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ALGORITHMS FOR CLASSIC DUAL TRIGONOMETRIC EQUATIONS†

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Abstract—The singular integral solutions to certain classic dual trigonometric equations provided by the formulas of Tranter and Bablojan are reduced to algorithms. A preliminary Fourier analysis is made of the data, and computational rules are derived by the systematic reduction of the singular integrals for each ordinary Fourier component of the data. Extensive numerical testing provides evidence for the correctness of both the original solutions and the resulting algorithms. The listing of programs in ANSI FORTRAN to implement the algorithms is appended.

1. BACKGROUND

We develop algorithms for the solution of the following classic dual trigonometric equations:

$$\sum_{n=0}^{\infty} \frac{A_n}{n+\frac{1}{2}} \cos(n+\tfrac{1}{2})x = f(x), \qquad 0<x<c \tag{1.1a}$$

$$\sum_{n=0}^{\infty} A_n \cos(n+\tfrac{1}{2})x = g(x), \qquad c<x<\pi \tag{1.1b}$$

$$\sum_{n=0}^{\infty} \frac{A_n}{n+\frac{1}{2}} \sin(n+\tfrac{1}{2})x = f(x), \qquad 0<x<c \tag{1.2a}$$

$$\sum_{n=0}^{\infty} A_n \sin(n+\tfrac{1}{2})x = g(x), \qquad c<x<\pi \tag{1.2b}$$

$$\sum_{n=1}^{\infty} \frac{A_n}{n} \sin nx = f(x), \qquad 0<x<c \tag{1.3a}$$

$$\sum_{n=1}^{\infty} A_n \sin nx = g(x), \qquad c<x<\pi \tag{1.3b}$$

where c is a fixed point and $f(x)$ and $g(x)$ are given functions.

Our starting point is an important set of formulas, based on double singular integrals, developed by Bablojan[1] and Tranter[2] for the solution of the above equations. The original analysis was given in[3] and subsequently simplified[1, 2, 4]. Other singular integral solutions are found in[5, 6]. The above equations are written in the canonical form described in[7]. In[2] solutions were not derived for the case $g \not\equiv 0$ in (1.1b) and $f \not\equiv 0$ in (1.2a) whereas in [1] solutions were not derived for (1.3a, b). Combining the results we have available singular integral solutions for all the equations (1.1a, b), (1.2a, b), (1.3a, b). The results here fulfill in part the suggestion made in [8] that a preliminary ordinary Fourier analysis would be the key to converting singular integral solutions into algorithms. (Naturally, the algorithmic resolution of ordinary Fourier analysis lies beyond the scope of this paper.)

These classic dual trigonometric equations occur in solving mixed boundary value problems in rectangular domains in the x–y plane [9, p. 150, 14, 20] and represent one of the simplest examples of dual orthogonality. Consequently, they have been studied in many investigations[8, 10, 11, 13, 15, 16, 18] and have found several applications especially in mixed boundary value problems in mechanical engineering[1, 21–23, 44] and are closely related to more general type dual series equations which occur in heat transfer theory, fracture mechanics, and wave guide design, e.g.[21, 24–26]. Moreover, because of the simple form of the dual orthogonal problem represented by the above equations, their solutions in the form of singular integrals have served as archetypal closed form solutions to dual Sturm–Liouville

†A copy of the computer code on punched cards or magnetic tape is available at cost from the first author.

problems providing the model for comparison and development of other solutions[26, 28, 29, 34, 44]. Also, because of their relation to canonical mixed boundary value problems in the plane, the harmonic functions associated with these dual series have served as bench marks against which to test other numerical methods of solving mixed boundary value problems[12, 17, 27, 34–38].

However, the fact that the solutions are expressed as double singular integrals has served as a barrier to automated numerical evaluation of the formulas and to their rigorous justification. These considerations make the present investigation timely for in it we reduce each singular integral to an algorithm which is implemented in a computer program. By algorithm we mean that given $f(x)$ and $g(x)$ each coefficient A_n can be computed to specified accuracy by a precisely defined, finite sequence of rules whose execution involves only a finite number of elementary arithmetic operations none of which is division by zero.

On the basis of extensive numerical testing we conclude that Bablojan and Tranter's formulas[1, 2] are correct for a wide variety of input functions $f(x)$ and $g(x)$. However, from such testing it is not possible to define the class of functions $f(x)$ and $g(x)$ for which the resulting solutions are rigorously valid.

