Annals of Pure and Applied Logic 36 (1987) 235–277
North-Holland

ON VARIETIES OF CYLINDRIC ALGEBRAS WITH APPLICATIONS TO LOGIC

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Communicated by D. Van Dalen
Received 31 May 1985; revised 3 March 1986

Mn_α, Mg_α, and Bg_α denote the classes of minimal, monadic-generated, and binary-generated cylindric algebras of dimension α respectively, and $\overline{Eq}K$ denotes the equational theory of the class K of algebras. In Theorem 2, we characterize those classes $K \subseteq Mg_\alpha$, $\alpha > 2$, for which $\overline{Eq}K$ is recursively enumerable (r.e.). As a corollary we obtain that $\overline{Eq}Mn_\alpha$ is not r.e.[1] iff $\alpha \geq \omega$, $\overline{Eq}Mg_\alpha$ is not r.e. iff $\alpha > 2$, $\overline{Eq}Bg_\alpha$ is r.e. for $\alpha \geq \omega$ and $\overline{Eq}Mn_\alpha = \overline{Eq}Mg_\alpha$ iff ($\alpha = 0$ or $\alpha \geq \omega$). These results solve Problems 4.11, 4.12 and the problem in item (1) on p. (ii) of the introduction of Part II of Henkin–Monk–Tarski [11] and continue the investigations started in Monk [22]. We discuss at length the logical meaning and consequences in the introduction and in Section 2. The proofs of the above results rely on the fact that the set of satisfiable Diophantine equations is not decidable. We also show that the equational theory of monadic-generated relation algebras is not r.e. Some further results can be found in Theorems 5 and 6: in Theorem 5 we give a single equation that characterizes being of characteristic 0 in Mg_ω, in Theorem 6 we investigate how big Mg_α is. We do some investigations concerning the lattice of varieties of CA_α's, $\alpha \geq \omega$.

Introduction

Boolean algebras (BA's) and cylindric algebras (CA's) are algebraizations of propositional and predicate (i.e., first-order) logic respectively. A CA is minimal, or monadic-generated resp., if it is generated by the empty set, or by a set of one-dimensional elements respectively. (One-dimensional elements correspond to formulas with at most one free variable. See the end of this introduction for precise definition.) The classes Mn_ω and Mg_ω of minimal and monadic-generated CA's respectively correspond to first-order logic having only equality (=), and to first-order logic having only unary predicate symbols (beside equality) called monadic logic respectively (for definitions of CA_ω, Mn_ω, Mg_ω see the end of this introduction). The set of theorems (i.e., valid formulas) of propositional logic is decidable while that of first-order logic is undecidable but recursively enumerable (r.e.). And indeed, the equational theory of BA's is decidable while that of the representable CA's is undecidable but r.e. It is known that monadic logic is

* Research supported by Hungarian National Foundation for Scientific Research grant No. 1810.
[1] After having completed this paper we learned that, independently of us, M. Rubin also proved that $\overline{Eq}Mn_\omega$ is not r.e.

decidable. Therefore one might expect $\overline{\text{EqMg}_\omega}$ to be decidable, too. And indeed, it was announced, mistakenly, in the 1971 edition of [11, p. 258] and in [22, Theorem 22] that the equational theories $\overline{\text{EqMn}_\omega}$ and $\overline{\text{EqMg}_\alpha}$ of Mn_ω and Mg_α are decidable. Later in [11, Problems 4.11, 4.12], these were asked as open problems. We prove in the present paper that $\overline{\text{EqMn}_\omega}$, $\overline{\text{EqMg}_\alpha}$ are not r.e. in spite of the facts that monadic and equality logics are decidable.

What logical meaning does this bear? To answer this, we define what a formula scheme is.

Definition 0.1. FmV denotes a countable set of formula variables (i.e., variables ranging over formulas) and $V = \{v_i : i \in \omega\}$ is our set of 'normal' variables. The set of *formula schemes* (or just schemes) is defined to be the smallest set satisfying

(i) φ is a scheme if $\varphi \in \text{FmV}$.
(ii) $v_i = v_j$ is a scheme if $i, j \in \omega$.
(iii) $\exists v_i \sigma$, $\neg\sigma$, $\sigma \wedge \xi$ are schemes if $i \in \omega$ and σ, ξ are schemes.

For example, $\varphi \wedge \psi \rightarrow \psi$ is a scheme if φ, ψ are formula variables. Another scheme is $\varphi \rightarrow \exists v_1 \varphi$ where $\varphi \in \text{FmV}$. (We use the derived connectives $\rightarrow$, $\vee$, $\forall v_i$ etc. the usual way.) In the everyday mathematical life we more often use formula schemes than formulas themselves. See, e.g., any axiomatization of first-order logic. We note that the formula schemes in ordinary mathematical life frequently have 'side-conditions', for example in "$\forall v_i(\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \forall v_i \psi)$, provided that v_i does not occur freely in φ".

In what follows, by a first-order formula we mean one without operation symbols. We say that a first-order formula φ is equality[2] (monadic) if the only atomic formulas occurring in φ are $v_i = v_j$ for $i, j \in \omega$ (all the atomic formulas occurring in φ are unary or $v_i = v_j$ for some $i, j \in \omega$).

Let σ be a formula scheme. An (equality, monadic) *instance* of σ is a first-order formula we get from σ by replacing the formula variables in σ with (equality, monadic) first-order formulas. We say that σ is (equality, monadic) *valid* if every (equality, monadic) instance of σ is a valid first-order formula.

Now we turn to the connection between formula schemes and cylindric equations. Recall that a CA_α is an algebra of the type $\langle A; +, \cdot, -, 0, 1, c_i, d_{ij}\rangle_{i,j\in\alpha}$ such that $\langle A; +, \cdot, -, 0, 1\rangle$ is a BA and c_i, d_{ij} are unary operations and constants resp.

Definition 0.2 (Scheme as a CA-equation). Our set of variables is $X = \{x_i : i \in \omega\}$ when we want to speak about CA's. Let $t : \text{FmV} \rightarrowtail X$ be arbitrary but one–one. We associate a CA_ω-term $\text{tr}(\sigma)$ to any scheme σ as follows.

(i) $\text{tr}(\varphi) = t(\varphi)$ if $\varphi \in \text{FmV}$.
(ii) $\text{tr}(v_i = v_j) = d_{ij}$ for $i, j \in \omega$.

[2] Well, "φ is equality" is a shortened version of the usual "φ is of the language of equality".

(iii) $\text{tr}(\exists v_i \sigma) = c_i \,\text{tr}(\sigma)$, $\text{tr}(\neg \sigma) = -\text{tr}(\sigma)$, $\text{tr}(\sigma \wedge \xi) = \text{tr}(\sigma) \cdot \text{tr}(\xi)$ if $i \in \omega$ and σ, ξ are schemes.

The CA_ω-equation $\text{eq}(\sigma)$ associated to the scheme σ is defined to be $\text{tr}(\sigma) = 1$.

Clearly, every CA_ω-term (written up by using $-$, $\cdot$, c_i, d_{ij}) is of form $\text{tr}(\sigma)$ for some scheme σ, hence every CA_ω-equation e is equivalent to $\text{eq}(\sigma)$ for some scheme σ. Rp_α denotes the class of representable CA_α's, for definition see the end of this introduction.

Proposition 0.3. *Let σ be a scheme. Then* (i)–(iii) *below hold.*

(i) *σ is equality valid iff* $\text{Mn}_\omega \models \text{eq}(\sigma)$.

(ii) *σ is monadic valid iff* $\text{Mg}_\omega \models \text{eq}(\sigma)$.

(iii) *σ is valid iff* $\text{Rp}_\omega \models \text{eq}(\sigma)$.

Proof. (i) and (ii) follow from $\text{Mn}_\omega \subseteq \text{Mg}_\omega \subseteq \mathbf{SP}\text{Cs}_\omega^{\text{reg}}$, see [11].

(iii) follows from $\text{EqRp}_\omega = \text{Eq}(\text{Cs}_\omega^{\text{reg}} \cap \text{Lf}_\omega)$, see [11]. The details are very similar to those of the proofs of 4.3.61–65 in [11, pp. 173–174]. Therefore we omit them. □

In the light of Proposition 0.3, the results that $\overline{\text{Eq}}\text{Mn}_\omega$, $\overline{\text{Eq}}\text{Mg}_\omega$ are not r.e. announced in the abstract imply the following: Though the set of valid equality (monadic) formulas is decidable, the set of equality (monadic) valid formula schemes is not even recursively enumerable. This happens in spite of the fact that equality logic does have an axiomatization using schemes only! Therefore the schemes derivable from this axiomatization are enough to yield all the valid equality formulas as instances but are far less than all the valid schemes. In a sense, we obtain that the set of valid schemes is much bigger than that of the derivable ones. It is impossible to give a sound inference system for monadic logic (or equality logic) by which all valid schemes (of this logic) would be provable. One might think that this is caused by some second-order behaviour of the schemes. But this is not the case, namely:

The set of valid schemes of first-order logic is recursively enumerable but not decidable. This follows from the theorem that $\overline{\text{Eq}}\text{Rp}_\omega$ is r.e. (Monk [23]) but undecidable (Tarski). For several different enumerations of the first-order valid schemes see section 4.1 of [11], more specifically 4.1.9, 4.1.15, 4.1.20 and Problem 4.1. Thus allowing only unary predicate symbols causes that we have much more valid schemes than when we allow binary predicates as well. Allowing no predicates at all does not imply more valid schemes than when unary predicates are allowed: the equality valid and monadic valid schemes co-incide. This follows from our theorem $\overline{\text{Eq}}\text{Mn}_\omega = \overline{\text{Eq}}\text{Mg}_\omega$.

How is it possible that the equality formulas are decidable but the schemes are not? When we want to decide a scheme, we have to enumerate all its instances and decide them one-by-one. That the schemes are not decidable means that when we want to decide a scheme, the 'structure' of the scheme (only finitely many variables occur in the scheme explicitly) does not tell us how many more variables (or 'structure') are involved in the possible non-validity of the scheme. In the first-order case the opposite is true: if we want to know whether a scheme is valid or not, the 'structure' of the scheme does tell us how complex instances we should check only. Namely: Let σ be a scheme. Assume that all the variables occurring in σ are among $v_0, \ldots, v_N$. Replace every formula variable $\varphi \in \mathrm{FmV}$ occurring in σ with the first-order formula $R_\varphi(v_0, \ldots, v_N)$. Then we get a first-order instance σ' of σ. Now [11, 4.3.62] states that $\mathrm{Rp}_\omega \models \mathrm{eq}(\sigma)$ if σ' is valid. (For some detail of this proof see Remark 1.7(a) in Section 1.) Thus σ is a first-order valid scheme iff σ' is a valid first-order formula. This gives an enumeration of all the valid schemes of first-order logic. As a contrast, in the cases of equality logic and monadic logic there is no general algorithm assigning such a formula σ' to every scheme σ.

Let $\alpha < \omega$ be an ordinal. An α-scheme is a scheme in which only v_i $(i \in \alpha)$ occur as (normal) variables. A formula of the first-order logic L_α using only α variables is a first-order formula in which only v_i $(i \in \alpha)$ occur as variables, but we require further that all the atomic formulas are either $v_i = v_j$ $(i, j \in \alpha)$ or of the form $R(v_0 \cdots v_n)$ where R is an $(n+1)$-ary relation symbol and $n < \alpha$. These logics L_α are well investigated, see e.g. [8–10, 13, 19, 25, 31]. We call an α-scheme α-valid if we arrive at valid formulas whenever we substitute formulas of L_α for the formula variables. Thus an α-valid scheme is a valid scheme of the first-order logic L_α using α variables. It can be proved analogously to Proposition 0.3 that an α-scheme σ is α-valid (equality, monadic α-valid) iff $\mathrm{Rp}_\alpha \models \mathrm{eq}(\sigma)$ (or $\mathrm{Mn}_\alpha \models \mathrm{eq}(\sigma)$, $\mathrm{Mg}_\alpha \models \mathrm{eq}(\sigma)$). Let $3 \leqslant \alpha < \omega$. Then the equality α-valid schemes are decidable while the monadic α-valid schemes are still not r.e. (The reason is that the unary predicates can somehow play the role of the missing variables v_i, $i \geqslant \alpha$.) The 2-valid schemes as well as the 2-valid monadic ones are decidable. But the reason is not that 2-logic is too simple: there are 2^ω many different monadic 'scheme-theories' (schemes valid in a fixed class of monadic models) in L_2.

For precise statements of the above mentioned logical results see Section 2.

For more general connections between logic and CA see Andréka–Németi–Sain [2] and Blok–Pigozzi [5].

Now we turn to defining the main cylindric algebraic notions we will use in the present paper.

Let α be any ordinal. The class CA_α is a variety defined by 7 simple schemes of equations in [11, p. 162] (we do not have to remember the specific forms of these herein). The symbol $\triangleq$ stands for "equals by definition". Let $\mathfrak{A} \in \mathrm{CA}_\alpha$ and $\beta \leqslant \alpha$. Then $\mathrm{Nr}_\beta \mathfrak{A} \triangleq \{x \in A : \Delta^{\mathfrak{A}} x \subseteq \beta\}$ where $\Delta^{\mathfrak{A}} x \triangleq \{i \in \alpha : c_i^{\mathfrak{A}} x \neq x\}$. If $X \subseteq A$ then $\mathrm{Sg}^{\mathfrak{A}} X$, or simply $\mathrm{Sg}X$, denotes the subuniverse of $\mathfrak{A}$ generated by X. Now

$$\mathrm{Mn}_\alpha \triangleq \{\mathfrak{A} \in \mathrm{CA}_\alpha : A = \mathrm{Sg}0\} \quad \text{and} \quad \mathrm{Mg}_\alpha \triangleq \{\mathfrak{A} \in \mathrm{CA}_\alpha : A = \mathrm{Sg}\mathrm{Nr}_1 \mathfrak{A}\}.$$

Thus $\mathrm{Mn}_\alpha \subseteq \mathrm{Mg}_\alpha \subseteq \mathrm{CA}_\alpha$. By a representable CA_α (an Rp_α), $\alpha \geqslant 2$, we mean a

CA_α isomorphic to a generalized cylindric set algebra (a Gs_α): A Gs_α is a Boolean set algebra with greatest element (i.e., with unit) a disjoint union V of Cartesian spaces ${}^\alpha U$ of dimension α. The nonboolean operations c_i, d_{ij} ($i, j \in \alpha$) are defined in terms of the structure of these spaces, namely for any $X \subseteq V$ we have

$$C_i^{[V]}X \triangleq \{s \in V : (\exists z \in X)(\forall j \in \alpha, j \neq i) s_j = z_j\} \quad \text{and} \quad D_{ij}^{[V]} \triangleq \{s \in V : s_i = s_j\}.$$

By a subbase of a Gs_α we understand the base U of one of the spaces ${}^\alpha U$ the union of which is the unit. A cylindric set algebra (a Cs_α) is a Gs_α with unit element a single Cartesian space. (For $\alpha \leqslant 1$, Rp_α is defined as $\mathbf{SP}Cs_\alpha$.) A fundamental theorem of CA-theory is that Rp_α is a variety and $Rp_\alpha \subsetneq CA_\alpha$ for $\alpha \geqslant 2$.

ω denotes the smallest infinite ordinal. We will extensively use the fact that every ordinal is the set of smaller ordinals. Thus ω is the set of all finite ordinals (natural numbers). For undefined notation and terminology we use in the present paper we refer the reader to [11]. However, we tried to make the paper understandable for that reader who, not wanting to use [11], simply ignores those sentences in which undefined notation occurs (but keeps on reading). At the end of the paper there is a list of notation. We note that the monograph [11] in itself contains all the material we rely on in the present paper. However, besides referring to [11], we usually quote the paper where the result in question appeared first.

In Section 1 we formulate the main results, in Section 2 we reformulate the results in their logical form and in Section 3 we give all the proofs. We number items in a section by giving first the number of the section then the number of the item, e.g. Theorem 2.7 is the seventh item in Section 2. We make an exception in Section 1: there we number the theorems separately from the rest and we do not give a section number, e.g. Theorem 3 is the third theorem in Section 1.

1. Formulating the results

Let α be an ordinal. Then Mn_α, Mg_α and Rp_α denote the classes of all minimal, monadic-generated, and representable cylindric algebras respectively (for definition see the end of the introduction). For any class K of algebras, $\overline{Eq}K$ and $\theta\rho K$ denote the equational theory and the first-order theory of K respectively.

It is proved in Monk [22] that $Mg_\alpha \subseteq Rp_\alpha$ and in [11, 4.2.1, 4.2.24, 4.2.23, 4.1.9–10, 4.2.18, 4.2.9] that $\theta\rho Mn_\alpha$ is decidable for $\alpha < \omega$, $\overline{Eq}CA_1$ is decidable (Comer [6]) but $\theta\rho CA_1$ is undecidable (Rubin [32]), $\overline{Eq}Rp_\alpha$ is r.e. (Monk [23]) but not decidable for $\alpha > 2$ (Tarski), decidable for $\alpha = 2$ (Scott [34]). All the above results can be found in [11].

For any class K of algebras, EqK, UnK and ElK denote the smallest equational, universal and first-order axiomatizable classes containing K resp., cf. [11, 4.1.1]. Then $\mathrm{Eq}K = \mathbf{HSP}K$, $\mathrm{Un}K = \mathbf{SUp}K$ and $\mathrm{El}K = \mathbf{UfUp}K$ where $\mathbf{I}K$, $\mathbf{H}K$, $\mathbf{S}K$, $\mathbf{P}K$, $\mathbf{Up}K$ and $\mathbf{Uf}K$ denote the classes of all isomorphic images, homomorphic images, subalgebras, algebras isomorphic to direct products, ultraproducts and ultrafactors of members of K respectively. $K \subset L$ denotes that K is a proper subclass of L. The first two statements of the following theorem are special corollaries of Theorem 2.

Theorem 1. (i) $\overline{\mathrm{Eq}}\mathrm{Mn}_\alpha$ *is not recursively enumerable (r.e.) iff* $\alpha \geqslant \omega$.

(ii) $\overline{\mathrm{Eq}}\mathrm{Mg}_\alpha$ *is not r.e. iff* $\alpha > 2$.

(iii) $\mathrm{EqMn}_\alpha = \mathrm{EqMg}_\alpha$ *iff* ($\alpha \geqslant \omega$ *or* $\alpha = 0$).

(iv) $\mathrm{EqMg}_\alpha \subset \mathrm{Rp}_\alpha$ *iff* $\alpha > 2$, *moreover* $\mathrm{ElMg}_2 \subset \mathrm{UnMg}_2 = \mathrm{Rp}_2$, $\mathrm{Mg}_1 = \mathrm{Rp}_1$.

Let $n \in \omega$. Then $\bar{d}(n \times n)$ denotes the CA_n-term $\prod\{-d_{ij} : i < j < n\}$. $|X|$ denotes the cardinality of the set X.

Definition 1.1. Let α be an ordinal.

(i) Let $K \subseteq \mathrm{CA}_\alpha$. Then K is said to be *bounded* iff $(\exists n \in \omega \cap (\alpha + 1))$ $K \models \bar{d}(n \times n) = 0$. K is said to be unbounded iff K is not bounded.

(ii) Let $n \in \omega$. Then $\mathfrak{A} \in \mathrm{Mg}_\alpha^n$ iff $(\exists X \subseteq \mathrm{Nr}_1\mathfrak{A})$ $[A = \mathrm{Sg}X$ and $|X| \leqslant n]$. (Cf. [11, 4.2.4].) Let $K \subseteq \mathrm{Mg}_\alpha$. Then K is said to be *boundedly generated* iff $(\exists n \in \omega)$ $K \subseteq \mathbf{SP}\mathrm{Mg}_\alpha^n$. K is said to be unboundedly generated iff K is not boundedly generated.

Remark 1.2. It can be proved that K is bounded iff there is $n \in (\alpha + 1) \cap \omega$ such that every element of K is isomorphic to a Gs_α with all subbases smaller than n. K is boundedly generated iff there is $n \in \omega$ such that every element of K is isomorphic to a subdirect product of cylindric set algebras generated by fewer than n monadic (i.e., 1-dimensional) generators. The above are easy to prove using [11].

Theorem 2. *Let* $\alpha > 2$, $K \subseteq \mathrm{Mg}_\alpha$. *Then* (i)–(iii) *below hold.*

(i) $\overline{\mathrm{Eq}}K$ *is either decidable or not r.e.*

(ii) *For* $\alpha \geqslant \omega$, $\overline{\mathrm{Eq}}K$ *is r.e. iff* K *is bounded.*

(iii) *For* $\alpha < \omega$, $\overline{\mathrm{Eq}}K$ *is r.e. iff* K *is boundedly generated.*

Remark 1.3. For $\alpha \leqslant 1$, $\overline{\mathrm{Eq}}K$ is decidable for every $K \subseteq \mathrm{Mg}_\alpha$. This follows from the proof of Monk's result [11, 4.1.22] (Monk [24]), since in the proof of [11, 4.1.22], all the subvarieties of CA_1 are described and it turns out that each proper subvariety of CA_1 is generated (as a variety) by one finite CA_1. There are $K \subseteq \mathrm{Mg}_2$ with $\overline{\mathrm{Eq}}K$ not r.e. This follows from the fact that there are 2^ω varieties of $\mathrm{EqMg}_2 = \mathbf{I}\mathrm{Gs}_2$ (a result of J. Johnson, see [11, 4.1.28]). We do not know whether there are $K \subseteq \mathrm{Mg}_2$ with $\overline{\mathrm{Eq}}K$ r.e. but not decidable. One might think that Theorem 2(i) is true because the set of all equations not valid in K is always r.e. for any $K \subseteq \mathrm{Mg}_\alpha$, $\alpha > 2$. This is not the case; a counterexample can be obtained by 'translating to CA_α' the example given in Remark 2.2(b). We also note that Theorem 2(ii) above generalizes to subclasses of CA_α in the following form: Let $\alpha \geq \omega$ and $K \subseteq \mathrm{CA}_\alpha$. Then $\overline{\mathrm{Eq}}K$ is decidable iff K is bounded. This is proved in [29].

Bg_α denotes the class of all binary-generated CA_α's, i.e., $\mathfrak{A} \in \mathrm{Bg}_\alpha$ iff $A = \mathrm{SgNr}_2\mathfrak{A}$.

Theorem 3. (i) $\overline{\mathrm{Eq}}\mathrm{Bg}_\alpha$ *is r.e. but not decidable for* $\alpha \geq \omega$.

(ii) $\mathrm{EqBg}_\alpha = \mathrm{EqBg}^1_\alpha = \mathrm{Rp}_\alpha$ *for* $\alpha \geq \omega$, *where* $\mathrm{Bg}^1_\alpha \triangleq \{\mathfrak{A} \in \mathrm{CA}_\alpha : (\exists x \in \mathrm{Nr}_2\mathfrak{A})\ A = \mathrm{Sg}\{x\}\}$.

Remark 1.4. (a) R. Maddux showed us that: $\overline{\mathrm{Eq}}\mathrm{Bg}_3$ is undecidable because in [17] actually the following is proved: If $3 \leq \alpha < \omega$ and $\mathrm{Bg}_\alpha \cap \mathrm{Rp}_\alpha \subseteq \mathrm{Eq}K \subseteq \mathrm{CA}_\alpha$, then $\overline{\mathrm{Eq}}K$ is undecidable. (To see this, one has to notice that $\mathrm{Rg}f \subseteq \mathrm{Nr}_2\mathfrak{C}$ in the last part of the proof in [17].) Therefore the second part of Theorem 3(i) can be sharpened by saying "$\overline{\mathrm{Eq}}\mathrm{Bg}_\alpha$ is undecidable iff $\alpha > 2$" (since for $\alpha \leq 2$, $\mathrm{Bg}_\alpha = \mathrm{CA}_\alpha$ and $\overline{\mathrm{Eq}}\mathrm{CA}_\alpha$ is decidable). We do not know whether $\overline{\mathrm{Eq}}\mathrm{Bg}_\alpha$ is r.e. or not for $2 < \alpha < \omega$.

(b) The condition $\alpha \geq \omega$ is necessary in Theorem 3(ii), since $\mathrm{Bg}_\alpha \nsubseteq \mathrm{Rp}_\alpha$ for all $1 < \alpha < \omega$. For $\alpha \geq 5$ this was shown by R. Maddux: Let $\alpha \geq 5$. It is proved in [20, Theorem 7], that there is a nonrepresentable relation algebra $\mathfrak{R}$ with an α-dimensional cylindrical basis. Hence $\mathfrak{R} \in \mathrm{SRa}^*\mathrm{Nr}_3\mathrm{CA}_\alpha$ by [20, Theorem 6]. Assume $\mathfrak{R} \subseteq \mathfrak{Ra}\mathfrak{Nr}_3\mathfrak{C}$ with $\mathfrak{C} \in \mathrm{CA}_\alpha$ and let $\mathfrak{B} \triangleq \mathfrak{Sg}^{(\mathfrak{C})}R$. Then $\mathfrak{B} \in \mathrm{Bg}_\alpha$ and $\mathfrak{R} \subseteq \mathfrak{Ra}\mathfrak{B}$, hence $\mathfrak{B}$ is not representable since $\mathfrak{R}$ is not representable. Clearly, $\mathrm{Bg}_2 = \mathrm{CA}_2 \nsubseteq \mathrm{Rp}_2$ (cf. [11]). Monk [22, p. 199] notes that $\mathrm{Bg}_3 \nsubseteq \mathrm{Rp}_3$. Also, $\mathrm{Bg}_4 \nsubseteq \mathrm{Rp}_4$ can be seen as follows: Let $\mathfrak{R}$ be a nonrepresentable relation algebra. Then by [11, 5.3.17] there is $\mathfrak{B} \in \mathrm{Bg}_4$ such that $\mathfrak{R} \subseteq \mathfrak{Ra}\mathfrak{Nr}_3\mathfrak{B}$. If $\mathfrak{B}$ were representable, so would be $\mathfrak{R}$. Hence $\mathfrak{B} \notin \mathrm{Rp}_4$.

