.$$

Let $\gamma_2 \in \Gamma_2 \cap R_\lambda$. Recall, $\gamma_2 = \gamma_1 \exp(\frac{1}{2}\alpha Z_1)$. Also, $Z_1 \in \mathfrak{r}_\lambda \Rightarrow \gamma_1 \in \Gamma_1 \cap R_\lambda$. Therefore,

$$\begin{aligned}\lambda(\log \gamma_2) &= \lambda(\log \gamma_1) + \frac{\alpha}{2}\lambda(Z_1) + \frac{\alpha}{4}\lambda([\log \gamma_1, Z_1]) \\ &= \lambda(\log \gamma_1) \in Z.\end{aligned}$$

Hence $\lambda(\log \Gamma_2 \cap \mathfrak{r}_\lambda) \subseteq Z$. Now applying Proposition 6 to $\lambda \in \mathcal{O}_\pi$ and Γ_2 we have shown that

$$\pi \in (\Gamma_2 \backslash G)\,\hat{}$$

proving Subclaim 1 for Case 2.

Case 3. $Z_1 \notin \mathfrak{r}_\lambda$ (This implies that $Y_1 \notin \mathfrak{r}_\lambda$ since $[X_1, Y_1] = Z_1$ and $\mathfrak{r}_\lambda$ is an ideal). Consider a rational polarizer $\mathfrak{h}$ for λ, containing $\mathfrak{g}^{(2)}$ and let $H = \exp \mathfrak{h}$. $\Gamma_1 \cap H$ being uniform in H, has a strong Malcev basis $\{A_1, ..., A_{s-1}, Z_1, R_1, ..., R_t\}$ where $R_i \in \log \Gamma_1 \cap \mathfrak{r}_\lambda$ on which $\Gamma_1 \cap H$ is strongly based. Now extend the basis $\{A_1, ..., A_{s-1}, Z_1, R_1, ..., R_t\}$ of $\mathfrak{h}$ to a basis of $\mathfrak{g}$ and define $\lambda_2 \in \mathfrak{g}^*$ by,

$$\lambda_2 \equiv \lambda(R_1)\, R_1^* + \cdots + \lambda(R_t)\, R_t^*.$$

Then $\lambda_2 \in \mathcal{O}_\lambda$, $\mathfrak{r}_{\lambda_2} = \mathfrak{r}_\lambda$ (by flatness of orbits) and $\mathfrak{h}$ (being an ideal) polarizes λ_2. Further $\lambda_2(\log \Gamma_1 \cap \mathfrak{h}) \subseteq Z$.

Let $\gamma_2 \in \Gamma_2 \cap R_\lambda$. Recall, $\gamma_2 = \gamma_1 \exp(\frac{1}{2}\alpha Z_1)$. Hence $\gamma_1 = \gamma_2 \exp(-\frac{1}{2}\alpha Z_1)$. Since $\gamma_2 \in R_\lambda \subseteq H$ and $Z_1 \in \mathfrak{h} = \log H$, it follows that $\gamma_1 \in H$. Therefore $\lambda_2(\log \gamma_1) \in Z$. This means,

$$\begin{aligned}\lambda_2(\log \gamma_2) &= \lambda_2(\log \gamma_1) + \frac{\alpha}{2}\lambda_2(Z_1) + \frac{\alpha}{4}\lambda_2([\log \gamma_1, Z_1]) \\ &= \lambda_2(\log \gamma_1) \in Z\end{aligned}$$

since $\lambda_2(Z_1)=0$ and $\log \gamma_1$, Z_1 are both in $\mathfrak{h}$. Hence $\lambda_2 \in \mathcal{O}_\lambda$ is such that $\lambda_2(\log \Gamma_2 \cap \mathfrak{r}_\lambda) = \lambda_2(\log \Gamma_2 \cap \mathfrak{r}_{\lambda_2}) \subseteq Z$. Also λ_2 is rational on $\mathfrak{g}_Q^{(2)}$ with respect to the rational structure given by $\log \Gamma_2$ since λ_2 is integral on

$$\begin{aligned}\log \Gamma_1 \cap \mathfrak{h} &\supseteq \log \Gamma_1 \cap \mathfrak{g}^{(2)} \\ &= \log \Gamma_2 \cap \mathfrak{g}^{(2)}.\end{aligned}$$

Applying Proposition 6 to λ_2 and Γ_2 yields Subclaim 1 for Case 3. Since the case $Z_1 \notin \mathfrak{r}_\lambda$ and $Y_1 \in \mathfrak{r}_\lambda$ is non-existent ($Y_1 \in \mathfrak{r}_\lambda \Rightarrow [X_1, Y_1] = Z_1 \in \mathfrak{r}_\lambda$), we have proved Subclaim 1 for all cases. Hence we are done with the proof of ($\Rightarrow$).