In Section 2 we list notation and formulas needed for describing the singular integral solutions. For completeness and ease of reference the singular integral solutions to (1.1a,b), (1.2a, b) and (1.3a, b) are listed in Section 3, and the correspondence between these equations in canonical form and those found in[1] and[2] is described. The algorithms are given in Section 4. The use of the software package and its technical description, as well as the results of numerical tests, are given in Section 5. Section 6 is devoted to the derivation of the formulas of Section 2, and in Section 7 the algorithms of Section 4 are derived. In the Appendix is listed the FORTRAN software which implements the algorithms.

2. PRELIMINARY DEFINITIONS AND FORMULAS FOR SUMS AND INTEGRALS

In so far as practicable we maintain notational consistency with[8]. k and n will always be used to denote nonnegative integers. We set $\beta_n = n(n+1)/(2n+1)$, and δ_{kn} is the Kronecker delta. Further $\gamma = \cos c$, $\zeta = \sin c$, $\phi = \cos(c/2)$, $\psi = \sin(c/2)$. By P_ν^μ we denote the Legendre function of the first kind of degree ν and order μ. We define $\mathcal{P}_n$ by

$\mathcal{P}_0(x) = 1 + x$ and $\mathcal{P}_n(x) = P_{n+1}(x) - P_{n-1}(x)$, $n = 1, 2, \ldots$. Note that $\mathcal{P}_n = (1-x^2)^{1/2} P_n^{\,1}/\beta_n$ [30, p. 171]. Let $N_{00}(\gamma) = 0$, and for other values of κ and ν we define three definite integrals $R_{\kappa\nu}$, $M_{\kappa\nu}$, and $N_{\kappa\nu}$ by

$$R_{\kappa\nu}(\gamma) = \int_{-1}^{\gamma} P_\kappa(x) P_\nu(x)\,\mathrm{d}x, \quad M_{\kappa\nu}(\gamma) = \int_{-1}^{\gamma} \frac{\mathcal{P}_\kappa(x)\mathcal{P}_\nu(x)}{1-x^2}\,\mathrm{d}x,$$

$$N_{\kappa\nu}(\gamma) = \int_{\gamma}^{1} \frac{\mathcal{P}_\kappa(x)\mathcal{P}_\nu(x)}{1-x^2}\,\mathrm{d}x.$$

$F(\theta, \eta)$, $K(\theta)$ and $E(\theta)$ denote respectively the incomplete elliptic integral of the first kind, the complete elliptic integral of the first kind and the complete elliptic integral of the second kind[30]. The derivation of the following ten formulas is discussed in Section 6. In these formulas any Legendre polynomial of negative degree is taken as zero.

$$R_{kn}(\gamma) = \frac{\beta_n \mathcal{P}_n(\gamma) P_k(\gamma) - \beta_k \mathcal{P}_k(\gamma) P_n(\gamma)}{n(n+1) - k(k+1)} \quad k, n = 0, 1, \ldots\ k \neq n, \tag{2.1}$$

$$\begin{aligned} R_{kk}(\gamma) = \frac{1}{2k+1}\Bigg[& 1 + \gamma(P_k(\gamma))^2 - \frac{2(k-1)}{2k-1} P_k(\gamma) P_{k-1}(\gamma) \\ & + 2\left\{\frac{P_{k-1}(\gamma)P_{k-2}(\gamma)}{(2k-1)(2k-3)} + \frac{P_{k-2}(\gamma)P_{k-3}(\gamma)}{(2k-3)(2k-5)} + \cdots + \frac{P_1(\gamma)P_0(\gamma)}{3\cdot 1}\right\}\Bigg] \quad k = 0, 1, \ldots, \end{aligned} \tag{2.2}$$

$$R_{-\frac{1}{2},k}(\gamma) = \frac{2}{\pi(k+\frac{1}{2})^2}\{[(\phi^2 + k\gamma)P_k(\gamma) - kP_{k-1}(\gamma)]K(\psi) - P_k(\gamma)E(\psi) + (-1)^k\}, \quad k = 0, 1, \ldots, \tag{2.3}$$

$$M_{00}(\gamma) = -\left[1 + \gamma + 2\log\left(\frac{1-\gamma}{2}\right)\right], \tag{2.4}$$

$$M_{kn}(\gamma) = \frac{1}{\beta_k}[(2n+1)R_{kn}(\gamma) - P_k(\gamma)\mathscr{P}_n(\gamma)] \quad k = 1, 2, \ldots, \quad n = 0, 1, 2, \ldots, \tag{2.5}$$

