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Spherical Functions and Harmonic Analysis on Free Groups*

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1. INTRODUCTION

Let $\mathbf{F}_r$, $r > 1$, be a free group with r generators. In this paper we study a principal series and a complementary series of *irreducible* unitary representations of $\mathbf{F}_r$, which are defined through the action of $\mathbf{F}_r$ on its Poisson boundary, relative to a simple random walk. We show that the regular representation of $\mathbf{F}_r$ can be written as a direct integral of the representations of the principal series and that the resulting harmonic analysis on the free group bears a close resemblance with the harmonic analysis of $SL(2, \mathbb{R})$.

In Section 2 we introduce the algebra of *radial* functions, i.e., functions which depend only on the length $|x|$ of a word x, and we define *spherical* functions as the radial eigenfunctions of the convolution operator by μ_1, where μ_1 denotes the probability distribution of the simple random walk. Spherical functions are naturally indexed by a complex number z, in such a way that, if ϕ_z is spherical, then $\mu_1 * \phi_z = \gamma(z)\,\phi_z$, with $\gamma(z) = [(2r-1)^z + (2r-1)^{1-z}]/2r$, and $\phi_z(e) = 1$. Thus the spherical function ϕ_z is uniquely determined by the corresponding eigenvalue of the convolution operator by μ_1. In our context, this operator plays the role of the Laplace–Beltrami operator on semisimple Lie groups [12]: for other aspects of this analogy, see [3, 10, 13]. We also show that there exist complex numbers c_z, c'_z such that $\phi_z(x) = c_z(2r-1)^{-z|x|} + c'_z(2r-1)^{(z-1)|x|}$ if $(2r-1)^{2z-1} \neq 1$, and $\phi_z(x) = (1 + |x|\,(r-1)/r)(2r-1)^{-z|x|}$, otherwise. As a consequence, ϕ_z is bounded if and only if $0 \leqslant \operatorname{Re} z \leqslant 1$, and it is positive definite if and only if $-1 \leqslant \gamma(z) \leqslant 1$. The decomposition of ϕ_z as a linear combination of exponentials is the analogue of Harish Chandra's asymptotic expansion for spherical functions on semisimple Lie groups (see Section 9.1 of [25], in particular Theorem 9.1.1.1 and Theorem 9.1.1.2).

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In Section 3 we introduce representations π_z of $\mathbf{F}_r$, acting on $L^2(\Omega, \nu)$, where (Ω, ν) is the Poisson boundary of $\mathbf{F}_r$, relative to μ_1. If $P(x, \omega) = d\nu_x/d\nu$ is the Poisson kernel relative to the action of $\mathbf{F}_r$ on Ω, and $\xi \in L^2(\Omega, \nu)$, we let $\pi_z(x)\,\xi(\omega) = P^z(x, \omega)\,\xi(x^{-1}\omega)$. Denoting by $\mathbf{1}$ the function identically one on Ω, we have $\phi_z(x) = (\pi_z(x)\,\mathbf{1}, \mathbf{1})$. The representations π_z are unitary if $-(2r-1)^{1/2}/r \leqslant \gamma(z) \leqslant (2r-1)^{1/2}/r$ (*principal series*) and unitarizable if $\operatorname{Im} z = k\pi/\ln(2r-1)$ (*complementary series*). We prove that if $\operatorname{Re} z \neq 0, 1$, the unitary representations of the principal and complementary series are *irreducible*. The main difference from the corresponding case for the representations of $SL(2, \mathbb{R})$ is due to the fact that the action of $\mathbf{F}_r$ on its Poisson boundary is not transitive. We base our reasoning on the heuristic argument that, since the free group can be realized as a lattice in $SL(2, \mathbb{R})$, for large integers n an appropriate average of the operators $\pi_z(x)$ over the words of length n should be similar to the average $\int_K \pi(k)\,dk$, where K is a maximal compact subgroup of $SL(2, \mathbb{R})$ and π belongs to the principal or complementary series of $SL(2, \mathbb{R})$. This integral average over K gives the projection on the function which are constant on the Poisson boundary of $SL(2, \mathbb{R})$. In our case, we define averages T_n of the operators $\pi_z(x)$, $|x| = n$, in such a way that, for every $\xi \in L^2(\Omega, \nu)$, $T_n \xi$ converges to the constant $\int_\Omega \xi\,d\nu$. Irreducibility then follows from the fact that $\mathbf{1}$ is a cyclic vector.

In Section 4 we give an explicit expression for the resolvent $(\mu_1 - \gamma(z))^{-1}$ as the operator of convolution by a constant multiple of $(2r-1)^{-z|x|}$. We also show that the regular representation decomposes as a direct integral of the representations of the principal series $\{\pi_{1/2+it}\}$. The corresponding Plancherel measure can be expressed in terms of the coefficients of the decomposition of the spherical function $\phi_{1/2+it}$ as a sum of exponentials. This is the analogue of the expression of the Plancherel measure of a semisimple Lie group in terms of Harish Chandra c-function (see Theorem 9.2.1.5 of [25]). The Plancherel formula is then applied to give a proof of the analogue of Herz's principe de majoration [7, 17]. We show that, for every coefficient $f * g$ of the regular representation of the free group, there exist functions $\xi, \eta \in L^2(\Omega)$ such that $f * g^*(x)| \leqslant (\pi_{1/2}(x)\,\xi, \eta)$, and $\|f\|_2 \|g\|_2 = \|\xi\|_{L^2(\Omega)} \|\eta\|_{L^2(\Omega)}$.

Spherical functions and radial functions on free groups have been studied by Cartier [3, 4] and Sawyer [23] in the context of random walks on homogeneous trees. In particular, the Plancherel measure was computed in [4, 23]. Later, but independently, Cohen studied the algebra of radial functions [5]. Cohen's work was used by Pytlik [21] to compute again the Plancherel measure; he also proved that the von Neumann algebra generated by radial functions acting on $l^2(\mathbf{F}_r)$ by convolution is maximal Abelian.

Cartier's point of view is different from ours. He is interested in the group G of isometries of a homogeneous tree (endowed with its natural metric). This group is the product of a compact subgroup K and a closed subgroup isomorphic to $\mathbf{F}_r$ [24]. Bi-invariant functions with respect to K restrict to radial functions on $\mathbf{F}_r$, and the Poisson boundary of $\mathbf{F}_r$ can be naturally identified with a homogeneous space for K. While all the theory of radial functions could be obtained within this framework, the irreducibility of the representation of the principal and complementary series of $\mathbf{F}_r$ cannot be deduced from the irreducibility of the corresponding representation of the group of isometries of the tree.

2. Spherical Functions

We denote by $|x|$ the length of the word $x \in \mathbf{F}_r$, i.e., the number of letters of the word x in its reduced form. A complex-valued function f on $\mathbf{F}_r$ will be called *radial* if it depends only on the length of a word, that is, if $f(x) = f(y)$ whenever $|x| = |y|$. The space of all finitely supported radial functions will be denoted by $\mathscr{A}$. Denote by e the identity in $\mathbf{F}_r$, let $\mu_0 = \delta_e$ be the Dirac function at e, and μ_n be the function which takes the value $1/2r(2r-1)^{n-1}$ on all words of length n, and zero otherwise. Clearly every element of $\mathscr{A}$ is a linear combination of the μ_n's. The following lemma is known [3, 4] and its proof is included for completeness.

LEMMA 1. *$\mathscr{A}$ is a commutative convolution algebra generated by μ_0 and μ_1; in fact*

$$\mu_1 * \mu_n = \frac{1}{2r}\mu_{n-1} + \frac{2r-1}{2r}\mu_{n+1}. \tag{1}$$

Proof. Observe that $\mu_1 * \mu_n(x) = (1/2r)\sum_{|y|=1}\mu_n(yx)$; on the other hand $|y| = 1$ and $\mu_n(yx) \neq 0$ implies $|x| = n-1$ or $n+1$. If $|x| = n-1$, there are $2r-1$ words y of length 1 such that $|yx| = n$; on the other hand, if $|x| = n+1$, there is only one word y such that $|y| = 1$ and $|yx| = n$. This yields the identity (1), which implies that $\mathscr{A}$ is the commutative algebra generated by $\mu_0 = \delta_e$ and μ_1.

DEFINITION. A function ϕ defined on $\mathbf{F}_r$ is callled *spherical* if:

(1) ϕ is radial;

(2) $\phi * f = c\phi$ for every $f \in \mathscr{A}$, where c is a constant depending on f and ϕ;

(3) $\phi(e) = 1$.

If f is any function on $\mathbf{F}_r$, we denote by $\mathscr{E}f$ the *radial* function whose value on the words of length n is $(1/2r(2r-1)^{n-1})\sum_{|x|=n} f(x)$. Since there are exactly $2r(2r-1)^{n-1}$ words of length n, $\mathscr{E}f$ is just the average of f on the words of equal length. In particular, if we adopt the notation

$$\langle f, g\rangle = \sum_{x \in F_r} f(x)\, g(x)$$

(whenever it makes sense), then $\langle \mathscr{E}f, \mathscr{E}g\rangle = \langle \mathscr{E}f, g\rangle = \langle f, \mathscr{E}g\rangle$. For any function f we write $(\lambda(x)f)(y) = f(x^{-1}y)$ and $(\rho(x)f)(y) = f(yx)$.