For the proof of ($\Leftarrow$) consider $\pi \in (\Gamma_2 \backslash G)^\wedge$. Note that every $\gamma_1 \in \Gamma_1$ can be written as

$$\gamma_1 = \gamma_2 \exp\left(\frac{\alpha}{2} Z_1\right)$$

for some $\gamma_2 \in \Gamma_2$ and $\alpha \in Z$. Now follow the sequence of steps given in the proof of ($\Rightarrow$) with Γ_2 in place of Γ_1.

This yields the proof of Subclaim 1.

Subclaim 2. *π occurs with the same multiplicity in both $L^2(\Gamma_1 \backslash G)$ and $L^2(\Gamma_2 \backslash G)$.*

Case 1. π is induced by $\lambda \in \mathfrak{g}^*$ such that $\lambda \equiv 0$ on $\mathfrak{g}^{(3)}$. For this we just note that by virtue of Theorem 8, $\bar{\Gamma}_1 \cong_R \bar{\Gamma}_2$ where

$$\bar{\Gamma}_1 = \Gamma_1/\Gamma_1 \cap G^{(3)} \qquad \text{and} \qquad \bar{\Gamma}_2 = \Gamma_2/\Gamma_2 \cap G^{(3)}$$

so that π occurs with the same multiplicity in both $L^2(\Gamma_1 \backslash G)$ and $L^2(\Gamma_2 \backslash G)$.

Case 2. π is induced by $\lambda \in \mathfrak{g}^*$ such that $\lambda \equiv 0$ on $\mathfrak{g}^{(3)}$.

Subcase 1. $\pi \in SI/Z$. We appeal to the multiplicity formula in Theorem 7. It is now crucial to note that $|P_{\Gamma_1}(\lambda)| = |P_{\Gamma_2}(\lambda)|$, where $|P_{\Gamma_i}(\lambda)|$ is the Pfaffian of λ with respect to the stong Malcev basis on which Γ_i is strongly based. To see this note that if Q is the change of basis matrix, B_i the matrix for which $P_{\Gamma_i}(\lambda) = (\det B_i)^{1/2}$ then $B_2 = QB_1Q^t$. Hence $\det B_2 = (\det Q)^2 \det B_1$. Q being a skew matrix it is obvious that $\det Q = 1$. Hence $m(\pi, \Gamma_1) = |P_{\Gamma_1}(\lambda)| = |P_{\Gamma_2}(\lambda)| = m(\pi, \Gamma_2)$.

This yields the proof of Subclaim 2.

Subcase 2. $\pi \notin SI/Z$. For this we factor out K_0 the connected component of the identity in $\mathrm{Ker}(\pi)$ to pass on to a square integrable representation $\bar{\pi}$ on the quotient, which has the same multiplicity as π. We apply the multiplicity formula in Theorem 7 to $\bar{\pi} \in SI/Z$. Then $m(\bar{\pi}, \bar{\Gamma}_1) = m(\bar{\pi}, \bar{\Gamma}_2)$ since the change of basis matrix $\bar{Q}$ in G/K_0 is still upper triangular with at most one non-zero term off the diagonal. Hence, we are through in this case.

Thus $\Gamma_1 \cong_R \Gamma_2$.