Relation algebras (RA's) form another algebraization of first-order logic, see e.g. Tarski–Givant [35] (and Remark 3.19 in Section 3 herein). Tarski proved that the equational theory of RA's as well as that of the representable RA's are undecidable but r.e. For RA theory see either one of Jónsson [14, 15], Maddux [16], Section 5.3 of [11] or Chapter 8 of [35]. Recall that in RA theory the semi-colon ';' denotes the operation of relation-composition.

Definition 1.5. We call a relation algebra $\mathfrak{R}$ monadic-generated iff $(\exists G \subseteq R)$ $[R = \mathrm{Sg}G$ and $(\forall x \in G)\ x\ ;\ 1 = x]$.

Theorem 4. *The equational theory of monadic-generated RA's is not r.e. Every monadic-generated RA is representable.*

Now we turn to subclasses of Mg_α which were touched upon in Theorem 2. Recall from [11, 2.4.61, 2.4.62] that a CA_α $\mathfrak{A}$ is of characteristic 0 iff $\mathfrak{A} \models \{c_{(n)}\bar{d}(n \times n) = 1 : n \in \omega \cap (\alpha + 1)\}$ and $|A| \neq 1$, where $c_{(n)}x \triangleq c_0 c_1 \cdots c_{n-1}x$. For $\mathfrak{A} \in \mathrm{Gs}_\alpha$, $\alpha \geq \omega$ this means that every subbase of $\mathfrak{A}$ is infinite (and $|A| \neq 1$). (This is in a sense the opposite of being bounded. Namely, $\mathfrak{A}$ is of characteristic 0 iff $\mathbf{H}\mathfrak{A}$ contains no (nondiscrete) bounded subclass.)

Notation (cf. [11, 3.1.5] for $\alpha \geq \omega$). For any $K \subseteq \mathrm{CA}_\alpha$ we denote

$$_\infty K \triangleq \{\mathfrak{A} \in K : \mathfrak{A} \text{ is of characteristic 0 or } |A| = 1\}.$$

$_\infty\mathrm{Gs}_\alpha$ or $_\infty\mathrm{CA}_\alpha$, $\alpha \geq \omega$ cannot be characterized inside Gs_α or CA_α by a single formula because there is a system of minimal Cs_α's with finite bases such that an ultraproduct of this system is of characteristic 0. Below we prove the opposite for Mg_α. Namely, we shall prove that within Mg_α the property of being of characteristic 0 can be expressed by a *single equation.* (We note that, because of the above ultraproduct reason, there is no Σ_1^0-sentence characterizing $_\infty\mathrm{Mg}_\alpha$ inside Mg_α.)

For a set Σ of formulas, $\mathrm{Mod}\,\Sigma$ denotes the class of all algebras in which Σ is valid.

Theorem 5. *There is a single equation e such that* $_\infty\mathrm{Mg}_\alpha = \mathrm{Mg}_\alpha \cap \mathrm{Mod}\{e\}$ *for every* $\alpha \geq \omega$, *hence* $_\infty\mathrm{Mn}_\omega = \mathrm{Mn}_\omega \cap \mathrm{Mod}\{e\}$.

We turn to formulating results to the effect that Mn and Mg are 'very large'. Their various closures contain all bounded classes of CA's or Lf's (depending on the closure). For the precise formulation we need some notation. For a Gs_α $\mathfrak{A}$, $\mathrm{Subb}(\mathfrak{A})$ denotes the set of all subbases of $\mathfrak{A}$ and $\mathrm{base}(\mathfrak{A}) \triangleq \bigcup \mathrm{Subb}(\mathfrak{A})$.

Definition 1.6.

$$\mathrm{Fb}'\mathrm{Gs}_\alpha \triangleq \{\mathfrak{A} \in \mathrm{Gs}_\alpha : |\mathrm{base}(\mathfrak{A})| < \omega\}.$$

$$\mathrm{Bb}'\mathrm{Gs}_\alpha \triangleq \{\mathfrak{A} \in \mathrm{Gs}_\alpha : (\exists n \in \omega)(\forall U \in \mathrm{Subb}(\mathfrak{A}))\, |U| < n\}.$$

Let K be any class of algebras similar to CA_α's. Then

$$\mathrm{F}K \triangleq \{\mathfrak{A} \in K : |A| < \omega\},$$

$$\mathrm{Fb}K \triangleq K \cap \mathbf{I}\mathrm{Fb}'\mathrm{Gs}_\alpha, \qquad \mathrm{Bb}K \triangleq K \cap \mathbf{I}\mathrm{Bb}'\mathrm{Gs}_\alpha.$$

(Here Fb refers to **f**inite **b**ase and Bb to **b**ounded sub-**b**ase.)

Note that for $\alpha \geq \omega$, $\mathfrak{A} \in \mathrm{BbCA}_\alpha$ iff $\{\mathfrak{A}\}$ is bounded (using [11, 3.2.11(vi)]).

Recall from [11] that $\mathrm{Lf}_\alpha \triangleq \{\mathfrak{A} \in \mathrm{CA}_\alpha : (\forall x \in A)\, |\Delta x| < \omega\}$.

Theorem 6. (i) $\mathrm{EqMg}_\alpha = \mathrm{EqMn}_\alpha = \mathrm{EqFbCs}_\alpha$ *for* $\alpha \geq \omega$.

(ii) $\mathbf{HSUp}\mathrm{Mn}_\alpha \subset \mathbf{HSUp}\mathrm{Mg}_\alpha$ *for* $\alpha \geq 2$, *i.e., there is a universal disjuntion of equations that holds in* Mn_α *but not in* Mg_α.

(iii) $\mathrm{UnMg}_\alpha = \mathrm{UnFbMg}_\alpha = \mathrm{UnBbGs}_\alpha$ *for any* α.

(iv) $\mathrm{BbLf}_\alpha \subseteq \mathbf{S}\mathrm{Mg}_\alpha$ *for any* α.

(v) *There is a* Π_2*-formula distinguishing the hereditarily nondiscrete* Mn_α*'s and* Mg_α*'s for* $\alpha \geq \omega$.

Remark 1.7. (a) We prove in this paper, when proving Theorem 2(ii), directly that $\overline{\mathrm{Eq}}\mathrm{Mn}_\omega$ is not r.e. (in Part (A) of that proof). By the first part $\mathrm{EqMg}_\omega = \mathrm{EqMn}_\omega$ of Theorem 6(i) we get a second proof: namely proving that $\overline{\mathrm{Eq}}\mathrm{Mn}_\omega$ is not r.e. as a corollary of "$\overline{\mathrm{Eq}}\mathrm{Mg}_\omega$ is not r.e." However, using the second part $\mathrm{EqMn}_\omega = \mathrm{EqFbCs}_\omega$ of Theorem 6(i), one can give still another proof for "$\overline{\mathrm{Eq}}\mathrm{Mn}_\omega$ is not r.e." (not using anything else). We sketch here this alternative proof.

We shall use the facts that the set of formulas valid in the finite models is not r.e., and that FbCs_ω corresponds somehow to the finite models. Let φ' be any (first-order) formula. We may assume that φ' is restricted by [11, 4.3.6]. Let the variables occurring in φ' be among $v_0, \ldots, v_N$. Replace each primitive subformula $R(v_0, \ldots, v_n)$ in φ' with $\forall v_{n+1} \cdots v_N R(v_0, \ldots, v_N)$. Then we get another formula φ such that each relation symbol occurring in φ has rank (arity) $M \triangleq N+1$, all variables occurring in φ are among $v_0, \ldots, v_N$ and [φ is valid in the finite models (FMod) iff φ' is valid in FMod]. From now on, the proof is basically the same as that of [11, 4.3.62]: Recall the cylindric term $\tau\mu'\varphi$ associated to φ from [11, 4.3.60]. We will show that $\mathrm{FMod} \models \varphi$ iff $\mathrm{FbCs}_\omega \models \tau\mu'\varphi = 1$. Assume $\mathfrak{M} \not\models \varphi$ for some $\mathfrak{M} \in \mathrm{FMod}$. Then $\mathfrak{Cs}^{\mathfrak{M}} \not\models \tau\mu'\varphi = 1$ and $\mathfrak{Cs}^{\mathfrak{M}} \in \mathrm{FbCs}_\omega$ can easily be seen, where $\mathfrak{Cs}^{\mathfrak{M}}$ is defined in [11, 4.3.4]. Assume $\mathfrak{C} \not\models \tau\mu'\varphi = 1$ for some $\mathfrak{C} \in \mathrm{FbCs}_\omega$. Then $\mathfrak{Rd}_M\mathfrak{C} \not\models \tau\mu'\varphi = 1$, where $\mathfrak{Rd}_M\mathfrak{C}$ is the M-dimensional reduct of $\mathfrak{C}$, hence $\mathfrak{C}' \not\models \tau\mu'\varphi = 1$ for some $\mathfrak{C}' \in \mathrm{FbCs}_M$ by Lemma 3.22(ii) in the proof of Theorem 6 herein. From this $\mathfrak{C}'$ then one can easily construct a model $\mathfrak{M} \in \mathrm{FMod}$ for which $\mathfrak{M} \not\models \varphi$. The above shows that $\overline{\mathrm{Eq}}\mathrm{FbCs}_\omega$ is not r.e.

The present direction of producing a simple proof for the special corollary Theorem 1(i) can be carried even further. Namely, in the above proof we used Theorem 6(i) which, in turn, is proved in Section 3. In Section 2, in Remark 2.6, we modify the proof of Theorem 6(i) by optimizing it with the simpleminded goal of obtaining a streamlined proof for the particular corollary Theorem 1(i) saying "$\overline{\mathrm{Eq}}\mathrm{Mn}_\omega$ is not r.e.", and trying to make this special proof as simple as possible.

(b) Theorem 6(iv) is not true for Lf_α in general, neither for those $\mathrm{Gs}_\alpha \cap \mathrm{Lf}_\alpha'$s with all subbases finite. To show this, let $\{U_i : i \in \omega\}$ be a set of disjoint sets such that $(\forall i \in \omega)\ |U_i| = i+2$. Let $V = \bigcup\{{}^\alpha U_i : i \in \omega\}$ and let $s \subseteq \bigcup\{{}^2 U_i : i \in \omega\}$ be a one–one function with no fix-point and with domain $\bigcup\{U_i : i \in \omega\}$. Let $X \triangleq \{z \in V : 2 \upharpoonright z \in s\}$, and $\mathfrak{B} \triangleq \mathfrak{Sg}^{(\mathfrak{Sb}V)}\{X\}$. Then $\mathfrak{B} \in \mathrm{Gs}_\alpha^{\mathrm{reg}} \cap \mathrm{Lf}_\alpha$, each subbase of which is finite. But $\mathfrak{B} \notin \mathbf{S}\mathrm{Mg}_\alpha$ by Lemma 3.3 in the proof of Theorem 2 in Section 3.

(c) We do not know whether there is a universal formula distinguishing the hereditarily nondiscrete Mn_α's and Mg_α's.

Remark 1.8. We know that the first-order theory $\theta\rho K$ is undecidable for every class $K \supseteq \mathrm{Rp}_\alpha$ of similar algebras, $\alpha \geqslant 1$, further $\theta\rho\mathrm{Mg}_\alpha$, $\theta\rho\mathrm{Bg}_\alpha$ and $\theta\rho\mathrm{Crs}_\alpha$ are undecidable for $\alpha \geqslant 1$. By Theorems 1, 3 and Remark 1.4, $\overline{\mathrm{EqMg}_\alpha}$ is decidable iff $\alpha \leqslant 2$ and the same holds for $\overline{\mathrm{EqBg}_\alpha}$. We proved in [28] that $\overline{\mathrm{EqCrs}_\alpha}$ is decidable for all α.

Proof (of the first sentence). Let $K \supseteq \mathrm{Rp}_\alpha$, $\alpha \geqslant 1$. We show that $\theta\rho K$ is undecidable. Let φ be any formula in the language of CA_1. Let $\bar{\varphi}(x)$ be the formula about $x \in \mathfrak{A} \in K$, saying $(\mathfrak{Rd}_1\mathfrak{Rl}_x\mathfrak{A} \in \mathrm{CA}_1 \rightarrow \mathfrak{Rd}_1\mathfrak{Rl}_x\mathfrak{A} \models \varphi)$. This $\bar{\varphi}(x)$ can be obtained as follows. Let $\gamma(x)$ say "$\mathfrak{Rd}_1\mathfrak{Rl}_x\mathfrak{A} \in \mathrm{CA}_1$" as follows: We translate, e.g., $c_0(c_0y \cdot z) = c_0y \cdot c_0z$ (this is C3) as follows. $(\forall y, z \leqslant x)$ $x \cdot c_0(x \cdot c_0y \cdot z) = x \cdot c_0y \cdot x \cdot c_0z$. Let us call this $x \upharpoonright \mathrm{C3}$. Then $\gamma(x)$ is $(x \upharpoonright \mathrm{C0} \wedge \cdots \wedge x \upharpoonright \mathrm{C7})$. We may assume that x does not occur in φ and that φ is a sentence. Then $x \upharpoonright \varphi$ is the relativization of φ to x, that is we replace c_0y by $x \cdot c_0y$ and $\exists y$ by $(\exists y \leqslant x)$ and $\forall y$ by $(\forall y \leqslant x)$. Now $\bar{\varphi}(x)$ is the formula $\gamma(x) \rightarrow x \upharpoonright \varphi$. Now we

Claim $\mathrm{CA}_1 \models \varphi$ iff $K \models \bar{\varphi}(x)$.

Proof. ($\Leftarrow$): Assume $\mathfrak{B} \in \mathrm{CA}_1$ and $\mathfrak{B} \not\models \varphi$. Then $\mathfrak{B} \subseteq \mathrm{P}_{i\in I}\mathfrak{C}_i$ with $\mathfrak{C}_i \in \mathrm{Cs}_1$. Let $U_i \triangleq \mathrm{base}(\mathfrak{C}_i) \dot{\cup} \{a_i\}$ be a disjoint union. Let $f_i = \langle \{(a_i : j \in \alpha)^0_b : b \in x\} : x \in C_i \rangle$ for $i \in I$. Then $f_i : C_i \rightarrow \mathrm{Sb}^\alpha U_i$. Let $hx = \langle f_i x_i : i \in I \rangle : B \rightarrow \mathrm{P}_{i\in I}(\mathrm{Sb}^\alpha U_i)$. Let $\mathfrak{A} = \mathfrak{Sg}\mathfrak{Sb}(\bigcup_{i\in I} {}^\alpha U_i)(h^*B)$. Let $X = h(1^{\mathfrak{B}})$. Then $\mathfrak{Rd}_1\mathfrak{Rl}_x\mathfrak{A} \cong \mathfrak{B}$, thus $\mathfrak{A} \not\models \varphi[X]$. Hence $K \not\models \bar{\varphi}(x)$ by $\mathfrak{A} \in K$.

($\Rightarrow$): Assume $\mathfrak{A} \in K$ and $\mathfrak{A} \not\models \bar{\varphi}[X]$, $X \in A$. Then $\mathfrak{Rd}_1\mathfrak{Rl}_x\mathfrak{A} \in CA_1$ and $\mathfrak{Rd}_1\mathfrak{Rl}_x\mathfrak{A} \not\models \varphi$. □ (Claim)

Since $\theta\rho\mathrm{CA}_1$ is undecidable, the above shows that $\theta\rho K$ is undecidable, too. For $\alpha \geqslant 1$, $\theta\rho\mathrm{Mg}_\alpha$ is undecidable, this is obvious for $\alpha \neq 2$ by the rest of this paper (and [11, 4.2.23] for $\alpha = 1$, since $\mathrm{Mg}_1 = \mathrm{CA}_1$), while the case $\alpha = 2$ follows from the fact that, in the language of Mg_2, we can speak about $\mathrm{Nr}_1\mathrm{Mg}_2 = \mathrm{CA}_1$. For $\alpha \leqslant 2$ we have $\mathrm{Bg}_\alpha = \mathrm{CA}_\alpha$ and $\theta\rho\mathrm{CA}_\alpha$ is undecidable by [11, 4.2.23, 4.2.25]. For $\alpha > 2$, $\theta\rho\mathrm{Bg}_\alpha$ is undecidable since $\overline{\mathrm{EqBg}_\alpha}$ is such by Theorem 3(i) and Remark 1.4. □

Related results are e.g. in Maddux [17] and in Schönfeld [33].

The results and techniques used in this paper give some information on the lattice of varieties of CA's. We turn briefly to this subject.

1.1. Lattice of varieties of CA_α's, $\alpha \geqslant \omega$

Let $\alpha \geqslant \omega$ and let Var denote the lattice[3] of varieties of CA_α's. The following notation will be useful. Let $n \in \omega$ and $K \subseteq \mathrm{CA}_\alpha$. Then ${}_{<n}K \triangleq K \cap \mathrm{Mod}(\bar{d}(n \times$

[3] There are set-theoretical inconveniences when we say that the elements of Var are *classes* or when we use a notation like $\{K : K \subseteq \mathrm{CA}_\alpha\}$. However, these problems are only of a notational character. Various ways of avoiding them are rather well known by now (e.g., one can use a conservative extension of Bernays–Gödel set theory in which classes of classes form a third sort). Therefore we simply ignore these problems here.

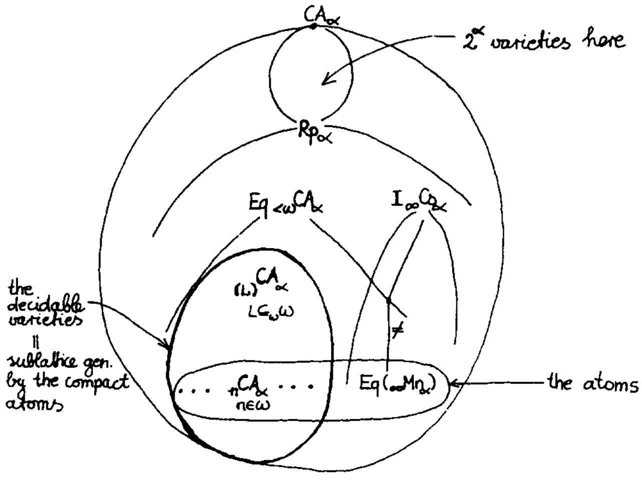

Fig. 1.

$n)=0)$, ${}_nK \triangleq {}_{<n+1}K \sim {}_{<n}K$, ${}_\omega K \triangleq {}_\infty K$ (cf. the notation preceding Theorem 5), ${}_{<\omega}K \triangleq \bigcup\{{}_nK : n \in \omega\}$ and more generally, if $L \subseteq \omega+1$, then ${}_{(L)}K \triangleq \bigcup\{{}_nK : n \in L\}$. Thus $K \subseteq \mathrm{CA}_\alpha$ is bounded iff $(\exists n \in \omega)\ K \subseteq {}_{<n}\mathrm{CA}_\alpha$. Further $\mathrm{BbGs}_\alpha = {}_{<\omega}\mathrm{Gs}_\alpha$, $\mathrm{BbCA}_\alpha = {}_{<\omega}\mathrm{CA}_\alpha$ and $\mathrm{FbCs}_\alpha = {}_{<\omega}\mathrm{Cs}_\alpha$. For $n \in \omega$, $n \neq 0$, ${}_nK$ is the class of members of K of characteristic n and ${}_n\mathrm{Cs}_\alpha$ is the class of all cylindric set algebras with bases of cardinality n.

Var is a distributive lattice since CA_α is a congruence-distributive variety.

About some important elements of Var: The most important subvariety of CA_α is Rp_α. It is known that $\mathrm{Rp}_\alpha = \mathbf{I}\mathrm{Gs}_\alpha = \mathrm{EqLf}_\alpha$. Another characteristic subvariety is $\mathbf{I}\,{}_\infty\mathrm{Cs}_\alpha = \mathbf{I}\,{}_\infty\mathrm{Gs}_\alpha = {}_\omega\mathrm{Rp}_\alpha$. Let $n \in \omega$ and $L \subseteq_\omega \omega$ be finite. Then ${}_{<n}\mathrm{CA}_\alpha$, ${}_n\mathrm{CA}_\alpha$ and ${}_{(L)}\mathrm{CA}_\alpha$ are subvarieties of Rp_α (by [11, 3.2.53]).

(1) *The atoms of* Var. The lattice Var is atomic[4] and has exactly ω many atoms. The atoms of Var are $\mathrm{Eq}({}_n\mathrm{Mn}_\alpha)$ for $n \leq \omega$. This can be seen as follows: Let $V \in$ Var be an atom. Let $\mathfrak{A} \in V$ be arbitrary and let $\mathfrak{M}$ be the minimal subalgebra of $\mathfrak{A}$. Then $\mathfrak{M} \in V$, hence $\mathrm{Eq}\{\mathfrak{M}\} = V$, since V is an atom, and $\mathbf{I}\{\mathfrak{M}\} = {}_n\mathrm{Mn}_\alpha$ for some $n \leq \omega$.

For $n < \omega$ we have 'good' characterizations[5] of the atoms $\mathrm{Eq}_n\mathrm{Mn}_\alpha : \mathrm{Eq}({}_n\mathrm{Mn}_\alpha) = \mathbf{I}\,{}_n\mathrm{Gs}_\alpha = {}_n\mathrm{CA}_\alpha$ (see Corollary 3.15).

For $n = \omega$ we do not know of a 'good' characterization; but we know the following.

Theorem 7. *Let* $\alpha \geq \omega$. *Then* (i)–(ii) *below hold.*

(i) $\mathrm{Eq}({}_\infty\mathrm{Mn}_\alpha) = \mathrm{Eq}({}_\infty\mathrm{Mg}_\alpha)$.

(ii) $\mathrm{Eq}({}_\infty\mathrm{Mn}_\alpha) \subset \mathrm{EqMn}_\alpha \cap \mathbf{I}\,{}_\infty\mathrm{Cs}_\alpha$.

Cf. also Theorem 5. Theorem 7(ii) implies that the characterization of the n-th atom does not generalize to the ω-th atom.

[4] Note that every lattice of subvarieties of a variety is atomic, see e.g. [3].

[5] By a 'good' characterization we mean one not involving 'Eq'.

(2) *Suprema of atoms in* Var. The supremum of all the atoms is EqMn_α. We have an 'almost good' characterization for EqMn_α:

$$\mathrm{EqMn}_\alpha = \mathrm{EqMg}_\alpha = \mathrm{EqFbGs}_\alpha = \mathrm{Eq}\ {}_{<\omega}\mathrm{CA}_\alpha.$$

The supremum of infinitely many atoms in Var always contains ${}_\infty\mathrm{Mn}_\alpha$, and is never simply a union (for proof see the proof of Theorem 7(ii)): Let $L \subseteq \omega + 1$ be infinite. Then

$$\operatorname*{Sup}_{\mathrm{Var}} \{\mathrm{Eq}({}_n\mathrm{Mn}_\alpha) : n \in L\} = \mathrm{Eq}({}_{(L\cup\{\omega\})}\mathrm{Mn}_\alpha).$$

This shows that $\mathrm{Eq}({}_\infty\mathrm{Mn}_\alpha)$ is not a compact atom in Var.

The supremum of finitely many of the other atoms, $\mathrm{Eq}({}_n\mathrm{Mn}_\alpha)$ for $n \in \omega$, is just their union: Let $L \subseteq \omega$ be finite. Then

$$\operatorname*{Sup}_{\mathrm{Var}} \{\mathrm{Eq}({}_n\mathrm{Mn}_\alpha) : n \in L\} = \bigcup \{\mathrm{Eq}({}_n\mathrm{Mn}_\alpha) : n \in L\}$$

$$= \mathrm{Eq}\ {}_{(L)}\mathrm{Mn}_\alpha = {}_{(L)}\mathrm{CA}_\alpha.$$

This follows from Lemma 3.17. Therefore $\mathrm{Eq}({}_n\mathrm{Mn}_\alpha)$ for $n \in \omega$ is a compact atom. Also, ${}_{<n}\mathrm{CA}_\alpha$, or more generally ${}_{(L)}\mathrm{CA}_\alpha$ for $L \subseteq_\omega \omega$, contains only finitely many varieties, namely $\{K \in \mathrm{Var} : K \subseteq {}_{<n}\mathrm{CA}_\alpha\} = \{{}_{(L)}\mathrm{CA}_\alpha : L \subseteq n\}$ (or more generally $\{K \in \mathrm{Var} : K \subseteq {}_{(L)}\mathrm{CA}_\alpha\} = \{{}_{(G)}\mathrm{CA}_\alpha : G \subseteq L\}$ for $L \subseteq {}_\omega\omega$).

(3) *Decidable varieties.* The set of all decidable varieties of CA_α's is exactly the sublattice generated by the compact atoms in Var, i.e., the decidable subvarieties of CA_α are exactly the finite unions of $\mathbf{I}\,{}_n\mathrm{Gs}_\alpha$'s, $n \in \omega$. This is proved in [29]. Thus $\{K \in \mathrm{Var} : K \subseteq \mathrm{CA}_\alpha$ and K is decidable$\} = \{{}_{(L)}\mathrm{CA}_\alpha : L \subseteq {}_\omega\omega\}$.