LEMMA 2. *If ϕ is not identically zero the following are equivalent:*

(1) *ϕ is spherical,*

(2) $\mathscr{E}(\lambda(x)\,\phi)(y) = \phi(x)\,\phi(y)$,

(3) *ϕ is radial and the functional $Lf = \langle f, \phi\rangle$ is multiplicative on the convolution algebra $\mathscr{A}$.*

Proof. Let ϕ be spherical, and $x, y \in \mathbf{F}_r$; define $\Phi_x(y) = \mathscr{E}(\lambda(x^{-1})\,\phi)(y)$. Set $F_y(x) = \mathscr{E}\delta_y(x)$, where δ_y is the Dirac function at y. Then $F_y \in \mathscr{A}$, and

$$\phi * F_y(x) = \langle \lambda(x^{-1})\,\phi, \mathscr{E}\delta_y\rangle = \langle \Phi_x, \delta_y\rangle = \Phi_x(y).$$

Since ϕ is spherical, $\Phi_x(y) = c\phi(x)$. By comparison with the obvious identity $\Phi_e(y) = \phi(y)$, it follows that $c = \phi(y)$ and $\Phi_x(y) = \phi(x)\,\phi(y)$. Thus (1) implies (2).

If (2) holds, choose x such that $\phi(x) \neq 0$, and write $\phi(y) = \mathscr{E}(\lambda(x)\,\phi)(y)/\phi(x)$; this shows that ϕ is radial. Furthermore, if $f, g \in \mathscr{A}$, then

$$\begin{aligned} L(f * g) &= \sum_{x,y} f(x)\, g(y)\, \phi(xy) = \sum_x f(x) \langle \lambda(x^{-1})\,\phi, g\rangle \\ &= \sum_x f(x) \langle \mathscr{E}(\lambda(x^{-1})\,\phi), g\rangle \\ &= \sum_{x,y} f(x)\, g(y)\, \phi(x)\, \phi(y) = Lf \cdot Lg. \end{aligned}$$

Thus (2) implies (3).

If (3) holds and $f \in \mathscr{A}$, then $\phi * f(x) = L(f * \delta_x) = \phi * f * \delta_x(e) = Lf \cdot L\delta_x = (Lf) \cdot \phi(x)$. Furthermore, choosing $f = \delta_e$, this means $\phi(x) = \phi(e)\,\phi(x)$, hence $\phi(e) = 1$. Thus ϕ is spherical, and (3) implies (1).

COROLLARY. *Let a be any word of length one in $\mathbf{F}_r$: then the values of the spherical function ϕ satisfy the identity*

$$\phi(a^{n+1}) = \frac{2r}{2r-1}\,\phi(a)\,\phi(a^n) - \frac{1}{2r-1}\,\phi(a^{n-1}). \tag{2}$$

In particular, a spherical function is uniquely determined by its value on the words of length 1.

Proof. Since $\phi(a^n) = \langle \phi, \mu_n \rangle$, and the functional defined by ϕ is multiplicative on $\mathscr{A}$, Lemma 1 yields

$$\langle \phi, \mu_1 * \mu_n \rangle = \phi(a)\, \phi(a^n) = \frac{1}{2r}\, \phi(a^{n-1}) + \frac{2r-1}{2r}\, \phi(a^{n+1})$$

from which (2) readily follows.

We now consider the Poisson boundary of $\mathbf{F}_r$, as in [10, 13]. Let Ω be the set of all infinite reduced words in the generators of $\mathbf{F}_r$ and their inverses. The space Ω is compact in the product topology, and $\mathbf{F}_r$ acts on Ω, by left multiplication, as a group of homeomorphisms. For each $x \in \mathbf{F}_r$, $|x| = n$, let $E(x)$ be the subset of Ω consisting of all infinite words whose first n letters coincide with the crresponding letters of x. Each $E(x)$ is open in Ω and the topology of Ω is generated by the basis $\{E(x), x \in \mathbf{F}_r\}$. Let us define a probability measure ν on Ω by the rule $\nu(E(x)) = 1/2r(2r-1)^{n-1}$ where $n = |x|$. The measure ν is quasi-invariant with respect to the action of $\mathbf{F}_r$ on Ω: for every $x \in \mathbf{F}_r$ and for every measurable subset A of Ω, its translate ν_x defined by $\nu_x(A) = \nu(x^{-1}A)$ is absolutely continuous with respect to ν. To compute the Radon–Nikodym derivative, let $x \in \mathbf{F}_r$, $|x| = n$, $\omega \in \Omega$, and consider the word ω_n of length n consisting of the first n letters of ω; define $\delta(x, \omega) = n - |x^{-1}\omega_n|$. Then it is clear that

$$\frac{d\nu_x}{d\nu}(\omega) = (2r-1)^{\delta(x,\omega)} \tag{3}$$

The action of $\mathbf{F}_r$ on Ω defines a convolution product of a measure on $\mathbf{F}_r$ and a measure on Ω; obviously $\mu_1 * \nu = (1/2r) \sum_{|x|=1} \nu_x = \nu$. The function $P(x, \omega) = d\nu_x(\omega)/d\nu$, defined on $\mathbf{F}_r \times \Omega$, is called the *Poisson kernel* of $\mathbf{F}_r$ associated to the simple random walk defined by μ_1.

The following cocycle identities are immediate:

$$P(xy, \omega) = P(y, x^{-1}\omega) \cdot P(x, \omega),$$
$$P(e, \omega) = 1.$$

The next result shows how spherical functions arise from the Poisson kernel.

THEOREM 1. *For each $z \in \mathbb{C}$ the function*

$$\phi_z(x) = \int_\Omega P^z(x, \omega)\, d\nu(\omega)$$

is spherical, and $\mu_1 * \phi_z = \gamma(z)\,\phi_z$, *with* $\gamma(z) = (2r)^{-1}((2r-1)^z + (2r-1)^{1-z})$. *Conversely, every spherical function can be obtained in this way. Furthermore,* $\phi_z(x) = \mathscr{E}(P^z(x, \omega))$.

Proof. By the cocycle identity,

$$\phi_z(xy) = \int_\Omega P^z(y, x^{-1}\omega)\, P^z(x, \omega)\, d\nu(\omega).$$

Clearly, for every $\omega \in \Omega$,

$$\sum_{|y|=1} P^z(y, \omega) = (2r-1)^z + (2r-1)^{1-z}$$

which is independent of ω, hence equal to $2r\langle \phi_z, \mu_1\rangle$. Therefore the cocycle indentity implies that $P^z(x, \omega) * \mu_1 = \langle \phi_z, \mu_1\rangle P^z(x, \omega)$, and

$$\begin{aligned} \phi_z * \mu_1(x) &= (2r)^{-1} \int_\Omega \sum_{|y|=1} P^z(y, x^{-1}\omega)\, P^z(x, \omega)\, d\nu(\omega) \\ &= \langle \phi_z, \mu_1\rangle\, \phi_z(x) = \gamma(z)\, \phi_z(x). \end{aligned}$$

Furthermore, ϕ_z is radial because the distribution of $P(x, \omega)$ with respect to ν depends only on the length of x. Since $\phi_z(e) = 1$ and δ_e and μ_1 generate the algebra $\mathscr{A}$, we conclude that ϕ_z is spherical. Conversely, let ϕ be any spherical function and let $|x| = 1$. Choose $z \in \mathbb{C}$ so that $\phi(x) = (2r)^{-1}[(2r-1)^z + (2r-1)^{1-z}]$ (this is possible because the function $\gamma(z)$ is surjective). Then $\phi_z(x) = \phi(x)$ when $|x| = 1$. By the corollary to Lemma 2, ϕ_z and ϕ coincide on the whole of $\mathbf{F}_r$. Finally, $\mathscr{E}(P^z(x, \omega)) * \mu_1 = \mathscr{E}(P^z(x, \omega) * \mu_1) = \langle \phi_z, \mu_1\rangle \mathscr{E}(P^z(x, \omega))$ and $\mathscr{E}(P^z(e, \omega)) = 1$, whence $\mathscr{E}(P^z(x, \omega)) = \phi_z(x)$ for every $x \in \mathbf{F}_r$.

Remark 1. The first part of the proof shows that $\phi_z(x) = \phi_{1-z}(x)$. More generally, $\phi_z = \phi_w$ if and only if $\gamma(z) = \gamma(w)$.

Let now $z \in \mathbb{C}$, and denote by ϕ_z the corresponding spherical function. Set, as above, $\gamma(z) = (2r)^{-1}[(2r-1)^z + (2r-1)^{1-z}]$. The next statement gives an expression of ϕ_z as a linear combination of exponentials.

THEOREM 2. (i) *If* $\sigma = \operatorname{Re} z$, *then, for every* $x \in \mathbf{F}_r$,

$$|\phi_z(x)| \leqslant \phi_\sigma(x).$$

(ii) *If* $(2r-1)^{2z-1} \neq 1$, *denote by* (c_z, c_z') *the solution of the linear system* $c + c' = 1$, $c(2r-1)^{-z} + c'(2r-1)^{z-1} = \gamma(z)$. *Then, for every* $x \in \mathbf{F}_r$,

$$\phi_z(x) = c_z(2r-1)^{-z|x|} + c_z'(2r-1)^{(z-1)|x|}.$$

In particular, $c_z = (2r)^{-1}[(2r-1)^{1-z} - (2r-1)^{z-1}]/[(2r-1)^{-z} - (2r-1)^{z-1}]$, $c'_z = c_{1-z}$ *and*

$$\phi_z = c_z h_z + c_{1-z} h_{1-z},$$

where $h_z(x) = (2r-1)^{-z|x|}$.