CLAIM 3. *$\Gamma_1 \ncong \Gamma_2$. Suppose there exists an isomorphism ψ of Γ_1 onto Γ_2. From* [M] *it follows that ψ lifts to an automorphism ψ_* of $\mathfrak{g}$. Also note that $\psi_*\colon \log \Gamma_1 \cap \mathfrak{g}^{(2)} \to \log \Gamma_2 \cap \mathfrak{g}^{(2)}$ is onto.*

Since $2X_1$, $Y_1 + \frac{1}{2}Z_1 \in \log \Gamma_2$, there exist $X, Y \in \log \Gamma_1$ such that

$$\psi_*(X) = 2X_1$$

$$\psi_*(Y) = Y_1 + \tfrac{1}{2}Z_1.$$

Also,

$$X = T_1 + Z$$

$$Y = T_2 + Z'$$

where the T_i's are in Z-span$\{2X_1, 4X_2, ..., 4X_{r-1}, Y_1, Z_1, 2Z_2, ..., 2Z_k\}$ and $Z, Z' \in \mathfrak{g}^{(3)}$.

$$\psi_*(X) = 2X_1 \Rightarrow \psi_*\left(\frac{X}{2}\right) = X_1.$$

Also,

$$\left[\psi_*\left(\frac{X}{2}\right), \psi_*(Y)\right] = \left[X_1, Y_1 + \frac{1}{2}Z_1\right] = Z_1 + \frac{W_1}{2}.$$

We note here that

$$\left[\psi_*\left(\frac{X}{2}\right), \psi_*(Y)\right]$$

has a fractional W_1-term.

Let

$$T_1 = a_1 \cdot 2X_1 + a_2 \cdot 4X_2 + \cdots + a_{r-1} \cdot 4X_{r-1} + a_r \cdot Y_1$$
$$+ b_1 \cdot Z_1 + b_2 \cdot 2Z_2 + \cdots + b_k \cdot 2Z_k,$$
$$T_2 = a'_1 \cdot 2X_1 + a'_2 \cdot 4X_2 + \cdots + a'_{r-1} \cdot 4X_{r-1} + a'_r \cdot Y_1$$
$$+ b'_1 \cdot Z_1 + b'_2 \cdot 2Z_2 + \cdots + b'_k \cdot 2Z_k.$$

But,

$$\left[\psi_*\left(\frac{X}{2}\right), \psi_*(Y)\right] = \psi_*\left(\left[\frac{X}{2}, Y\right]\right) = \psi_*\left(\left[\frac{T_1}{2}, T_2\right]\right)$$
$$= \psi_*\Big(\Big[\frac{1}{2} \cdot (a_1 \cdot 2X_1 + a_2 \cdot 4X_2 + \cdots + a_{r-1} \cdot 4X_{r-1}$$
$$+ a_r \cdot Y_1 + b_1 \cdot Z_1 + b_2 \cdot 2Z_2 + \cdots + b_k \cdot 2Z_k), (a'_1 \cdot 2X_1$$
$$+ a'_2 \cdot 4X_2 + \cdots + a'_{r-1} \cdot 4X_{r-1} + a'_r \cdot Y_1$$
$$+ b'_1 \cdot Z_1 + b'_2 \cdot 2Z_2 + \cdots + b'_k \cdot 2Z_k)\Big]\Big)$$
$$= \psi_*\Big(\Big[\Big(a_1 \cdot X_1 + 2a_2 \cdot X_2 + \cdots + 2a_{r-1} \cdot X_{r-1}$$
$$+ \frac{1}{2} a_r \cdot Y_1 + \frac{1}{2} b_1 \cdot Z_1 + b_2 \cdot Z_2 + \cdots + b_k \cdot Z_k\Big),$$
$$(a'_1 \cdot 2X_1 + a'_2 \cdot 4X_2 + \cdots + a'_{r-1} \cdot 4X_{r-1} + a'_r$$
$$\cdot Y_1 + b'_1 \cdot Z_1 + b'_2 \cdot 2Z_2 + \cdots + b'_k \cdot 2Z_k)\Big]\Big)$$

has only integral terms of W_1, since

$$\{X_1, \ldots, X_{r-1}, Y_1, \underbrace{Z_1, \ldots, Z_k, \overbrace{W_1, \ldots, W_n}^{\text{span } \mathfrak{g}^{(3)}}}_{\text{span } \mathfrak{g}^{(2)}}\}$$

is a chain basis for $\mathfrak{g}$. We remind ourselves (with the understanding that we are calling the vector X_r in the original chain basis as Y_1) that the chain basis $\{X_1, \ldots, X_{r-1}, Y_1, Z_1, \ldots, Z_k, W_1, \ldots, W_n\}$ has the following properties:

(i) $[X_1, Y_1] = Z_1$; $[X_1, Z_1] = W_1$

(ii) $[Y_1, Z_1] \in Z\text{-span}\{4W_2, \ldots, 4W_n\}$

(iii) $[X_i, X_j]$, $[X_i, Z_t]$, $[Y_1, X_j]$ and $[Y_1, Z_t]$ are all in Z-span $\{Z_1, ..., Z_k, W_1, ..., W_n\}$.