$$N_{kn}(\gamma) = \frac{2}{\beta_n}(\delta_{kn} - \delta_{0k}) - M_{kn}(\gamma) \quad k = 0, 1, \ldots, \quad n = 1, 2, \ldots \tag{2.6}$$

$$\sum_{n=0}^{\infty} \frac{P_n(\gamma)}{2n+1} = \frac{1}{2}K(\phi). \tag{2.7}$$

$$\int_c^{\pi} K\left(\cos\frac{\theta}{2}\right)\sin\theta \, d\theta = 4(E(\phi) - \psi^2 K(\phi)). \tag{2.8}$$

$$\sum_{n=1}^{\infty} \frac{\beta_n N_{kn}(\gamma)}{2n+1} = \frac{1}{2}\{\mathscr{P}_k(\gamma)K(\phi) + (-1)^k \pi (k + \tfrac{1}{2})R_{-\frac{1}{2},k}(-\gamma)\}, \quad k = 0, 1, \ldots, \tag{2.9}$$

$$\sum_{n=0}^{\infty} \frac{R_{0n}(\gamma)}{2n+1} = \frac{\pi}{4}\left(\frac{8}{\pi} - R_{-\frac{1}{2},0}(-\gamma)\right). \tag{2.10}$$

3. SOLUTIONS AS SINGULAR INTEGRALS

To convert the dual equations of Section 1 into the forms given in [1] and [2] requires only simple transformations of the independent variable and A_n [9, p. 150]. The correspondence between the equations here and those given in [2] is this: (1.1a,b) here corresponds to the dual eqn (8) and (9) in [2]; (1.2a,b) corresponds to the dual eqn (1) and (2); (1.3a,b) corresponds to the union of the dual eqn (19) and (20) and the dual eqn (26) and (27). The dual equations of concern in [1] are (18) and (23). Note that in Bablojan's notation (18) in [1] with $p = -1$ and (23) with $p = 1$ both correspond to (1.2a,b) here. Dual eqn (18) with $p = 1$ and dual eqn (23) with $p = -1$ correspond to (1.1a,b) here.

By using an observation of Gordon [31] we can simplify the analysis by assuming $g \equiv 0$ without loss of generality. Let $G(x)$ be defined by $G = g$ on $c < x < \pi$, whereas on $0 < x < c$ let $G(x)$ be defined arbitrarily (suggestions for doing this conveniently are given in Section 4). Let B_n, C_n, and $F(x)$ be defined by

$$B_n = \frac{2}{\pi}\int_0^{\pi} G(x)\cos(n + \tfrac{1}{2})x \, dx \tag{3.1}$$

$$F(x) = f(x) + \sum_{n=0}^{\infty} \frac{B_n}{n + \frac{1}{2}}\cos(n + \tfrac{1}{2})x, \qquad 0 < x < c, \tag{3.2}$$

$$A_n = C_n - B_n. \tag{3.3}$$

Then $\{C_n\}$ satisfies the dual trigonometric equation

$$\sum_{n=0}^{\infty} \frac{C_n}{n + \frac{1}{2}}\cos(n + \tfrac{1}{2})x = F(x), \qquad 0 < x < c, \tag{3.4a}$$

$$\sum_{n=0}^{\infty} C_n \cos(n + \tfrac{1}{2})x = 0, \qquad c < x < \pi, \tag{3.4b}$$

if, and only if, $\{A_n\}$ satisfies (1.1a,b).

If we now define B_n and $F(x)$ by

$$B_n = \frac{2}{\pi}\int_0^{\pi} G(x)\sin(n + \tfrac{1}{2})x \, dx \tag{3.5}$$

$$F(x) = f(x) + \sum_{n=0}^{\infty} \frac{B_n}{n + \frac{1}{2}}\sin(n + \tfrac{1}{2})x, \qquad 0 < x < c, \tag{3.6}$$

then $\{A_n\}$ satisfies the dual trigonometric eqn (1.2a,b) if, and only if, $\{C_n\}$ satisfies

$$\sum_{n=0}^{\infty} \frac{C_n}{n+\frac{1}{2}} \sin(n+\tfrac{1}{2})x = F(x), \qquad 0<x<c, \tag{3.7a}$$

$$\sum_{n=0}^{\infty} C_n \sin(n+\tfrac{1}{2})x = 0, \qquad c<x<\pi. \tag{3.7b}$$

Finally, let $\{A_n\}$ satisfy the dual eqn (1.3a,b) and define B_n and $F(x)$ by

$$B_n = \frac{2}{\pi}\int_0^{\pi} G(x) \sin nx \, dx \tag{3.8}$$

$$F(x) = f(x) + \sum_{n=1}^{\infty} \frac{B_n}{n} \sin nx, \qquad 0<x<c. \tag{3.9}$$