(4) *On the subvarieties of* $\mathrm{Eq}\ {}_{<\omega}\mathrm{CA}_\alpha$. $\{\mathrm{Eq}K : K \subseteq {}_{<\omega}\mathrm{CA}_\alpha\} = \{\mathrm{Eq}\ {}_{(L)}\mathrm{Mn}_\alpha : L \subseteq \omega\} \subset \{\mathrm{Eq}K : K \subseteq \mathrm{EqMn}_\alpha\}$. I.e., there is $K \in \mathrm{Var}$ such that $K \subseteq \mathrm{EqMn}_\alpha$ but $\mathrm{Eq}(K \cap \mathrm{Mn}_\alpha) \subset K$. An example of such a K is $\mathrm{EqMn}_\alpha \cap \mathbf{I}\,{}_\infty\mathrm{Cs}_\alpha$, see Theorem 7(ii). For the first equality see Lemma 3.17. Thus $|\{\mathrm{Eq}K : K \subseteq {}_{<\omega}\mathrm{CA}_\alpha\}| = 2^\omega$. We do not know whether $|\{K \in \mathrm{Var} : K \subseteq \mathrm{EqMn}_\alpha\}| = 2^\alpha$ or not.

(5) *On the number of subvarieties.* It is proved in [11, 4.1.24–28], a result of J. Johnson, that there are $\geq 2^\omega$ varieties below Rp_α, for[6] every $\alpha \geq 2$. This gave rise to the problem stated as Problem 4.2 in [11], whether there are 2^α varieties below Rp_α or not. In [30] we show that there are 2^α varieties of CA_α containing Rp_α. About the logical meaning we note the following: the number of subvarieties of CA_α corresponds, roughly, to the number of (syntactical) scheme-theories. Concerning 'normal' first-order theories, we do not have more than 2^ω theories (in a countable similarity type) even if we allow more than ω many individual

[6] We note that M. Rubin proved in 1985 that there are $\geq 2^\omega$ varieties below ${}_\infty\mathrm{Rp}_\alpha$, too.

variables. But if $|\alpha| > \omega$, then there are strictly more than 2^ω scheme-theories, by the above mentioned result in [30].

Remark 1.9. In [30], the lattice of subvarieties of CA_α is investigated (for both finite and infinite α). The following are proved, among others, in [30]: For any ordinal α, let Var_α denote the lattice of subvarieties of CA_α.

(a) Let $\alpha \geqslant 3$. If $n \in \omega \cap (\alpha+1)$, then ${}_nCA_\alpha$ has a complement variety $-{}_nCA_\alpha$ in Var_α. The center $Z(Var_\alpha)$ of the lattice Var_α is the sublattice generated by $\{{}_nCA_\alpha, -{}_nCA_\alpha : n \in \omega \cap (\alpha+1)\}$.

(b) Let $\alpha > 1$. There are infinitely many co-atoms in Var_α. Actually, let $\mathfrak{Sb}({}^\alpha n)$ denote the Cs_α with unit ${}^\alpha n$ and universe the powerset of ${}^\alpha n$. Then $\mathfrak{Sb}({}^\alpha n)$ is a splitting algebra and the conjugate variety of $\mathfrak{Sb}({}^\alpha n)$ is a co-atom of Var_α, for every $n \in \omega$. (For these notions see, e.g., Jónsson [14].)

(c) Let $\alpha < \omega$. Define $\mathrm{div} \triangleq \bar{d}_\alpha + \sum \{\bar{d}_n - c_{(\alpha)}\bar{d}_{n+1} : n < \alpha\}$ where $\bar{d}_n \triangleq \bar{d}(n \times n)$. Then $c_{(\alpha)}(\mathrm{div} \cdot x) \cdot c_{(\alpha)}(\mathrm{div} - x) = 0$ is an equational basis for Mn_α. Further, $\{c_{(\alpha)}\bar{d}_\alpha = 1,\ c_{(\alpha)}(\bar{d}_\alpha \cdot x) \cdot c_{(\alpha)}(\bar{d}_\alpha - x) = 0\}$ is an equational basis for ${}_\infty Mn_\alpha$.

Related results on lattices of varieties are in Blok [3], [4] and in Jónsson [14].

2. Formulating the results in their logical form

In the introduction we introduced all the machinery needed for stating the theorems of this paper in a purely logical form (and for investigating things further from a logical point of view, too).

Let $\mathfrak{M}$ be an arbitrary model (of an arbitrary first-order language) and let σ be an arbitrary scheme. Then we say that $\mathfrak{M} \models \sigma$ iff $\mathfrak{M} \models \sigma'$ for every instance σ' of σ which is in the language of $\mathfrak{M}$. Let K be any class of models (perhaps of different languages). Then the scheme-theory $S\theta\rho K$ of K is defined to be $\{\sigma : \sigma$ is a scheme and $(\forall \mathfrak{M} \in K)\ \mathfrak{M} \models \sigma\}$. The 'normal' first-order theory $\theta\rho K$ of K, if K is a class of similar models, is defined to be $\{\varphi : \varphi$ is a formula of the language of K and $K \models \varphi\}$. Now we define some classes of models. (If $\mathfrak{M}$ is a model, then M denotes its universe or carrier set.)

$$\mathrm{Equmd} \triangleq \{\langle M, =\rangle : M \text{ is a set}\},$$

$$\mathrm{Monmd} \triangleq \{\mathfrak{M} : \mathfrak{M} \text{ is a model with unary relations only}\}$$

$$\text{1-Binmd} \triangleq \{\langle M, R\rangle : R \subseteq {}^2M\},$$

$$\mathrm{Mod} \triangleq \{\mathfrak{M} : \mathfrak{M} \text{ is a model}\},$$

$$\mathrm{FMod} \triangleq \{\mathfrak{M} \in \mathrm{Mod} : |M| < \omega\}.$$

Theorem 2.1. *Statements* (i)–(v) *below hold.*

(i) $S\theta\rho(\mathrm{Equmd}) = S\theta\rho(\mathrm{Monmd}) = S\theta\rho(\mathrm{FMod})$ *is not r.e.*[7]

[7] Independently of us, M. Rubin also proved that $S\theta\rho(\mathrm{Equmd}) = S\theta\rho(\mathrm{FMod})$ is not r.e.

(ii) *$S\theta\rho$(1-Binmd) = $S\theta\rho$(Mod) is r.e.*
(iii) *There is a scheme σ such that for every $\mathfrak{M} \in$ Monmd, $\mathfrak{M} \models \sigma$ iff $|M| \geq \omega$.*
(iv) *Let $K \subseteq$ Monmd. Then (a)–(b) below hold.*
  (a) *$S\theta\rho(K)$ is either decidable or not r.e.*
  (b) *$S\theta\rho(K)$ is r.e. iff $(\exists n \in \omega)(\forall \mathfrak{M} \in K)\, |M| \leq n$.*

Remark 2.2. (a) In Theorem 2.1: (i) follows from Theorem 1(i) + Theorem 6(i), (ii) follows from Theorem 3, (iii) follows from Theorem 5, (iv) follows from Theorem 2(ii). We give a direct, logical proof for Theorem 2.1(i) in Remark 2.6. We note that by using the theorem proved in [29], the following generalization of Theorem 2.1(iv)(b) is also true:

(*) Let $K \subseteq$ Mod. Then $S\theta\rho K$ is decidable iff $(\exists n \in \omega)(\forall \mathfrak{M} \in K)\, |M| \leq n$.

(b) One would think that the fact that $S\theta\rho$(FMod) is not r.e. might be a trivial corollary of the fact that $\theta\rho$(FMod) is not r.e. This is not so. Shortly we turn to investigating the connection between $S\theta\rho(K)$ and $\theta\rho K$, where we prove $S\theta\rho K$ is r.e. $\not\Rightarrow \theta\rho K$ is r.e., for $K \subseteq$ Mod. The assumption $\mathfrak{M} \in$ Monmd is necessary in Theorem 2.1(iii), cf. the remark following the definition of ${}_\infty K$ in Section 1. Concerning Theorem 2.1(iv)(a), there is $K \subseteq$ Equmd such that $N(K) \triangleq \{\sigma : \sigma$ is a scheme and $K \not\models \sigma\}$ is not r.e.: Let $N \subseteq \omega$ be such that N is not r.e. Define $K \triangleq \{\langle n, =\rangle : n \in N\}$. For every $n \in \omega$ let $\sigma_n \triangleq$ "there exist n elements" $\rightarrow$ "there exist $n+1$ elements". Then $(\forall n \in \omega)\,(K \not\models \sigma_n$ iff $n \in N)$, showing that $N(K)$ is not r.e.

Now, we turn to investigating a bit the connection between the scheme-theory $S\theta\rho K$ and the 'normal' first-order theory $\theta\rho K$ of a class K of similar models. As we have already seen, $\theta\rho K$ decidable $\not\Rightarrow S\theta\rho K$ r.e., a counterexample is $K =$ Equmd. In the other direction, first we note that the obvious way of turning a hypothetical enumeration of $S\theta\rho K$ into an enumeration of $\theta\rho K$ does not work; namely there is a valid monadic formula φ such that φ is an instance of no monadic valid formula scheme $\bar{\varphi}$. E.g., $\exists v_1 R(v_0) \leftrightarrow R(v_0)$ is such a monadic formula. (But here being monadic is not necessary, e.g., $\exists v_2 R(v_0 v_1) \leftrightarrow R(v_0 v_1)$ is such a formula, too.) And indeed, next we will show that "$S\theta\rho K$ r.e. $\not\Rightarrow \theta\rho K$ r.e.". We do not know whether "$S\theta\rho K$ decidable $\Rightarrow \theta\rho K$ r.e." holds or not.

Proposition 2.3. (i) *There is a class K of similar models such that $\theta\rho K$ is not r.e. while $S\theta\rho K$ is r.e. Moreover, K has only one binary relation symbol.*

(ii) *There is a model $\mathfrak{M}$, with $\theta\rho\mathfrak{M}$ not r.e. but $\overline{\mathrm{Eq}}\mathfrak{Cs}^{\mathfrak{M}}$ r.e. where $\mathfrak{Cs}^{\mathfrak{M}}$ is the* Cs_ω *associated to $\mathfrak{M}$ in* [11, § 4.3] *and $\mathfrak{M}$ has only one binary relation symbol.*

Proof. Let U be the set of all hereditarily finite sets and let $\mathfrak{A} \triangleq \langle U, \in\rangle$. Then $\theta\rho\mathfrak{A}$ is well known to be not r.e. Let $\mathfrak{M} \triangleq \langle U; \in, R : R \subseteq {}^2U\rangle$. Then the two projection functions $U \times U \rightarrow U$ are in $\mathfrak{M}$, hence $(\forall n \in \omega)\,(\forall T \subseteq {}^nU)\, T$ is definable without parameters in $\mathfrak{M}$. Therefore the same schemes are valid in $\mathfrak{M}$ as in $_{\infty}\text{Mod}$, where $_{\infty}\text{Mod} = \{\mathfrak{M} \in \text{Mod}: |M| \geq \omega\}$. Namely, if $\mathfrak{N} \in {}_{\infty}\text{Mod}$ and $\mathfrak{N} \nvDash \sigma$ (for some scheme σ), then there is a finite reduct $\langle N, R_1, \ldots, R_n\rangle \nvDash \sigma$ of $\mathfrak{N}$ with the same property. We may assume $N = U$ by the Löwenheim–Skolem theorems. By definability of $R_1, \ldots, R_n$ in $\mathfrak{M}$ we have $\mathfrak{M} \nvDash \sigma$. Since there are only countably many schemes, we need only countably many of the relations in $\mathfrak{M}$. By using techniques similar to the ones in the proof of Theorem 3(ii), we can code up all these relations of $\mathfrak{M}$ together with epsilon into a single binary $B \subseteq M \times M$. Hence $\langle M, B\rangle$ has the desired properties. We have proved $\mathfrak{M} \vDash \sigma \Rightarrow {}_{\infty}\text{Mod} \vDash \sigma$. The other direction is trivial. Since $\theta\rho({}_{\infty}\text{Mod})$ is r.e., by Corollary 2.5 below the schemes valid in $_{\infty}\text{Mod}$ and therefore those valid in $\mathfrak{M}$ are r.e. Obviously, $\theta\rho\mathfrak{M}$ is not r.e. since $\theta\rho\mathfrak{A}$ is not such. □

However, in some special cases, when K is defined in a 'simple' way, recursive enumerability (and also decidability) of $S\theta\rho K$ does imply recursive enumerability of $\theta\rho K$, cf. Corollary 2.5 below. We begin with some simple facts.

Lemma 2.4. *Let φ be a formula. Then there is a scheme $\bar{\varphi}$ such that for every cardinal κ we have $\{\mathfrak{M}: |M| = \kappa$, $\mathfrak{M}$ is a model of the language of $\varphi\} \vDash \varphi$ iff $\{\mathfrak{M} \in \text{Mod}: |M| = \kappa\} \vDash \bar{\varphi}$.*

Moreover, $\bar{\varphi}$ can be computed recursively from φ.

The proof of Lemma 2.4 can be recovered from the proof of [11, 4.3.62] together with Remark 1.7(a).

Let $L \subseteq \text{Cardinals}$ and let Λ be any first-order language. Then $_L\text{Mod} \triangleq \{\mathfrak{M} \in \text{Mod}: |M| \in L\}$ and $_L\text{Mod}_\Lambda \triangleq \{\mathfrak{M} \in {}_L\text{Mod}: \mathfrak{M}$ is a model of the language $\Lambda\}$.

Corollary 2.5. *Let $L \subseteq \text{Cardinals}$ and let Λ be any first-order language. Consider statements* (i)–(iii) *below. Then* (i) ⇒ (ii) *and* (i) ⇔ (iii) *hold. Further, if there are relation symbols of arbitrarily large finite arities in Λ, then* (i) ⇔ (ii) *holds, too.*

(i) *The set of schemes valid in $_L\text{Mod}$ is r.e. (decidable).*
(ii) *The set of formulas valid in $_L\text{Mod}_\Lambda$ is r.e. (decidable).*
(iii) $\overline{\text{Eq}}\{\mathfrak{A} \in \text{Cs}_\omega : |\text{base}(\mathfrak{A})| \in L\}$ *is r.e. (decidable).*

We conjecture that (i) ⇔ (ii) in Corollary 2.5 holds for arbitrary non-monadic language Λ.

Proof of Corollary 2.5. (i) ⇒ (ii) follows from Lemma 2.4.

(i) ⇔ (iii). Let $_L\text{Cs}_\omega \triangleq \{\mathfrak{A} \in \text{Cs}_\omega : |\text{base}(\mathfrak{A})| \in L\}$. Let σ be a scheme. We will show that $_L\text{Mod} \vDash \sigma$ iff $_L\text{Cs}_\omega \vDash \text{eq}(\sigma)$. If $_L\text{Mod} \nvDash \sigma$, then $_L\text{Cs}_\omega \nvDash \text{eq}(\sigma)$ is easy to see by using the definitions. Assume that $_L\text{Cs}_\omega \nvDash \text{eq}(\sigma)$, say $\mathfrak{A} \nvDash eq(\sigma)$ for $\mathfrak{A} \in \text{Cs}_\omega$ with $U \triangleq \text{base}(\mathfrak{A})$ and $|U| \in L$. Then there are $\bar{a}: X \to A$ and $z \in 1^{\mathfrak{A}}$ such that $z \notin \text{tr}(\sigma)^{\mathfrak{A}}(\bar{a})$. Let $N \subseteq \omega$ be such that all the indices occurring in $\text{tr}(\sigma)$ are among N. Recall that $t: \text{FmV} \rightarrowtail X$. For every $\varphi \in \text{FmV}$ let $r_\varphi \triangleq \{s \in {}^N U : s \cup (\omega \sim$

$N) \restriction z \in \bar{a}(t\varphi)\}$. Let σ' be the instance of σ where we replace each formula-variable $\varphi \in \mathrm{FmV}$ with $R_\varphi(v_0, \ldots, v_{N-1})$ where R_φ is an N-ary relation symbol and let $\mathfrak{M} \triangleq \langle U, r_\varphi \rangle_{\varphi \in \mathrm{FmV}}$. Then $\mathfrak{M} \nvDash \sigma'[z]$ can be shown by an easy induction, using the fact that $z \notin \mathrm{tr}(\sigma)^{\mathfrak{A}}(\bar{a})$. Since $|U| \in L$ we have $\mathfrak{M} \in {}_L\mathrm{Mod}$, thus ${}_L\mathrm{Mod} \nvDash \sigma$. (i)$\Leftrightarrow$(iii) has been proved.

If Λ has relation symbols of arbitrarily large finite arities, then the above chain of thought can be modified to show (i)$\Leftrightarrow$(ii), as follows. Let σ be a scheme, let N be the set of (normal) variables occurring in σ and let σ' be an instance of σ where each formula-variable $\varphi \in \mathrm{FmV}$ is replaced with $R_\varphi(v_0, \ldots, v_m)$ where R_φ is a relation symbol of arity $1 + m \geq N$ and different formula-variables are replaced with different formulas. Then one can show that ${}_L\mathrm{Mod} \vDash \sigma$ iff ${}_L\mathrm{Mod}_\Lambda \vDash \sigma'$. □

Remark 2.6. Now, using the above Corollary 2.5, we give here a simple proof for Theorem 2.1(i). The proof we give here is an 'optimization' of the proof given for Theorem 6(i) in Section 3, adjusted specifically for the goal of proving Theorem 2.1(i) directly.

First we prove $S\theta\rho(\mathrm{Equmd}) = S\theta\rho(\mathrm{Monmd}) = S\theta\rho(\mathrm{FMod})$. Let σ be a scheme and assume $\mathrm{FMod} \nvDash \sigma$. We will show $\mathrm{Equmd} \nvDash \sigma$. Assume that the formula variables occurring in σ are among $\varphi_1, \ldots, \varphi_n \in \mathrm{FmV}$. Let $\sigma' = \sigma(\varphi_i/\Phi_i)$ be an instance of σ and $\mathcal{M} \in \mathrm{FMod}$ be such that $\mathcal{M} \nvDash \sigma'$. We may assume $M \in \omega$. Assume that the variables (bound and free) occurring in σ' are among $v_0, \ldots, v_{N-1} \in V$. For every $a \in {}^N M$ define $m(a) \triangleq \bigwedge \{v_i = v_{N+a_i} : i \in N\}$ and define $\eta_i \triangleq \bigvee \{m(a) : a \in {}^N M$ and $\mathcal{M} \vDash \Phi_i[a]\}$. Then η_i is an equality formula for every $1 \leq i \leq n$. We will show that $\langle M, = \rangle \nvDash \sigma(\varphi_i/\eta_i)$. Let $k : \{v_N, \ldots, v_{N+M-1}\} \to M$ be such that $k(v_{N+i}) = i$ for every $i \in M$. Now the following can be shown by induction on the structure of the scheme ξ: "Let ξ be any scheme with formula variables among $\varphi_1, \ldots, \varphi_n$ and with (normal) variables among $v_0, \ldots, v_{N-1}$. Then for every $a \in {}^N M$ *we have*

$$\mathcal{M} \vDash \xi(\varphi_i/\Phi_i)[a] \quad \text{iff} \quad \langle M, = \rangle \vDash \xi(\varphi_i/\eta_i)[a \cup k]."$$

Then by $\mathcal{M} \nvDash \sigma(\varphi_i/\Phi_i)$ we will have $\langle M, = \rangle \nvDash \sigma(\varphi_i/\eta_i)$.. Thus $\mathrm{FMod} \nvDash \sigma$ implies $\mathrm{Equmd} \nvDash \sigma$. Clearly, $\mathrm{Equmd} \nvDash \sigma$ implies $\mathrm{Monmd} \nvDash \sigma$. Assume $\mathrm{Monmd} \nvDash \sigma$. Then $\mathrm{Monmd} \nvDash \sigma'$ for some monadic instance σ' of σ. It is known that then $\mathcal{M} \nvDash \sigma'$ for a finite $\mathcal{M} \in \mathrm{Monmd}$, too. (For completeness, we note that this can be proved, e.g., by the techniques of Monk [22].) Thus $\mathrm{FMod} \nvDash \sigma$. By the above we have seen $S\theta\rho(\mathrm{Equmd}) = S\theta\rho(\mathrm{Monmd}) = S\theta\rho(\mathrm{FMod})$. Let Λ be the first-order language having only one binary relation symbol. It is known that the first-order formulas valid in the finite models with one binary relation is not r.e., i.e. that $\theta\rho({}_\omega\mathrm{Mod}_\Lambda)$ is not r.e. Then $S\theta\rho({}_\omega\mathrm{Mod})$ is not r.e. by part (i)$\Rightarrow$(ii) of Corollary 2.5 (which part is a direct corollary of Lemma 2.4), hence $S\theta\rho(\mathrm{FMod})$ is not r.e., since ${}_\omega\mathrm{Mod} = \mathrm{FMod}$ by definition.

So far we dealt with 'usual' first-order logics, i.e., first-order logics having infinitely many variables. Now we turn to first-order logics having only finitely many variables. Let $\alpha < \omega$. Then $S\theta\rho_\alpha K$ is the set of α-schemes α-valid in K, i.e., if $\mathcal{M} \in \text{Mod}$ and σ is an α-scheme, then $\mathcal{M} \models_\alpha \sigma$ iff $\mathcal{M} \models \sigma'$ for every instance σ' of σ which is a formula of the logic L_α using only α variables, and then $S\theta\rho_\alpha K$ is defined the usual way. (For L_α and its literature see the second part of the introduction.)

Theorem 2.7. *Let* $2 < \alpha \leq \omega$. *Then* (i)–(iv) *below hold.*

(i) $S\theta\rho_\alpha(\text{Equmd})$ *is decidable.*

(ii) $S\theta\rho_\alpha(\text{Monmd}) = S\theta\rho_\alpha(\text{FMod})$ *is not r.e.*

(iii) $S\theta\rho_\alpha(\text{Monmd}) \supset S\theta\rho_\alpha(\text{Mod})$

(iv) *Let* $K \subseteq \text{Monmd}$. *Then* (a)–(b) *below hold.*

(a) $S\theta\rho_\alpha(K)$ *is either decidable or not r.e.*

(b) $S\theta\rho_\alpha(K)$ *is decidable iff there is a finite monadic language* Λ *such that every* $\mathcal{M} \in K$ *is definitionally equivalent to some model of* Λ.

Remark 2.8. In Theorem 2.7: (i) follows from [11, 4.2.1]; (ii) follows from Theorem 6(iii) + Theorem 1(ii); (iii) follows from Theorem 1(iv), and (iv) follows from Theorem 2(i), (iii). Most parts of Theorem 2.7 generalize to $\alpha \leq 2$.

Problem 2.9. Find a 'nice' axiomatization of $S\theta\rho\text{Mod}$! This would be relevant to solving an old central problem of algebraic logic which is restated as Problem 4.1 in [11].

3. Proofs

We shall prove the theorems in the following order: 2, 4, 6, 1, 3, 5, 7. The following notation will be frequently used in the proofs:

$X \sim Y \triangleq \{a \in X : a \notin Y\}$ is the difference of the sets X and Y.

$X \subseteq_\omega Y$ means that X is a *finite* subset of Y.

$\text{Sb}U$ denotes the powerset of U, $\text{Sb}U \triangleq \{X : X \subseteq U\}$.

$\mathfrak{Sb}V$ denotes the full cylindric-relativised set algebra with unit V, i.e., $\mathfrak{Sb}V = \langle \text{Sb}V, \cup, \cap, \sim, 0, V, C_i^{[V]}, D_{ij}^{[V]} \rangle_{i,j \in \alpha}$ if $V \subseteq {}^\alpha U$ for some U. For $C_i^{[V]}$, $D_{ij}^{[V]}$ see the end of the introduction.

$1^{\mathfrak{A}}$ denotes the unit of the CA_α $\mathfrak{A}$.

$s_j^i x \triangleq c_i(d_{ij} \cdot x)$ in any CA_α $\mathfrak{A}$, for $i, j \in \alpha$ and $x \in A$.

$\text{Do}f$, $\text{Rg}f$ denote the domain and range of the function f.

$f^*X \triangleq \{f(x) : x \in X\}$ is the f-image of X, for any function f.

f_i denotes $f(i)$ if f is a function.

(a, b) denotes the same as $\langle a, b \rangle$ (the pair of a and b).

$f : A \rightarrowtail B$ denotes that f is one–one.

$f: A \rightarrowtail\!\!\!\twoheadrightarrow B$ denotes that f is bijective.

${}^{A}B$ denotes the set of all functions mapping A into B.

$A \upharpoonleft f = \{(u, v) \in f : u \in A\}$ is the function f restricted to A.

Thus if $s \in {}^{n}U$ and $f \in {}^{\alpha}U$, then $s \cup (\alpha \sim n) \upharpoonleft f$ denotes the function that agrees with s on n and with f on $\alpha \sim n$.

Let $k \in {}^{\alpha}U$, i.e., let $k : \alpha \rightarrow U$ and $i \in \alpha$. Then $k(i/u)$ or k^{i}_{u} denote the function we get by changing the i-th value to u, i.e., $k(i/u) = \{(i, u)\} \cup (k \sim \{(i, k(i))\})$.

Undefined terminology or notation is taken from [11].

Proof of Theorem 2. The difficult part is to show when $\overline{\text{Eq}}K$ is not r.e. We begin with these parts.

(A) Let $\alpha \geq \omega$ and assume $K \subseteq \text{Mg}_{\alpha}$ is unbounded. We will show that $\overline{\text{Eq}}K$ is not r.e. We shall prove the following theorem. Let $\boldsymbol{\omega} \triangleq \langle \omega, +, \cdot, 0, 1 \rangle$ be the standard model of arithmetic.

Theorem 3.1. *There is a recursive function ε mapping the set of number-theoretic equations into the set of equations of CA_{ω} such that for all number-theoretic equations $e(\bar{x})$ we have*

$$\boldsymbol{\omega} \models \neg e(\bar{x}) \quad \textit{iff} \quad K \models \varepsilon(e(\bar{x})),$$

where $K \subseteq \text{Mg}_{\alpha}$ is unbounded, $\alpha \geq \omega$.

Since the set of insatisfiable Diophantine equations is not r.e., Theorem 3.1 will imply that $\overline{\text{Eq}}K$ is not r.e. Now we turn to proving Theorem 3.1.