This leads to a contradiction and proves Claim 3. Hence the theorem.

4. THE NON-FLAT ORBIT EXAMPLE

In order to investigate the role of flatness of coadjoint orbits in the flat orbit phenomenon that was exhibited in Section 4 we examined the lowest dimensional example of a nilpotent Lie group with non-flat coadjoint orbits. We denote this group by G_3 and the corresponding Lie algebra by $\mathfrak{g}_3$. More precisely, $\mathfrak{g}_3$ is the four dimensional Lie algebra spanned by the vectors X, Y, Z, W with non-zero brackets given by

$$[X, Y] = Z; \qquad [X, Z] = W.$$

G_3 is called the *three-step chain group.*

As the results that we obtained for this group are of a computational nature, we simply state an auxiliary result (classification of discrete cocompact subgroups of G_3) and the main result, without giving a proof for either.

Lemma 1. *$\Gamma \subseteq G_3$ is a uniform subgroup of G_3 if and only if there exists a strong Malcev basis $\{X_1, Y_1, Z_1, W_1\}$ for $\mathfrak{g}_3$ such that*

(i) $[X_1, Y_1] = a_1 Z_1 + s W_1$; $[X_1, Z_1] = a_2 W_1$; $[Y_1, Z_1] = 0$; *for some $a_1, a_2 \in Z$ and $s \in Q$ satisfying $s + \frac{1}{2} a_1 a_2 \in Z$.*

(ii) $\Gamma = \{\exp(\alpha_1 X_1) \exp(\alpha_2 Y_1) \exp(\alpha_3 Z_1) \exp(\alpha_4 W_1) \colon \alpha_i \in Z\}$.

We now state the main result for discrete cocompact subgroups of G_3 for which $s = 0$.

Theorem 10. *Let Γ_1 be a uniform subgroup of G_3 such that*

$$\log \Gamma_1 \colon X_1, Y_1, Z_1, W_1$$

where $[X_1, Y_1] = a_1 Z_1$; $[X_1, Z_1] = a_2 W_1$; for some $a_1, a_2 \in Z$. If $\Gamma_1 \cong_R \Gamma_2$ for some uniform subgroup Γ_2 of G_3 then $\Gamma_1 \cong \Gamma_2$.

5. SOME ASSORTED ADDITIONAL RESULTS

In this section we prove some short results for three-step nilpotent Lie algebras with one-dimensional center.

PROPOSITION 7. *Suppose* $\mathfrak{g}$ *is a three-step nilpotent Lie algebra with all co-adjoint orbits flat and* $\dim \mathfrak{z}(\mathfrak{g}) = 1$. *Then* $\forall \lambda \in \mathfrak{g}^*$ *such that* $\lambda \not\equiv 0$ *on* $\mathfrak{z}(\mathfrak{g})$, $\mathfrak{r}_\lambda = \mathfrak{z}(\mathfrak{g})$.

Proof. As $\mathfrak{r}_\lambda$ is an ideal (flat orbit condition),

$$[V, Y] \in \mathfrak{r}_\lambda \qquad \forall Y \in \mathfrak{g}.$$

Hence, for every $X \in \mathfrak{g}$, $\lambda([[V, Y], X]) = 0$. Now, since $\mathfrak{g}$ is three-step $[[V, Y], X]$ is central and $\lambda \not\equiv 0$ on RW,

$$[[V, Y], X] = 0.$$

This means $[V, Y]$ is central for $Y \in \mathfrak{g}$. Since $V \in \mathfrak{r}_\lambda$, $[V, Y] = 0$, otherwise

$$[V, Y] = \alpha W, \qquad \alpha \neq 0 \Rightarrow \lambda([V, Y]) = \alpha \lambda(W) \neq 0$$

a contradiction to the fact that $V \in \mathfrak{r}_\lambda$. Hence $[V, Y] = 0$ for all $Y \in \mathfrak{g}$ and therefore $V \in \mathfrak{z}(\mathfrak{g})$.

