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Lim-inf convergence in partially ordered sets ☆

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Abstract

The lim-inf convergence in a complete lattice was introduced by Scott to characterize continuous lattices. Here we introduce and study the lim-inf convergence in a partially ordered set. The main result is that for a poset P the lim-inf convergence is topological if and only if P is a continuous poset. A weaker form of lim-inf convergence in posets is also discussed.

Keywords: Lim-inf-convergence; Continuous posets; Lim-$\inf_2$-convergence; α-Continuous posets

1. Introduction

Let P be a partially ordered set (or poset, for short). The Birkhoff–Frink–McShane definition of order-convergence in P is defined as follows (see [1–3]): a net $(x_i)_{i\in I}$ in P is said to o-converge to $y \in P$ if there exist subsets M and N of P such that

(1) M is up-directed and N is down-directed,

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(2) $y = \sup M = \inf N$, and
(3) for each $a \in M$ and $b \in N$, there exists $k \in I$ such that $a \leqslant x_i \leqslant b$ hold for all $i \geqslant k$.

In general, o-convergence is not topological, i.e., the poset P may not be topologized so that nets o-converge if and only if they converge with respect to the topology. One basic problem here is: for what posets is the o-convergence topological? Although it has long been known that in every completely distributive lattice the o-convergence is topological, one still has not been able to find a satisfactory necessary and sufficient condition for o-convergence to be topological.

In [4] the lim-inf-convergence in complete lattices is introduced. A net $(x_i)_{i\in I}$ in a complete lattice lim-inf-converges to x if $x \leqslant \underline{\lim}_{i\in I} x_i = \inf\{\sup_{i\geqslant k}\{x_i\}\colon k \in I\}$. It was proved that for a complete lattice L, the lim-inf-convergence is topological if and only if L is a continuous lattice. The notion of continuous lattice was introduced by Dana Scott as a generalization of algebraic lattices and has found its applications in many fields such as computer science, topology and logic. Later on, continuous direct complete posets (or continuous dcpos) was introduced as an appropriate generalization of continuous lattices (see [5]). In this note we consider the lim-inf-convergence in an arbitrary partially ordered set. We prove that the lim-inf-convergence in a poset is topological if and only if the poset is a continuous poset. The definition of continuous poset is similar to that of continuous dcpo. We shall also consider another type of lim-inf-convergence, the counterpart of the o_2-convergence studied in [6,7], and prove a similar characterization of the poset for which this convergence is topological.

2. Lim-inf-convergence and continuous partially ordered sets

A net $(x_i)_{i\in I}$ in a complete lattice is said to lim-inf-converge to an element x if $x = \underline{\lim}_{i\in I} x_i = \sup\{\inf\{x_i\colon i \geqslant k\}\colon k \in I\}$. Since in a poset the infimum of a subset need not exists, thus we have to define the lim-inf-convergence in an arbitrary poset in a different way. Throughout of the paper we shall use $\bigvee A$ and $\bigwedge A$ to denote $\sup A$ and $\inf A$, respectively.

Definition 1. A net $(x_i)_{i\in I}$ in a poset P is said to lim-inf-converge to an element $y \in P$ if there exists an up-directed subset M of P such that

(A1) $\bigvee M$ exists with $\bigvee M \geqslant y$, and
(A2) for any $m \in M$, $x_i \geqslant m$ holds eventually (that is, there exists $k \in I$ such that $x_i \geqslant m$ for all $i \geqslant k$).

In this case we write $y \equiv \text{lim-inf}\, x_i$.

Remark 1.

(1) Let $P = \{a, b\} \cup \{b_i\colon i \in \mathbb{N}\}$, where $\mathbb{N}$ denotes the set of all natural numbers. The order on P is defined by $a < b$, $b_1 < b_2 < \cdots < b$. Let $S = (x_i)_{i\in\mathbb{N}}$ be the net where

$x_i = b_i, \forall i \in \mathbb{N}$. Take $M = \{b_i\colon i \in \mathbb{N}\}$. Then M is an up-directed subset of P with $\bigvee M = b > a$. In addition, for each $m \in M$ there is $k \in \mathbb{N}$ such that $x_i \geqslant m$ whenever $i \geqslant k$. Thus S lim-inf-converges to the element a. However, there is no up-directed set $M \subseteq P$ such that $\bigvee M = a$ and for every $x \in M$, $x_i \geqslant x$ holds eventually. Hence the inequality for $\bigvee M \geqslant y$ in condition (A1) cannot be reduced to the equality $\bigvee M = y$.