The idea of the translation ε: Let x, y, z be variables in the language of CA_{α}'s. (They can be thought of as formula variables.) We can express, by a cylindrical algebraic equation $\tau_1(x) = 1$, about x that "x is a one–one unary function with no fix-point" (cf. τ_1 in Definition 3.2 below). Lemma 3.3 says that in Mg_{α}, the domain of such an x is always finite. (It is not so in CA_{α} or in Bg_{α}.) Hence x is the successor function restricted to a finite initial segment N of ω. Then we can express that y, z are addition and multiplication restricted to this N. (See τ_2 and τ_3 in Definition 3.2 and Lemma 3.4.) Having 0, suc, $+$, $\cdot$ we can then translate number-theoretic equations to cylindric algebraic equations.

The formulas we use to express that x, y, z are successor, addition and multiplication are as follows. (These formulas will be coded as CA_{ω}-terms in Definition 3.2(i) below.)

$$x(v_0v_1) \wedge x(v_0v_2) \rightarrow v_1 = v_2 \quad \text{(i.e., } x \text{ is a function)},$$

$$x(v_0v_1) \wedge x(v_2v_1) \rightarrow v_0 = v_2 \quad (x \text{ is one–one}),$$

$$x(v_0v_1) \rightarrow v_0 \neq v_1 \quad (x \text{ has no fix-point}).$$

Define

$d(v_6) \Leftrightarrow \exists v_0\,(v_0 = v_6 \wedge \exists v_1 x(v_0 v_1))$ ($v_6 \in \mathrm{Do}x$),

$r(v_6) \Leftrightarrow \exists v_1\,(v_1 = v_6 \wedge \exists v_0 x(v_0 v_1))$ ($v_6 \in \mathrm{Rg}x$),

$n(v_6) \Leftrightarrow d(v_6) \wedge \neg r(v_6)$ (v_6 is a starting point of x).

$\exists v_6 n(v_6) \wedge \forall v_0 v_6 (n(v_6) \wedge n(v_0) \rightarrow v_6 = v_0)$ (There is exactly one starting point in x).

$y(v_0 v_1 v_2) \wedge y(v_0 v_1 v_3) \rightarrow v_2 = v_3$ (y is a function),

$y(v_0 v_1 v_2) \rightarrow (d(v_0) \wedge d(v_1) \wedge d(v_2))$ (y is on the domain of x).

We shall write $u + v = w$ and $u + 1 = v$ instead of $y(uvw)$ and $x(uv)$ resp.

$n(v_0) \wedge d(v_1) \rightarrow y(v_0 v_1 v_1)$ ($0 + u = u$ for $u \in \mathrm{Do}x$).

$$\exists v_3[y(v_3 v_1 v_4) \wedge x(v_0 v_3)] \leftrightarrow \exists v_2[x(v_2 v_4) \wedge y(v_0 v_1 v_2)]$$

$$((v + 1) + u = w \leftrightarrow w = (v + u) + 1).$$

Similarly we can express that z is multiplication:

$z(v_0 v_1 v_2) \wedge z(v_0 v_1 v_3) \rightarrow v_2 = v_3$ (z is a (partial) function),

$z(v_0 v_1 v_2) \rightarrow (d(v_0) \wedge d(v_1) \wedge d(v_2))$ (z is on the domain of x),

$n(v_0) \wedge d(v_1) \rightarrow z(v_0 v_1 v_0)$ ($0 \cdot u = 0$),

$$\exists v_3[z(v_3 v_1 v_4) \wedge x(v_0 v_3)] \leftrightarrow \exists v_2[y(v_2 v_1 v_4) \wedge z(v_0 v_1 v_2)]$$

$$((v + 1) \cdot u = w \leftrightarrow w = (v \cdot u) + u).$$

In Definition 3.2(i) below, the above formulas are coded as cylindric terms.

Definition 3.2. (i) τ_1 is defined to be the CA_7-term

$$-c_{(3)}(x \cdot s^1_2 x - d_{12}) - c_{(3)}(x \cdot s^0_2 x - d_{02}) - c_{(2)}(x \cdot d_{01}) \cdot -c_{(7)}(c_{(7\sim 2)} x - x).$$

Let $d(x) \triangleq s^0_6 c_1 x$, and $n(x) \triangleq d(x) - s^1_6 c_0 x$.
$\sigma(x)$ is defined to be the term

$$c_6 n(x) - c_0 c_6 (n(x) \cdot s^6_0 n(x) - d_{06}).$$

τ_2 is defined to be the CA_7-term

$$-c_{(4)}(y \cdot s^2_3 y - d_{23}) \cdot$$
$$-c_{(3)}(y - [s^6_0 d(x) \cdot s^6_1 d(x) \cdot s^6_2 d(x)]) \cdot$$
$$-c_{(3)}(s^6_0 n(x) \cdot s^6_1 d(x) \cdot d_{12} - y) \cdot$$
$$-c_{(5)}(c_3[s^0_3 s^2_4 y \cdot s^1_3 x] \oplus c_2[s^0_2 s^1_4 x \cdot y]) \cdot$$
$$-c_{(7)}(c_{(7\sim 3)} y - y).$$

τ_3 is defined to be the CA_7-term

$$-c_{(4)}(z \cdot s_3^2 z - d_{23}) \cdot$$
$$-c_{(3)}(z - [s_0^6 d(x) \cdot s_1^6 d(x) \cdot s_2^6 d(x)]) \cdot$$
$$-c_{(3)}(s_0^6 n(x) \cdot s_1^6 d(x) \cdot d_{02} - z) \cdot$$
$$-c_{(5)}(c_3[s_3^0 s_4^2 z \cdot s_3^1 x] \oplus c_2[s_2^0 s_4^2 y \cdot z]) \cdot$$
$$-c_{(7)}(c_{(7\sim 3)} z - z).$$

$\varphi(x, y, z)$ is defined to be the term $\tau_1 \cdot \tau_2 \cdot \tau_3 \cdot \sigma(x)$.

(ii) Let $V \triangleq \{v_i : i \in \omega, i > 6\}$ be our set of variables when speaking about $\boldsymbol{\omega}$. Let $e(\bar{x})$ be a number-theoretic equation with free variables $x_0, \ldots, x_n \in V$. There is an algorithm which to each number-theoretic equation with free variables $x_0, \ldots, x_n \in V$ associates a formula $\exists y_0 \cdots y_k(b_0 \wedge \cdots \wedge b_m)$ equivalent to $e(\bar{x})$ in $\boldsymbol{\omega}$ and such that $y_0, \ldots, y_k \in V$ and each b_i has the form $u+1=v$, $u+v=w$, $u \cdot v = w$ or $u=0$ for some $u, v, w \in \{x_0, \ldots, x_n, y_0, \ldots, y_k\}$. (Cf. Malcev [21, Section 7.1, Theorem 4].) Denote $\bar{x} = \langle x_0, \ldots, x_n \rangle$, $\bar{y} = \langle y_0, \ldots, y_k \rangle$. Let $\exists \bar{y}\,(b_0 \wedge \cdots \wedge b_m)$ be associated to $e(\bar{x})$ by the above algorithm. For each $l \leq m$ define the cylindric term β_l as follows: β_l is defined to be $s_i^0 s_j^1 x$, $s_i^0 s_j^1 s_k^2 y$, $s_i^0 s_j^1 s_k^2 z$ or $s_i^6 n(x)$ respectively if b_l is $v_i + 1 = v_j$, $v_i + v_j = v_k$, $v_i \cdot v_j = v_k$, or $v_i = 0$ (for $i, j, k > 6$) respectively.

Now we define $\varepsilon(e(\bar{x}))$ to be $\varphi(x, y, z) \cdot \prod \{\beta_l : l \leq m\} = 0$. □

We are going to show $\boldsymbol{\omega} \models \exists \bar{x}\, e(\bar{x})$ iff $K \not\models \varepsilon(e(\bar{x}))$. But first we need some lemmas.

Notation. Let $\mathfrak{A} \in Gs_\alpha$ with base U, $R \in A$, $k \in 1^{\mathfrak{A}}$ and $n \in \alpha$. Then $R[\![k, n]\!] \triangleq \{s \in {}^nU : s \cup [(\alpha \sim n) \upharpoonright k] \in R\}$. E.g., $R[\![k, 2]\!]$ is the following binary relation on $U : R[\![k, 2]\!] = \{(u, v) \in {}^2U : k_{uv}^{01} \in R\}$.

Let $R \subseteq {}^nU$ and let $\equiv$ be an equivalence relation on U. Then $R/{\equiv} \triangleq \{(u_1/{\equiv}, \ldots, u_n/{\equiv}) : (u_1, \ldots, u_n) \in R\}$.

Let $\mathfrak{A} \in Gs_\alpha$ and $x \in A$. We recall from [11, 3.1.1] that x is regular in $\mathfrak{A}$ iff $(\forall q, k \in 1^{\mathfrak{A}})[(1 \cup \Delta^{\mathfrak{A}}(x)) \upharpoonright q \subseteq k \Rightarrow (q \in x \text{ iff } k \in x)]$ and $\mathfrak{A}$ is regular if all of its elements are regular. Gs_α^{reg} denotes the class of all regular Gs_α's.

The following lemma says, roughly, that in any $\mathfrak{A} \in Mg_\alpha \cap Gs_\alpha^{reg}$ if $r \triangleq R[\![k, 2]\!]$ (with $R \in A$, $k \in 1^{\mathfrak{A}}$) is a "function between its blocks without fixpoints", then $\mathrm{Rg}\, r$ contains only finitely many blocks. We shall use the following lemma in most cases when the equivalence relation $\equiv$ in it is the identity.

Lemma 3.3. *Let $\alpha \geq 2$, $\mathfrak{A} \in Mg_\alpha \cap Gs_\alpha^{reg}$ and $R \in A$. Then there is $n \in \omega$ with the following property: If $k \in 1^{\mathfrak{A}}$ and $\equiv$ is an equivalence relation on* base($\mathfrak{A}$) *such that $R[\![k, 2]\!]/{\equiv}$ is a function with no fixpoint, then $|\mathrm{Rg}\, R[\![k, 2]\!]/{\equiv}| < n$.*

Proof. Let $\Gamma \triangleq 1 \cup \Delta R$. By $\mathrm{Mg}_\alpha \subseteq \mathrm{Lf}_\alpha$, $m \triangleq |\Gamma| + 3$ is finite. Since $\mathfrak{A} \in \mathrm{Mg}_\alpha$, there is a finite $G \subseteq \mathrm{Nr}_1\mathfrak{A}$ generating R, such that G is a partition of $1^{\mathfrak{A}}$. Let $n > m \cdot |G|$ finite. Now let $k \in 1^{\mathfrak{A}}$ and let $\equiv$ be an equivalence relation on base($\mathfrak{A}$). Assume that $R[\![k, 2]\!]/\equiv$ is a function with no fixpoint. Let $L \triangleq \mathrm{Rg}\, R[\![k, 2]\!]$.

(*) Assume $|L/\equiv| \geq n$.

There is $U \in \mathrm{Subb}(\mathfrak{A})$ with $k \in {}^{\alpha}U$. By regularity, G induces a partition of U, namely $G_0 = \{\{f_0 : f \in X\} \cap U : X \in G\}$ is one. By $|L/\equiv| \geq n$ and $L \subseteq U$, there is $Y \in G_0$ with $|(L \cap Y)/\equiv| \geq m$. Let $Y^+ \triangleq (L \cap Y) \sim k^*\Gamma$. By $|(L \cap Y)/\equiv| \geq m = |\Gamma| + 3$ we have $|Y^+/\equiv| \geq 3$. Let $t \in Y^+$. By $Y^+ \subseteq L$, there is $a \in U$ with $(a, t) \in R[\![k, 2]\!]$. Then $a \neq t$ because $R[\![k, 2]\!]/\equiv$ has no fixpoint. Let $e \in Y^+ \sim \{a, t\}/\equiv$. Such an e exists by $|Y^+/\equiv| > 2$. Let $T \triangleq \mathrm{base}(\mathfrak{A})$ and $f : T \rightarrowtail\!\!\!\!\twoheadrightarrow T$ be a permutation of T interchanging e and t and leaving the rest fixed. I.e., $f(e) = t$ and $(\forall x \in T \sim \{e, t\}) f(x) = x$. Now $f(a) = a$ by $a \notin \{e, t\}$. Then f induces a base-automorphism $\tilde{f} \in \mathrm{Is}(\mathfrak{Sb}1^{\mathfrak{A}}, \mathfrak{Sb}1^{\mathfrak{A}})$ by [11]. Since $\{e, t\} \subseteq Y \in G_0$ and G consists of mutually disjoint regular elements, we have $G \upharpoonright \tilde{f} \subseteq \mathrm{Id}$. Thus $\tilde{f}R = R$. Now $(a, t) \in R[\![k, 2]\!] \Rightarrow k^{01}_{at} \in R \Rightarrow f \circ k^{01}_{at} \in \tilde{f}R = R$, which by $[e, t \notin k^*\Gamma \Rightarrow \Gamma \upharpoonright (f \circ k^{01}_{at}) = \Gamma \upharpoonright k^{01}_{ae}]$ and by regularity of R implies $k^{01}_{ae} \in R$, thus $(a, e) \in R[\![k, 2]\!]$. Since $e \not\equiv t$, this means that $R[\![k, 2]\!]/\equiv$ is not a function. A contradiction, disproving our assumption (*). □

Lemma 3.4. *Let* $\mathfrak{A} \in \mathrm{Mg}_\alpha \cap \mathrm{Gs}^{\mathrm{reg}}_\alpha$, $\alpha \geq 7$, $k \in 1^{\mathfrak{A}}$ *and* $X, Y, Z \in A$. *Set* $s \triangleq X[\![k, 2]\!]$, $a \triangleq Y[\![k, 3]\!]$, $m \triangleq Z[\![k, 3]\!]$. *Assume* $k \notin c_{(7)}(c_{(7\sim 2)}X - X)$. *Then* (i)–(ii) *below hold.*

(i) $k \in \tau_1(X)^{\mathfrak{A}}$ *iff* (s *is a finite one–one function with no fixpoints*), *and* $k \in \sigma(X)^{\mathfrak{A}}$ *iff* $|\mathrm{Do}s \sim \mathrm{Rg}s| = 1$.

(ii) *Assume further that* $k \notin \sum \{c_{(7)}(c_{(7\sim 3)}W - W) : W \in \{Y, Z\}\}$. *Then* (a) *and* (b) *below are equivalent*:

(a) $k \in \varphi(X, Y, Z)^{\mathfrak{A}}$.

(b) *There are* $N \in \omega$ *and* $n : N + 1 \rightarrowtail \mathrm{base}(\mathfrak{A})$ *such that*

$$s = \{\langle ni, n(i+1)\rangle : i < N\},$$
$$a = \{\langle ni, nj, n(i+j)\rangle : i, j, i+j < N\},$$
$$m = \{\langle ni, nj, n(i \cdot j)\rangle : i, j, i \cdot j < N\}.$$

Proof. Let everything be as in the statement of Lemma 3.4. Assume $k \notin c_{(7)}(c_{(7\sim 2)}X - X)$. This means that

(*) ($k'^{01}_{uv} \in X$ iff $k^{01}_{uv} \in X$) holds for every u, v and $k' \in c_{(7)}\{k\}$.

(1) $k \notin c_{(3)}(X \cdot s^1_2 X - d_{12})$ iff s is a function.

For, assume $k \in c_{(3)}(X \cdot s^1_2 X - d_{12})$. Then there are u, v, w such that $k^{012}_{uvw} \in X \cdot c_1(d_{12} \cdot X) - d_{12}$. Thus $v \neq w$ and $k^{012}_{uww} \in X$. Then $k^{01}_{uv}, k^{01}_{uw} \in X$ by (*). Hence $(u, v), (u, w) \in X[\![k, 2]\!] = s$ and $v \neq w$ show that s is not a function. On the other hand, assume that s is not a function. Then there are u, v, w such that $(u, v), (u, w) \in s$ and $v \neq w$. Thus $k^{01}_{uv}, k^{01}_{uw} \in X$, therefore by (*) we have $k^{012}_{uvw}, k^{012}_{uww} \in X$, showing $k^{012}_{uvw} \in X \cdot s^1_2 X - d_{12}$. One can prove similarly the statements (2), (3) and (4).

(2) $k \notin c_{(3)}(X \cdot s^0_2 X - d_{02})$ iff s is one–one (i.e., $(u, v), (w, v) \in s \Rightarrow u = w$).

(3) $k \notin c_{(2)}(X \cdot d_{01})$ iff s has no fixpoint (i.e., $(\forall u)\,(u, u) \notin s$).

(4) $k \in \sigma(X)$ iff $|\mathrm{Do}s \sim \mathrm{Rg}s| = 1$.

Assume $k \in \tau_1(X)^{\mathfrak{A}}$. Then s is a one–one function with no fixpoints, by (1)–(3) above. By Lemma 3.3 then Rgs is finite, hence s is finite, too. Conversely, if s is a finite one–one function with no fixpoints, then $k \in \tau_1(X)^{\mathfrak{A}}$, by (1)–(3) above. (i) has been proved. Assume now

(**) $k \notin c_{(7)}([(c_{(7\sim 2)}X - X) + (c_{(7\sim 3)}Y - Y) + (c_{(7\sim 3)}Z - Z)]$.

Let s be a finite one–one function with no fixpoints and with $|\mathrm{Do}s \sim \mathrm{Rg}s| = 1$. Let $F \triangleq \mathrm{Do}s \cup \mathrm{Rg}s$. Let $N \triangleq |F| - 1$. Then, by the properties of s, there exists a $n : N + 1 \twoheadrightarrow F$ such that $n0 \in \mathrm{Do}s \sim \mathrm{Rg}s$ and $n(i+1) = s(ni)$ for every $i < N$. Then $s = \{\langle ni, n(i+1)\rangle : i < N\}$ holds. The converse clearly holds, hence by (i) we proved

(5) $k \in \tau_1 \cdot \sigma(X)^{\mathfrak{A}}$ iff $(\exists N \in \omega)(\exists n : N + 1 \rightarrowtail \mathrm{base}(\mathfrak{A}))$

$$s = \{\langle ni, n(i+1)\rangle : i < N\}.$$

Assume *from now on* that $k \in \tau_1 \cdot \sigma(X)^{\mathfrak{A}}$ and s, N and n are as in (5) above. Let $D \triangleq \mathrm{Do}s$ and let $\mathbf{0}$ denote the unique element of $D \sim \mathrm{Rg}s$. Then $n0 = \mathbf{0}$ and $u \in D$ iff $(\exists i < N)\, u = ni$.

(6) Let $k' \in c_{(7)}\{k\}$. Then

$k' \in d(X)$ iff $k'(6) \in D$ and $k' \in n(X)$ iff $k'(6) = \mathbf{0}$.

For, $k' \in d(X) = c_0(d_{06} \cdot c_1 X)$ iff $k'^{0}_{u} \in c_1 X$ where $u = k'(6)$, and $k'^{0}_{u} \in c_1 X$ if $(\exists v)\, k'^{01}_{uv} \in X$ iff (by (**)) $(\exists v)\, k^{01}_{uv} \in X$ iff $u \in \mathrm{Do}s = D$. $k' \in s^1_0 c_0(X)$ iff $k'(6) \in \mathrm{Rg}$ can be proved similarly, hence $k' \in n(X)$ iff $k'(6) \in D \sim \mathrm{Rg}s$ iff $k'(6) = \mathbf{0}$.

(7) $k \notin c_{(4)}(Y \cdot s^2_3 Y - d_{23})$ iff a is a function, i.e.,

$$(u, v, w), (u, v, z) \in a \Rightarrow w = z,$$

can be proved analogously to (1).

(8) $k \notin c_{(3)}(Y - [s^6_0 d(X) \cdot s^6_1 d(X) \cdot s^6_2 d(X)])$ iff $[(u, v, w) \in a \Rightarrow u, v, w \in D]$.

For, assume $k \in c_{(3)}(Y - s^6_0 d(X))$. Then there are u, v, w such that $k^{012}_{uvw} \in Y - c_6(d_{06} \cdot d(X))$. Then $(u, v, w) \in a$ and $k^{0126}_{uvwu} \notin d(X)$, therefore $u \notin D$ by (6). Similarly $(u, v, w) \in a$ and $u \notin D$ implies $k^{012}_{uvw} \in Y - s^6_0 d(X)$. The remaining part is completely analogous.

From *now on,* assume that a is a (partial) binary function on D, i.e., that $a:P\to D$ for some $P\subseteq {}^2D$. We shall write $u+v=w$ instead of $(u,v,w)\in a$.

(9) $k\notin c_{(3)}(s^6_0n(X)\cdot s^6_1d(X)\cdot d_{12}-Y)$ iff $\mathbf{0}+u=u$ for every $u\in D$.

For, assume $k\in c_{(3)}(s^6_0n(X)\cdot s^6_1d(X)\cdot d_{12}-Y)$. Then there are u, v, w such that $k^{012}_{uvw}\in s^6_0n(X)\cdot s^6_1d(X)\cdot d_{12}-Y$. Then $v=w$ and $(u,v,w)\notin a$. By $k^{012}_{uvw}\in c_6(d_{06}\cdot n(X))$ we have $k^{0126}_{uvwu}\in n(X)$, thus $u=\mathbf{0}$ by (6). Similarly, $k^{012}_{uvw}\in s^6_1d(X)$ implies $v\in D$. We have seen $v\in D$ and $\mathbf{0}+v\neq v$. The other direction, $\mathbf{0}+v\neq v$ for some $v\in D\Rightarrow k\in c_{(3)}(s^6_0n(X)\cdot\cdots)$ is analogous.

(10) $k\notin c_{(5)}(c_3[s^0_3s^2_4Y\cdot s^1_3X]\oplus c_2[s^0_2s^1_4X\cdot Y])$ iff

$(sv+u=w\leftrightarrow w=s(v+u))$ for every u, v, $w\in D$.

For, assume $k\in c_{(5)}(\cdots\oplus\cdots)$. Then $k'\in s^0_3s^2_4Y\cdot s^1_3X$ but $k'\notin c_2[s^0_2s^1_4X\cdot Y]$, or the other way round, for some $k'\in c_{(5)}\{k\}$. Assume the first case. Let $(v,u,q,p,w)=5\upharpoonright k'$. Then $k'^{0}_{p}\in s^2_4Y$, hence $k'^{02}_{pw}\in Y$, therefore $p+u=w$. Also, by $k'\in s^1_3X$ we have $k'^{1}_{p}\in X$, thus $s(v)=p$. Thus $s(v)+u=w$. By $k'\notin c_2[s^0_2s^1_4X\cdot Y]$ we have that for every q, either $s(q)\neq w$ or $v+u\neq q$. This means $s(v+u)\neq w$ (either not defined or unequal). The other parts are similar, we omit them.

Now by (7)–(10) we have $k\in\tau_2(X,Y)^{\mathfrak{A}}$ iff $a=\{\langle ni,nj,n(i+j)\rangle:i,j,i+j<N\}$ as follows. Assume $k\in\tau_2(X,Y)^{\mathfrak{A}}$. Let $i,j,i+j<N$. Then $\langle n0,nj,nj\rangle\in a$ by (9). By $\Leftarrow$ of (10) then, by induction, $\langle ni,nj,n(i+j)\rangle\in a$ since $s^{n(i)}nj=n(i+j)$ by $s=\{\langle ni,n(i+1)\rangle:i<N\}$. This proves the inclusion $a\supseteq\{\cdots\}$. To see the other inclusion, assume $\langle ni,nj,nk\rangle\in a$ for some $i,j,k<N$. By $\Rightarrow$ of (10) then $\langle n0,nj,n(k-i)\rangle\in a$, hence $nj=n(k-i)$ by (9) and (7), thus $j=k-i$, i.e., $k=i+j$. Conversely, assume $a=\{\langle ni,nj,n(i+j)\rangle:i,j,i+j<N\}$. Then $k\in\tau_2(X,Y)^{\mathfrak{A}}$ by (7)–(10) and by our assumption (**).

Assume $k\in\tau_1\cdot\tau_2\cdot\sigma(X,Y)^{\mathfrak{A}}$. Then the proof of

(11) $k\in\tau_3(X,Y,Z)^{\mathfrak{A}}$ iff $m=\{\langle ni,nj,n(i\cdot j)\rangle:i,j,i\cdot j<N\}$

is similar to the above, therefore we omit it. □

Let $e(\bar{x})$ be a number-theoretic equation. Let $\exists\bar{y}\bigwedge B$, where $B=\{b_0,\ldots,b_m\}$, $W\triangleq\{x_0,\ldots,x_n,y_0,\ldots,y_k\}\subseteq V$ and $\{\beta_l:l\leqslant m\}$ be associated to $e(\bar{x})$ as in Definition 3.2(ii).

Lemma 3.5. *$\omega\models\exists\bar{x}\,e(\bar{x})$ implies $K\not\models\varepsilon(e(\bar{x}))$ for every unbounded $K\subseteq\mathrm{Mg}_\alpha$, $\alpha\geqslant\omega$.*

Proof. Assume $\omega\models\exists\bar{x}\,e(\bar{x})$. Then $\omega\models\exists\bar{x}\,\exists\bar{y}\bigwedge B$. Let $h\in{}^W\omega$ be such that $\omega\models\bigwedge B[h]$. Let $N\in\omega$ be such that $h^*W\subseteq N$ and let $Q\in\omega$ be such that $W\subseteq\{v_i:i<Q\}$. Let $N'\triangleq N+1$. Since K is unbounded, there is $\mathfrak{M}\in K$ with $\mathfrak{M}\not\models\bar{d}(N'\times N')=0$. By $K\subseteq\mathrm{Mg}_\alpha\subseteq\mathbf{I}\mathrm{Gs}_\alpha=\mathbf{SP}\mathrm{Cs}^{\mathrm{reg}}_\alpha$, we may assume $\mathfrak{M}\in\mathbf{SP}\mathrm{Cs}^{\mathrm{reg}}_\alpha$.