This means $\mathfrak{z}(\mathfrak{g}) \subseteq \mathfrak{r}_\lambda$ and we are done with the proof of Proposition 7.

Remark 2. Note that for three-step algebras $\mathfrak{g}$ with one-dimensional center, any polarizer for $\lambda \not\equiv 0$ on $\mathfrak{z}(\mathfrak{g})$ that contains $\mathfrak{g}^{(2)}$, has to be contained in $cent(\mathfrak{g}^{(2)})$. Hence for such an algebra if $cent(\mathfrak{g}^{(2)}) = \mathfrak{g}^{(2)}$, then $\mathfrak{g}^{(2)}$ itself would serve as a polarizer. Given below is an example of an algebra $\mathfrak{g}$ for which $cent(\mathfrak{g}^{(2)}) = \mathfrak{g}^{(2)}$.

EXAMPLE 1. Consider the seven dimensional algebra $\mathfrak{g}$ spanned by the vectors $X_1, Y_1, Y_2, Z_1, Z_2, Z_3, W_1$ whose non-zero Lie brackets are given by:

$$\begin{aligned} &[X_1, Y_1] = Z_1; && [X_1, Y_2] = Z_2; \\ &[Y_2, Y_1] = Z_3; && [X_1, Z_1] = W_1; \\ &[Y_1, Z_1] = W_1; && [Y_2, Z_2] = W_1; \\ &[X_1, Z_3] = W_1; && [Y_2, Z_1] = W_1. \end{aligned}$$

Note that $\mathfrak{g}$ is three step. Further,

$$\mathfrak{g}^{(2)} = R\text{-span}\{Z_1, Z_2, Z_3, W_1\} \qquad \text{and}$$
$$cent(\mathfrak{g}^{(2)}) = R\text{-span}\{Z_1, Z_2, Z_3, W_1\}.$$

PROPOSITION 8. *Suppose* $\mathfrak{g}$ *is a three-step nilpotent Lie algebra with* $\dim \mathfrak{z}(\mathfrak{g}) = 1$. *If* $cent(\mathfrak{g}^{(2)}) = \mathfrak{g}^{(2)}$ *then* $\mathfrak{g}$ *has flat orbits.*

Proof.

Case 1. $\lambda(W)=0$. Since λ can be identified with a functional on $\bar{\mathfrak{g}}^*$ where $\bar{\mathfrak{g}}=\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ and $\bar{\mathfrak{g}}$ is two-step, it follows that $\mathcal{O}_\lambda$ is flat.

Case 2. $\lambda(W)\neq 0$. Consider a basis $\{X_1, ..., X_r, \underbrace{Z_1, ..., Z_{k-1}, W}_{\text{span } \mathfrak{g}^{(2)}}\}$ of $\mathfrak{g}$.

Let $X\in \mathfrak{r}_\lambda$. Since there exists a polarizer containing $\mathfrak{g}^{(2)}$, $\mathfrak{r}_\lambda\subseteq \mathfrak{g}^{(2)}$. As $\mathfrak{g}$ is three-step, this forces $[X, Y]=0$ for every $Y\in\mathfrak{g}$. Hence $X\in\mathfrak{z}(\mathfrak{g})$. By the flat orbit condition $\mathcal{O}_\lambda$ is flat.

This proves Proposition 8.

Proposition 9. *Let $\mathfrak{g}$ be any nilpotent Lie algebra. The cent$(\mathfrak{g}^{(2)})$ can be at most a two-step algebra.*

Proof. If $X\in\mathfrak{g}$, it follows by the Jacobi identity that

$$[Y_1, [Y_2, X]]+[Y_2, [X, Y_1]]+[X, [Y_1, Y_2]]=0.$$

Since Y_1 and Y_2 commute with $\mathfrak{g}^{(2)}$ it follows that $[X, [Y_1, Y_2]]=0$. Consequently, $[Y_1, Y_2]\in\mathfrak{z}(\mathfrak{g})$. Hence *cent*$(\mathfrak{g}^{(2)})$ is at most a two-step algebra.

This proves Proposition 9.

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