(2) Let $(x_i)_{i\in I}$ be a net in P such that $x = \inf\{x_i\colon i \in I\}$ exists. The singleton $M = \{x\}$ is an up-directed set, $\bigvee M = x$ and $x_i \geqslant x$ holds for all i, so $(x_i)_{i\in I}$ lim-inf-converges to x.

(3) If $(x_i)_{i\in I}$ lim-inf-converges to x, then it lim-inf-converges to every y with $y \leqslant x$. Thus the lim-inf-limits of a net is generally not unique.

For a poset P, the way-below relation $\ll$ on P can be defined in the same way as for dcpos. We say that x is *way below* y, $x \ll y$, if for any up-directed set $D \subseteq P$ for which $\bigvee D$ exists and $y \leqslant \bigvee D$, then there is $d \in D$ such that $x \leqslant d$.

From the definition of the way-below relation we see easily that if $x \leqslant y \ll z \leqslant w$, then $x \ll w$, and if $x \ll y$, then $x \leqslant y$.

Lemma 1. *If x and y are two elements of a poset P, then $x \ll y$ if and only if for any net $(x_i)_{i\in I}$ which lim-inf-converges to y, $x_i \geqslant x$ holds eventually.*

Proof. Suppose $x \ll y$ and $(x_i)_{i\in I}$ lim-inf-converges to y. Then there exists an up-directed set M such that $y \leqslant \bigvee M$ and for each $a \in M$, $x_i \geqslant a$ holds eventually. Since $x \ll y$, there is $a \in M$ with $x \leqslant a$. Hence $x_i \geqslant a \geqslant x$ holds eventually.

Conversely, suppose the condition is satisfied. If D is an up-directed subset with $\bigvee D \geqslant y$, then the net $(x_d)_{d\in D}$ lim-inf-converges to y, where $x_d = d$ for each $d \in D$. By the assumption, there is $x_d \in D$ such that $x_d \geqslant x$. Thus $x \ll y$. □

Definition 2. A poset P is called a continuous poset if for each $a \in P$, the set $\{x \in P\colon x \ll a\}$ is an up-directed set and $a = \bigvee\{x \in P\colon x \ll a\}$.

It can be seen that P is continuous if and only it for each $a \in P$ there is an up-directed subset D of $\{x \in P\colon x \ll a\}$ such that $\bigvee D = a$. The way-below relation $\ll$ on a continuous poset is interpolating, i.e., if $x \ll y$, then there is z with $x \ll z \ll y$ (see [4] for the proof of the interpolating property of continuous dcpos).

Example 1. For any set X, let $\mathcal{P}_0(X)$ be the set of all finite subsets of X. Then $(\mathcal{P}_0(X), \subseteq)$ is a continuous poset. This follows from the observation that for each $A \in \mathcal{P}_0(X)$, $A \ll A$. However, $\mathcal{P}_0(X)$ is not direct complete unless X is a finite set.

The lemma below follows from Lemma 1.

Lemma 2. *If P is a continuous poset, then a net $(x_i)_{i\in I}$ in P lim-inf-converges to y if and only if for each $x \ll y$, $x_i \geqslant x$ holds eventually.*

Let $\mathcal{L}$ be the class consisting of all the pairs $((x_i)_{i\in I}, x)$ of a net $(x_i)_{i\in I}$ and an element x in a poset P with $x \equiv \text{lim-inf}\, x_i$. The class $\mathcal{L}$ is called topological if there is a topology τ on P such that $((x_i)_{i\in I}, x) \in \mathcal{L}$ if and only if the net $(x_i)_{i\in I}$ converges to x with respect to the topology τ. By Kelley [8], $\mathcal{L}$ is topological if and only if it satisfies the following four conditions:

(Constants) If $(x_i)_{i\in I}$ is a constant net with $x_i = x$, $\forall i \in I$, then $((x_i)_{i\in I}, x) \in \mathcal{L}$.
(Subnets) If $((x_i)_{i\in I}, x) \in \mathcal{L}$ and $(y_j)_{j\in J}$ is a subnet of $(x_i)_{i\in I}$, then $((y_j)_{j\in J}, x) \in \mathcal{L}$.
(Divergence) If $((x_i)_{i\in I}, x)$ is not in $\mathcal{L}$, then there exists a subnet $(y_j)_{j\in J}$ of $(x_i)_{i\in I}$ which has no subnet $(z_k)_{k\in K}$ so that $((z_k)_{k\in K}, x)$ belongs to $\mathcal{L}$.
(Iterated limits) If $((x_i)_{i\in I}, x) \in \mathcal{L}$, and if $((x_{i,j})_{j\in J(i)}, x_i) \in \mathcal{L}$ for all $i \in I$, then $((x_{i,f(i)})_{(i,f)\in I\times M}, x) \in \mathcal{L}$, where $M = \prod_{i\in I} J(i)$.