Then by $\mathfrak{M} \nvDash \bar{d}(N' \times N') = 0$, $\mathfrak{M}$ has a subdirect factor $\mathfrak{C} \in \mathrm{Cs}_\alpha^{\mathrm{reg}} \cap \mathrm{Mg}_\alpha$ with base U such that $|U| \geq N'$. We may assume $N' \subseteq U$. It is enough to show $\mathfrak{C} \nvDash \varepsilon(e(\bar{x}))$, since this implies $K \ni \mathfrak{M} \nvDash \varepsilon(e(\bar{x}))$. Let $k \in {}^\alpha\omega$ be such that $h \subseteq k$ and $(\forall i \in N')\, k(Q+i) = i$. Define

$$X \triangleq \sum \{d^{\mathfrak{C}}_{0,Q+i} \cdot d^{\mathfrak{C}}_{1,Q+i+1}: 0 \leq i \leq N\},$$

$$Y \triangleq \sum \{d^{\mathfrak{C}}_{0,Q+i} \cdot d^{\mathfrak{C}}_{1,Q+j} \cdot d_{2,Q+i+j}: i, j \in \omega,\, i+j < N\},$$

$$Z \triangleq \sum \{d^{\mathfrak{C}}_{0,Q+i} \cdot d^{\mathfrak{C}}_{1,Q+j} \cdot d^{\mathfrak{C}}_{2,Q+i\cdot j}: i, j \in \omega,\, i \cdot j < N\}.$$

Then $X[\![k, 2]\!] = \{\langle i, i+1\rangle : i < N\}$, $Y[\![k, 3]\!] = \{\langle i, j, i+j\rangle : i, j, i+j < N\}$, and $Z[\![k, 3]\!] = \{\langle i, j, i\cdot j\rangle : i, j, i \cdot j < N\}$ therefore $k \in \varphi(X, Y, Z)^{\mathfrak{C}}$ by Lemma 3.4. Also, by $h \subseteq k$ and $\boldsymbol{\omega} \vDash \bigwedge B[h]$, by the definition of β_l's, we have $k \in \prod\{\beta_l : l \leq m\}$ (with x, y, z substituted by X, Y, Z in $\mathfrak{C}$). Thus $k \in \varphi(X, Y, Z)^{\mathfrak{C}} \cap \prod\{\beta_l : l \leq m\}$ showing $\mathfrak{C} \nvDash \varepsilon(e(\bar{x}))$. Therefore $K \nvDash \varepsilon(e(\bar{x}))$. □

Lemma 3.6. $\mathrm{Mg}_\alpha \nvDash \varepsilon(e(\bar{x}))$ *implies* $\boldsymbol{\omega} \vDash \exists \bar{x}\, e(\bar{x})$, *for* $\alpha \geq \omega$.

Proof. Let $\mathfrak{M} \in \mathrm{Mg}_\alpha$ be such that $\mathfrak{M} \nvDash \varepsilon(e(\bar{x}))$. We may assume $\mathfrak{M} \in \mathrm{Gs}_\alpha^{\mathrm{reg}} \cap \mathrm{Mg}_\alpha$. By $\mathfrak{M} \nvDash \varepsilon(e(\bar{x}))$, there are $X, Y, Z \in M$ and $k \in 1^{\mathfrak{M}}$ such that $k \in \varphi(X, Y, Z)^{\mathfrak{M}} \cap \prod\{\beta_j : j \leq m\}$. Let $s \triangleq X[\![k, 2]\!]$, $a \triangleq Y[\![k, 3]\!]$ and $m \triangleq Z[\![k, 3]\!]$. Let $N \in \omega$ and $n : N+1 \rightarrowtail \mathrm{base}(\mathfrak{M})$ be such that $s = \{\langle ni, n(i+1)\rangle : i < N\}$, $a = \{\langle ni, nj, n(i+j)\rangle : i, j, i+j < N\}$, $m = \{\langle ni, nj, n(i\cdot j)\rangle : i, j, i \cdot j < N\}$. Such N and n exist by Lemma 3.4(ii) and by $k \in \varphi(X, Y, Z)^{\mathfrak{M}}$. Let $h : W \to \omega$ be defined by $(\forall v_j \in W)$

$$h(v_j) \triangleq \begin{cases} n^{-1}(k_j) & \text{if } kj \in \mathrm{Rg}n, \\ 0 & \text{otherwise.} \end{cases}$$

We will show $\boldsymbol{\omega} \vDash \bigwedge B[h]$. Let $b_l \in B$. Assume b_l is $v_i + 1 = v_j$. Then β_l is $s_i^0 s_j^1 x$, hence $k \in s_i^0 s_j^1 X$ by $k \in \prod\{\beta_l : l \leq m\}$. Then $\langle k(i), k(j)\rangle \in s$ by $i, j \notin 2$. Hence $ki, kj \in \mathrm{Rg}n$ and $h(v_j) = h(v_i) + 1$. Thus $\boldsymbol{\omega} \vDash b_l[h]$. The other cases are completely analogous, hence we omit their proofs. We have seen $\boldsymbol{\omega} \vDash \bigwedge B[h]$. Therefore $\boldsymbol{\omega} \vDash \exists \bar{x}\, e(\bar{x})$. □

Now Lemmas 3.5, 3.6 imply $\boldsymbol{\omega} \vDash \neg e(\bar{x})$ iff $K \vDash \varepsilon(e(\bar{x}))$ for all unbounded $K \subseteq \mathrm{Mg}_\alpha$. Thus Theorem 3.1 has been proved and $\overline{\mathrm{Eq}}K$ is not r.e.

(B) Again, we will use that the set of unsatisfiable Diophantine equations is not r.e.

Theorem 3.7. *There is a recursive function* η *mapping the set* NTE *of all number-theoretic equations into the set of equations of* CA_ω *such that for all*

$e(\bar{x}) \in \mathrm{NTE}$ *we have*

$$\omega \models \neg \exists \bar{x}\, e(\bar{x}) \quad \textit{iff} \quad K \models \eta(e(\bar{x})),$$

where $3 \leqslant \alpha < \omega$ *and* $K \subseteq \mathrm{Mg}_\alpha$ *is unboundedly generated.*

To prove Theorem 3.7, assume $3 \leqslant \alpha < \omega$. First we show that $\overline{\mathrm{Eg}}\mathrm{Mg}_\alpha$ is not r.e. and then we will modify the proof to show that $\overline{\mathrm{Eq}}K$ is not r.e. whenever $K \subseteq \mathrm{Mg}_\alpha$ is unboundedly generated.

(B1) The proof will be similar to the one in (A) – only the associated CA-equation $\varepsilon(e(\bar{x}))$ will be different.

The idea of the modification: The main idea is that we will simulate variables v_i in $e(\bar{x})$ by 'constant' elements (monadic generators) instead of treating them as variables ('indices', i.e., members of α). This will immediately settle the case $\alpha \geqslant 7$. To be able to express $\varphi(X, Y, Z)$ for all $\alpha \geqslant 3$ (and not only for $\alpha \geqslant 7$), we will use the 'projection functions (or pairing function)' technique, see Tarski–Givant [35] or Maddux [18]. Cf. also Remark 3.19. Now the formulas we use to express that p_0, p_1 are projection functions and x, y, z are successor, addition and multiplication, using only 3 variables, are as follows (these formulas will be coded as cylindric terms in Definition 3.8 below):

Express that x is a one–one function with no fix-point, as before. Express also that $|\mathrm{Do}x \sim \mathrm{Rg}x| = 1$.

Expressing that p_0, p_1 are 'projection functions':

$$p_i(v_0 v_1) \wedge p_i(v_0 v_2) \rightarrow v_1 = v_2 \quad \text{for } i \in 2,$$
$$v_0 \in \mathrm{Do}x \wedge v_1 \in \mathrm{Do}x \rightarrow \exists v_2[p_0(v_2 v_0) \wedge p_1(v_2 v_1)].$$

Using p_0, p_1 we can code 'addition' as follows:

$$y(v_0 v_1) \wedge y(v_0 v_2) \rightarrow v_1 = v_2,$$
$$y(v_0 v_1) \rightarrow [p_0 v_0 \in \mathrm{Do}x \wedge p_1 v_0 \in \mathrm{Do}x \wedge v_1 \in \mathrm{Do}x],$$
$$p_0 v_0 = 0 \wedge p_1 v_0 = v_1 \wedge v_1 \in \mathrm{Do}x \rightarrow y(v_0 v_1) \qquad (0 + u = u),$$
$$\exists v_1[x(p_0 v_0, p_0 v_1) \wedge p_1 v_0 = p_1 v_1 \wedge y(v_1 v_2)]$$
$$\leftrightarrow \exists v_1[y(v_0 v_1) \wedge x(v_1 v_2)] \qquad ((v+1) + u = w \leftrightarrow w = (v+u) + 1),$$
$$p_0 v_0 = p_0 v_1 \wedge p_1 v_0 = p_1 v_1 \rightarrow (y(v_0 v_2) \leftrightarrow y(v_1 v_2)).$$

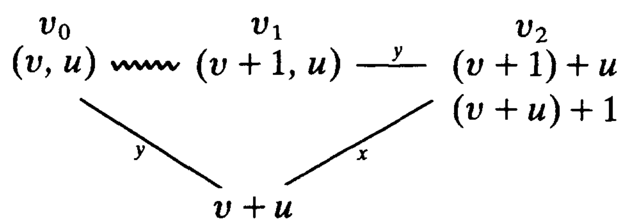

Illustration for the definition of y

Here we can express $x(p_0 v_0, p_0 v_1)$ by

$$\exists v_2(p_0(v_1 v_2) \wedge \exists v_1[p_0(v_0 v_1) \wedge x(v_1 v_2)])$$

and $p_1v_0 = p_1v_1$ can be expressed by

$$\exists v_2(p_1(v_0v_2) \wedge p_1(v_1v_2)).$$

Expressing that z is 'multiplication' goes as follows:

$$z(v_0v_1) \wedge z(v_0v_2) \rightarrow v_1 = v_2,$$

$$z(v_0v_1) \rightarrow [p_0v_0 \in \mathrm{Do}x \wedge p_1v_0 \in \mathrm{Do}x \wedge v_1 \in \mathrm{Do}x],$$

$$v_1 = 0 \wedge p_0v_0 = v_1 \wedge p_1v_0 \in \mathrm{Do}x \rightarrow z(v_0v_1) \qquad (0 \cdot u = 0),$$

$$\exists v_1[x(p_0v_0, p_0v_1) \wedge p_1v_0 = p_1v_1 \wedge z(v_1v_2)]$$
$$\leftrightarrow \exists v_1(p_1v_1 = p_1v_0 \wedge \exists v_2[z(v_0v_2) \wedge p_0(v_1v_2)] \wedge y(v_1v_2))$$
$$((v+1) \cdot u = w \leftrightarrow w = (v \cdot u) + u),$$

$$p_0v_0 = p_0v_1 \wedge p_1v_0 = p_1v_1 \rightarrow (z(v_0v_2) \leftrightarrow z(v_1v_2)).$$

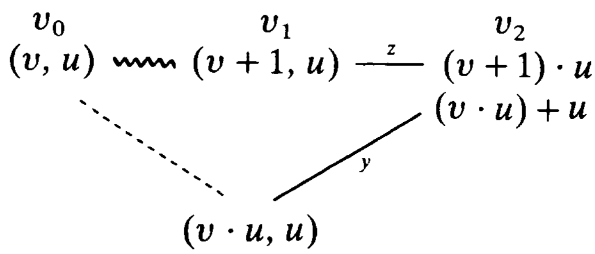

Illustration for the definition of z

Now to every variable w let us associate a constant q_w. That q_w is a constant can be expressed as follows:

$$q_w(v_0) \wedge q_w(v_1) \rightarrow v_0 = v_1,$$

$$\exists v_0 q_w(v_0).$$

Let u, v, w be variables. Then $u + 1 = v$, $u + v = w$ and $u = 0$ can be expressed as follows

$$v_0 = q_u \wedge v_1 = q_v \rightarrow x(v_0v_1),$$

$$p_0v_0 = q_u \wedge p_1v_0 = q_v \wedge v_1 = q_w \rightarrow y(v_0v_1),$$

$$q_u \in \mathrm{Do}x \wedge q_u \notin \mathrm{Rg}x. \quad \square$$

Definition 3.8. (i) τ_1 is defined to be the CA_3-term

$$-c_{(3)}(x \cdot s_2^1 x - d_{12}) - c_{(3)}(x \cdot s_2^0 x - d_{02}) - c_{(2)}(x \cdot d_{01}) - c_{(3)}(c_2 x - x).$$

π is defined to be the CA_3-term

$$-c_{(3)}(p_0 \cdot s_2^1 p_0 - d_{12}) - c_{(3)}(p_1 \cdot s_2^1 p_1 - d_{12}).$$
$$-c_{(3)}(c_1 x \cdot s_1^0 c_1 x - c_2[s_0^1 s_2^0 p_0 \cdot s_2^0 p_1]).$$
$$-c_{(3)}(c_2 p_0 - p_0) - c_{(3)}(c_2 p_1 - p_1).$$

Let $n(x) \triangleq c_1 x - s_0^1 c_0 x$ and $\sigma \triangleq c_0 n(x) - c_0 c_1(n(x) \cdot s_1^0 n(x) - d_{01})$.
τ_2 is defined to be the CA_3-term

$$\begin{aligned}
&-c_{(3)}(y \cdot s_2^1 y - d_{12}) \cdot \\
&-c_{(3)}(y - [c_1(p_0 \cdot s_1^0 c_1 x) \cdot c_1(p_1 \cdot s_1^0 c_1 x) \cdot s_1^0 c_1 x]) \cdot \\
&-c_{(3)}(c_1[p_0 \cdot s_1^0 n(x)] \cdot p_1 \cdot s_1^0 c_1 x - y) \cdot \\
&-c_{(3)}(c_1[c_2(s_1^0 s_2^1 p_0 \cdot c_1[p_0 \cdot s_1^0 s_2^1 x]) \cdot c_2(s_2^1 p_1 \cdot s_1^0 s_2^1 p_1) \cdot s_1^0 s_2^1 y] \\
&\qquad \oplus c_1[y \cdot s_1^0 s_2^1 x]) \cdot \\
&-c_{(3)}[c_2(s_2^1 p_0 \cdot s_1^0 s_2^1 p_0) \cdot c_2(s_2^1 p_1 \cdot s_1^0 s_2^1 p_1) \cdot (s_2^1 y \oplus s_1^0 s_2^1 y)] \cdot \\
&-c_{(3)}(c_2 y - y).
\end{aligned}$$

τ_3 is defined to be the CA_3-term

$$\begin{aligned}
&-c_{(3)}(z \cdot s_2^1 z - d_{12}) \cdot \\
&-c_{(3)}(z - [c_1(p_0 \cdot s_1^0 c_1 x) \cdot c_1(p_1 \cdot s_1^0 c_1 x) \cdot s_1^0 c_1 x]) \cdot \\
&-c_{(3)}(s_1^0 n(x) \cdot p_0 \cdot c_1(p_1 \cdot s_1^0 c_1 x) - z) \cdot \\
&-c_{(3)}(c_1[c_2(s_1^0 s_2^1 p_0 \cdot c_1[p_0 \cdot s_1^0 s_2^1 x]) \cdot c_2(s_2^1 p_1 \cdot s_1^0 s_2^1 p_1) \cdot s_1^0 s_2^1 z] \\
&\qquad \oplus c_1[c_2(s_2^1 p_1 \cdot s_1^0 s_2^1 p_1) \cdot c_2(s_2^1 z \cdot s_1^0 s_2^1 p_0) \cdot s_1^0 s_2^1 y]) \cdot \\
&-c_{(3)}[c_2(s_2^1 p_0 \cdot s_1^0 s_2^1 p_0) \cdot c_2(s_2^1 p_1 \cdot s_1^0 s_2^1 p_1) \cdot (s_2^1 z \oplus s_1^0 s_2^1 z)] \cdot \\
&-c_{(3)}(c_2 z - z).
\end{aligned}$$

Let $\psi(x, y, z, p_0, p_1) \triangleq \tau_1 \cdot \pi \cdot \tau_2 \cdot \tau_3 \cdot \sigma$.

(ii) Let $e(\bar{x})$ be a number-theoretic equation. Let V, $W = \{x_0, \ldots, x_n, y_0, \ldots, y_k\} \subseteq V$, and $b_0, \ldots, b_m$ be associated to $e(\bar{x})$ as in Definition 3.2(ii). To every $w \in W$ we associate a variable q_w in the language of CA_3. Define

$$\kappa(W) \triangleq \prod \{-c_{(3)}(c_1 c_2 q_w - q_w) - c_{(3)}(q_w \cdot s_1^0 q_w - d_{01}) \cdot c_0 q_w : w \in W\}.$$

For each $l \leq m$ define the cylindric term ξ_l as follows: ξ_l is defined to be

$$\begin{aligned}
&-c_{(2)}(q_u \cdot s_1^0 q_v - x), \\
&-c_{(2)}(c_1[p_0 \cdot s_1^0 q_u] \cdot c_1[p_1 \cdot s_1^0 q_v] \cdot s_1^0 q_w - y), \\
&-c_{(2)}(c_1[p_0 \cdot s_1^0 q_u] \cdot c_1[p_1 \cdot s_1^0 q_v] \cdot s_1^0 q_w - z), \quad \text{or} \\
&-c_{(2)}(q_u - n(x))
\end{aligned}$$

according to whether b_l is $u + 1 = v$, $u + v = w$, $u \cdot v = w$ or $u = 0$.

Now we define $\eta(e(\bar{x}))$ to be

$$\psi(x, y, z, p_0, p_1) \cdot \kappa(W) \cdot \prod \{\xi_l : l \leq m\} = 0. \quad \square$$

Lemma 3.9. *Let* $\alpha \geq 3$, $\mathfrak{A} \in \mathrm{Mg}_\alpha \cap \mathrm{Gs}_\alpha^{\mathrm{reg}}$, $k \in 1^{\mathfrak{A}}$ *and* X, Y, Z, P_0, $P_1 \in A$. *Set* $s \triangleq X[\![k, 2]\!]$, $a \triangleq Y[\![k, 2]\!]$, $m \triangleq Z[\![k, 2]\!]$, $p_0 \triangleq P_0[\![k, 2]\!]$, $p_1 \triangleq P_1[\![k, 2]\!]$, $D \triangleq \mathrm{Dos}$ *and*

$P \triangleq \{u \in \mathrm{Dop}_0 \cap \mathrm{Dop}_1 : p_0 u \in D \wedge p_1 u \in D\}$. *Assume* $k \notin \sum \{c_{(3)}(c_2 q - q) : q \in \{X, Y, Z, P_0, P_1\}\}$.

(i) *Then* (a) *and* (b) *below are equivalent.*

(a) $k \in \psi(X, Y, Z, P_0, P_1)^{\mathfrak{A}}$.

(b) p_0, p_1 *are unary functions,* $(\forall u, v \in D)(\exists w \in P)(p_0 w = u \wedge p_1 w = v)$, $a : S_1 \to D$, $m : S_2 \to D$ *for some* $S_1, S_2 \subseteq P$, *and there are* $N \in \omega$ *and* $n : N + 1 \rightarrowtail \mathrm{base}(\mathfrak{A})$ *such that* $s = \{\langle ni, n(i+1)\rangle : i < N\}$ *and for every* $q \in P$ *and* $u \in D$ *we have*

$$(q, u) \in a \quad \textit{iff} \quad (\exists i, j \in N)[i + j \in N \wedge ni = p_0 q \wedge nj = p_1 q \wedge n(i + j) = u],$$

$$(q, u) \in m \quad \textit{iff} \quad (\exists i, j \in N)[i \cdot j \in N \wedge ni = p_0 q \wedge nj = p_1 q \wedge n(i \cdot j) = u].$$

(ii) *Let* $g : \{q_w : w \in W\} \to A$. *Assume* $k \notin \sum \{c_{(3)}(c_1 c_2 g(q_w) - g(q_w)) : w \in W\}$. *Then* (a) *and* (b) *below hold.*

(a) $k \in \kappa(W)^{\mathfrak{A}}[g]$ *iff* $|g(q_w)[\![k, 1]\!]| = 1$ *for every* $w \in W$.

(b) *Assume* $k \in \psi(X, Y, Z, P_0, P_1)^{\mathfrak{A}}$. *Let* N, n, *be as in* (i)(b). *Assume* $k \in \kappa(W)^{\mathfrak{A}}[g]$. *For every* $w \in W$ *let* $\{c_w\} = g(q_w)[\![k, 1]\!]$. *Let* $h : W \to N$ *be defined by* $h_w = n^{-1}(c_w)$ *if* $c_w \in \mathrm{Rg}n$, $h_w = 0$ *otherwise. Then*

$$k \in \prod \{\xi_l : l \leq m\}[X, Y, Z, P_0, P_1, g] \quad \textit{iff} \quad (\omega \models \bigwedge B[h] \wedge (\forall w \in W)\, c_w \in \mathrm{Rg}n).$$

The proof of Lemma 3.9 is similar to that of Lemma 3.4. The proof of the last two statements of (i)(b) goes as follows: Let $i, j \in N$ be such that $p_0 q = ni$, $p_1 q = nj$. Then both directions are proved by induction on i. We omit the rest of the proof.

Let $e(\bar{x})$ be a number-theoretic equation. Let $\exists \bar{y} \bigwedge B$ where $B = \{b_0, \ldots, b_m\}$, $W = \{x_0, \ldots, x_n, y_0, \ldots, y_k\} \subseteq V$ and $\{\xi_l : l \leq m\}$ be associated to $e(\bar{x})$ as in Definitions 3.2, 3.8.

Lemma 3.10. *Let* $3 \leq \alpha$. *Then* $\omega \models \exists \bar{x}\, e(\bar{x})$ *iff* $\mathrm{Mg}_\alpha \not\models \eta(e(\bar{x}))$.

Proof. The proof of Lemma 3.10 is very similar to the proofs of Lemma 3.5, 3.6, using Lemma 3.9 instead of Lemma 3.4. Because of this, we will be more sketchy here, in proving Lemma 3.10. Assume $\omega \models \exists \bar{x}\, e(\bar{x})$. Then $\omega \models \exists \bar{x}\, \exists \bar{y} \bigwedge B$. Let $h \in {}^W\omega$ be such that $\omega \models \bigwedge B[h]$. Let $N \in \omega$ be such that $h^* W \subseteq N$. Let $U \triangleq (N + 1) \cup {}^2N$. Then U is finite. For every $u \in U$ let $Q(u) \triangleq \{s \in {}^\alpha U : s_0 = u\}$. Let $\mathfrak{M} \triangleq \mathfrak{S}\mathrm{g}^{(\mathfrak{S}b^\alpha U)}\{Q(u) : u \in U\}$. Then $\mathfrak{M} \in \mathrm{Cs}_\alpha^{\mathrm{reg}} \cap \mathrm{Mg}_\alpha$. Let

$$\mathrm{P}_i \triangleq \{s \in {}^\alpha U : s_0 \in {}^2N \text{ and } s_1 = pj_i(s_0)\} \quad \text{for } i \in 2,$$

$$X \triangleq \{s \in {}^\alpha U : s_0 \in N \text{ and } s_1 = s_0 + 1\},$$

$$Y \triangleq \{s \in {}^\alpha U : s_0 \in {}^2N \text{ and } s_1 = pj_0(s_0) + pj_1(s_0) < N\},$$

$$Z \triangleq \{s \in {}^\alpha U : s_0 \in {}^2N \text{ and } s_1 = pj_0(s_0) \cdot pj_1(s_0) < N\}.$$

Then $X, Y, Z, P_0, P_1 \in M$ since U is finite. (E.g., $P_0 = \Sigma\{Q(\langle m, n\rangle) \cdot s_1^0 Q(m): m, n \in N\}$.) Let $g(q_w) \triangleq Q(h_w)$ for every $w \in W$. Let $k \in 1^{\mathfrak{M}}$. Let $g': \{x, y, z, p_0, p_1\} \cup \{q_w: w \in W\} \to M$ be defined by $g \subseteq g'$ and $g'(x) = X$, $g'(y) = Y$, $g'(z) = Z$, $g'(p_i) = P_i$ for $i \in 2$. Now $k \in (\psi(x, y, z, p_0, p_1) \cdot \kappa(W))^{\mathfrak{M}}[g']$ by Lemma 3.9 and by inspecting the above definitions. Also, $k \in \prod\{\xi_l : l \leq m\}[g']$ by Lemma 3.9(ii) and since $[\boldsymbol{\omega} \models \bigwedge B[h]$ and $(\forall w \in W) h_w \in N]$. These statements show $\mathfrak{M} \not\models \eta(e(\bar{x}))$.

Conversely, assume $\mathrm{Mg}_\alpha \not\models \eta(e(\bar{x}))$. Then $\mathfrak{M} \not\models \eta(e(\bar{x}))$ for some $\mathfrak{M} \in \mathrm{Gs}_\alpha^{\mathrm{reg}} \cap \mathrm{Mg}_\alpha$, since $\mathrm{Mg}_\alpha \subseteq \mathbf{I}\mathrm{Gs}_\alpha^{\mathrm{reg}}$ by [22, 11]. Let $g: \{x, y, \ldots\} \to M$ be such that $\mathfrak{M} \not\models \eta(e(\bar{x}))[g]$. Let $k \in 1^{\mathfrak{M}}$ be such that $k \in (\psi \cdot \kappa(W) \cdot \prod\{\xi_l : l \leq m\})[g]$. Then $\boldsymbol{\omega} \models \bigwedge B[h]$ for some $h: W \to \omega$ by Lemma 3.9(ii)(b). Thus $\boldsymbol{\omega} \models \exists \bar{x}\, e(\bar{x})$ and we are done. □

By the above we have seen that $\overline{\mathrm{Eq}\mathrm{Mg}_\alpha}$ is not r.e. for $\alpha \geq 3$.