(iii) *If* $(2r-1)^{2z-1} = 1$, *then, for every* $x \in \mathbf{F}_r$,

$$\phi_z(x) = (1 + |x|(r-1)/r)(2r-1)^{-z|x|}.$$

Proof. Observe that $P(x, \omega)$ is positive, and, if $\sigma = \operatorname{Re} z$, then $|P^z(x, \omega)| = P^\sigma(x, \omega)$: this proves (i). To prove (ii), suppose $(2r-1)^{2z-1} \neq 1$. Then $(2r-1)^{-z} - (2r-1)^{z-1} \neq 0$, hence the linear system $c + c' = 1$, $c(2r-1)^{-z} + c'(2r-1)^{z-1} = \gamma(z)$ is nonsingular. Let (c_z, c'_z) be the solution of this system and form the function $f(x) = c_z(2r-1)^{-z|x|} + c'_z(2r-1)^{(z-1)|x|}$. Then f is radial, $f(e) = 1$ and $f(x) = \gamma(z) = \phi_z(x)$ whenever $|x| = 1$. As a consequence, $\mu_1 * f(e) = \gamma(z)$. By the corollary to Lemma 2, in order to prove that $f = \phi_z$ it remains to show that $\mu_1 * f(x) = \gamma(z) f(x)$ for every $x \neq e$. Set $h_z(x) = (2r-1)^{-z|x|}$: we now show that, if $x \neq e$, $\mu_1 * h_z(x) = \gamma(z)\, h_z(x)$. Indeed, when $|y| = 1$, $h_z(xy) = (2r-1)^{-z}\, h_z(x)$ if $|xy| = |x| + 1$, and $h_z(xy) = (2r-1)^z\, h_z(x)$ if $|xy| = |x| - 1$. Therefore

$$\begin{aligned}\mu_1 * h_z(x) &= (2r)^{-1} \sum_{|y|=1} h_z(xy) = (2r)^{-1}[(2r-1)^{1-z} + (2r-1)^z] \cdot h_z(x) \\ &= \gamma(z) \cdot h_z(x).\end{aligned}$$

Since $f = c_z h_z + c'_z h_{1-z}$ and $\gamma(z) = \gamma(1-z)$, (ii) is proved. To prove (iii), suppose $(2r-1)^{2z-1} = 1$, and let $h_z(x) = (2r-1)^{-z|x|}$, $k_z(x) = |x|\, h_z(x)$ and $q(x) = (1 + |x|\,(r-1)/r)\, h_z(x) = h_z(x) + r^{-1}(r-1)\, k_z(x)$. Then $q(e) = 1$, and if $|x| = 1$, one has

$$q(x) = (2r-1)^{-z}\,(1 + (r-1)/r) = r^{-1}(2r-1)^{1-z} = \gamma(z)$$

because $(2r-1)^z = (2r-1)^{1-z}$, by the hypothesis. Therefore $\mu_1 * q(e) = \gamma(z)$. It remains to show that $\mu_1 * q(x) = \gamma(z)\, q(x)$ for every $x \neq e$. By the proof of part (ii), we know that this convolution equation is satisfied by the function h_z; therefore it suffices to show that k_z satisfies the same equation, i.e., $\mu_1 * k_z(x) = \gamma(z)\, k_z(x)$ for every $x \neq e$. Indeed, when $x \neq e$,

$$\begin{aligned}(2r)^{-1} \sum_{|y|=1} k_z(xy) &= (2r)^{-1}\,[(2r-1)^{1-z}(|x|+1) + (2r-1)^z(|x|-1)]\, h_z(x) \\ &= \gamma(z) \cdot k_z(x),\end{aligned}$$

using again the fact that $(2r-1)^{1-z} = (2r-1)^z$.

Denote by $l^1_\#$ the completion of $\mathscr{A}$ in the l^1 norm. An immediate consequence of Lemma 2 is that a spherical function ϕ determines a continuous multiplicative functional on $l^1_\#$ if and only if ϕ is bounded. Let $z \in \mathbb{C}$, and let ϕ_z be the corresponding spherical function. By means of Theorem 2, we can now characterize the subset of $\mathbb{C}$ associated to bounded spherical functions.

COROLLARY. (i) *The spherical function ϕ_z is bounded if and only if* $0 \leqslant \operatorname{Re} z \leqslant 1$.

(ii) *For every $p > 2$ $(p \neq \infty)$, $\phi_z \in l^p(\mathbf{F}_r)$ if and only if* $1/p < \operatorname{Re} z < 1 - 1/p$.

Proof. If $0 \leqslant \operatorname{Re} z \leqslant 1$, it follows immediately by parts (ii) and (iii) of Theorem 2 that ϕ_z is bounded. Conversely, if $\operatorname{Re} z < 0$ or $\operatorname{Re} z > 1$, then Theorem 2, part (ii) implies that $\lim_{|x| \to \infty} |\phi_z(x)| = \infty$. This proves (i). A similar argument proves (ii). Indeed, let $\sigma = \operatorname{Re} z$, and for a given p such that $2 < p < \infty$, suppose $1/p < \sigma < 1 - 1/p$. If $(2r-1)^{2z-1} \neq 1$, then $1 - p\sigma$ and $1 + p(\sigma - 1)$ are both negative, and by Theorem 2 one obtains the estimate

$$\begin{aligned}\|\phi_z\|_p^p &= 1 + 2r \sum_{n=1}^{\infty} (2r-1)^{n-1} |\phi_z(a^n)|^p \\ &< C \sum_{n=1}^{\infty} \left[(2r-1)^{n-np\sigma} + (2r-1)^{n+np(\sigma-1)}\right] < \infty,\end{aligned}$$

where a is any word of length 1 and C is a constant. On the other hand, if $(2r-1)^{2z-1} = 1$, then $\operatorname{Re} z = \frac{1}{2}$ and by part (iii) of Theorem 2 one has

$$\|\phi_z\|_p^p < C \sum_{n=1}^{\infty} n(2r-1)^{n-np/2} < \infty$$

since $2 < p < \infty$. Vice versa, if $\sigma = \operatorname{Re} z < 1/p$, then

$$\|\phi_z\|_p^p \sim \sum_{n=1}^{\infty} (2r-1)^n |\phi_z(a^n)|^p = \infty$$

because the terms of the series behave asymptotically as $(2r-1)^{n(1-p\sigma)}$. An analogous estimate holds for $\operatorname{Re} z > 1 - 1/p$, and (ii) is proved.

Remark 2. We have already observed that $\phi_{z_1} = \phi_{z_2}$ if $\gamma(z_1) = \gamma(z_2)$. Thus the Gelfand spectrum Σ of the Banach algebra $l^1_\#$ can be identified with the image under γ of the strip $S = \{0 \leqslant \operatorname{Re} z \leqslant 1\}$. Since $l^1_\#$ has an identity, Σ is compact: indeed, γ is a periodic function of $\operatorname{Im} z$, that is, $\gamma(z + 2\pi i/\ln(2r-1)) = \gamma(z)$. It is easy to see that $\Sigma = \gamma(S) = \{z\colon (\operatorname{Re} z)^2 + ((r/(r-1)) \operatorname{Im} z)^2 \leqslant 1\}$. The same ellipse was found by other methods by Cartier [4]; see also [9, 21].

It is convenient to describe the map $\gamma: S \to \Sigma$ in some detail. The central axis of the strip, $\operatorname{Re} z = \frac{1}{2}$, is mapped by γ onto the segment I connecting the two foci of Σ, $I = \{z: \operatorname{Im} z = 0, -(2r-1)^{1/2}/r \leqslant \operatorname{Re} z \leqslant (2r-1)^{1/2}/r\}$. The segment $\{\operatorname{Im} z = 0, 0 \leqslant \operatorname{Re} z \leqslant 1\} \subset S$ is mapped onto $\{\operatorname{Im} z = 0, (2r-1)^{1/2}/r < \operatorname{Re} z \leqslant 1\} \subset \Sigma$, and the segment $\{\operatorname{Im} z = 0, -1 \leqslant \operatorname{Re} z < -(2r-1)^{1/2}/r\}$ is the image under γ of $\{\operatorname{Im} z = i\pi/\ln(2r-1), 0 \leqslant \operatorname{Re} z \leqslant 1\}$. We also remark that the substrip $\{1/p < \operatorname{Re} z < 1 - 1/p\}$ maps onto the subellipse $\{(\operatorname{Re} z)^2 + ((r/(r-1)) \operatorname{Im} z)^2 \leqslant 1 - 1/p\}$.

A phenomenon similar to the periodicity of the function γ arises in the study of bi-K-invariant functions defined on the group $G = SL(2, Q_p)$, with respect to a suitable compact subgroup K ([14]; see also [22]). Indeed, as in the case of $l^1_{\#}$, the commutative convolution algebra of bi-K-invariant L^1-functions on G contains an identity, and therefore has compact spectrum.