Lemma 3.

(1) *For every poset the class $\mathcal{L}$ satisfies the axioms* (Constants) *and* (Subnets).
(2) *If P is a continuous poset, then $\mathcal{L}$ also satisfies the axioms* (Divergence) *and* (Iterated limits).

Proof. (1) The axiom (Constants) is clearly satisfied.

(Subnets) Suppose now that $((x_i)_{i\in I}, x) \in \mathcal{L}$ and D is up-directed such that $x \leqslant \bigvee D$ and for each $a \in D$, $x_i \geqslant a$ holds eventually. Thus for any subnet $(y_j)_{j\in J}$ of $(x_i)_{i\in I}$ and every $a \in D$, $y_j \geqslant a$ also holds eventually. Thus $((y_j)_{j\in J}, x) \in \mathcal{L}$.

(2) Now assume that P is continuous.

(Divergence) Suppose that $((x_i)_{i\in I}, x)$ is not in $\mathcal{L}$. Since the set $D = \{z \in P\colon z \ll x\}$ is an up-directed set whose supremum equals x, there exists $z \in D$ such that for any $i \in I$ there is a $j \in I$ with $j \geqslant i$ and $x_j \ngeqslant z$. Let J be the subset of I consisting of all $k \in I$ such that $x_k \ngeqslant z$. Then J is co-final in I and $(x_j)_{j\in J}$ is a subnet of $(x_i)_{i\in I}$. In addition, from Lemma 1 it follows that there is no subnet $(z_k)_{k\in K}$ of $(x_j)_{j\in J}$ such that $((z_k)_{k\in K}, x) \in \mathcal{L}$. Hence axiom (Divergence) is satisfied.

(Iterated limits) Assume $((x_i)_{i\in I}, x) \in \mathcal{L}$ and $((x_{i,j})_{j\in J(i)}, x_i) \in \mathcal{L}$ for every $i \in I$. We show that $((x_{i,f(i)})_{(i,f)\in I\times M}, x) \in \mathcal{L}$, where $M = \prod_{i\in I} J(i)$. By Lemma 2, it is enough to show that for each $z \ll x$, $x_{i,f(i)} \geqslant z$ holds eventually. Choose w such that $z \ll w \ll x$. There exists i_0 such that $x_i \geqslant w$ for all $i \geqslant i_0$. Thus $z \ll x_i$ for all $i \geqslant i_0$. Again as $(x_{i,j})_{j\in J(i)}$ lim-inf-converges to x_i, so for each $i \geqslant i_0$ there exists $g(i) \in J(i)$ such that if $j \in J(i)$ and $j \geqslant g(i)$ then $x_{i,j} \geqslant z$. Define $h \in \prod_{i\in I} J(i)$ such that $h(i) = g(i)$ if $i \geqslant i_0$ and $h(i)$ is any element in $J(i)$ otherwise. Now if $(i, f) \in I \times M$ and $(i, f) \geqslant (i_0, h)$, then $x_{i,f(i)} \geqslant z$. The proof is complete. □

Lemma 4. *If, in a poset P, the class $\mathcal{L}$ satisfies the conditions* (Iterated limits)*, then P is continuous.*

Proof. Let $a \in P$ and let $\mathcal{D}_a = \{\{x_{i,j}\}_{j\in J(i)}\colon i \in I\}$ be the family of all directed subsets of P whose supremum exist and is above a. For each $i \in I$, let $x_i = \bigvee\{x_{i,j}\colon j \in J(i)\}$. Then for each $i \in I$, we have $x_i \geqslant a$. Moreover, since $\{a\}$ is a member of $\mathcal{D}_a$, we have