(B2) *The idea* of the modification of the proof in (B1): The problem is that if $K \subseteq \mathrm{Mg}_\alpha$ is unbounded, then we do not necessarily have 'constants' in some element of K – though these constants are needed for satisfying $\kappa(W) \neq 0$. Indeed, let $K \triangleq \{\mathfrak{A} \in \mathrm{Cs}_\alpha : (\exists \text{ partition } P \text{ of base}(\mathfrak{A}))\ [(\forall p \in P)\, |p| \geq 2 \wedge \mathfrak{A} = \mathfrak{Sg}\{\bar{p} : p \in P\}]\}$, where $\bar{p} \triangleq \{s \in 1^{\mathfrak{A}} : s_0 \in p\}$. Then K is unbounded but $K \models \kappa(W) = 0$ for any W, hence $K \models \eta(e(\bar{x}))$ for any number-theoretic equation $e(\bar{x})$. But, as we shall see below, this is the only shortcoming and it can be overcome by changing the formulation of $\eta(e(\bar{x}))$ as follows.

e is an equivalence relation:

$$e(v_0v_0), \qquad e(v_0v_1) \to e(v_1v_0), \qquad e(v_0v_1) \wedge e(v_1v_2) \to e(v_0v_2).$$

x, y, z, p_0, p_1 do not 'separate' e: Let $\xi \in \{x, y, z, p_0, p_1\}$ and let $w \in W$.

$$\xi(v_0v_1) \wedge e(v_0v_2) \to \xi(v_2v_1),$$
$$\xi(v_0v_1) \wedge e(v_1v_2) \to \xi(v_0v_2),$$
$$q_w(v_0) \wedge e(v_0v_1) \to q_w(v_1).$$

The rest of the formulas are the same, except that we replace $v_i = v_j$ everywhere with $e(v_iv_j)$. Below we formalize the above in the language of CA_3.

Definition 3.11. (i) β is defined to be the CA_3-term

$$-c_{(2)}(d_{01} - e) - c_{(2)}(e - s_0^2 s_1^0 s_2^1 e) - c_{(3)}(e \cdot s_1^0 s_2^1 e - s_2^1 e) - c_{(3)}(c_2 e - e).$$

For every $\xi \in \{x, y, z, p_0, p_1\}$, σ_ξ is the term

$$-c_{(3)}(\xi \cdot s_2^1 e - s_2^0 \xi) - c_{(3)}(\xi \cdot s_1^0 s_2^1 e - s_2^1 \xi).$$

Let γ be the following term

$$\beta \cdot \prod\{\sigma_\xi : \xi \in \{x, y, z, p_0, p_1\}\} \cdot \prod\{-c_{(2)}(q_w \cdot e - s_1^0 q_w) : w \in W\}.$$

(ii) Let $\psi'(e, x, y, z, p_0, p_1)$ and $\kappa'(e, W)$ be the terms we obtain by replacing d_{01}, d_{02}, d_{12} respectively with β, $s_2^1\beta$, $s_1^0 s_2^1\beta$ everywhere in the terms $\psi(x, y, z, p_0, p_1)$ and $\kappa(W)$ defined in Definition 3.8. We define $\delta(e(\bar{x}))$ to be

$$\gamma \cdot \psi'(e, x, y, z, p_0, p_1) \cdot \kappa(e, W) \cdot \prod \{\xi_l : l \leqslant m\} = 0$$

where the terms ξ_l $(l \leqslant m)$ are as defined in Definition 3.8(ii). □

Now we state (without proof) the lemma analogous to Lemma 3.9.

Lemma 3.12. *Let* $\alpha \geqslant 3$, $\mathfrak{A} \in \mathrm{Mg}_\alpha \cap \mathrm{Gs}_\alpha^{\mathrm{reg}}$, $k \in 1^{\mathfrak{A}}$ *and* $E, X, Y, Z, P_0, P_1 \in A$. *Set* $e \triangleq E[\![k, 2]\!]$, $s \triangleq X[\![k, 2]\!], \ldots, p_1 \triangleq P_1[\![k, 2]\!]$. *Let* $Q_w \in A$ *for every* $w \in W$. *Set* $q_w \triangleq Q_w[\![k, 1]\!]$. *Assume that* $k \notin \sum \{c_{(3)}(c_2 q - q) : q \in \{E, X, Y, Z, P_0, P_1\}\} + \sum \{c_{(3)}(c_1 c_2 Q_w - Q_w) : w \in W\}$. *Then* (i)–(iii) *below hold.*

(i) (a) *and* (b) *below are equivalent.*

(a) $k \in \gamma(E, X, Y, Z, P_0, P_1, \langle Q_w : w \in W\rangle)^{\mathfrak{A}}$.

(b) *e is an equivalence relation on* base($\mathfrak{A}$) *and s, a, m, p_0, p_1, q_w do not separate e.*

(ii) *Assume that* $k \in \gamma(E, X, Y, Z, P_0, P_1, \langle Q_w : w \in W\rangle)^{\mathfrak{A}}$. *Then* (a) *and* (b) *below are equivalent. Let D and P be as in Lemma* 3.9.

(a) $k \in \psi'(E, X, Y, Z, P_0, P_1)^{\mathfrak{A}} \cdot \kappa'(E, \langle Q_w : w \in W\rangle)^{\mathfrak{A}}$.

(b) $\bar{p}_0 \triangleq p_0/e$ *and* $\bar{p}_1 \triangleq p_1/e$ *are unary functions,* $(\forall u, v \in D)(\exists w \in P)(wp_0u \wedge wp_1v)$, *$a/e$, m/e are partial functions from P/e to D/e; there are $N \in \omega$ and* $n : N + 1 \rightarrowtail \mathrm{base}(\mathfrak{A})/e$ *such that* $s/e = \{(ni, n(i+1)) : i < N\}$ *and for every* $q \in P$ *and* $u \in D$

$$(q, u) \in a \quad \mathit{iff} \quad (\exists i, j \in N)[i + j \in N \wedge ni = \bar{p}_0(q/e) \wedge n_j = \bar{p}_1(u/e) \wedge n(i + j) = u/e],$$

$$(q, u) \in m \quad \mathit{iff} \quad (\exists i, j \in N)[i \cdot j \in N \wedge ni = \bar{p}_0(q/e) \wedge n_j = \bar{p}_1(u/e) \wedge n(i \cdot j) = u/e],$$

and further $|q_w/e| = 1$ *for every* $w \in W$.

(iii) *Assume that* $k \in (\gamma \cdot \psi' \cdot \kappa'(W))(E, X, \ldots, \langle Q_w : w \in W\rangle)^{\mathfrak{A}}$. *Let n, N be as in* (ii)(b). *Let* $h : W \to N$ *be defined as* $h(w) \triangleq n^{-1}(c_w)$ *if* $q_w/e = \{c_w\}$ *and* $c_w \in \mathrm{Rg}n$, $h(w) \triangleq 0$ *otherwise. Then* $k \in \prod \{\xi_l : l \leqslant m\}[X, Y, Z, P_0, P_1, \langle Q_w : w \in W\rangle]$ *iff* $(\boldsymbol{\omega} \models \bigwedge B[h]$ & $(\forall w \in W) c_w \in \mathrm{Rg}n)$.

Now one can prove a lemma analogous to Lemma 3.10, but using Lemma 3.12 instead of Lemma 3.9. We sketch the proof of one direction of the modified Lemma 3.10.

Lemma 3.13. *Let* $3 \leqslant \alpha$ *and* $K \subseteq \mathrm{Mg}_\alpha$ *be unboundedly generated. Let $e(\bar{x})$ be any number-theoretic equation. Then* $\boldsymbol{\omega} \models \exists \bar{x}\, e(\bar{x})$ *implies* $K \not\models \delta(e(\bar{x}))$.

Proof. Assume $\omega \models \exists \bar{x}\, e(\bar{x})$. Let h, N, and U be as in the beginning of the proof of Lemma 3.10. Let $n \triangleq 2^{|U|}$. Then $K \nsubseteq \mathbf{SP}\mathrm{Mg}_\alpha^n$, hence there is $\mathfrak{M} \in K \sim \mathbf{SP}\mathrm{Mg}_\alpha^n$. We may assume $\mathfrak{M} \in \mathrm{Cs}_\alpha^{\mathrm{reg}} \sim \mathrm{Mg}_\alpha^n$. Then there is $Q: U \rightarrow (\mathrm{Nr}_1\mathfrak{M} \sim \{0\})$ such that $Q_u \cap Q_v = 0$ whenever $u \neq v$. Let $V \triangleq \mathrm{base}(\mathfrak{M})$. Let e be an equivalence relation on V such that $\{Q_u : u \in U\} \subseteq V/e$. Define

$$E \triangleq \{s \in {}^\alpha V : (s_0, s_1) \in e\},$$
$$P_0 \triangleq \{s \in {}^\alpha V : (\exists n, m \in N)(s_0 \in Q(\langle n, m\rangle) \wedge s_1 \in Q(n))\},$$
$$P_1 \triangleq \{s \in {}^\alpha V : (\exists n, m \in N)(s_0 \in Q(\langle n, m\rangle) \wedge s_1 \in Q(m))\},$$
$$X \triangleq \{s \in {}^\alpha V : (\exists n \in N)(s_0 \in Q(n) \wedge s_1 \in Q(n+1))\},$$
$$Y \triangleq \{s \in {}^\alpha V : (\exists n, m \in N)(s_0 \in Q(\langle n, m\rangle) \wedge s_1 \in Q(n+m) \wedge n+m \in N)\},$$
$$Z \triangleq \{s \in {}^\alpha V : (\exists n, m \in N)(s_0 \in Q(\langle n, m\rangle) \wedge s_1 \in Q(n \cdot m) \wedge n \cdot m \in N)\}.$$

From now on the proof goes almost exactly as the proof of Lemma 3.10. □

By the above, Theorem 3.7 has been proved. Hence $\overline{\mathrm{Eq}}K$ is not r.e. if $3 \leq \alpha < \omega$ and $K \subseteq \mathrm{Mg}_\alpha$ is unboundedly generated.

(C) Now we start proving the cases when $\overline{\mathrm{Eq}}K$ is decidable.

(C1) Let $\alpha \geq \omega$. We shall prove more, namely we shall consider classes $K \subseteq \mathrm{CA}_\alpha$, too, and not only classes $K \subseteq \mathrm{Mg}_\alpha$. We will show that if $K \subseteq \mathrm{CA}_\alpha$ is bounded, then $\overline{\mathrm{Eq}}K$ is decidable. We note that the converse of this statement is also true: If $\overline{\mathrm{Eq}}K$ for $K \subseteq \mathrm{CA}_\alpha$, $\alpha \geq \omega$, is decidable, then K is bounded. This is proved in [29]. We shall use the following lemmas, which also give information on the lattice of varieties of CA_α's. Recall the notation ${}_n\mathrm{Gs}_\alpha$, ${}_n\mathrm{Mn}_\alpha$, ${}_{(L)}\mathrm{Mn}_\alpha$ and ${}_{<n}\mathrm{CA}_\alpha$ from the end of Section 1. Let $\mathfrak{A} \in \mathrm{CA}_\beta$, $\beta \geq \alpha$. Then $\mathfrak{Rd}_\alpha\mathfrak{A}$ denotes the α-dimensional reduct of $\mathfrak{A}$, i.e., $\mathfrak{Rd}_\alpha\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, c_i^{\mathfrak{A}}, d_{ij}^{\mathfrak{A}}\rangle_{i,j\in\alpha}$.

Lemma 3.14. *Let* $1 < \alpha < \omega > n$ *and* $\beta \geq \alpha + n$. *Let* $\mathfrak{A} \in {}_n\mathrm{Gs}_\beta$. *Then* $\mathbf{HSP}\{\mathfrak{Rd}_\alpha\mathfrak{A}\} = \mathbf{I}\,{}_n\mathrm{Gs}_\alpha$.

Proof. Assume $\mathfrak{A} \in {}_n\mathrm{Gs}_\beta$. We may assume $1^{\mathfrak{A}} = {}^\beta n$ (since $\mathbf{HRd}\mathfrak{A} \supseteq \mathrm{Rd}\mathbf{H}\mathfrak{A}$). Let $H \triangleq (\beta \sim \alpha)$ and $t \in {}^H n$ be such that $(\forall i < n)\, t(\alpha + i) = i$. To every $s \in {}^\alpha n$ there is

(+) $x_s \in A$ such that $(\forall q \in {}^\beta n)[t \subseteq q \Rightarrow (q \in x_s \Leftrightarrow s \subseteq q)]$.

To define this x_s we need only the d_{ij}'s with $i < \alpha$ and $\alpha \leq j < \alpha + n$, namely $x_s = \prod \{d_{i,\alpha+s(i)} : i < \alpha\}$. Let $h = \langle \{\alpha \upharpoonright q : t \subseteq q \in Y\} : Y \in A\rangle$. Now $h \in \mathrm{Hom}(\mathfrak{Rd}_\alpha\mathfrak{A}, \mathfrak{Sb}\,{}^\alpha n)$ is easy to verify, see [12, 4.7.1.2(ii)]. Let $\mathfrak{B} = h^*\mathfrak{Rd}_\alpha\mathfrak{A}$. By (+) above and since $h(x_s) = \{s\}$ for all $s \in {}^\alpha n$, we have $\{\{s\} : s \in 1^{\mathfrak{B}}\} \subseteq B$ which by $n, \alpha < \omega$ implies that $\mathfrak{B}$ is full. Then clearly $\mathbf{SP}\mathfrak{B} = \mathbf{SP}\,{}_n\mathrm{Cs}_\alpha \supseteq {}_n\mathrm{Gs}_\alpha$. Thus $\mathbf{SPH}\{\mathfrak{Rd}_\alpha\mathfrak{A}\} \supseteq {}_n\mathrm{Gs}_\alpha$. It was proved in [12, 7.18(i)] that $\mathbf{I}\,{}_n\mathrm{Gs}_\alpha$ is a variety (and this is easy to prove based on results in [11]), and $\mathrm{Rd}_\alpha\,{}_n\mathrm{Gs}_\beta \subseteq \mathbf{I}\,{}_n\mathrm{Gs}_\alpha$ by pp. 53–54 of [11]. □

Corollary 3.15. *Let $n<\omega\leqslant\alpha$. Then $\mathbf{I}_n\mathrm{Gs}_\alpha$ has no nontrivial subvariety.*

Proof. Let $\mathfrak{A}\in{}_n\mathrm{Gs}_\alpha$ be arbitrary. We will show $\mathbf{HSP}\{\mathfrak{A}\}=\mathbf{I}_n\mathrm{Gs}_\alpha$. Let e be any equation in the language of CA_α and assume that ${}_n\mathrm{Gs}_\alpha\nvDash e$. Then there is $\Gamma\subseteq_\omega\alpha$ and $\mathfrak{B}\in{}_n\mathrm{Gs}_\alpha$ such that $\mathfrak{Rd}_\Gamma\mathfrak{B}\nvDash e$. By Lemma 3.14 we have $\mathbf{HSP}\{\mathfrak{Rd}_\Gamma\mathfrak{B}\}=\mathbf{I}_n\mathrm{Gs}_\Gamma=\mathbf{HSP}\{\mathfrak{Rd}_\Gamma\mathfrak{A}\}$, hence $\mathfrak{Rd}_\Gamma\mathfrak{A}\nvDash e$, i.e., $\mathfrak{A}\nvDash e$. □

Remark 3.16. Corollary 3.15 is not true for $\alpha<\omega$, and $n>1$, because ${}_n\mathrm{Mn}_\alpha\vDash c_{(\alpha\sim1)}x=c_{(\alpha)}x\,\#\,{}_n\mathrm{Cs}_\alpha$ for $n>1$, hence $\mathrm{Eq}({}_n\mathrm{Mn}_\alpha)$ is a proper nontrivial subvariety of $\mathbf{I}_n\mathrm{Gs}_\alpha$, which is a variety for $0<\alpha$, $n<\omega$ by [12, II.7.18].

Lemma 3.17. (i) *Let $K\subseteq\mathrm{FbGs}_\alpha$. Then $\mathrm{Eq}K=\mathrm{Eq}({}_{(L)}\mathrm{Mn}_\alpha)$ for some $L\subseteq\omega$.*
(ii) *Let $K\subseteq{}_{<n}\mathrm{CA}_\alpha$. Then $\mathrm{Eq}K=\mathrm{Eq}({}_{(L)}\mathrm{Mn}_\alpha)$ for some $L\subseteq n$.*

Proof. Let $\mathfrak{A}\in K$, $n\triangleq|\mathrm{base}(\mathfrak{A})|$ and let $\mathfrak{M}$ be the minimal subalgebra of $\mathfrak{A}$. Let $L\triangleq\{n\in\omega:K\cap{}_n\mathrm{Gs}_\alpha\neq0\}$. Then $n\in L$ and $\mathfrak{M}\in{}_{(L)}\mathrm{Mn}_\alpha\cap\mathrm{Eq}K$. Also, $\mathfrak{A}\in\mathbf{I}_n\mathrm{Gs}_\alpha=\mathrm{Eq}\{\mathfrak{M}\}$ by Corollary 3.15. This shows $\mathrm{Eq}K=Eq({}_{(L)}\mathrm{Mn}_\alpha)$. If $K\subseteq{}_{<n}\mathrm{CA}_\alpha$, then $K\subseteq\mathrm{FbGs}_\alpha$ by [11, 4.2.53] and $L\subseteq n$. □

Lemma 3.18. *Let $L\subseteq\omega$ be finite and $\alpha\geqslant\omega$. Then $\overline{\mathrm{Eq}}({}_{(L)}\mathrm{Mn}_\alpha)$ is decidable.*

Proof. Let $\mathfrak{A}\in{}_n\mathrm{Cs}_\alpha\cap\mathrm{Mn}_\alpha$ and $\mathfrak{A}\nvDash e$. Then $\mathfrak{Rd}_\Gamma\mathfrak{A}\nvDash e$ for some finite $\Gamma\subseteq\omega$. (*) Actually, Γ is the set of all indices occurring in e. Since $\mathfrak{Rd}_\Gamma\mathfrak{A}\in\mathbf{I}_n\mathrm{Gs}_\Gamma$ by [12, 4.7.1.2] (or equivalently by the proof of [11, 3.1.118]) we have $\mathbf{SP}_n\mathrm{Cs}_\Gamma=\mathbf{I}_n\mathrm{Gs}_\Gamma\nvDash e$. We have proved (**) ${}_n\mathrm{Mn}_\alpha\nvDash e\Rightarrow{}_n\mathrm{Cs}_\Gamma\nvDash e$. In the other direction, assume ${}_n\mathrm{Cs}_\Gamma\nvDash e$. Then ${}_n\mathrm{Cs}_\alpha\nvDash e$ by the proof of [11, 3.1.121], so by Corollary 3.15, $\mathbf{HSP}_n\mathrm{Mn}_\alpha=\mathbf{I}_n\mathrm{Gs}_\alpha\nvDash e$, hence ${}_n\mathrm{Mn}_\alpha\nvDash e$. Together with (**) this proves (***) ${}_n\mathrm{Mn}_\alpha\vDash e\Leftrightarrow{}_n\mathrm{Cs}_\Gamma\vDash e$. Clearly, ${}_n\mathrm{Cs}_\Gamma$ has only finitely many finite elements (note that $|{}^\Gamma n|<\omega$), hence given e and Γ we can effectively decide whether ${}_n\mathrm{Cs}_\Gamma\vDash e$ holds. By (*), Γ is effectively computable from e. (*[4]) This provides us with a decision procedure for ${}_n\mathrm{Mn}_\alpha$. Let $L\subseteq\omega$ be finite. Then $\overline{\mathrm{Eq}}({}_{(L)}\mathrm{Mn}_\alpha)=\overline{\mathrm{Eq}}\cup\{{}_k\mathrm{Mn}_\alpha:k\in L\}=\bigcap\{\overline{\mathrm{Eq}}_k\mathrm{Mn}_\alpha:k\in L\}$ provides us with a decision procedure using (*[4]) and finiteness of L. □

(C2) Assume $2<\alpha<\omega$ and $K\subseteq\mathrm{Mg}_\alpha$ is boundedly generated. Then $K\subseteq\mathbf{SP}\mathrm{Mg}_\alpha^n$ for some $n\in\omega$. We have $\mathrm{Mg}_\alpha^n\subseteq\mathbf{SP}(\mathrm{Mg}_\alpha^n\cap\mathrm{Cs}_\alpha)$ since $\mathrm{Mg}_\alpha\subseteq\mathbf{I}\mathrm{Gs}_\alpha$. Then $K\subseteq\mathbf{SP}(\mathrm{Mg}_\alpha^n\cap\mathrm{Cs}_\alpha)\subseteq\mathbf{P}_s\mathbf{S}(\mathrm{Mg}_\alpha^n\cap\mathrm{Cs}_\alpha)$, where $\mathbf{P}_sL$ denotes the class of all subdirect products of members of L. Thus there is $L\subseteq\mathbf{S}(\mathrm{Mg}_\alpha^n\cap\mathrm{Cs}_\alpha)$ with $\mathrm{Eq}K=\mathrm{Eq}L$. By [11, 2.2.26] we have $\mathbf{S}(\mathrm{Mg}_\alpha^n\cap\mathrm{Cs}_\alpha)$ is a finite set of finite algebras, hence so is L and we can decide $\overline{\mathrm{Eq}}L$. □ (Theorem 2)

Proof of Theorem 4. Let $\mathfrak{R}$ be a monadic-generated RA. Then $\mathfrak{R}=\mathfrak{Ra}\mathfrak{A}$ for some $\mathfrak{A}\in\mathbf{SNr}_3\mathrm{CA}_4$ by [11, 5.3.17]. Let $R=\mathrm{Sg}G$ where $(\forall x\in G)\,x\,;1=x$. Then

$(\forall x \in G)\, \Delta^{\mathfrak{A}} x \subseteq 1$ can easily be seen. We may assume that $A = \mathrm{Sg}^{\mathfrak{A}} G$ by [11, 5.3.12]. Hence $\mathfrak{A} \in \mathrm{Mg}_3$, therefore $\mathfrak{A}$ is representable by Monk [22, Theorem 21] (or by [11, 3.2.12]). Then $\mathfrak{R}$ is representable, too. The second statement of Theorem 4 has been proved. By the above we also see that an analog of Lemma 3.3 holds for the class MRA of monadic-generated RA's.

Let MRA denote the class of all monadic-generated RA's. The proof of "$\overline{\mathrm{Eq}}$MRA is not r.e." is practically the same as that of "$\overline{\mathrm{Eq}}\mathrm{Mg}_3$ is not r.e.". The only difference is that instead of $\eta(e(\bar{x}))$ we will now use a relation algebraic correspondent $\rho(e(\bar{x}))$ of the number-theoretic equation $e(\bar{x})$. We give here the translation. Let $e(\bar{x})$ be a number-theoretic equation with free variables $x_0, \ldots, x_n \in V = \{v_i : i \in \omega, i > 6\}$. Let $e(\bar{x})$ be equivalent in $\boldsymbol{\omega}$ to $\exists y_0 \cdots y_k(b_0 \wedge \cdots \wedge b_m)$ such that $y_0, \ldots, y_k \in V$ and each b_i has the form $u+1=v$, $u+v=w$, $u \cdot v = w$ or $u=0$ for some $u, v, w \in W \triangleq \{x_0, \ldots, x_n, y_0, \ldots, y_k\}$. First we translate the formulas "x is a one–one function with no fix-point..." to RA-theoretic inequalities and non-equalities:

(1) $x^{\cup} ; x \leq 1'$ (x is a function)

(2) $x ; x^{\cup} \leq 1'$ (x is one–one).

(3) $x \leq -1'$ (x has no fix-point).

(4) $[(x;1) - (1;x)^{\cup}] \cdot [(x;1)^{\cup} - (1;x)] \leq 1'$ ($|\mathrm{Do}x \sim \mathrm{Rg}x| \leq 1$).

(5) $(x;1) - (1;x)^{\cup} \neq 0$. ($|\mathrm{Do}x \sim \mathrm{Rg}x \neq 0$).

(6) $p_i^{\cup} ; p_i \leq 1'$ for $i \in 2$

(7) $(x;1) \cdot (x;1)^{\cup} \leq p_0^{\cup} ; p_1$ $(\forall v_0, v_1 \in \mathrm{Do}x)\, \exists v_2(p_0 v_2 = v_0 \wedge p_1 v_2 = v_1)$.

(8) $y^{\cup} ; y \leq 1'$.

(9) $y ; 1 \leq p_i ; x ; 1$ for $i \in 2$ $(v_0 \in \mathrm{Do}y \rightarrow p_i v_0 \in \mathrm{Do}x)$.

(10) $(1;y)^{\cup} \leq x ; 1$ $(\mathrm{Rg}y \subseteq \mathrm{Do}x)$.

(11) $[p_0 ; (x ; 1 - (1;x)^{\cup})] \cdot p_1 \cdot (x;1)^{\cup} \leq y$ ($0 + u = u$, see the formula preceding Definition 3.8).

(12) $[(p_0 ; x ; p_0^{\cup}) \cdot (p_1 ; p_1^{\cup})] ; y = y ; x$ $((v+1)+u = w \leftrightarrow w = (v+u)+1)$.

(12)′ $(p_0 ; p_0^{\cup}) \cdot (p_1 ; p_1^{\cup}) \leq 1'$ ('pairs' are unique).

(13) $z^{\cup} ; z \leq 1'$.

(14) $z ; 1 \leq p_i ; x ; 1$ for $i \in 2$.

(15) $(1;z)^{\cup} \leq x ; 1$.

(16) $[x ; 1 - (1;x)^{\cup}]^{\cup} \cdot p_0 \cdot [p_1 ; x ; 1] \leq z$ $(0 \cdot u = 0)$.