3. Principal and Complementary Series of Representations

For each complex number z, with $0 \leqslant \operatorname{Re} z \leqslant 1$, we define a representation π_z of $\mathbf{F}_r$ on the Hilbert space $L^2(\Omega, \nu)$ in the following way: for every $\xi \in L^2(\Omega)$,

$$(\pi_z(x)\,\xi)(\omega) = P^z(x, \omega)\,\xi(x^{-1}\omega).$$

The fact that π_z is a homomorphism follows from the cocycle identity. The spherical function ϕ_z is a matrix coeffiient of π_z:

$$\phi_z(x) = (\pi_z(x)\,\mathbf{1}, \mathbf{1}),$$

where $\mathbf{1}$ denotes the function identically one on Ω. It is easy to see that π_z is a unitary representation if and only if $\operatorname{Re} z = \frac{1}{2}$ (however, if $\operatorname{Re} z \neq \frac{1}{2}$, π_z extends to an isometric representation on $L^p(\Omega, \nu)$, for $p = (\operatorname{Re} z)^{-1}$). More generally, ϕ_z is positive definite whenever $\gamma(z)$ is real, i.e., if $\operatorname{Re} z = \frac{1}{2}$ or $\operatorname{Im} z = k\pi/\ln(2r-1)$ for some integer k. Indeed, for every z such that $\operatorname{Im} z = k\pi/\ln(2r-1)$, ϕ_z is positive definite as a consequence of Theorem 2, because both $(2r-1)^{-t|x|}$ and $(2r-1)^{-(1-t)|x|}$ are positive definite functions when $0 \leqslant t \leqslant 1$ [8, 15]. The fact that ϕ_z is positive definite if $\gamma(z)$ is real can also be proved directly: it suffices to notice that ϕ_z defines a positive functional on $l^1_{\#}$, and to show that the map $\mathscr{E}: l^1 \to l^1_{\#}$ preserves positivity. This approach was used in [6]. On the other hand, ϕ_z is not positive definite if $\gamma(z)$ is not real because a radial positive definite function is necessarily real.

We presently show how, for $-1 < \gamma(z) < 1$, the representation π_z can be unitarized. Let $\mathscr{K}_z$ be the linear subspace of $L^2(\Omega)$ generated by the constant function $\mathbf{1}$ under the action of π_z: in other words, $\mathscr{K}_z$ consists of the linear combinations of the function $\omega \to P^z(x, \omega)$. A Hilbert space norm on $\mathscr{K}_z$ can be defined, when ϕ_z is positive definite, by

$$\|\Sigma c_i P^z(x_j, \omega)\|_z^2 = \sum_{i,j} c_i \bar{c}_j \phi_z(x_i x_j^{-1}).$$

Then π_z is isometric with respect to this norm and extends to a unitary representation π'_z on the completion $\mathscr{H}_z$ of $\mathscr{K}_z$. We shall prove that, for $z \neq k\pi i/\ln(2r-1)$, $k \in \mathbb{Z}$, the vector $\mathbf{1}$ is cyclic for π_z, in other word that $\mathscr{K}_z$ is dense in $L^2(\Omega)$. In particular, for $\operatorname{Re} z = \frac{1}{2}$, this implies $\mathscr{H}_z = L^2(\Omega)$ and $\pi_z = \pi'_z$, because π_z is unitary.

PROPOSITION 1. *If $z \neq k\pi i/\ln(2r-1)$, $k \in \mathbb{Z}$, then the function $\mathbf{1}$ is a cyclic vector for π_z.*

Proof. As in Section 2, let $E(x) \subset \Omega$ be the set of infinite words whose first $|x|$ letters are the same as the letters of x. Since $E(x) \cap E(y) = E(y)$ if $E(x) \cap E(y) \neq \varnothing$ and $|x| < |y|$, linear combinations of the characteristic functions χ_x of the sets $E(x)$ are dense in $L^2(\Omega)$. Therefore it suffices to show that, for every $x \in \mathbf{F}_r$, χ_x belongs to the linear space $\mathscr{K}_z$ of linear combinations of functions of the type $\pi_z(y)\,\mathbf{1}$, $y \in \mathbf{F}_r$. We shall prove this fact by induction on the length of x. Suppose that $\chi_y \in \mathscr{K}_z$ for all $|y| < n$, and let $|x| = n$, $x = x_1 \cdots x_n$. For $j = 1,\ldots, n$ let $y_j = x_1 \cdots x_j$. Write $B_j = E(y_j)$. Then, by (3), the function $P^z(x, \omega)$ (regarded as a function on Ω) is constant on the sets B_n, $\Omega - B_1$ and $B_j - B_{j+1}$, for $j = 1,\ldots, n-1$. Furthermore, if (and only if) $z \neq k\pi i/\ln(2r-1)$, $P^z(x, \omega)$ takes on different values on these sets. In particular, one has $P^z(x, \omega) = (2r-1)^{nz}$ if and only if $\omega \in B_n = E(x)$. Therefore χ_x is a linear combination of $P^z(x, \omega)$ and of characteristic functions of the set B_j, $j = 1,\ldots, n-1$, and $\Omega - B_1$. Now these sets are finite unions of sets of the type $E(y)$, with $|y| < n$, and the proof is complete.

The previous proposition, applied to the representations π_z such that $-1 < \gamma(z) < 1$, yields the fact that such representations are unitarizable:

THEOREM 3. *Let $z \neq k\pi i/\ln(2r-1)$, $k \in \mathbb{Z}$, and suppose that ϕ_z is positive definite. There exists a densely defined injective linear operator J_z mapping $\mathscr{H}_z$ into $L^2(\Omega)$ such that J_z^{-1} is densely defined and $\pi'_z = J_z^{-1}\pi_z J_z$ on the domain of J_z.*

Proof. It is enough to define J_z as the identity map on $\mathscr{K}_z$.

In analogy with the current terminology for semisimple Lie groups, the family of unitary representations π_z with $\operatorname{Re} z = \frac{1}{2}$ will be called the *principal series* of representations of $\mathbf{F}_r$, while the unitary representations π'_z with

$\operatorname{Re} z \neq 0, 1, \frac{1}{2}$ and $\operatorname{Im} z = k\pi/\ln(2r-1)$, $k \in \mathbb{Z}$, will be called the *complementary series*.

We shall use an alternative realization of the space $\mathscr{H}_z$, as a Hilbert space of functions on $\mathbf{F}_r$. Indeed, for every square-integrable function ξ on Ω, we define its *Poisson transform*

$$P_z \xi(x) = (\pi_z(x)\, \xi, \mathbf{1}) = \int_\Omega P^z(x, \omega)\, \xi(x^{-1}\omega)\, d\nu(\omega).$$

The transform P_z maps the linear space $\mathscr{K}_z$ to the space of linear combinations of left translates of ϕ_z. We shall prove that, if $z \neq 1 + k\pi i/\ln(2r-1)$, $k \in \mathbb{Z}$, then P_z is injective. When P_z is injective and ϕ_z is positive definite, $\mathscr{H}_z$ is isomorphic with the completion $\tilde{\mathscr{H}}_z$ of the space of linear combinations of left translates $\lambda(x)\,\phi_z$ in the norm $\|\sum_i c_i \lambda(x_i)\,\phi_z\|_z^2 = \sum_{i,j} c_i \bar{c}_j \phi_z(x_i x_j^{-1})$, and P_z extends to an isometry of $\mathscr{H}_z$ onto $\tilde{\mathscr{H}}_z$ which intertwines π_z' with the representation by left translation on $\tilde{\mathscr{H}}_z$: this isometry will be denoted again by P_z.

PROPOSITION 2. *Let $0 \leqslant \operatorname{Re} z \leqslant 1$. Then the Poisson transform P_z is injective on $L^2(\Omega)$ if and only if $z \neq 1 + k\pi i/\ln(2r-1)$, $k \in \mathbb{Z}$. Therefore, if $-1 < \gamma(z) < 1$, P_z is injective on $\mathscr{H}_z$.*

Proof. It is easy to show that $\pi_z(x)^* = \pi_{1-\bar{z}}(x^{-1})$. Thus, if $\xi \in L^2(\Omega)$ and $P_z \xi = 0$, it follows, for every $x \in \mathbf{F}_r$:

$$(\xi, \pi_{1-\bar{z}}(x^{-1})\, \mathbf{1}) = (\pi_z(x)\, \xi, \mathbf{1}) = P_z \xi(x) = 0.$$

By Proposition 1, this implies that $\xi = 0$, unless $z = 1 + k\pi i/\ln(2r-1)$. Conversely, suppose $z = 1 + k\pi i/\ln(2r-1)$. If k is even, then $\gamma(z) = 1$, ϕ_z is identically one and P_z maps the linear subspace $\mathscr{K}_z$ generated by $\{\pi_z(x)\,\mathbf{1}, x \in \mathbf{F}_r\}$ onto the constant functions on $\mathbf{F}_r$; on the other hand, if k is odd, then $\gamma(z) = -1$, ϕ_z is the nontrivial radial character χ of $\mathbf{F}_r$ defined by $\chi(x) = (-1)^{|x|}$, and P_z is not injective on $\mathscr{K}_z$ because χ is invariant under translation by words of even length.

This argument shows that P_z is injective on $\mathscr{K}_z$ if and only if $z \neq 1 + k\pi i/\ln(2r-1)$. Therefore, if $-1 < \gamma(z) < 1$, P_z is injective on $\mathscr{K}_z$ and ϕ_z is positive definite: thus P_z extends to an isometry of $\mathscr{H}_z$ onto $\tilde{\mathscr{H}}_z$, and is injective on the whole of $\mathscr{H}_z$.

Remark 3. The Poisson transform P_z maps $L^2(\Omega)$ into a space of functions on $\mathbf{F}_r$ that are eigenvectors of the convolution operator on the right by μ_1, with eigenvalue $\gamma(z)$: this space is obviously stable under left translations. As observed in the Introduction, the convolution operator by μ_1 on the free group plays the role of the Laplace–Beltrami operator on semisimple Lie groups. Thus the realization of the representation π_z' given above corresponds with the construction of representations of a semisimple Lie group on eigenspaces of the Laplace–Beltrami operator [12, 16].