$\inf\{x_i\colon i \in I\} = a$. Let the set I be equipped with the largest pseudo order on I, that is $i \leqslant k$ holds for any two $i, k \in I$. The order is transitive and reflexive but need not be antisymmetric. Then $(x_i)_{i\in I}$ is a net and, by Remark 1(2) it lim-inf-converges to a. For each $\{x_{i,j}\colon j \in J(i)\} \in \mathcal{D}_a$, define an order on $J(i)$ by $j_1 \leqslant j_2$ if $x_{i,j_1} \leqslant x_{i,j_2}$. Then $J(i)$ is a directed set and $(x_{i,j})_{j\in J(i)}$ is a net in P, which obviously lim-inf-converges to x_i. Now since the condition (Iterated limits) is satisfied, the net $(x_{i,f(i)})_{(i,f)\in I\times M}$ lim-inf-converges to a, where $M = \prod_{i\in I} J(i)$. By the definition of lim-inf limit, there exists an up-directed subset $D \subseteq P$ such that $\bigvee D \geqslant a$ and for each $d \in D$, $x_{i,f(i)} \geqslant d$ holds eventually. We now show that $D \subseteq \{x \in P\colon x \ll a\}$. Let $d \in D$. For any directed set $A \subseteq P$ with $\bigvee A \geqslant a$, $A = \{x_{m,j}\colon j \in J(m)\}$ for some $m \in I$. There exists (i_d, f_d) such that $(i, f) \geqslant (i_d, f_d)$ implies $x_{i,f(i)} \geqslant d$. Now $m \geqslant i_d$ (note that the order on I is the biggest pseudo order, that is $i \leqslant k$ for any $i, k \in I$), hence $x_{m,f_d(m)} \geqslant d$. Note that $x_{m,f_d(m)} \in A$, thus $d \ll a$. Thus D is an up-directed subset of $\{x \in P\colon x \ll a\}$ and $\bigvee D \geqslant a$. Thus $a \leqslant \bigvee\{x \in P\colon x \ll a\} \leqslant a$, which implies $a = \bigvee\{x \in P\colon x \ll a\}$. Hence P is continuous. □

The combination of Lemmas 3 and 4 deduces the following theorem.

Theorem 1. *For any poset P the lim-inf-convergence is topological if and only if P is a continuous poset.*

3. Lim-inf$_2$-convergence

In [7], the o_2-convergence was considered (in [6] this convergence is called 2-convergence). This convergence can be generalized by replacing directed subsets with arbitrary subsets. Here we consider only the lim-inf$_2$-convergence, a part of generalized o_2-convergence for lim-inf-convergence. We shall establish a characterization for this convergence to be topological.

Definition 3. A net $(x_i)_{i\in I}$ in a poset P is said to lim-inf$_2$-converge to $x \in P$ if there exists a subset $M \subseteq P$, such that

(B1) $\bigvee M$ exists and $x \leqslant \bigvee M$, and
(B2) for each $m \in M$, $x_i \geqslant m$ holds eventually.

Obviously if $(x_i)_{i\in I}$ lim-inf-converges to x, then it lim-inf$_2$-converges to x.

In [9] Raney established a characterization of completely distributive lattices using the long-below relation $\lhd$.

A complete lattice L is completely distributive if and only if for every $a \in L$, $a = \bigvee\{x \in L\colon x \lhd a\}$, where $x \lhd y$ if for any subset $A \subseteq L$ with $\bigvee A \geqslant y$, there exists $z \in A$ such that $x \leqslant z$.

For any two elements x and y in a poset P, we define $x \lhd y$, if for any subset $A \subseteq P$ for which $\bigvee A$ exists and $y \leqslant \bigvee A$, there exists $z \in A$ with $x \leqslant z$. It is easy to verify that (i) the relation $\lhd$ is transitive; (ii) $x \lhd y$ implies $x \leqslant y$. If P is a chain, then $x < y$ implies $x \lhd y$.

A poset P is called *supercontinuous* if for each $a \in P$, $a = \bigvee\{x \in P\colon x \lhd a\}$.

Example 2.

(1) Every chain $(P, \leqslant)$ is supercontinuous. In this case, for every $a \in P$, if $x < a$, then $x \lhd a$. If $\bigvee\{x \in P\colon x < a\} < a$ then $a \lhd a$. Hence it follows that $a = \bigvee\{x \in P\colon x \lhd a\}$ holds for every $a \in P$.
(2) Given a set X. Let $\mathcal{P}_0(X)$ be the set of all finite subsets of X. Then $(\mathcal{P}_0(X), \subseteq)$ is supercontinuous. Again, $\mathcal{P}_0(X)$ is generally not a complete lattice. In general, if m is a cardinal, then $\mathcal{P}_m(X) = \{A \subseteq X\colon |A| \leqslant m\}$ is supercontinuous with respect to $\subseteq$. This follows from the observation that $\{x\} \lhd A$ holds for every $x \in A$, $A \in \mathcal{P}_m(X)$.
(3) Let P be a supercontinuous poset and $A = \downarrow A = \{x \in L\colon x \leqslant y,\ \text{for some } y \in A\}$. Suppose for any $D \subseteq A$ for which $\bigvee D$ exists in A then $\bigvee D$ is the supremum of D in L. Then A is also a supercontinuous poset.