(17) $[(p_0 ; x ; p_0^{\cup}) \cdot (p_1 ; p_1^{\cup})] ; z = [(p_1 ; p_1^{\cup}) \cdot (z ; p_0^{\cup})] ; y$
$((v+1) \cdot u = w \leftrightarrow w = (v \cdot u) + u)$.

$$\left.\begin{array}{ll}(18) & q_w = q_w ; 1 \\ (19) & q_w \cdot q_w^{\cup} \leq 1' \\ (20) & q_w \neq 0\end{array}\right\} \quad \text{for every } w \in W \qquad (q_w \text{ is a constant}).$$

Let $u, v, w \in W$. Then

$$\xi(u+1=v) \triangleq (q_u \cdot q_v^{\cup} \leq x),$$
$$\xi(u+v=w) \triangleq [(p_0 ; q_u) \cdot (p_1 ; q_v) \cdot q_w^{\cup} \leq y],$$
$$\xi(u \cdot v=w) \triangleq [(p_0 ; q_u) \cdot (p_1 ; q_v) \cdot q_w^{\cup} \leq z],$$
$$\xi(u=0) \triangleq [q_u \leq (x ; 1-(1 ; x)^{\cup})].$$

Now to each statement (i) ($1 \leq i \leq 20$) we associate an RA-term τ_i such that for every simple $\mathfrak{R} \in \mathrm{RA}$ and evaluation k of the variables we have

(*) $\mathfrak{R} \models (i)[k]$ iff $\mathfrak{R} \models \tau_i \neq 0[k]$ iff $\mathfrak{R} \models \tau_i = 1[k]$.

E.g., for τ_1 we can take $\tau_1 \triangleq -(1 ; (x^{\cup} ; x-1') ; 1)$. Indeed, in a simple RA we have $\tau_1 \neq 0$ iff $\tau_1 = 1$ iff $1 ; (x^{\cup} ; x-1') ; 1 = 0$ iff $(x^{\cup} ; x-1') = 0$ iff $x^{\cup} ; x \leq 1'$. We can also associate such terms $\tau(u+1=v)$, etc. to $\xi(u+1=v)$, etc.

Now we define $\rho(e(\bar{x}))$ to be $\prod\{\tau_i : 1 \leq i \leq 20\} \cdot \prod\{\tau(b_i) : 0 \leq i \leq m\} = 0$. We will show

$$\boldsymbol{\omega} \models \exists \bar{x}\, e(\bar{x}) \quad \text{iff} \quad \mathrm{MRA} \not\models \rho(e(\bar{x})).$$

Assume $\boldsymbol{\omega} \models \exists \bar{x}\, e(\bar{x})$. Let $\boldsymbol{\omega} \models \bigwedge B[h]$ and let $h^*W \subseteq N$ for $N \in \omega$ as in the proof of Lemma 3.10. Let $U \triangleq (N+1) \cup {}^2N$. Let $\mathfrak{R}$ denote the full relation set algebra with base U (i.e., $R = \mathrm{Sb}(U \times U)$). Then $\mathfrak{R} \in \mathrm{MRA}$ and there are $X, Y, Z, P_0, P_1, Q_w : w \in W$ in R for which (1)–(20) together with $\bigwedge\{\xi(b_i) : 0 \leq i \leq m\}$ hold. Therefore $\mathfrak{R} \not\models \rho(e(\bar{x}))$ by (*). Assume $\mathfrak{R} \not\models \rho(e(\bar{x}))$ for some $\mathfrak{R} \in \mathrm{MRA}$. We may assume that $\mathfrak{R}$ is simple and representable. Then by (*) there are $X, Y, Z, P_0, P_1, Q_w : w \in W$ in $\mathfrak{R}$ for which (1)–(20) together with $\bigwedge\{\xi(b_i) : 0 \leq i \leq m\}$ hold. Now Lemma 3.3 and [11, 5.3.17] imply that X is finite. Therefore $X, Y, Z, Q_w : w \in W$ provide a solution for $e(\bar{x})$ in $\boldsymbol{\omega}$. □ (Theorem 4).

Remark 3.19. We note that there are deeper reasons why we could translate these sentences to RA-terms: (1) If projection functions are available, then every first-order formula with free variables v_0, v_1 can be translated to a formula with free variables v_0, v_1 but using only the (bound or free) variables v_0, v_1, v_2; and (2) every formula of the latter shape can be translated to an RA-term (with the same meaning of course). See Tarski–Givant [35, Theorem (ix) in Chapter 6], or stated and proved precisely in the above form in [27, Lemmas 1, 2].

Proof of Theorem 6. First we prove Theorem 6(iv). Proof of $\mathrm{BbLf}_\alpha \subseteq \mathbf{SMg}_\alpha$: Let $\mathfrak{C} \in \mathrm{BbLf}_\alpha$. This means $\mathfrak{C} \cong \mathfrak{B} \in \mathrm{Bb'Gs}_\alpha \cap \mathrm{Lf}_\alpha$ for some $\mathfrak{B}$. If $\alpha < \omega$, then $\mathfrak{B}$ is regular. If $\alpha \geq \omega$, then $\mathfrak{Gs}_\alpha \subseteq \mathbf{I}\mathrm{Gs}_\alpha^{\mathrm{reg}}$ and by $\mathfrak{B} \in \mathrm{BbGs}_\alpha$ we have $c_{(n)}\bar{d}(n \times n) = 0$

in $\mathfrak{B}$, hence $c_{(n)}\bar{d}(n \times n) = 0$ in every $\mathfrak{B}'$ isomorphic to $\mathfrak{B}$ (for some n). Thus we may assume $\mathfrak{B} \in \mathrm{Gs}_\alpha^{\mathrm{reg}} \cap \mathrm{Lf}_\alpha$. We will show $\mathfrak{B} \in \mathbf{S}\mathrm{Mg}_\alpha$. Let V be the unit of $\mathfrak{B}$ and $U \triangleq \mathrm{base}(\mathfrak{B})$. For every $S \subseteq U$ define $\bar{S} \triangleq \{s \in V : s_0 \in S\}$. Let $\mathfrak{A} \triangleq \mathfrak{S}\mathfrak{g}^{(\mathfrak{S}\mathfrak{b}V)}\{\bar{S} : S \subseteq U\}$. Then $\mathfrak{A} \in \mathrm{Mg}_\alpha$. Let $b \in B$ be arbitrary. Let $\Delta \triangleq \Delta b$ and $b_\Delta \triangleq \{\Delta \upharpoonright s : s \in b\}$. Then $|\Delta| < \omega$ and $b = \{s \in V : \Delta \upharpoonright s \in b_\Delta\}$ by $\mathfrak{B} \in \mathrm{Gs}_\alpha^{\mathrm{reg}} \cap \mathrm{Lf}_\alpha$. Let $N \in \omega$ be an upper bound of the sizes of $\mathfrak{B}$'s subbases. (Exists by $\mathfrak{B} \in \mathrm{Bb}'\mathrm{Gs}_\alpha$.) Let $n = |{}^\Delta N|$. Then $(\forall W \in \mathrm{Subb}(\mathfrak{B}))\, |b_\Delta \cap {}^\Delta W| \leq n$, hence there are $e_0, \ldots, e_n$ such that $b_\Delta = e_0 \cup \cdots \cup e_n$ and

$$(*) \quad (\forall i \leq n)(\forall W \in \mathrm{Subb}(\mathfrak{B}))\, |e_i \cap {}^\Delta W| \leq 1.$$

For every $i \leq n$ let $b_i \triangleq \{s \in V : \Delta \upharpoonright s \in e_i\}$. Then $b = b_0 \cup \cdots \cup b_n$. Let $i \leq n$ be fixed. We will show that $b_i \in A$. For every $j \in \Delta$ define $S_j \triangleq \{s_j : s \in b_i\}$. Then $b_i = \prod \{s_j^0 \bar{S}_j : j \in \Delta\}$ holds by $(*)$, showing $b_i \in A$. Hence $b \in A$, too, by $b = b_0 \cup \cdots \cup b_n$. Therefore $B \subseteq A$, hence $\mathfrak{B} \subseteq \mathfrak{A}$. □ (Theorem 6(iv))

For proving the rest of Theorem 6, we formulate some lemmas. Lemma 3.20 below is taken, with some reformulation, from Monk [22].

Lemma 3.20. *Let $\alpha < \omega$. Then* (i)–(ii) *below hold.*

(i) *Every finite* Mg_α *can be represented as a* Gs_α *with a finite base.*

(ii) *Every finitely generated subalgebra of a* Mg_α *is finite.*

Proof. (ii) is stated as Theorem 13 in [22] and it can easily be derived from [11, 2.2.24]. (i) is an easy consequence of Theorems 17, 20 of [22], but can also be proved by using [11] as follows. The proof of [11, 2.5.61] shows that a finitely generated free monadic-generated CA_α (i.e., $\mathfrak{Fr}_X^{(\Delta)}\mathrm{CA}_\alpha$ with $|X| < \omega$ and $\mathrm{Rg}\Delta = \{1\}$) with α finite is a subdirect product of finitely many Cs_α's with finite bases, hence is in FbGs_α. It is easy to see that $\mathbf{H}\mathrm{FbGs}_\alpha \subseteq \mathrm{FbGs}_\alpha$ if $\alpha < \omega$ (since if $\mathfrak{A} \in \mathrm{Gs}_\alpha$ and $\alpha < \omega$, $|\mathrm{base}(\mathfrak{A})| < \omega$, then every ideal of $\mathfrak{A}$ is generated by a single zero-dimensional element z in $\mathfrak{A}$, and this z is a union of some subunits of $\mathfrak{A}$). Hence every finite Mg_α is in FbGs_α, if $\alpha < \omega$. □

Remark 3.21. (i) Lemma 3.20(i) is not true for $\mathrm{Cs}_\alpha \cap \mathrm{Lf}_\alpha$ for $\alpha \geq 3$ in general, not even for a Cs_α generated by a single 2-dimensional element. For counterexample see [11, 3.1.38]. But one can show that every finite $\mathrm{Mg}_\alpha \cap \mathrm{Gs}_\alpha$ $(\alpha < \omega)$ is actually *ext*-isomorphic to one with finite base.

(ii) As a corollary of Lemmas 3.20 and Theorem 6(iv) we get that every finitely generated subalgebra of a BbLf_α is finite.

The next lemma is a corollary of results in [12]. If α is not an ordinal, but an arbitrary set, then by a CA_α, Mg_α etc. we understand the natural thing.

Lemma 3.22. *Let* $0 \in \Delta \subseteq \alpha$. *Then* (i)–(ii) *below hold.*

(i) $\mathrm{FbMg}_\Delta \subseteq \mathbf{SIRd}_\Delta\mathrm{FbMg}_\alpha$.

(ii) $\mathrm{Rd}_\Delta\mathrm{BbGs}_\alpha \subseteq \mathrm{BbGs}_\Delta$.

Proof. (i) Let $\mathfrak{M} \in \mathrm{FbMg}_\Delta$. Then $\mathfrak{M}$ is isomorphic to a $\mathfrak{B} \in \mathrm{Gs}_\Delta \cap \mathrm{Mg}_\Delta$ with a finite base U. Let $\{U_j : j \in J\}$ be the set of subbases of $\mathfrak{B}$. Define $V \triangleq \bigcup \{{}^\alpha U_j : j \in J\}$. Then V is a Gs_α-unit. For every $b \in B$ define $f(b) \triangleq \{s \in V : \Delta \upharpoonright s \in b\}$. Let $\mathfrak{C} \triangleq \mathfrak{S}\mathrm{g}^{(\mathfrak{S}\mathfrak{b}V)} f^*B$. Then it is not difficult to check that $f : \mathfrak{B} \rightarrowtail \mathfrak{R}\mathfrak{d}_\Delta\mathfrak{C}$. Clearly, $\mathfrak{C} \in \mathrm{FbGs}_\alpha$ since $\mathrm{base}(\mathfrak{C}) = \mathit{base}(\mathfrak{B})$ is finite. It remains to show that $\mathfrak{C} \in \mathrm{Mg}_\alpha$. Let $G \subseteq B$ be a set of monadic generators for $\mathfrak{B}$. Then f^*G generates f^*B in $\mathfrak{C}$ by $f : \mathfrak{B} \rightarrowtail \mathfrak{R}\mathfrak{d}_\Delta\mathfrak{C}$, hence f^*G generates $\mathfrak{C}$ by $C = \mathrm{Sg}f^*B$. Clearly, $\Delta^{\mathfrak{C}}(fg) = 1$ for every $g \in G$. (i) has been proved. (ii) follows from the proof of [11, 3.1.125] namely, the function rd^ρ defined in [11, 3.1.124] does not change the sizes of the subbases by [11, 3.1.125(iii)]. □

We are ready to prove the rest of Theorem 6. First we prove (iii). Proof of $\mathrm{Mg}_\alpha \subseteq \mathbf{SUp}\mathrm{FbMg}_\alpha$: Let $\mathfrak{M} \in \mathrm{Mg}_\alpha$. Let $G \subseteq M$ be a set of monadic generators for $\mathfrak{M}$. For every $0 \in \Delta \subseteq \alpha$, Δ finite and $G_0 \subseteq G$, G_0 finite define $\mathfrak{R} \triangleq \mathfrak{R}(\Delta, G_0) \triangleq \mathfrak{S}\mathrm{g}^{(\mathfrak{R}\mathfrak{d}_\Delta\mathfrak{M})} G_0$. Then $\mathfrak{R} \in \mathrm{FbMg}_\Delta$ by Lemma 3.20. Let $\mathfrak{G}(\Delta, G_0) \in \mathrm{FbMg}_\alpha$ be such that $\mathfrak{R} \subseteq \mathfrak{R}\mathfrak{d}_\Delta\mathfrak{G}(\Delta, G_0)$. Such an FbMg_α exists by Lemma 3.22(i). Since for every finite $X \subseteq M$ there are finite $\Delta \subseteq \alpha$ and $G_0 \subseteq G$ such that $X \subseteq R(\Delta, G_0)$, we have that $\mathfrak{M} \in \mathbf{SUp}\{\mathfrak{G}(\Delta, G_0) : 0 \in \Delta \subseteq_\omega \alpha, G_0 \subseteq_\omega G\} \subseteq \mathbf{SUp}\mathrm{FbMg}_\alpha$.

Proof of $\mathrm{BbGs}_\alpha \subseteq \mathbf{SUp}\mathrm{Mg}_\alpha$: $\mathrm{BbGs}_\alpha \subseteq \mathbf{SUp}\mathrm{BbLf}_\alpha$, because if $\alpha < \omega$ then $\mathrm{Gs}_\alpha \subseteq \mathrm{Lf}_\alpha$ and if $\alpha \geqslant \omega$ then $\mathrm{Gs}_\alpha \subseteq \mathbf{SUp}\mathrm{Lf}_\alpha$ by [11, 2.6.52, 3.2.10]; and by using $\mathrm{Gs}_\alpha \subseteq \mathbf{SUp}\mathrm{Lf}_\alpha$ it is easy to prove that if $\mathfrak{B} \in \mathrm{Gs}_\alpha$, $\mathfrak{B} \models c_{(n)}\bar{d}(n \times n) = 0$ then $\mathfrak{B} \in \mathbf{SUp}\{\mathfrak{A} \in \mathrm{Lf}_\alpha : \mathfrak{A} \models c_{(n)}\bar{d}(n \times n) = 0\} \subseteq \mathbf{SUp}\mathrm{BbLf}_\alpha$. By Theorem 6(iv) $\mathrm{BbLf}_\alpha \subseteq \mathbf{S}\mathrm{Mg}_\alpha$, hence $\mathrm{BbGs}_\alpha \subseteq \mathbf{SUp}\mathrm{BbLf}_\alpha \subseteq \mathbf{SUp}\mathrm{Mg}_\alpha$. Clearly, $\mathrm{FbMg}_\alpha \subseteq \mathrm{BbGs}_\alpha$, hence (iii) of Theorem 6 has been proved.

Proof of Theorem 6(i): Let $\alpha \geqslant \omega$. We want to prove $\mathrm{Mg}_\alpha \subseteq \mathrm{EqMn}_\alpha$. Let e be an equation and assume $\mathrm{Mg}_\alpha \not\models e$. Then $\mathrm{BbGs}_\alpha \not\models e$ by Theorem 6(iii). Then there is a finite $\Delta \subseteq \alpha$ such that $\mathrm{BbGs}_\Delta \not\models e$, by Lemma 3.22(ii). Then $\mathrm{FbCs}_\Delta \not\models e$. Let $\mathfrak{C} \in \mathrm{Cs}_\Delta$ with a finite base U such that $\mathfrak{C} \not\models e$. Let $\mathfrak{M}$ be the minimal Cs_α with base U. Let $w : U \rightarrowtail \alpha \sim \Delta$. For every $s \in {}^\Delta U$ define

$$m(s) \triangleq \prod \{d_{i, w(si)} : i \in \Delta\}.$$

For every $a \in C$ define $f(a) \triangleq \sum \{m(s) : s \in a\}$. Then $f : C \to M$ since ${}^\Delta U$ is finite. The next argument is extracted from [12, II.4.7.1.2] or [11, 3.1.124, 3.1.125]. Let $k \in {}^{(\alpha \sim \Delta)}U$ be such that $(\forall u \in U)\, k(wu) = u$. For any $X \in M$ let $gX = \{t \in {}^\Delta U : t \cup k \in X\}$. Then it is easily verified that $g : \mathfrak{R}\mathfrak{d}_\Delta\mathfrak{M} \to \mathfrak{S}\mathfrak{b}({}^\Delta U)$. Moreover, $gfa = a$ for all $a \in C$, so $\mathfrak{C} \subseteq g^*\mathfrak{R}\mathfrak{d}_\Delta\mathfrak{M}$. Since $\mathfrak{C} \not\models e$, it follows that $\mathfrak{R}\mathfrak{d}_\Delta\mathfrak{M} \not\models e$, hence $\mathfrak{M} \not\models e$. We have seen $\mathrm{Mn}_\alpha \not\models e$. We have seen $\mathrm{Mn}_\alpha \models e \Rightarrow \mathrm{Mg}_\alpha \models e$. By $\mathrm{Mn}_\alpha \subseteq \mathrm{Mg}_\alpha$ this implies $\mathrm{EqMn}_\alpha = \mathrm{EqMg}_\alpha$. $\mathrm{EqMg} = \mathrm{EqFbCs}_\alpha$ follows from Theorem 6(iii). Theorem 6(i) has been proved.

Proof of Theorem 6(ii): Let δ denote the formula $\forall x\,(x \cdot c_0^{\partial} d_{01} = 0 \bigvee x \geqslant c_0^{\partial} d_{01})$. Then clearly $\mathrm{Mn}_\alpha \models \delta$ (cf. [11, 2.1.20(ii)]), but $\mathrm{Mg}_\alpha \not\models \delta$ for $\alpha \geqslant 2$. This proves Theorem 6(ii).

Proof of Theorem 6(v): Let φ denote the following Π_2-formula

$$\forall x\, \exists y\, ([\sigma(y) - c_0(c_1 y \oplus c_1 x)] + (\sigma(y) - c_0(c_1 y \oplus -c_1 x)] \\ + \beta(c_1 x) + \beta(-c_1 x) \neq 0),$$

where

$$\sigma(y) \triangleq -c_{(3)}(y \cdot s_2^1 y - d_{12}) - c_{(3)}(y \cdot s_2^0 y - d_{02}) - c_{(2)}(y \cdot d_{01}) \cdot -c_{(7)}(c_{(7\sim 2)} y - y),$$

and $\beta(z) \triangleq -c_{(2)}(z \cdot s_1^0 z - d_{01})$. Roughly speaking, φ expresses that either $\mathrm{Do}(c_1 x)$ or the complement of $\mathrm{Do}(c_1 x)$ is finite. We will show that $\mathrm{Mn}_\alpha \models \varphi$ while $\mathfrak{M} \not\models \varphi$ for some hereditarily nondiscrete Mg_α $\mathfrak{M}$. Let $\mathfrak{M} \in \mathrm{Mn}_\alpha \cap \mathrm{Gs}_\alpha$ and $X \in M$, $s \in 1^{\mathfrak{M}}$ be arbitrary. Let $x \triangleq c_1 X$. Define $D_0 \triangleq \{u : s(0/u) \in x\}$ and $D_1 \triangleq \{u : s(0/u) \in -x\}$. We will show that

(*) either $|D_0| < \omega$ or $|D_1| < \omega$.

Assume both D_0 and D_1 are infinite. Let $\Delta \triangleq \Delta x$ and $S \triangleq \{s_i : i \in \Delta\}$. Then $|S| < \omega$. Let $u \in D_0 \sim S$, $v \in D_1 \sim S$ and let $f : \mathrm{base}(\mathfrak{M}) \twoheadrightarrow \mathrm{base}(\mathfrak{M})$ be the function interchanging u and v and leaving all the other elements fixed. Then $\tilde{f}x = x$ for the induced base-isomorphism $\tilde{f}$ since $\mathfrak{M} \in \mathrm{Mn}_\alpha$. Let $s' \triangleq f \circ s$. Then $\Delta \upharpoonright s' = \Delta \upharpoonright s$ since f is identity on S, hence $s(0/w) \in x$ iff $s'(0/w) \in x$ for every w by the regularity of x (every $\mathrm{Mn}_\alpha \cap \mathrm{Gs}_\alpha$ is regular (see [11, 3.1.63]). Now $s(0/u) \in x$, hence $s'(0/u) \in x$, therefore $f \circ (s'(0/u)) \in \tilde{f}(x) = x$, but $f \circ (s'(0/u)) = s(0/v)$, contradicting $s(0/v) \in -x$. (*) has been proved.

Assume $|D_0| \leqslant 1$. Then $s \in \beta(x)$ and we are done. Assume now $1 < |D_0| < \omega$. Let $w : D_0 \rightarrowtail \alpha \sim \Delta$. Let $k \in 1^{\mathfrak{M}}$ be such that $\Delta \upharpoonright s \subseteq k$ and $(\forall u \in D_0)\, k(wu) = u$. Let g be any one–one function without fixpoints and with domain and range D_0. Define $y \triangleq \sum \{d_{0,wu} \cdot d_{1,w(gu)} : u \in D_0\}$. Then $y[\![k, 2]\!] = g$, hence $k \in \sigma(y)$ by Lemma 3.4. By $\Delta \upharpoonright s \subseteq k$ we have $D_0 = \{u : s(0/u) \in x\} = \{u : k(0/u) \in x\}$, hence $k \notin c_0(c_1 y - c_1 x)$. The other case, $|D_1| < \omega$, is completely analogous. We have seen $\mathrm{Mn}_\alpha \models \varphi$. Let V, W be disjoint infinite sets and $U \triangleq V \cup W$, $X \triangleq \{s \in {}^\alpha U : s_0 \in V\}$. Let $\mathfrak{M} \triangleq \mathfrak{S}\mathrm{g}^{(\mathfrak{S}\mathrm{b}^{\alpha} U)}\{X\}$. Then $\mathfrak{M} \in \mathrm{Mg}_\alpha$. Assume $Y \in M$ and $k \in {}^\alpha U$ is such that

$$k \in [\sigma(Y) - c_0(c_1 Y \oplus c_1 X)] + [\sigma(Y) - c_0(c_1 Y \oplus -c_1 X)],$$

say $k \in \sigma(Y) - c_0(c_1 Y \oplus c_1 X)$. (Note that $\beta(c_1 X) + \beta(-c_1 X) = 0$.) Let $R \triangleq Y[\![k, 2]\!]$. Then R is finite by $k \in \sigma(Y)$, see Lemma 3.4. By $k \notin c_0(c_1 Y - c_1 X)$ we have $\mathrm{Do} R = V$, hence R is infinite since V is infinite. Contradiction. □ (Theorem 6)

Proof of Theorem 1. Proof of Theorem 1(iii): $\mathrm{EqMn}_0 = \mathrm{EqMg}_0$ since Mn_0 consists of the one- and two-element BA's and $\mathrm{Mg}_0 = \mathrm{BA}$. $\mathrm{EqMn}_1 \neq \mathrm{EqMg}_1$ since $\mathrm{Mn}_1 \models c_0 x = x$ while $\mathrm{Mg}_1 \not\models c_0 x = x$. For $2 \leqslant \alpha < \omega$, $\mathrm{EqMn}_\alpha \neq \mathrm{EqMg}_\alpha$ since EqMn_α

is r.e. while $\overline{\mathrm{Eq}}\mathrm{Mg}_\alpha$ is not. A concrete equation distinguishing them is, e.g., $c_{(\alpha-1)}x = c_{(\alpha)}x$. For $\alpha \geq \omega$, $\mathrm{EqMn}_\alpha = \mathrm{EqMg}_\alpha$ is proved in Theorem 6(i).

Proof of Theorem 1(iv): $\mathrm{Mg}_1 = \mathrm{CA}_1$ by definition and $\mathrm{Rp}_1 = \mathrm{CA}_1$ by [11, 3.2.55].