It is important to notice that, if $-1 < \gamma(z) = \gamma(w) < 1$, then the representations π_z' and π_w' are equivalent. In fact, under this assumption, the Poisson transform P_z interwines π_z' with the representation acting by left translations on $\tilde{\mathscr{H}}_z$, having cyclic vector $\phi_z = P_z \mathbf{1}$ (Proposition 2 and remarks preceding it). Since $\gamma(z) = \gamma(w)$, then $\phi_z = \phi_w$, and the representations π_z' and π_w' are both equivalent to the representation by left translations on the same Hilbert space of functions on $\mathbf{F}_r$.

Let us consider, in particular, the representations of the principal series. Since the function $t \to \gamma(1/2 + it)$ is an even periodic function on R, with period $2\pi/\ln(2r-1)$, we can choose as a set of representatives for the principal series the subset $\{\pi_z, z \in J\}$ where $J = \{1/2 + it, 0 \leqslant t \leqslant \pi/\ln(2r-1)\}$. On the other hand, it is easy to see that a set of representatives for the complementary series consists of the family $\{\pi_z', z \in J_1 \cup J_2\}$, with $J_1 = \{z\colon 1/2 < \operatorname{Re} z < 1, \operatorname{Im} z = 0\}$, $J_2 = \{z\colon 1/2 < \operatorname{Re} z < 1, \operatorname{Im} z = \pi/\ln(2r-1)\}$. As observed in Remark 2, these sets of parameters are mapped under γ onto the following subsets of the ellipse $(\operatorname{Re} z)^2 + ((r/(r-1)) \operatorname{Im} z)^2 \leqslant 1$: the principal series corresponds to the segment connecting the two foci, while the complementary series corresponds to the two real segments $\gamma(J_1) = ((2r-1)^{1/2}/r, 1)$ and $\gamma(J_2) = (-1, -(2r-1)^{1/2}/r)$.

In the remainder of this section we shall prove that the representations π_z' of the principal and complementary series are irreducible.

We need first a few preliminary results. Let $x, y \in \mathbf{F}_r$ and let $q_n(j; x, y)$ be the probability that $|xwy| = n + |x| + |y| - 2j$ when w is a random word of length $n > j$.

LEMMA 3. *For every* $x, y \in \mathbf{F}_r$, $\lim_n q_n(j; x, y) = p_j(x, y)$ *exists and depends only on* j, $|x|$ *and* $|y|$. *Moreover*

$$q_n(j; x, y) - p_j(x, y) = O((2r-1)^{-n}).$$

Proof. Let $A_n(j; x, y) = \{w\colon |w| = n, |xwy| = n + |x| + |y| - 2j\}$: then $q_n(j; x, y) = |A_n(j; x, y)|/(2r)(2r-1)^{n-1}$. Let $|x| = k$ and $|y| = m$. The estimate for $|A_n(j; x, y)|$ is slightly different in the cases: $j < \min(k, m)$, $k \leqslant j < m$, $k \leqslant m \leqslant j < m + k$, $j = m + k$. We shall restrict attention to the case $j < \min(k, m)$; the other cases are treated with the same methods. We observe that $A_n(j; x, y)$ is the union of $j+1$ disjoint subsets: indeed, if $x = x_1 \cdots x_k$ and $y = y_1 \cdots y_m$, then $A_n(j; x, y) = \bigcup_{t=1}^{j+1} B_t$, where B_t is the subset of all words w which have $t-1$ cancellations on the right with y and $j - (t-1)$ cancellations on the left with x. Observe also that B_t is in one-to-one correspondence with the set $A_{n-j}(0; x_{k-j+t}^{-1}, y_t^{-1})$. Now, if $|u| = |v| = 1$, one has

$$|A_{2h}(0;u,v)| = 1 + (2r-2)\sum_{i=1}^{h}(2r-1)^{2i-1} \qquad (\text{if } u \neq v^{-1}),$$

$$|A_{2h}(0;u,u^{-1})| = (2r-2)\sum_{i=1}^{h}(2r-1)^{2i-1},$$

$$|A_{2h+1}(0;u,v)| = (2r-2)\sum_{i=0}^{h}(2r-1)^{2i} \qquad (\text{if } u \neq v),$$

$$|A_{2h+1}(0;u,u)| = 1 + (2r-2)\sum_{i=0}^{h}(2r-1)^{2i}.$$

These formulas are proved by an induction argument on the index h, as a consequence of the equality $A_{h+1}(0;u,v) = \bigcup\{A_h(0;u,v')\,v' : |v'| = 1, v' \neq v^{-1}\}$. This shows that, if $h = [(n-j)/2]$, then

$$(2r-2)\sum_{i=1}^{h}(2r-1)^{2i-1} \leqslant |B_t| \leqslant \sum_{i=0}^{h}(2r-1)^{2i+1};$$

therefore

$$(j+1)(2r-2)\sum_{i=1}^{h}(2r-1)^{2i-1} \leqslant |A_n(j;x,y)| \leqslant (j+1)\sum_{i=0}^{h}(2r-1)^{2i+1}.$$

As a consequence, for each j, the limit

$$p_j(x,y) = \lim_n q_n(j;x,y) = \lim_n |A_n(j;x,y)|/2r(2r-1)^{n-1}$$

exists and depends only on $|x|$ and $|y|$; moreover, $q_n(j;x,y) - p_j(x,y) = O((2r-1)^{-n})$.

Let $\pi = \pi_z'$ be a representation of the principal or complementary series, and $\phi = \phi_z$ be the corresponding spherical function. From now on the proof of the irreducibility of π_z' splits into two different cases. On the one hand, we have the representations π_z with $-1 < \gamma(z) \leqslant -(2r-1)^{1/2}/r$ or $(2r-1)^{1/2}/r \leqslant \gamma(z) < 1$ (i.e., the representations of the complementary series together with the boundary points of the principal series), on the other hand, we have the remaining representations of the principal series. The difference between these two cases lies in the asymptotic behaviour of the corresponding spherical functions: indeed, Theorem 2 implies that, in the former case, the product $\phi_z(x)(2r-1)^{|x|/2}$ is unbounded, while in the latter case the same product oscillates between -1 and $+1$. This is reflected in the statement of the following lemma.

LEMMA 4. *Let $\pi = \pi_z$ be a unitary representation of the principal or complementary series, acting on the Hilbert space $\mathscr{H} = \mathscr{H}_z$. Let $\phi = \phi_z$ be the* *corresponding spherical function. Denote by $\phi(n)$ the value of ϕ on the words of length n. Then, if $z \neq 1/2 + k\pi i/\ln(2r-1)$, $k \in \mathbb{Z}$,*

$$(\pi(\mu_n)\,\pi(x)\,\mathbf{1}, \pi(\mu_n)\,\pi(y)\,\mathbf{1})_{\mathscr{H}} = O((2r-1)^{\beta n}), \tag{4}$$

where

$$\beta = |2\,\mathrm{Re}\,z - 1| - 1.$$

Furthermore:

(i) *if* $\mathrm{Im}\,z = k\pi/\ln(2r-1)$, $k \in \mathbb{Z}$, *then*

$$\lim_n \phi(n)^{-1}\,(\pi(\mu_n)\,\pi(x)\,\mathbf{1}, \pi(y)\,\mathbf{1})_{\mathscr{H}} = \phi(x)\,\phi(y)$$

and

$$\lim_n \phi(n)^{-2}\,(\pi(\mu_n)\,\pi(x)\,\mathbf{1}, \pi(\mu_n)\,\pi(y)\,\mathbf{1})_{\mathscr{H}} = \phi(x)\,\phi(y),$$

(ii) *if* $\mathrm{Re}\,z = 1/2$ *and* $\mathrm{Im}\,z \neq k\pi/\ln(2r-1)$, *there exists a subsequence* n_k *such that, for each x and y,*

$$\lim_k \phi(n_k)^{-1}\,(\pi(\mu_{n_k})\,\pi(x)\,\mathbf{1}, \pi(y)\,\mathbf{1}) = \phi(x)\,\phi(y).$$

Proof. Throughout this proof, we shall denote by (,) the inner product in $\mathscr{H}$. We write $\mu_n * \mu_n = \sum_{i=0}^{n} a_i(n)\,\mu_{2i}$, where

$$\begin{aligned} a_0(n) &= 1/2r(2r-1)^{n-1}, \qquad & a_n(n) &= (2r-1)/2r, \\ a_i(n) &= (r-1)/r(2r-1)^{n-i} & &\text{for } 0 < i < n. \end{aligned} \tag{5}$$

This formula can be proved by direct computation. Then

$$\begin{aligned} (\pi(\mu_n)\,\pi(x)\,\mathbf{1}, \pi(\mu_n)\,\pi(y)\,\mathbf{1}) &= (\pi(\mu_n * \mu_n)\,\pi(x)\,\mathbf{1}, \pi(y)\,\mathbf{1}) \\ &= \sum_{i=0}^{n} a_i(n)(\pi(\mu_{2i})\,\pi(x)\,\mathbf{1}, \pi(y)\,\mathbf{1}). \end{aligned}$$