Although in every poset, $x \lhd y$ implies $x \ll y$, but a supercontinuous poset need not be a continuous poset.

Example 3. Let $\mathcal{E}(\mathbb{N}) = \{A \subseteq \mathbb{N}\colon |A| \leqslant 1 \text{ or } |A| = \infty\}$. Then as a subposet of $\mathcal{P}(\mathbb{N})$, $\mathcal{E}(\mathbb{N})$ is a supercontinuous poset. Indeed, for each $x \in \mathbb{N}$ one can easily see that $\{x\} \lhd \{x\}$ and $A = \bigvee\{\{x\}\colon x \in A\}$ holds for every $A \in \mathcal{E}(\mathbb{N})$. On the other hand, $A \ll \mathbb{N}$ if and only if A is a singleton. But the set of all singletons is not a directed set, that is the set $\{A \in \mathcal{E}(\mathbb{N})\colon A \ll \mathbb{N}\}$ is not a directed set. So $(\mathcal{E}(\mathbb{N}), \subseteq)$ is not a continuous poset. Notice that this poset is a dcpo.

Definition 4. Let P be a poset.

(1) Let $x, y \in P$. Define $x \ll_\alpha y$ if for every net $(x_i)_{i\in I}$ which lim-inf$_2$-converges to y, $x_i \geqslant x$ holds eventually.
(2) A poset P is called α-continuous if $a = \bigvee\{x \in P\colon x \ll_\alpha a\}$ holds for every $a \in P$.

Remark 2.

(1) Obviously every supercontinuous poset is α-continuous. The converse is not true. It is easy to check that every finite lattice is α-continuous. But a finite lattice is supercontinuous if and only if it is distributive.
A poset is constructed at the end of the paper which is continuous but not α-continuous (see Example 4).
(2) If P is α-continuous, then for each $a \in P$, $a = \bigvee\{x \in P\colon \exists z \in P, x \ll_\alpha z \ll_\alpha a\}$. This is because $a = \bigvee\{y \in P\colon y \ll_\alpha a\}$ and for each $y \ll_\alpha a$, $y = \bigvee\{x \in P\colon x \ll_\alpha y\}$.

Lemma 5. *If P is a complete lattice, then $x \ll y$ if and only if $x \ll_\alpha y$.*

Proof. Suppose $x \ll y$ and $(x_i)_{i\in I}$ is a net that lim-inf$_2$-converges to y. It then follows that $\bigvee\{\inf\{x_i\colon i \geqslant k\}\colon k \in I\} \geqslant y$. Since $\{\inf\{x_i\colon i \geqslant k\}\colon k \in I\}$ is a directed set and $x \ll y$, there exists $k_0 \in I$ such that $\inf\{x_i\colon i \geqslant k_0\} \geqslant x$. So $x_i \geqslant x$ holds for all $i \geqslant k_0$. Thus $x \ll_\alpha y$.

The converse implication is true for all posets. □

Note that if L is a complete lattice, then $\{x \in L\colon x \ll a\}$ is a directed set for every $a \in L$. Thus it follows that a complete lattice is continuous if and only if it is α-continuous.

Now let $\mathcal{S}$ be the class consisting of all pairs $((x_i)_{i\in I}, x)$, where $(x_i)_{i\in I}$ is a net that lim-inf$_2$-converges to x. Again one can easily show that for any poset P, the class $\mathcal{S}$ satisfies the axioms (Constants) and (Subnets).

Proposition 1. *If P is α-continuous, then the class $\mathcal{S}$ satisfies the axioms* (Divergence) *and* (Iterated limits).

Proof. (Divergence) Suppose that $((x_i)_{i\in I}, x)$ is not in $\mathcal{S}$. Since $\bigvee\{y \in P\colon y \ll_\alpha x\} = x$, there is $y \ll_\alpha x$ such that $x_i \geqslant y$ does not hold eventually. Put $J = \{i \in I\colon x_i \ngeqslant y\}$. Then $(x_j)_{j\in J}$ is a subnet of $(x_i)_{i\in I}$ which has no subnet lim-inf$_2$-convergent to x.