Proof of $\mathrm{Rp}_2 \subseteq \mathbf{SUp}\mathrm{Mg}_2$: $\mathrm{Rp}_2 \subseteq \mathbf{SUp}\mathrm{FRp}_2$ by[8] [11, 4.2.8], $\mathrm{FRp}_2 \subseteq \mathrm{FbRp}_2$ by Henkin's result [11, 3.2.66], and $\mathrm{FbRp}_2 \subseteq \mathbf{S}\mathrm{Mg}_2$ by Theorem 6(iv). $\mathrm{Mg}_2 \subseteq \mathbf{SUp}\mathrm{Rp}_2$ by Monk [22, Theorem 21] or by [11, 3.2.12], and $\mathbf{SUp}\mathrm{Rp}_2 = \mathrm{Rp}_2$ by [11, 3.1.97]. $\mathrm{UnMg}_2 = \mathrm{Rp}_2$ has been proved. To see $\mathrm{ElMg}_2 \subset \mathrm{Rp}_2$, let

$$\varphi \triangleq \exists x\, (c_0 x = 1 \wedge c_1 x = 1 \wedge x < -d_{01}) \rightarrow \exists y\, (c_0 y > y = c_1 y).$$

Now $\mathrm{Mg}_2 \models \varphi$ but ${}_\kappa\mathrm{Cs}_2 \not\models \varphi$ if $\kappa > 1$. Let $\alpha > 2$. Then $\overline{\mathrm{Eq}}\mathrm{Mg}_\alpha$ is not r.e. by Theorem 2(ii) while $\overline{\mathrm{Eq}}\mathrm{Rp}_\alpha$ is r.e. (by, e.g., [11, 4.1.15–16]), hence $\mathrm{EqMg}_\alpha \neq \mathrm{EqRp}_\alpha = \mathrm{Rp}_\alpha$. A concrete equation showing $\mathrm{EqMn}_\alpha \subset \mathrm{Rp}_\alpha$ for $\alpha > 2$ is given in [11, 4.1.32]; that equation works for showing $\mathrm{EqMg}_\alpha \subset \mathrm{Rp}_\alpha$, too. An alternative equation, using the techniques of the present paper (see the proof of Theorem 2), is the CA-equational formulation of "x is a one–one function without fix-points and $\mathrm{Do}x \sim \mathrm{Rg}x \neq 0$ implies that $\mathrm{Rg}x \sim \mathrm{Do}x \neq 0$". Theorem 1(iv) has been proved.

Proof of Theorem 1(i)–(ii): For $\alpha > 2$, Theorem 1(i)–(ii) follow from Theorem 2 since Mn_α, Mg_α $\alpha \geq \omega$ are not bounded and for $\alpha < \omega$, Mn_α is boundedly generated while Mg_α is not. Let $\alpha \leq 2$. Then $\overline{\mathrm{Eq}}\mathrm{Mn}_\alpha$ is decidable by [11, 4.2.1], $\overline{\mathrm{Eq}}\mathrm{Mg}_2 = \overline{\mathrm{Eq}}\mathrm{Rp}_2$ is decidable by[9] [11, 4.2.9]. Let $\alpha = 1$. Then $\mathrm{Mg}_1 = \mathrm{CA}_1$. By Comer [6, p. 176], the elementary theory of FCA_1 is decidable (see also [11, 4.2.24]). Since $\mathrm{CA}_1 = \mathrm{EqFCA}_1$ by [11, 2.5.6], the equational theory of CA_1, hence $\overline{\mathrm{Eq}}\mathrm{Mg}_1$ also, is decidable. The equational theory of $\mathrm{Mg}_0 = \mathrm{BA}$ is obviously decidable. □ (Theorem 1)

Proof of Theorem 3. Proof of Theorem 3(ii): Let $\alpha \geq \omega$. We want to show $\mathrm{EqBg}^1_\alpha = \mathrm{Rp}_\alpha$. Now $\mathrm{Bg}^1_\alpha \subseteq \mathrm{Bg}_\alpha \subseteq \mathrm{Lf}_\alpha \subseteq \mathrm{Rp}_\alpha$ by [11, 3.2.8], thus $\mathrm{EqBg}^1_\alpha \subseteq \mathrm{EqBg}_\alpha \subseteq \mathrm{Rp}_\alpha$. By [11, 3.1.123] we have $\mathrm{Rp}_\alpha = \mathrm{Eq}(\mathrm{Cs}^{\mathrm{reg}}_\alpha \cap \mathrm{Lf}_\alpha)$. Thus it is enough to show $\mathrm{Cs}^{\mathrm{reg}}_\alpha \cap \mathrm{Lf}_\alpha \subseteq \mathbf{HSP}\mathrm{Bg}^1_\alpha$. Let $\mathfrak{A} \in \mathrm{Cs}^{\mathrm{reg}}_\alpha \cap \mathrm{Lf}_\alpha$. Let $U \triangleq \mathrm{base}(\mathfrak{A})$. Let $\mathfrak{R}$ denote the greatest regular Lf-subalgebra of $\mathfrak{Sb}\,{}^\alpha U$. Then $\mathfrak{A} \subseteq \mathfrak{R} \in \mathrm{Cs}^{\mathrm{reg}}_\alpha \cap \mathrm{Lf}_\alpha$. Assume $|U| < \omega$. We will show that $\mathfrak{R} \in \mathrm{Bg}^1_\alpha$. We may assume $U \in \omega$. Define $X \triangleq \{s \in {}^\alpha U : s_1 = s_0 + 1\}$. For $i \in U$ define $Y_i \triangleq \{s \in {}^\alpha U : s_0 = i\}$. Now $Y_0 = c_1(d_{01} - c_0 X)$ and $Y_{i+1} = c_1(d_{01} \cdot c_0(Y_i \cdot X))$ if i, $i+1 \in U$. Using the Y_i's, it is not difficult to see that $\mathfrak{R} = \mathfrak{Sg}\{X\}$, hence $\mathfrak{R} \in \mathrm{Bg}^1_\alpha$. Assume now $|U| \geq \omega$. We will show that $\mathfrak{R} \in \mathrm{EqBg}^1_\alpha$. To this end it is enough to show that $\mathfrak{Sg}X \in \mathbf{S}\mathrm{Bg}^1_\alpha$ for every finite $X \subseteq R$. Let $X \subseteq_\omega R$. We may assume that the elements of X are disjoint. First we note that we may assume that $|X| = 1$. For we may assume that $\Delta x \cap \Delta y = 0$ for

[8] We note that a simple short proof, analogous to [11, 2.5.4] and not using it, can be found in [1].

[9] We note that the proof of decidability of $\overline{\mathrm{Eq}}\mathrm{Gs}_2$ in Scott [34] is based on a claim of Gödel which has been disproved in the meantime (see [7]), hence the proof in [34] does not work.

distinct $x, y \in X$ (hint: use the substitution functions s^i_j). Then if we set $Z \triangleq \bigcup X$ and $\theta \triangleq \bigcup\{\Delta x : x \in X\}$, we have $x = c_{(\theta\sim\Delta x)}Z$ for all $x \in X$, as desired. So, suppose that $X = \{Z\}$. We may assume without loss of generality that $\Delta Z = m+1 \in \omega$ and $m \geqslant 2$. Now we use the well-known method of interpreting an m-ary relation in a binary one; see, e.g. [26, proof of 16.51]. Let $S = \{a \in {}^mU : a \subseteq x \in Z \text{ for some } x\}$. By [11, 3.1.112], $\mathfrak{N}$ is subisomorphic to some $\mathfrak{C} \in \mathrm{Cs}^{\mathrm{reg}}_\alpha \cap \mathrm{Lf}_\alpha$ with base W such that $|W| = |U|^+$. Let $\mathfrak{C}'$ be the greatest regular locally finite subalgebra of $\mathfrak{Sb}({}^\alpha W)$. Note that the isomorphism f of $\mathfrak{N}$ into $\mathfrak{C}'$ is given by $ft = \{a \in {}^\alpha W : \Delta t \upharpoonright a \subseteq x \in t \text{ for some } x\}$. We now define a symmetric binary relation (graph) T on W. For each $a \in S$, choose distinct elements $e^a_0, e^a_1, \ldots e^a_m$ in $W \sim U$ and put the following diagram in T:

$$\begin{array}{ccccc} e^a_0 & \text{---} & e^a_1 & \text{---} \cdots \text{---} & e^a_m \\ | & & | & & | \\ a_0 & & a_1 & \cdots & a_m \end{array}$$

(Distinct a's get distinct e^a_i's.) Then there is a formula $\varphi(v_0, \ldots, v_m)$ in the language of $\langle W, T\rangle$ which defines S, that is, such that $S = \{s \in {}^mW : \langle W, T\rangle \models \varphi[a]\}$. E.g.,

$$\varphi(v_0, \ldots, v_m) \triangleq \exists v_{m+1} \cdots v_{2m+2}\Big(\bigwedge_{i\leqslant m} Tv_iv_{i+m+1} \wedge \bigwedge_{i<m} Tv_{i+m+1}v_{i+m+2}\Big)$$

will do. We may assume that φ is restricted. With each formula ψ in the language of $\langle W, T\rangle$ we associate a cylindric term ψ' as follows:

$$(Rv_0v_1)' \triangleq v_0, \qquad (v_i = v_j)' \triangleq d_{ij},$$
$$(\neg\psi)' \triangleq -\psi', \qquad (\psi \vee \chi)' \triangleq \psi' + \chi', \qquad (\exists v_i\psi)' \triangleq c_i\psi'.$$

Let $T' \triangleq \{a \in {}^\alpha W : 2 \upharpoonright a \in T\}$. Then $\varphi'^{[\mathfrak{C}]}T' = fZ$. Thus $f^*\mathfrak{Sg}\{Z\} \subseteq \mathfrak{Sg}\{T'\}$, as desired. $\mathrm{EqBg}^1_\alpha = \mathrm{Rp}_\alpha$ has been proved.

Proof of Theorem 3(i): That $\overline{\mathrm{Eq}}\mathrm{Bg}_\alpha$ is r.e. for $\alpha \geqslant \omega$ follows from Theorem 3(ii). Next we prove that $\overline{\mathrm{Eq}}\mathrm{Bg}_\alpha$ is not decidable for $\alpha > 3$. Let $\alpha > 3$. Recall from [11, §5.3] that $\mathrm{Ra}^*\mathrm{CA}_\alpha = \mathrm{Ra}^*\mathrm{Nr}_3\mathrm{CA}_\alpha \subseteq \mathrm{RA}$. By Theorem 5.3.16 of [11, p. 220], we have $\mathrm{RRA} \subseteq \mathrm{RA}^*\mathrm{SNr}_3\mathrm{CA}_\alpha = \mathrm{SRa}^*\mathrm{CA}_\alpha$. Since $\mathrm{Ra}^*\mathrm{CA}_\alpha = \mathrm{Ra}^*\mathrm{Bg}_\alpha$, we have $\mathrm{RRA} \subseteq \mathrm{SRa}^*\mathrm{Bg}_\alpha \subseteq \mathrm{RA}$. Theorem 1 of Chapter 12 of Maddux [16, p. 220] says $\forall K\,(\mathrm{RRA} \subseteq K \subseteq \mathrm{RA} \Rightarrow \overline{\mathrm{Eq}}\mathrm{K}$ is undecidable). Therefore $\overline{\mathrm{Eq}}\mathrm{SRa}^*\mathrm{Bg}_\alpha = \overline{\mathrm{Eq}}\mathrm{Ra}^*\mathrm{Bg}_\alpha$ is undecidable. Since $\mathfrak{Ra}\mathfrak{A}$ is a (generalized) reduct of $\mathfrak{A}$, this means that $\overline{\mathrm{Eq}}\mathrm{Bg}_\alpha$ is undecidable, too. □ (Theorem 3)

Proof of Theorem 5. Let $\tau(x)$ denote the following cylindric term

$$-c_{(3)}(x \cdot s^1_2x - d_{12}) - c_{(3)}(x \cdot s^0_2x - d_{02}) - c_{(2)}(x \cdot d_{01}) - c_{(3)}(c_2x - x) \cdot c^\partial_0c_1x.$$

Let e be the equation $c^\partial_0d_{01} + \tau(x) = 0$. Let $\alpha \geqslant \omega$ and $\mathfrak{A} \in \mathrm{Mg}_\alpha$. We may assume $\mathfrak{A} \in \mathrm{Gs}^{\mathrm{reg}}_\alpha$. Assume $\mathfrak{A} \not\models e$. If $\mathfrak{A} \not\models c^\partial_0d_{01} = 0$, then clearly $\mathfrak{A} \notin {}_\infty\mathrm{Mg}_\alpha$. Assume $\mathfrak{A} \models c^\partial_0d_{01} = 0$. By $\mathfrak{A} \not\models e$ then there are $X \in A$ and $k \in 1^{\mathfrak{A}}$ such that $k \in \tau(X)$. Let

$R \triangleq X[\![k, 2]\!]$. Then by Lemma 3.4, R is a one–one function with no fix-point, hence DoR is finite by Lemma 3.3. Let U be the subbase of $\mathfrak{A}$ for which $k \in {}^\alpha U$. Assume $u \in U$. Then $k^0_u \in c_1 X$ by $k \in c^\partial_0 c_1 X$. Thus $u \in$ DoR, showing $U =$ DoR. Since DoR is finite, this shows $\mathfrak{A} \notin {}_\infty\mathrm{Mg}_\alpha$. Assume now $\mathfrak{A} \notin {}_\infty\mathrm{Mg}_\alpha$. Then there is a homomorphic image $\mathfrak{B}$ of $\mathfrak{A}$ such that $\mathfrak{B} \in \mathrm{Cs}_\alpha$ with a finite base U. It is enough to show $\mathfrak{B} \not\models e$, since then $\mathfrak{A} \not\models e$, too. We may assume $U \in \omega$. If $U = 1$, then $\mathfrak{A} \not\models c^\partial_0 d_{01} = 0$ and we are done. Assume $U \geqslant 2$. Let $X \triangleq \sum \{d_{0,3+i} \cdot d_{1,3+(i\oplus 1)} : i \in U\}$ where $\oplus$ means addition modulo U, and let $k \in {}^\alpha U$ be such that $(\forall i \in U)\, k(3+i) = i$. Let $R \triangleq X[\![k, 2]\!]$. Then $R = \{(i, i \oplus 1) : i \in U\}$, hence R is a one–one function with no fix-point and Do$R = U$, showing $k \in \tau(X)$. □ (Theorem 5)

Proof of Theorem 7. Proof of Theorem 7(i): It is enough to show ${}_\infty Mg_\alpha \cap \mathrm{Cs}_\alpha \subseteq \mathrm{Eq}\,{}_\infty\mathrm{Mn}_\alpha$. Let $\mathfrak{A} \in {}_\infty\mathrm{Mg}_\alpha \cap \mathrm{Cs}_\alpha$ and let e be a CA_α-equation such that $\mathfrak{A} \not\models e(a_0, \ldots, a_n)$ for some $a_0, \ldots, a_n \in A$. We will show that ${}_\infty\mathrm{Mn}_\alpha \not\models e$. Let $\Gamma \subseteq_\omega \alpha$ and $G \subseteq {}_\omega\mathrm{Nr}_1\mathfrak{A}$ be such that all the indices occurring in e are contained in Γ and $\{a_0, \ldots, a_n\} \subseteq R$ where $\mathfrak{R} \triangleq \mathfrak{Sg}^{(\mathfrak{Rd}_\Gamma \mathfrak{A})} G$. Then $\mathfrak{R} \not\models e(a_0, \ldots, a_n)$ and $\mathfrak{R} \in {}_\infty\mathrm{Mg}_\Gamma$. If $G = 0$, then we are done. Assume $G \neq 0$. For every $g \in G$ let $\bar{g} \triangleq \{s_0 : s \in g\}$. We may assume that $\{\bar{g} : g \in G\}$ is a partition of base($\mathfrak{R}$). Fix an element $\gamma \in G$ with $|\bar{\gamma}| \geqslant \omega$. For every $g \in G$ define $\bar{g}'$ as $\bar{g}$ if $|\bar{g}| < |\Gamma|$ or if g is γ, otherwise let $\bar{g}' \subseteq \bar{g}$ be such that $|\bar{g}| = |\Gamma|$. Let $U \triangleq \bigcup \{\bar{g}' : g \in G\}$. Define $\mathfrak{R}' \triangleq \mathfrak{Rl}({}^\alpha U)\mathfrak{R}$. Exactly as in the proof of Lemma 3.20, one can show that $\mathfrak{R} \cong \mathfrak{R}'$. Let $\bar{G} \triangleq \{\bar{g}' : g \in G\}$ and $W \triangleq \bigcup \{\bar{g}' : g \in G, g \neq \gamma\}$. Then $W \subseteq_\omega U$. For every $z \in {}^\Gamma\bar{G}$ define $\hat{z} \triangleq \{s \in {}^\Gamma U : (\forall i \in \Gamma)\, s_i \in z_i\}$. Then it is not difficult to show that $(\forall a \in R)\,(\exists Z \subseteq {}^\Gamma\bar{G})\, a = \bigcup \{\hat{z} : z \in Z\}$. Let $w : W \rightarrowtail \alpha \sim \Gamma$ be arbitrary. For every $z \in {}^\Gamma\bar{G}$ define

$$m(z) \triangleq \prod \left\{ \sum \{d_{i,wu} : u \in z_i\} : i \in \Gamma,\, z_i \neq \gamma \right\} \cdot \prod \{-d_{i,wu} : i \in \Gamma,\, z_i = \gamma,\, u \in W\}.$$

For every $a \in R$ define $f(a) \triangleq \sum \{m(z) : z \in Z\}$, where $a = \bigcup \{\hat{z} : z \in Z\}$. From now on the proof is basically the same as that of ($\mathrm{FbCs}_\Delta \not\models e \Rightarrow \mathrm{Mn}_\alpha \not\models e$) in the proof of Theorem 6(i). Therefore we omit it.

Proof of Theorem 7(ii): Let e be the equation we defined in the proof of Theorem 5. Then $\mathrm{Eq}({}_\infty\mathrm{Mn}_\alpha) \models e$ by Theorem 5. We will show that $\mathrm{EqMn}_\alpha \cap \mathbf{I}\,{}_\infty\mathrm{Cs}_\alpha \not\models e$. Let $I \triangleq \omega \sim 2$ and let U be any non-principal ultrafilter on I. For every $n \in I$ let $\mathfrak{C}_n \triangleq \mathfrak{Sb}\,{}^\alpha n$ and define $\mathfrak{C} \triangleq P\langle \mathfrak{C}_n : n \in I \rangle / U$. Then $\mathfrak{C} \in \mathbf{Up}\mathrm{FbCs}_\alpha \subseteq \mathrm{EqMn}_\alpha$ by Theorem 6(i). For every $n \in \omega$, $\mathfrak{C} \models c_{(n)}\bar{d}(n \times n) = 1$ since $(\forall m \geqslant n)\, \mathfrak{C}_m \models c_{(n)}\bar{d}(n \times n) = 1$. Thus $\mathfrak{C}$ is of characteristic 0 by Theorem 2.4.63(i) of [11]. Hence $\mathfrak{C} \in \mathbf{I}\,{}_\infty\mathrm{Cs}_\alpha$ by [11, 3.1.108–109]. For every $n \in I$ let $f_n : n \rightarrowtail\!\!\!\!\rightarrow n$ be a permutation of n with no fix-point and define $b_n \triangleq \{s \in {}^\alpha n : s_1 = f_n(s_0)\}$, $b \triangleq \langle b_n : n \in I \rangle / U$. Then $b_n \in C_n$ and $\tau(b_n) = 1$ for every $n \in I$, hence $\tau(b) = 1$ in

$\mathfrak{C}$, too, where τ is the term in the definition of e. This shows $\mathfrak{C} \not\models e$. □ (Theorem 7)

List of notation

FmV set of formulavariables
$S\theta\rho K$ set of formula-schemes valid in K
Equmd class of models with equality only
Monmd class of models with only unary relations
1-Binmd class of models with one binary relation
Mod class of all models
FMod class of all finite models
$\mathrm{tr}(\sigma)$ translation of the formula-scheme σ to a CA-term
$\mathrm{eq}(\sigma)$ CA-equation corresponding to the scheme σ
$\mathrm{Mod}\,\Sigma$ class of models of Σ
$\boldsymbol{\omega} = \langle \omega, +, \cdot, 0, 1 \rangle$ the standard model of arithmetic
EqK, UnK, ElK least class containing K and axiomatizable by equations, universal formulas, first-order formulas resp.
$\overline{\mathrm{Eq}K}$, $\theta\rho K$ set of all equations, first-order formulas resp. valid in K
IK, **H**K, **S**K, **P**K, **Up**K, **Uf**K class of all isomorphic copies, homomorphic images, subalgebras, direct products, ultraproducts, ultrafactors resp. of members of K
ω least infinite ordinal
$|X|$ cardinality of X
$X \sim Y = \{a \in X : a \notin Y\}$
$X \subseteq_\omega Y$ X is a finite subset of Y
SbU set of all subsets of U
Dof, Rgf domain and range resp. of f
f^*X f-image of X
f_i, $f(i)$ the value of f at place i
$f(i/u)$ function f changed at place i to u
$f : A \rightarrowtail B$, $f : A \twoheadrightarrow B$ f is one–one, bijection resp.
${}^AB$ set of all functions mapping A into B
$A \upharpoonright f$ f domain-restricted to A
$R/\equiv = \{(u_1/\equiv, \ldots, u_n/\equiv) : (u_1, \ldots, u_n) \in R\}$
$R[\![k, n]\!]$ n-ary relation defined by $R \in \mathfrak{A} \in \mathrm{Gs}_\alpha$, $k \in 1^{\mathfrak{A}}$ (see above Lemma 3.3)
$\Delta^{\mathfrak{A}}x = \{i : c_i^{\mathfrak{A}}x \neq x\}$, dimension set of x
$\mathrm{Nr}_\beta\mathfrak{A} = \{x \in A : \Delta^{\mathfrak{A}}x \subseteq \beta\}$
$C_i^{[V]}X = \{s \in V : (\exists u)s(i/u) \in X\}$
$D_{ij}^{[V]} = \{s \in V : s_i = s_j\}$
$\mathfrak{Sb}V = \langle \mathrm{Sb}V, \cup, \cap, \sim, 0, V, C_i^{[V]}, D_{ij}^{[V]} \rangle_{i,j<\alpha}$, full cylindric set algebra with unit V

$1^{\mathfrak{A}}$ unit of $\mathfrak{A}$
$s^i_j x = c_i(d_{ij} \cdot x)$
$\mathrm{Subb}(\mathfrak{A})$ set of all **subb**ases of $\mathfrak{A}$
$\mathrm{base}(\mathfrak{A}) = \bigcup \mathrm{Subb}(\mathfrak{A})$, base of $\mathfrak{A}$
$\bar{d}(n \times n) = \prod \{-d_{ij} : i < j < n\}$
$c_{(\{i_1,\ldots,i_n\})}x = c_{i_1} \cdots c_{i_n} x$
$\mathfrak{Rd}_\alpha \mathfrak{A}$ α-dimensional reduct of $\mathfrak{A} \in \mathrm{CA}_\beta$, $\beta \geqslant \alpha$
$\mathrm{Rd}_\alpha K = \{\mathfrak{Rd}_\alpha \mathfrak{A} : \mathfrak{A} \in K\}$
$\mathrm{Sg}^{\mathfrak{A}} X$ subset of $\mathfrak{A}$ generated by X
$\mathfrak{Sg}^{\mathfrak{A}} X$ subalgebra of $\mathfrak{A}$ generated by X
CA_α class of α-dimensional cylindric algebras
$\mathrm{Mn}_\alpha = \{\mathfrak{Sg}^{\mathfrak{A}} 0 : \mathfrak{A} \in \mathrm{CA}_\alpha\}$, class of **min**imal CA's
$\mathrm{Mg}_\alpha = \{\mathfrak{Sg}^{\mathfrak{A}} X : X \subseteq \mathrm{Nr}_1 \mathfrak{A},\ \mathfrak{A} \in \mathrm{CA}_\alpha\}$, class of **m**onadic-**g**enerated CA_α's
$\mathrm{Mg}^n_\alpha = \{\mathfrak{Sg}^{\mathfrak{A}} X : \mathfrak{A} \in \mathrm{CA}_\alpha,\ X \subseteq \mathrm{Nr}_1 \mathfrak{A},\ |X| \leqslant n\}$
$\mathrm{Bg}_\alpha = \{\mathfrak{Sg}^{\mathfrak{A}} X : \mathfrak{A} \in \mathrm{CA}_\alpha,\ X \subseteq \mathrm{Nr}_2 \mathfrak{A}\}$, class of **b**inary-**g**enerated CA_α's
$\mathrm{Bg}^1_\alpha = \{\mathfrak{Sg}^{\mathfrak{A}} \{x\} : \mathfrak{A} \in \mathrm{CA}_\alpha,\ x \in \mathrm{Nr}_2 \mathfrak{A}\}$
Rp_α class of **rep**resentable CA_α's
Gs_α class of **g**eneralized cylindric **s**et algebras
$\mathrm{Gs}^{\mathrm{reg}}_\alpha$ class of all regular Gs_α's
Cs_α class of **c**ylindric **s**et algebras
$\mathrm{Lf}_\alpha = \{\mathfrak{A} \in \mathrm{CA}_\alpha : (\forall x \in A)\ |\Delta^{\mathfrak{A}} x| < \omega\}$, class of **l**ocally **f**inite CA_α's
$\mathrm{Fb'Gs}_\alpha$ class of all Gs_α's with **f**inite **b**ase
$\mathrm{Bb'Gs}_\alpha$ class of all Gs_α's with **b**ounded sub**b**ases
$\mathrm{F}K$ class of **f**inite members of K
$\mathrm{Fb}K = K \cap \mathbf{I}\mathrm{Fb'Gs}_\alpha$
$\mathrm{Bb}K = K \cap \mathbf{I}\mathrm{Bb'Gs}_\alpha$
${}_{<n}K = K \cap \mathrm{Mod}(\bar{d}(n \times n) = 0)$
${}_nK = {}_{<n+1}K \sim {}_{<n}K$, ${}_\omega K = {}_\infty K$
${}_{<\omega}K = \bigcup \{{}_nK : n \in \omega\}$, ${}_{(L)}K = \bigcup \{{}_nK : n \in L\}$

Acknowledgements

The author is grateful to Balázs Bíró, Stan Burris, László Csirmaz, Roger Maddux, Don Monk, Matti Rubin for helpful discussions. Very special thanks are due to Don Monk and Matti Rubin whose invaluable help and patience were crucial in this work. In particular, the connections with logic (formula schemes) and their importance were demonstrated to the author by Matti Rubin. I am indebted to the referee for his precious help.

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