In this case $\{C_n\}$ satisfies the dual equation

$$\sum_{n=1}^{\infty} \frac{C_n}{n} \sin nx = F(x), \qquad 0<x<c, \tag{3.10a}$$

$$\sum_{n=1}^{\infty} C_n \sin nx = 0, \qquad c<x<\pi. \tag{3.10b}$$

The solution to the dual eqn (3.4a,b) based on formula (18) in[2] or formula (25) in[1] is given by

$$C_n = C_0 P_n(\gamma) + \frac{1}{2}\int_0^c H(\theta) \frac{d}{d\theta}(P_n(\cos\theta))\, d\theta, \qquad n = 1, 2, \ldots, \tag{3.11}$$

where

$$r(x,\theta) = (\cos x - \cos\theta)^{1/2}$$

$$H(\theta) = \frac{2\sqrt{2}}{\pi}\int_0^{\theta} \frac{dF(x)}{dx} \cdot \frac{\sin x}{r(x,\theta)}\, dx, \qquad 0<\theta<c, \tag{3.12}$$

and C_0 is found by substituting into (3.4a).

The solution to the dual eqn (3.7a,b) based on (22) in[1] is

$$C_n = (n+\tfrac{1}{2})\int_0^c J(\theta) P_n(\cos\theta) \sin\theta \, d\theta, \qquad n = 0, 1, \ldots, \tag{3.13}$$

where

$$J(\theta) = \frac{\sqrt{2}}{\pi}\int_0^{\theta} \frac{F'(x)}{r(x,\theta)}\, dx, \qquad 0<\theta<c. \tag{3.14}$$

The solution to the dual eqn (3.9a,b) based on (30) and (31) in[2] is

$$C_1 = \int_0^c I(\theta) \sin\theta \, d\theta \tag{3.15}$$

$$C_n = -C_{n-1} + (2n-1)\int_0^c I(\theta) P_{n-1}(\cos\theta) \sin\theta \, d\theta, \qquad n = 2, 3, \ldots, \tag{3.16}$$

where

$$I(\theta) = \frac{\sqrt{2}}{\pi}\int_0^c \frac{dF(x)}{dx} \cdot \frac{\cos(x/2)}{r(x,\theta)}\, dx, \qquad 0<\theta<c. \tag{3.17}$$

4. ALGORITHMS

In developing the algorithms we perform an ordinary Fourier decomposition of $F(x)$ and find the algorithm for each ordinary Fourier component. Combining this with (3.1) and (3.3) dual Fourier analysis is made to depend upon ordinary Fourier decomposition. In other words, one cannot hope that dual Fourier decomposition is simpler than ordinary Fourier decomposition so that one seeks to make dual Fourier analysis depend upon ordinary Fourier analysis as simply as possible. Since F is defined only over part of the interval $0<x<\pi$, there are infinitely many ways in which to obtain an ordinary Fourier decomposition representing F on $0<x<c$ in terms of functions orthogonal on $0<x<\pi$. Naturally, one seeks the decomposition so that the ordinary Fourier components decrease rapidly. This is achieved by introducing a function $\Phi(x)$ such that $\Phi = F$ on $0<x<c$ and Φ is defined on $c<x<\pi$ in such a way that Φ is as smooth as F, and Φ and a specified number of its derivatives are zero at $x=\pi$. Similar remarks hold for constructing $G(x)$. Further consideration of this point is not warranted here since it lies in the domain of ordinary Fourier analysis and can be found in the literature, e.g.[32, 33].

Suppose, then, that Φ is defined by $\Phi = F$ on $0<x<c$ and $\Phi = h$ on $c<x<\pi$ where h is a function chosen in accordance with the remarks above. Then

$$\Phi(x) = \sum_{k=0}^{\infty} \lambda_k \cos(k+\tfrac{1}{2})x \tag{4.1}$$

$$\lambda_k = \frac{2}{\pi}\int_0^c f(x)\cos(k+\tfrac{1}{2})x\,dx + \frac{2}{\pi}\int_c^{\pi} h(x)\cos(k+\tfrac{1}{2})x\,dx. \tag{4.2}$$

Consider the dual series problem given by (3.4b) and

$$\sum_{n=0}^{\infty} \frac{C_n}{n+\frac{1}{2}}\cos(n+\tfrac{1}{2})x = \sum_{k=0}^{\infty} \lambda_k \cos(k+\tfrac{1}{2})x, \qquad 0<x<c. \tag{4.3}$$