(Iterated limits) Suppose $(x_i)_{i\in I}$ lim-inf$_2$-converges to x, and for each $i \in I$, $(x_{i,j})_{j\in J(i)}$ lim-inf$_2$-converges to x_i. By Remark 2(2), $x = \bigvee\{y \in P\colon \exists z \in P, y \ll_\alpha z \ll_\alpha x\}$. Thus in order to show that the net $(x_{i,f(i)})_{i\in I}$ lim-inf$_2$-converges to x, it is enough to verify that if $y \ll_\alpha z \ll_\alpha x$, then $x_{i,f(i)} \geqslant y$ holds eventually. But this is similar to the proof of the case for lim-inf-convergence, so we omit it. □

Lemma 6. *If P is a poset such that the class $\mathcal{S}$ satisfies the axiom* (Iterated limits), *then P is α-continuous.*

Proof. The proof is similar to that of Lemma 4. For any $a \in P$, consider the collection $\{(x_{i,j})_{j\in J(i)}\colon i \in I\}$ of nets $(x_{i,j})_{j\in J(i)}$ that lim-inf$_2$-converges to a. Let $(x_i)_{i\in I}$ be the constant net in which $x_i = a$, $\forall i \in I$. So for each $i \in I$, $(x_{i,j})_{j\in J(i)}$ lim-inf$_2$-converges to x_i. Thus by the assumption, the net $(x_{i,f(i)})_{(i,f)\in I\times M}$ lim-inf$_2$-converges to a, where $M = \Pi_{i\in I} J(i)$ and I is equipped with the pseudo orders that $k \leqslant i$ holds for any $k, i \in I$. Thus there is a subset A of P such that $\bigvee A \geqslant a$ and $x_{i,f(i)} \geqslant y$ holds eventually for any $y \in A$. Then one can verify that $\bigvee A = a$ and $A \subseteq \{x \in P\colon x \ll_\alpha a\}$. Thus P is α-continuous. □

Theorem 2. *For any poset P the lim-inf$_2$-convergence is topological if and only if P is α-continuous.*

Remark 3. Suppose P is a lattice and $(x_i)_{i\in I}$ is a net in P that lim-inf$_2$-converges to x. Then there is a subset M of P with $\bigvee M \geqslant x$ and for each $m \in M$, $x_i \geqslant m$ holds eventually. Put $K = \{\bigvee D\colon D$ is a finite subset of $M\}$. Then K is up-directed and for each $k \in K$, $x_i \geqslant k$ holds eventually. Hence $(x_i)_{i\in I}$ lim-inf-converges to x. Hence in a lattice the two convergences are equivalent.

The following is an example of a poset in which the two convergences are not equivalent.

Example 4. The following example is a modification of one in [5]. Let $P = \{T\} \cup \{a_1, a_2, \ldots\} \cup \{b_1, b_2, \ldots\}$. The order $\leqslant$ on P is defined as follows:

(1) $a_i < T$, $b_i < T$ for all $i = 1, 2, 3, \ldots$;
(2) if $k \geqslant i$, then $a_k \geqslant b_i$.

By definition, if $i \neq j$ then a_i and a_j are incomparable and b_i and b_j are incomparable too. Note that T is the top element. Let $B = \{b_1, b_2, \ldots\}$. Then clearly $\bigvee B = T$. Since for each $b_i \in B$, $a_n \geqslant b_i$ whenever $n \geqslant i$, thus the net $(a_i)_{i\in\mathbb{N}}$ lim-inf$_2$-converges to T. However $(a_i)_{i\in\mathbb{N}}$ is not lim-inf-convergent to T because there exists no up-directed set D with $\bigvee D = T$ and for each $d \in D$, $a_i \geqslant d$ holds eventually.

One can easily check that $T \ll T$, $a_i \ll a_i$ and $b_i \ll b_i$ for all i. Thus P is a continuous poset (actually a continuous dcpo).

On the other hand, this P serves also as an example of poset which is continuous but not α-continuous. Indeed, consider the element a_1 of P. Since the net $(a_i)_{i\in\mathbb{N}}$ lim-inf$_2$-converges to T, it lim-inf$_2$-converges to a_1 as well. But $a_i \geqslant a_1$ does not hold eventually, thus $a_1 \ll_\alpha a_1$ does not hold. The only element x satisfying $x \ll_\alpha a_1$ is b_1. So $\bigvee\{x \in P\colon x \ll_\alpha a_1\} = b_1 \neq a_1$, hence P is not α-continuous.

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