Let $m = |x| + |y|$ and $s = [m/2] + 1$. For $2i > m$,

$$(\pi(\mu_{2i})\,\pi(x)\,\mathbf{1}, \pi(y)\,\mathbf{1}) = \sum_{j=0}^{m} q_{2i}(j, x, y)\,\phi(2i + m - 2j),$$

where the $q_{2i}(j, x, y)$ are as in Lemma 3. Since $|a_i(n)| \leqslant (2r-1)^s\,(2r-1)^{-n}$ for $i \leqslant s$, it suffices to show that

$$\sum_{i=s+1}^{n} a_i(n)(\pi(\mu_{2i})\,\pi(x)\,\mathbf{1}, \pi(y)\,\mathbf{1}) = O((2r-1)^{\beta n}).$$

We write the above sum as

$$\sum_{i=s+1}^{n} a_i(n) \sum_{j=0}^{m} q_{2i}(j; x, y)\, \phi(2i+m-2j)$$
$$= \sum_{i=s+1}^{n} a_i(n) \sum_{j=0}^{m} p_j(x, y)\, \phi(2i+m-2j)$$
$$+ \sum_{i=s+1}^{n} a_i(n) \sum_{j=0}^{m} [q_{2i}(j; x, y) - p_j(x, y)]\, \phi(2i+m-2j).$$

Now, by Lemma 3, $q_{2i}(j; x, y) - p_j(x, y) = O((2r-1)^{-2i})$; by Theorem 2, $\phi(2i+m-2j) = O((2r-1)^{\beta i})$; furthermore $a_i(n) \leqslant (2r-1)^{-n+i}$. Therefore the second sum in the right-hand side is $O((2r-1)^{\beta n})$. It remains to estimate the first sum:

$$\sum_{i=s+1}^{n} a_i(n) \sum_{j=0}^{m} p_j(x, y)\, \phi(2i+m-2j)$$
$$= \sum_{j=0}^{m} p_j(x, y) \sum_{i=s+1}^{n} a_i(n)\, \phi(2i+m-2j).$$

The above equality imply that we must only estimate

$$\sum_{i=s+1}^{n} a_i(n)\, \phi(2i+m-2j). \tag{6}$$

We can assume that $m-2j=2l$ is an even number. Indeed, by (2), the value of ϕ on words of odd length is a linear combination of the two values on the words of nearest even length. It follows from (5) that $a_{h-l}(n) = a_h(n+l)$. Therefore, the identity $\phi(n)^2 = \langle \mu_n * \mu_n, \phi \rangle = \sum_{h=0}^{n} a_h(n) \langle \mu_{2h}, \phi \rangle$ implies

$$\sum_{i=s+1}^{n} a_i(n)\, \phi(2i+m-2j) = \sum_{h=s+l+1}^{n+l} a_{h-l}(n)\, \phi(2h)$$
$$= \sum_{h=s+l+1}^{n+l} a_h(n+l)\, \phi(2h)$$
$$= \phi(n+l)^2 - \sum_{h=0}^{s+l} a_h(n+l)\, \phi(2h).$$

Since $\phi(n+l)^2$ and $a_h(n+l)$ are both $O((2r+1)^{\beta n})$, so is the sum (6).

To prove (i) we observe that

$$(\pi(\mu_n)\, \pi(x)\, \mathbf{1}, \pi(y)\, 1) = \sum_{j=0}^{m} q_n(j; x, y)\, \phi(n+m-2j),$$

where $m = |x| + |y|$. Since $q_n(j; x, y)$ converges to $p_j(x, y$ (Lemma 3) and $\phi(n + m - 2j)/\phi(n)$ converges under the hypotheses of (i) (Theorem 2), we obtain that $\phi(n)^{-1} (\pi(\mu_n) \pi(x) \mathbf{1}, \pi(y) \mathbf{1})$ converges for each x and y. One could at this point compute directly the limit $f(x, y)$ of $f_n(x, y) = \phi(n)^{-1} (\pi(\mu_n) \pi(x) \mathbf{1}, \pi(y) \mathbf{1})$. It is simpler, however, to notice that, for fixed x, $f(x, y)$ is a radial function of y, and that, for $|w| = n$, $f_n(x, e) \phi(n) = \mathscr{E}(\lambda(x) \phi)(w) = \phi(x) \phi(n)$, which implies $f(x, e) = \phi(x)$. On the other hand, if $\phi = \phi_z$, $\lambda(wx) \phi * \mu_1 = \lambda(wx)(\phi * \mu_1) = \gamma(z) \lambda(wx) \phi$. Therefore $f_n * \mu_1 = \gamma(z) f_n$ and $f * \mu_1 = \gamma(z) f$. Thus, by Theorem 1, for fixed x, f is a constant multiple of ϕ and $f(x, y) = \phi(x) \phi(y)$. To complete the proof of (i) write, as before,

$$\begin{aligned}
&\phi(n)^{-2} (\pi(\mu_n) \pi(x) \mathbf{1}, \pi(\mu_n) \pi(y) \mathbf{1}) \\
&\quad = \phi(n)^{-2} \sum_{h=0}^{n} a_h(n)(\pi(\mu_{2h}) \pi(x) \mathbf{1}, \pi(y) \mathbf{1}) \\
&\quad = \sum_{h=0}^{n} a_h(n) \phi(n)^{-2} \phi(2h) f_{2h}(x, y).
\end{aligned}$$

Fix $\varepsilon > 0$, and choose k so that, for all $h > k$, $|f_{2h}(x, y) - \phi(x) \phi(y)| < \varepsilon$. Recall that $\phi(n)^{-2} \sum_{h=0}^{n} a_h(n) \phi(2h) = 1$, and observe that $\phi(2h) \geqslant 0$ (by the hypothesis on z and Theorem 2). Therefore,

$$\begin{aligned}
&\limsup_n \left| \sum_{h=0}^{n} a_h(n) \phi(n)^{-2} \phi(2h) f_{2h}(x, y) - \phi(x) \phi(y) \right| \\
&\quad \leqslant \lim_n \sum_{h=0}^{k} a_h(n) \phi(n)^{-2} \phi(2h) |f_{2h}(x, y) - \phi(x) \phi(y)| \\
&\qquad + \limsup_n \sum_{h=k+1}^{n} a_h(n) \phi(n)^{-2} \phi(2h) |f_{2h}(x, y) - \phi(x) \phi(y)| < \varepsilon.
\end{aligned}$$

Indeed, for fixed k, the hypothesis on z and Theorem 2 imply

$$\sum_{h=0}^{k} a_h(n) \phi(n)^{-2} \phi(2h) \leqslant k(2k + 1)(2r - 1)^{-n} \phi(n)^{-2} \to 0.$$

Finally, to prove (ii), let n_k be a sequence of positive integers such that the limit $\lim_k \phi(n_k)(2r - 1)^{n_k/2}$ exists and is nonzero: such a sequence exists because of the hypothesis on z and Theorem 2. As in the proof of (i), we let

$$(\pi(\mu_{n_k}) \pi(x) \mathbf{1}, \pi(y) \mathbf{1}) = \sum_{j=0}^{m} q_{n_k}(j; x, y) \phi(n_k + m - 2j),$$

where $m = |x| + |y|$. Again $q_{n_k}(j; x, y)$ converges to $p_j(x, y)$ (Lemma 3), and we have only to prove that $\phi(n_k + m - 2j)/\phi(n_k)$ converges as $k \to \infty$.

By Theorem 2, letting $c = c_{\frac{1}{2}+it}$, one can write

$$\phi(n_k + m - 2j)/\phi(n_k)$$
$$= (2r-1)^{j-m/2}\,\mathrm{Re}(c(2r-1)^{-it(n_k+m-2j)})/\mathrm{Re}(c(2r-1)^{-itn_k}).$$

Now $\mathrm{Re}(c(2r-1)^{-itn_k}) = \phi(n_k)(2r-1)^{n_k/2}$ converges by the choice of n_k, and the proof is complete.

Let us consider the functional $\xi \to I(\xi) = \int_\Omega \xi\, dv$, defined on the dense subspace $\mathscr{K}_z$ of $\mathscr{H}_z$. Since the functional I has norm one, it extends to the whole of $\mathscr{H}_z$. If $\mathrm{Re}\, z = 1/2$, $\mathscr{H}_z$ coincides with $L^2(\Omega)$ and $I(\xi)$ is simply the integral of ξ, for every $\xi \in L^2(\Omega)$. We shall now prove that $I(\xi)$ can be approximated through the action of $\mathbf{F}_r$ on Ω.

THEOREM 4. *Let z be as in Lemma* 4(i), *and define $T_n = \phi_z(n)^{-1}\pi_z(\mu_n)$; then, for each $\xi \in \mathscr{H} = \mathscr{H}_z$,* $\lim T_n\xi = I(\xi)\,\mathbf{1}$, *in the norm of $\mathscr{H}$. On the other hand, let z and n_k be as in Lemma* 4(ii): *then* $\lim_k T_{n_k}(\xi) = I(\xi)\,\mathbf{1}$ *in the weak topology of $\mathscr{H} = L^2(\Omega)$, for each $\xi = \sum c_j\pi_z(x_j)\,\mathbf{1}$.*

Proof. To prove the first part of the statement, let $\phi = \phi_z$, $\pi = \pi_z$ and $\xi = \sum c_j\pi(x_j)\,\mathbf{1}$; then, by Lemma 4(i)

$$\|T_n\xi\|^2 = \phi(n)^{-2}\sum_{i,j} c_i\bar{c}_j(\pi(\mu_n)\,\pi(x_i)\,\mathbf{1}, \pi(\mu_n)\,\pi(x_j)\,\mathbf{1})$$

converges to $\sum c_j\bar{c}_j\phi(x_i)\,\phi(x_j) = |\sum c_i\phi(x_i)|^2 = |I(\xi)|^2$. By the same token, $(T_n\xi, \eta)$ converges to $I(\xi)\,I(\eta)$ if $\eta = \sum d_j\pi(y_j)\,\mathbf{1}$. Therefore $\lim_n \|T_n\xi - I(\xi)\,\mathbf{1}\| = 0$. Since $\limsup \|T_n\xi\| \leqslant |I(\xi)| \leqslant \|\xi\|$, we have $\limsup_n \|T_n\| \leqslant 1$, hence $\lim_n \|T_n\xi - I(\xi)\,\mathbf{1}\| = 0$ for every $\xi \in \mathscr{H}$.