Clearly $\{C_n\}$ is a solution of the dual series eqn (3.4a,b) if, and only if, it is a solution to the dual series eqn (4.3, 3.4b). This suggests solving (3.4a,b) separately for each ordinary Fourier component, i.e. when F is given by

$$F(x) = \cos(k+\tfrac{1}{2})x, \qquad 0<x<c. \tag{4.4}$$

In so doing we obtain the following algorithm. For the case $k=0$ the solution to (3.4a,b) is

$$C_0 = \frac{1}{2K(\phi)}\left[\psi^2 K(\phi) + 2 - \frac{\pi}{4}R_{-\frac{1}{2},0}(-\gamma)\right] \tag{4.5}$$

$$C_n = C_0 P_n(\gamma) - \frac{1}{2}\left(\psi^2 P_n(\gamma) + \frac{1}{2}R_{0n}(\gamma)\right), \qquad n = 1, 2, \ldots. \tag{4.6}$$

For the case $k = 1, 2, \ldots$, the solution to (3.4a,b) is

$$C_0 = \frac{1}{2K(\phi)}\{2-(k+\tfrac{1}{2})[\mathcal{P}_k(\gamma)K(\phi) + (-1)^k\pi(k+\tfrac{1}{2})R_{-\frac{1}{2},k}(-\gamma)]\} \tag{4.7}$$

$$C_n = C_0 P_n(\gamma) + \frac{1}{2}(k+\tfrac{1}{2})\beta_n N_{kn}(\gamma), \qquad n = 1, 2, \ldots. \tag{4.8}$$

For the dual eqn (3.7a,b) with F given by

$$F(x) = \sin(k+\tfrac{1}{2})x \tag{4.9}$$

the algorithm is

$$C_n = (k+\tfrac{1}{2})(n+\tfrac{1}{2})\left[\frac{\delta_{kn}}{n+\frac{1}{2}} - R_{kn}(\gamma)\right], \qquad k, n = 0, 1, \ldots. \tag{4.10}$$

For the dual eqn (3.10a,b) with F given by

$$F(x) = \sin kx \tag{4.11}$$

the algorithm is

$$C_1 = \frac{-k}{2}(R_{0k}(\gamma) + R_{k-1,0}(\gamma) - 2\delta_{0,k-1}), \qquad k = 1, 2, \ldots, \tag{4.12}$$

$$C_{n+1} = -C_n + k[\delta_{kn} + \delta_{k-1,n} - (n + \tfrac{1}{2})\{R_{kn}(\gamma) + R_{k-1,n}(\gamma)\}], \qquad k, n = 1, 2, \ldots. \tag{4.13}$$

Let us examine the solution to (3.4a,b) given by (4.3) and (4.6) or (4.7) and (4.8) and consider the determination of a coefficient C_n to specified accuracy. This explicitly involves the execution of 16 or less elementary arithmetic operations. Next it requires the evaluation of the first $n+1$ Legendre polynomials at γ. These were computed by setting $P_0(\gamma) = 1$, $P_1(\gamma) = \gamma$ and using the recursion relation

$$P_{n+1}(\gamma) = \frac{1}{n+1}[(2n+1)\gamma P_n(\gamma) - nP_{n-1}(\gamma)]. \tag{4.14}$$

Beyond this finite number of rational operations two evaluations each of the sine, cosine, logarithm, and the first two complete elliptic integrals are required. There are several algorithms available for computing the trigonometric and logarithmic functions[39, 40]. For the evaluation of the elliptic integrals we used the Chebyshev type algorithms of Cody[41]. It follows, therefore, that formulas (3.1), (3.3) and (4.5)–(4.8) constitute an algorithm in the sense enumerated above for computing $\{A_n\}$ the solution to (1.1a,b). Similar remarks are valid for (1.2a,b) and (1.3a,b).

5. SOFTWARE USAGE AND COMPUTER TESTS

The software package is used by the statement

```
CALL BBLTRN(KASE,NB TRMS,K,SMALL C,C0,CN,IERR,AP)
```

(i) KASE (input) has allowed values 1, 2, or 3. If KASE = 1, the dual eqn (3.4a,b) is solved with F given by (4.4). If KASE = 2, the dual eqn (3.7a,b) is solved with F given by (4.9). If KASE = 3, the dual eqn (3.10a,b) is solved with F given by (4.11).

(ii) NB TRMS (input) denotes the number of Fourier coefficients to be computed, i.e. $\{C_n: n = 0, 1, \ldots, \text{NB TRMS}\}$ is calculated. For KASE = 3, BBLTRN returns C_0 with the value zero. NB TRMS must be nonnegative.

(iii) K (input) corresponds to k on the right hand sides of (4.4), (4.9), and (4.11). K must be nonnegative.

(iv) SMALL C (input) corresponds to the breakpoint c. It must satisfy $0 \leq \text{SMALL C} \leq \pi$.

(v) C0 (output) is the first Fourier coefficient C_0.

(vi) CN (output) is a one-dimensional array of the remaining Fourier coefficients, *viz.* CN(N) corresponds to C_N.

(vii) IERR (output) is an error parameter. IERR = 0 is the first executable statement in BBLTRN. If an error in the input data is discovered, IERR is incremented as shown in Table 1 and control is returned to the calling unit without the coefficients being calculated. If no errors are discovered, IERR retains the value zero. The user should test IERR immediately upon returning from BBLTRN, since this is the only means of detecting an error in the input data.

(viii) AP is a one-dimensional array used for work space. It should be dimensioned to 4 greater than the larger of NB TRMS and K.

We define a degenerate condition as one which implies that all the Fourier coefficients are zero. Tests are made for three such conditions. If at least one of them is true, C0 and CN(N), $N = 1, 2, \ldots, \text{NB TRMS}$, are set equal to zero and a return is executed. If all three degeneracy checks are negative, the program evaluates C0 and CN by implementing the algorithms. The three conditions are: (i) $c = 0$; (ii) KASE = 3 and $k = 0$; (iii) $\psi^2 = 0$ (which can result from underflow even if c is positive).

Table 1. The input parameters to SUBROUTINE BBLTRN are listed in column 1. If an error is discovered for a given parameter, IERR is incremented by the corresponding value in column 2.

Input Parameters	IERR Increment
KASE	1
NB TRMS	2
K	4
SMALL C	8

Test cases were run for all three dual equations as follows: $c = 0.2\pi$, $c = 0.5\pi$ and $c = 0.75\pi$ and $k = 0, 1, 2, 3, 4, 5, 6, 8$, and 16 with NB TRMS = 200; $c = 0.01\pi$, 0.02π, 0.98π and 0.99π, and $k = 0, 3$, and 11 with NB TRMS = 128. In all cases agreement could be considered close between the series approximations using 129 or 201 terms and the right hand sides of the dual equations. Pointwise errors were of the order 10^{-4} over $0 < x < c$ and of the order 10^{-2} over $c < x < \pi$. In regards to these latter values, one should note that the second series in each pair of dual equations had maximum absolute values greater than 10 and often 100 when evaluated over $0 < x < c$. As the number of terms in the series approximations was increased, say from 8 to NB TRMS, consistent improvements in the approximations were observed. As expected, there was somewhat greater deviation within a distance of 0.02 on either side of c. We take these results as empirical evidence for the correctness of Bablojan and Tranter's formulas.

In the range of NB TRMS between 0 and 200 it took about 1 millisecond to compute a given Fourier coefficient. If NB TRMS was small, fewer computations were required per coefficient, but this was offset by the higher overhead costs per coefficient. These results depend on k and are averaged over the values of k given above.

Estimates of round-off error indicate that for NB TRMS and k less than 200 and $c \geq 0.01\pi$, the coefficients are computed with relative error less than 10^{-8}. As c approaches zero, all the coefficients approach zero and though the absolute error with which they are computed remains less than 10^{-8}, the relative error increases. However, checking the computer output by a hand calculation indicates that for c as small as 0.00001π, the relative error is smaller than 10^{-5}.

The technical specifications are as follows.

Special condition. None.

Common blocks. /BT/, length 5.

Precision. Single.

I/0. None.

Portability. American National Standards Institute FORTRAN[45]. In units BBLTRN, ELLIP E and ELLIP K there are machine dependent constants initialized in DATA statements.

Space required. 890_{10}.

Required resident routines. None.

Specifications. If NB TRMS ≥ 1, the calling unit must specify CN to be a one-dimensional array of length at least NB TRMS. The calling unit must specify AP to be a one-dimensional array of length at least 4 + MAX0(NB TRMS,K).

The estimations of round-off error, requirements for space, and times for execution listed above were determined on the CDC 6400 using the SCOPE 3.3 operating system and the FTN compiler with optimization level 1[46, 47].