For the second part, observe that, for each fixed $\xi = \sum c_j\pi(x_j)\,\mathbf{1}$, (4) yields that $\|T_{n_k}\xi\|$ is bounded. In addition, by Lemma 4(ii), $(T_{n_k}\xi, \eta)$ converges to $I(\xi)\,I(\eta)$ whenever $\eta = \sum d_j\pi(y_j)\,\mathbf{1}$. It follows that $T_{n_k}\xi$ converges weakly to $I(\xi)\,\mathbf{1}$.

We are now ready to prove the irreducibility of the representations.

THEOREM 5. *Suppose that $-1 < \gamma(z) < 1$, or equivalently, that $\pi = \pi'_z$ is either a principal or a complementary series representation. Then π is irreducible.*

Proof. Let Q be a projection on $\mathscr{H} = \mathscr{H}_z$ such that $Q\pi(x) = \pi(x)\,Q$ for every $x \in \mathbf{F}_r$, and let $\xi = Q\,\mathbf{1}$. Then $\pi(\mu_n)\,\xi = Q\pi(\mu_n)\,\mathbf{1} = \phi(n)\,Q\,\mathbf{1} = \phi(n)\,\xi$. If z satisfies the hypothesis of Lemma 4(ii), denote by n_k the subsequence whose existence is asserted in Lemma 4(ii); on the other hand, if z satisfies the hypothesis of Lemma 4(i), let $n_k = k$. Then $T_{n_k}\xi = \xi$; therefore, denoting by P_z the Poisson transform (see the remarks preceding Proposition 2),

$$P_z\xi(x^{-1}) = (T_{n_k}\xi, \pi(x)\,\mathbf{1}) = (\xi, T_{n_k}\pi(x)\,\mathbf{1}) = (\xi, 1)\,\phi_z(x),$$

because, by Theorem 4, $T_{n_k}\pi(x)\,\mathbf{1}$ converges to $\phi_z(x)\,\mathbf{1}$, at least in the weak topology of $\mathscr{H}$. Since P_z is injective when $-1 < \gamma(z) < 1$, this means that ξ is a multiple of $\mathbf{1}$. Thus either $Q\,\mathbf{1} = 0$ or $Q\,\mathbf{1} = \mathbf{1}$.

In the first case $Q\pi(x)\,\mathbf{1} = \pi(x)\,Q\,\mathbf{1} = 0$ for every $x \in \mathbf{F}_r$, thus $Q = 0$, because $\mathbf{1}$ is cyclic. In the latter case $(\mathbb{I} - Q)\,\mathbf{1} = 0$ and Q is the identity. Therefore only trivial projections commute with every $\pi(x)$, and π is irreducible.

Remark 4. The proof of Theorem 5 shows that, if $-1 < \gamma(z) < 1$ and $u \in \tilde{\mathscr{H}}_z$ (the completion of linear combinations of translates of ϕ_z), then $\mu_1 * u = \gamma(z)\,u$ implies that $u = \phi_z$. Indeed, if $\mu_1 * u = \gamma(z)\,u$ and $u = P_z\xi$, then $T_{n_k}\xi = \xi$, and the argument used in the proof of the theorem yields that $\xi = \mathbf{1}$.

Remark 5. Since $\pi_{1-\bar{z}}(x) = \pi_z(x^{-1})^*$, the representations π_z and $\pi_{1-\bar{z}}$ are dual representations: if $\mathscr{M} \subset L^2(\Omega)$ is an invariant subspace for the former, then $\mathscr{M}^\perp$ is invariant for the latter, and conversely. For instance, let us consider the representations π_z such that $\gamma(z) = \pm 1$, i.e., $z = \sigma + it$, with $\sigma = 0$ or 1 and $t = k\pi/\ln(2r-1)$, $k \in \mathbb{Z}$. If $\sigma = 0$, then the one-dimensional subspace $\mathscr{M}$ of constant functions on Ω is invariant under π_z, which acts on it as the trivial representation of $\mathbf{F}_r$ for even values of k, and as the nontrivial radial character $\chi(x) = (-1)^{|x|}$ for odd values of k. On the other hand, if $\sigma = 1$, then $\mathscr{M}^\perp$ is an invariant subspace of codimension one for π_z.

We have seen that π'_z and π'_{1-z} are unitarily equivalent if $-1 < \gamma(z) < 1$. Conversely, π'_z and π'_{1-z} are not equivalent if $\gamma(z) = 1$. Indeed, for $z = k\pi i/\ln(2r-1)$, π'_z is an irreducible one-dimensional representation: in fact, $\mathscr{H}_z$ is generated by the vector $\mathbf{1}$, and π'_z is the trivial character for even k, and the nontrivial radial character for odd k. On the other hand, if $z = 1 + k\pi i/\ln(2r-1)$, then π'_z is a reducible infinite dimensional representation: it admits an invariant subspace of codimension one, then annihilator of the vector $\mathbf{1}$.

4. The Plancherel Formula

In this section we shall determine the spectrum of the convolution operator by μ_1 and the Plancherel measure. We shall first explicitly exhibit the resolvant of μ_1. If $z \in \mathbb{C}$, it is easy to see that, for $x \neq e$, the function $h_z(x) = (2r-1)^{-z|x|}$ satisfies the identity

$$(\mu_1 - \gamma(z)\,\delta_e) * h_z(x) = 0.$$

On the other hand,

$$(\mu_1 - \gamma(z)\,\delta_e) * h_z(e) = (2r)^{-1}\,((2r-1)^{-z} - (2r-1)^z);$$

thus, if $z \neq k\pi i/\ln(2r-1)$, $k \in \mathbb{Z}$, the function

$$k_z(x) = 2r((2r-1)^{-z} - (2r-1)^z)^{-1}\, h_z(x)$$

satisfies the identity

$$(\mu_1 - \gamma(z)\,\delta_e) * k_z = \delta_e.$$

Let $k_z^{(n)}$ be the truncation of k_z to words shorter than n. If $\operatorname{Re} z = 1/2$, then $k_z \notin l^2(\mathbf{F}_r)$, but the sequence $(\mu_1 - \gamma(z)\,\delta_e) * k_z^{(n)}$ is bounded in $l^2(\mathbf{F}_r)$. It follows that the spectrum $\sigma(\mu_1)$ of μ_1 contains the set $\{\gamma(z)\colon \operatorname{Re} z = 1/2\}$. On the other hand, the norm of μ_1 as a convolution operator on $l^2(\mathbf{F}_r)$ is $(2r-1)^{1/2}/r$ [18]. Therefore $\sigma(\mu_1) = \{\gamma(z)\colon \operatorname{Re} z = 1/2\} = [-(2r-1)^{1/2}/r, (2r-1)^{1/2}/r]$.

On the spectrum $\sigma(\mu_1)$ there exists a positive measure ("Plancherel measure") which yields an inversion formula for functions in $l^1_\#$:

$$f(e) = \int_{\sigma(\mu_1)} \langle f, \phi_z \rangle \, dq(\tau),$$

where $\tau = \gamma(z) \in \sigma(\mu_1)$. This follows from the fact that the map $f \to f(e)$, defined on $l^1_\#$, extends, via the Gelfand transform, to a positive linear functional on the involutive algebra $\mathscr{C}(\sigma(\mu_1))$ of all continuous functions on $\sigma(\mu_1)$. Since $f(e) = (\mathscr{E}f)(e)$, the inversion formula is also valid for nonradial functions $f \in l^1(\mathbf{F}_r)$. For our purposes, it is more convenient to define a *Plancherel measure* m on the line $\operatorname{Re} z = 1/2$ as $m = q \circ \gamma$. Since γ is periodic, it is enough to restrict m to an appropriate segment of this line, say (as in Remark 3) the segment $J = \{1/2 + it\colon 0 \leqslant t \leqslant \pi/\ln(2r-1)\}$. Denoting again by m the measure on J thus obtained, the inversion formula reads

$$f(e) = \int_J \langle f, \phi_z \rangle \, dm(z) \tag{7}$$

for every $f \in l^1(\mathbf{F}_r)$.

We shall prove now, for the measure m, an analogue of Harish Chandra's theorem relating the Plancherel measure of a semisimple Lie group to the coefficients appearing in the asymptotic expansion of spherical functions (see Theorem 9.2.1.5 of [25]).

This result could also be obtained using the explicit computations of the Plancherel measure on $\sigma(\mu_1)$ given in [4, 23]; our direct approach is simpler.