6. DERIVATION OF FORMULAS FOR SUMS AND INTEGRALS

The formulas (2.1) and (2.2) for R_{kn} and formulas (2.4) and (2.5) for M_{kn} are given in[8], while formula (2.6) for N_{kn} was given in[42]. Let us establish formula (2.3) for $R_{-\frac{1}{2},k}$ for $k \neq 0$ (the derivation for $k = 0$ is similar). Using the definition for $R_{-\frac{1}{2},k}$, integrating by parts twice, and employing Legendre's differential equation

$$P_\nu(x)=\frac{-1}{\nu(\nu+1)}\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}(P_\nu(x)\right] \tag{6.1}$$

to replace first P_k and then $P_{-\frac{1}{2}}$, we obtain

$$(k+\tfrac{1}{2})^2R_{-\frac{1}{2},k}(\gamma)=\lim_{\epsilon=+0}\,[(1-x^2)(P_{-\frac{1}{2}}(x)P_k'(x)-P_{-\frac{1}{2}}'(x)P_k(x))]_\gamma^{-1+\epsilon}.$$

Remembering that [30, p. 171]

$$(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}(P_{-\frac{1}{2}}(x))=\frac{1}{2}[xP_{-\frac{1}{2}}(x)-P_{\frac{1}{2}}(x)] \tag{6.2}$$

we find

$$\begin{aligned}(k+\tfrac{1}{2})^2R_{-\frac{1}{2},k}(\gamma)&=\zeta^2[P_{-\frac{1}{2}}'(\gamma)P_k(\gamma)-P_{-\frac{1}{2}}(\gamma)P_k'(\gamma)]\\&\quad+\frac{(-1)^k}{2}\lim_{\epsilon=+0}\,[P_{-\frac{1}{2}}(-1+\epsilon)+P_{\frac{1}{2}}(-1+\epsilon)].\end{aligned} \tag{6.3}$$

The functions $P_{\pm\frac{1}{2}}$ are expressible as elliptic integrals [43, p. 337], *viz.*,

$$P_{-\frac{1}{2}}(\gamma)=\frac{2}{\pi}K(\psi)\quad\text{and}\quad P_{\frac{1}{2}}(\gamma)=\frac{2}{\pi}(2E(\psi)-K(\psi)). \tag{6.4}$$

Substituting from (6.4) into (6.3) combined with some algebra yields the desired formula (2.3).

Next we turn to the summation formula (2.7). From [30, p. 232]

$$2\sum_{n=0}^{\infty}\frac{(-1)^nP_n(\cos\theta)}{2n+1}x^{2n+1}=F\left(\sin\frac{\theta}{2},\eta\right)$$

where $x=\tan(\eta/2)$. We obtain the summation formula (2.7) by setting $\eta=\pi/2$ and $\theta=\pi-c$. Likewise, formula (2.8) follows readily from formula (610.01) in [48].

Next we establish the summation formula (2.9). Now

$$\sum_{n=1}^{\infty}\frac{\beta_nN_{kn}(\gamma)}{2n+1}=-\int_\gamma^1\left[\mathscr{P}_k(\mu)\sum_{n=1}^{\infty}\frac{1}{2n+1}\frac{\mathrm{d}P_n(\mu)}{\mathrm{d}\mu}\right]\mathrm{d}\mu.$$

Using (2.7) we see that

$$\sum_{n=1}^{\infty}\frac{\beta_nN_{kn}(\gamma)}{2n+1}=-\frac{1}{2}\int_\gamma^1\mathscr{P}_k(\mu)\frac{\mathrm{d}}{\mathrm{d}\mu}\left(K\left(\left[\frac{1+\mu}{2}\right]^{1/2}\right)\right)\mathrm{d}\mu$$

One now integrates by parts and uses (6.4) and

$$P_n'(x)=-(\beta_n\mathscr{P}_n(x))/(1-x^2). \tag{6.5}$$

A little algebra then gives the desired result.

Finally, we derive formula (2.10). Let Z denote the series on the left hand side of (2.10). From the definition of R_{on} we see that

$$Z=\int_{-1}^{\gamma}\left(\sum_{n=0}^{\infty}\frac{P_n(\mu)}{2n+1}\right)\mathrm{d}\mu.$$

Therefore from (2.7)

$$Z=\frac{1}{2}\int_{-1}^{\gamma}K\left(\left[\frac{1+\mu}{2}\right]^{1/2}\right)\mathrm{d}\mu.$$

From (6.4) the integrand above is $\pi P_{-\frac{1}{2}}(-\mu)/2$. Thus we see that

$$Z = \frac{\pi}{4}[R_{-\frac{1}{2},0}(1) - R_{-\frac{1}{2},0}(-\gamma)].$$

Using (2.8) to evaluate $R_{-\frac{1}{2},0}(1)$, we obtain formula (2.10).