THEOREM 6. *Let* $\phi_z = c_z h_z + c_{1-z} h_{1-z}$, *as in Theorem* 2. *For* $0 < t \leqslant \pi/\ln(2r-1)$, *let* $c(1/2 + it) = c_{1/2+it} = \overline{c_{1/2-it}}$: *then* $dm(1/2 + it) = ((2r-1)/4r)\,|c(1/2 + it)|^{-2}\, dt$.

Proof. Theorem 1 and the expression (3) for the Poisson kernel yield the following identity, for $|x| = k$:

$$\begin{aligned}\phi_z(x) &= (2r)^{-1}(2r-1)((2r-1)^{(z-1)k} + (2r-1)^{-zk}) \\ &\quad + r^{-1}(r-1)(2r-1)^{(z-1)k} \sum_{j=1}^{k-1} (2r-1)^{(1-2z)j}. \end{aligned} \tag{8}$$

This formula can also be verified directly from the expansion of ϕ_z given in Theorem 2, or by induction from the Corollary to Lemma 2. On the other hand, by (7),

$$\delta_e(x) = \int_J \phi_z(x)\, dm(z). \tag{9}$$

For $n \in \mathbb{Z}$, let

$$\hat{m}(n) = \int_J (2r-1)^{-int}\, dm(1/2 + it).$$

Then one has $\hat{m}(0) = 1$, and, by (8),

$$\hat{m}(1) + \hat{m}(-1) = 0. \tag{10}$$

Furthermore, for each integer n, (8) and (9) imply

$$\hat{m}(n) + \hat{m}(-n) = -(2r-2)/(2r-1) \sum_{j=1}^{n-1} \hat{m}(2j-n).$$

Observe that $\hat{m}$ is real valued, because $\gamma(1/2 + it) = \gamma(1/2 - it)$ and $m = q \circ \gamma$. Furthermore, m is positive, thus $\hat{m}(n) = \hat{m}(-n)$. Hence

$$\hat{m}(n) = -(r-1)/(2r-1) \sum_{j=1}^{n-1} \hat{m}(2j-n). \tag{11}$$

Therefore, by (10), $\hat{m}(2k+1) = 0$ for every $k \in \mathbb{Z}$; on the other hand, if $k > 1$,

$$\begin{aligned}\hat{m}(2k) - \hat{m}(2k-2) &= -(r-1)/(2r-1)(\hat{m}(2k-2) + \hat{m}(-2k+2)) \\ &= -2(r-1)/(2r-1)\, \hat{m}(2k-2),\end{aligned}$$

whence $\hat{m}(2k) = (2r-1)^{-1}\, \hat{m}(2k-2)$. Since, by (11), $\hat{m}(2) = -(r-1)/(2r-1)$, it follows

$$\hat{m}(2k) = -(r-1)(2r-1)^{-k} \qquad \text{if} \quad k \geqslant 1.$$

For $0 < t \leqslant \pi/\ln(2r-1)$, write $dm(t) = u(t)\,dt$: one obtains

$$\begin{aligned} u(t) &= \sum_{k=-\infty}^{\infty} \hat{m}(k)(2r-1)^{ikt} \\ &= 1-(r-1)\left[\sum_{k=1}^{\infty}(2r-1)^{k(2it-1)} + \sum_{k=1}^{\infty}(2r-1)^{k(-2it-1)}\right] \\ &= 1+2(r-1)\operatorname{Re}[(2r-1)^{2it-1}((2r-1)^{2it-1}-1)^{-1}] \\ &= \operatorname{Re}[((2r-1)^{2it}-1)((2r-1)^{2it-1}-1)^{-1}]. \end{aligned}$$

On the other hand, Theorem 2 implies

$$c(1/2+it) = (2r)^{-1}((2r-1)^{1-2it}-1)((2r-1)^{-2it}-1)^{-1}.$$

An elementary verification now shows that

$$u(t) = (2r-1)(4r)^{-1}\,|c(1/2+it)|^{-2}.$$

We shall now write the Plancherel theorem in a form suitable for the proof of the following Proposition 3. For every $t \in \mathbb{R}$ and every finitely supported function f on $\mathbf{F}_r$, we define an element $f_t \in L^\infty(\Omega)$ by

$$f_t(\omega) = \sum_{x \in \mathbf{F}_r} f(x)\,P^{1/2+it}(x,\omega) = \sum f(x)\,\pi_{1/2+it}(x)\,\mathbf{1}.$$

From now on, we shall write the Plancherel measure as $dm(t)$ instead of $dm(\frac{1}{2}+it)$.

THEOREM 7. *Let f, g be finitely supported functions.*

(i) $f * g^*(x) = \int_J (\pi_{1/2+it}(x^{-1})f_t, g_t)_{L^2(\Omega)}\,dm(t)$,

(ii) $\|f\|_2^2 = \int_J \|f_t\|^2_{L^2(\Omega)}\,dm(t)$.

Proof. Observe that

$$\begin{aligned} (\pi_{1/2+it}(x^{-1})f_t, g_t) &= \sum_{y,w} f(y)\,\overline{g(w)}(\pi_{1/2+it}(x^{-1}y)\,\mathbf{1}, \pi_{1/2+it}(w)\,\mathbf{1}) \\ &= \sum f(y)\,\overline{g(w)}\,\phi_{1/2+it}(w^{-1}x^{-1}y). \end{aligned}$$

Therefore, by (9), the right-hand side of (i) equals

$$\sum f(y)\,\overline{g(w)}\int_J \phi_{1/2+it}(w^{-1}x^{-1}y)\,dm(t) = f * g^*(x).$$

Part (ii) follows from (i).

It follows from Theorem 7 that the map $f \to f_t$ extends to an isometry of $l^2(\mathbf{F}_r)$ to the space $L^2(J, m, L^2(\Omega))$ of square-integrable functions on J with values in $L^2(\Omega)$. In other terms,

$$l^2(\mathbf{F}_r) \simeq \int_J^{\oplus} L^2(\Omega)\, dm(t)$$

and

$$\lambda \simeq \int_J^{\oplus} \pi_{1/2+it}\, dm(t).$$

We now prove the analogue of Herz's principe de majoration [7, 17], relative to the representation $\pi_{1/2}$.

PROPOSITION 3. *For every $f, g \in l^2(\mathbf{F}_r)$, there exist $\xi, \eta \in L^2(\Omega)$ such that $|f * g^*(x)| \leqslant (\pi_{1/2}(x)\, \xi, \eta)$ and $\|f\|_2 \, \|g\|_2 = \|\xi\|_{L^2} \|\eta\|_{L^2}$.*

Proof. Let $\xi(\omega) = [\int_J |f_t(\omega)|^2\, dm(t)]^{1/2}$ and $\eta(\omega) = [\int_J |g_t(\omega)|^2\, dm(t)]^{1/2}$. By the Plancherel formula, $\|\xi\|_{L^2} = \|f\|_2$ and $\|\eta\|_{L^2} = \|g\|_2$; furthermore, by Schwarz's inequality

$$\begin{aligned}
|f * g^*(x)| &\leqslant \int_J |(\pi_{1/2+it}(x) f_t, g_t)|\, dm(t) \\
&= \int_\Omega P^{1/2}(x, \omega) \int_J |f_t(x^{-1}\omega)\, g_t(\omega)|\, dm(t)\, d\nu(\omega) \\
&\leqslant \int_\Omega P^{1/2}(x, \omega)\, \xi(x^{-1}\omega)\, \eta(\omega)\, d\nu(\omega) = (\pi_{1/2}(x)\, \xi, \eta).
\end{aligned}$$

COROLLARY. *For every positive function h with finite support, $\|\pi_{1/2}(h)\| = \|\lambda(h)\|$.*

Proof. Observe that $\lambda(h) = \sup\{|\langle h, f * g^*\rangle| : \|f\|_2, \|g\|_2 \leqslant 1\}$, and, by the proposition,

$$\begin{aligned}
|\langle h, f * g^*\rangle| &\leqslant \langle h, (\pi_{1/2}(x)\, \xi, \eta)\rangle \leqslant (\pi_{1/2}(h)\, \xi, \eta) \\
&\leqslant \|\pi_{1/2}(h)\| \, \|f\|_2 \, \|g\|_2.
\end{aligned}$$

The converse inequality is immediate, since π is weakly contained in λ.

Remark 6. The principe de majoration is used in [17] to prove that the coefficients of the regular representation of a semisimple Lie group G with finite center satisfy the inequality

$$\int_K \int_K |u(k_1 x k_2)|^2\, dk_1\, dk_2 \leqslant \phi^2(x)\, \|u\|^2_{A(G)}, \tag{12}$$

where K is a maximal compact subgroup, $A(G)$ is the Fourier algebra of G as defined in [11], and $\phi_{1/2}(x)$ is the Legendre function (i.e., the spherical function associated to the quasi-regular representation). Denoting by A the Fourier algebra of the free group, the analogue of (12), in our context, would be the inequality

$$\mathscr{E}(|u|^2)(x) \leqslant \phi_{1/2}^2(x) \, \|u\|_A^2 \tag{13}$$

for coefficients of the regular representation. Let $n = |x|$; then, if $\phi_{1/2}^2(x) = (1 + ((r-1)/r)\, n)^2 \, (2r-1)^{-n}$ is replaced by $((2r-1)/2r)(1+n)^2 \, (2r-1)^{-n}$, inequality (13) can be deduced from Lemma 1.4 of [15]. The results of [1], which improve earlier work of Leinert [20] and Bozejko [2], show that (13) is true for $|x| = 1$. We believe that (13) is true in general, but we have not been able to provide a proof.

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