7. DERIVATION OF ALGORITHMS

We start with the derivation of formulas (4.5)–(4.8). Let us denote by $H_k(\theta)$ the value of $H(\theta)$ when $F(x)$ is given by (4.4). Using Mehler's representation for the Legendre polynomials [30, p. 235], *viz.*

$$P_i(\cos\theta) = \frac{\sqrt{2}}{\pi}\int_0^\theta \frac{\cos(i+\frac{1}{2})x}{r(x,\theta)}\,\mathrm{d}x = \frac{\sqrt{2}}{\pi}\int_\theta^\pi \frac{\sin(i+\frac{1}{2})x}{r(\theta,x)}\,\mathrm{d}x, \quad i = 0, 1, \ldots,$$

we see (3.12) implies that

$$H_0 = \frac{1}{2}(\cos\theta - 1) \quad \text{and} \quad H_k = (k+\tfrac{1}{2})\mathscr{P}_k(\cos\theta), \quad k = 1, 2, \ldots. \tag{7.1}$$

Substituting into (3.11) we obtain

$$C_n = C_0P_n(\gamma) + \frac{1}{2}\int_0^c \frac{\mathrm{d}}{\mathrm{d}\theta}(P_n(\cos\theta))(\cos\theta - 1)\,\mathrm{d}\theta, \quad k = 0 \quad \text{and} \quad n = 0, 1, \ldots, \tag{7.2}$$

$$C_n = C_0P_n(\gamma) + (k+\tfrac{1}{2})\int_0^c \frac{\mathrm{d}}{\mathrm{d}\theta}(P_n(\cos\theta))\mathscr{P}_k(\cos\theta)\,\mathrm{d}\theta, \quad k = 1, 2, \ldots, \quad n = 0, 1, \ldots. \tag{7.3}$$

Integrating by parts we see that

$$\int_0^c \frac{\mathrm{d}}{\mathrm{d}\theta}(P_n(\cos\theta))(\cos\theta - 1)\,\mathrm{d}\theta = -(1-\gamma)P_n(\gamma) - R_{0n}(\gamma).$$

This relation and (7.2) give (4.6). Using (6.5) we see that

$$\int_0^c \frac{\mathrm{d}}{\mathrm{d}\theta}(P_n(\cos\theta))\mathscr{P}_k(\cos\theta)\,\mathrm{d}\theta = -\beta_nN_{kn}.$$

Now we turn to the question of obtaining C_0. We substitute from (4.6) into (3.4a) and evaluate at $x = 0$. This gives for $k = 0$

$$\left(C_0 - \frac{\psi^2}{2}\right)\sum_{n=0}^\infty \frac{P_n(\gamma)}{2n+1} - \frac{1}{4}\sum_{n=0}^\infty \frac{R_{0n}(\gamma)}{2n+1} = 1. \tag{7.4}$$

Replacing the sums in (7.4) from (2.7) and (2.10) gives (4.5). Substituting from (4.8) into (3.4a) at $x = 0$ we obtain

$$C_0K(\phi) + (k+\tfrac{1}{2})\sum_{n=1}^\infty \frac{\beta_nN_{kn}(\gamma)}{2n+1} = 1.$$

Using (2.9) and some algebra we obtain (4.7).

If we denote by $J_k(\theta)$ the value of $J(\theta)$ when $F(x)$ is given by (4.9), we find after using Mehler's representation that

$$J_k(\theta) = (k+\tfrac{1}{2})P_k(\cos\theta).$$

Substituting into (3.13) and rearranging we obtain (4.10).

Finally we consider the derivation of (4.12) and (4.13). Let us denote by $I_k(\theta)$ the value of $I(\theta)$ when $F(x)$ is given by (4.11). Using Mehler's representation and the addition formula for cosines we see that

$$I_k(\theta) = \frac{k}{2}(P_k(\cos\theta) + P_{k-1}(\cos\theta)).$$

Equations (4.12) and (4.13) follow upon inserting this expression into (3.15) and (3.16) respectively.

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```
C*****************************      APPENDIX      ******************************C
C
      SUBROUTINE EBLTRN(KASE,NB TRMS,K,SMALL C,C0,CN,IERR,AP)
COMMENTS. THIS SOFTWARE PACKAGE IS THE APPENDIX TO THE PAPER
C     KELMAN,R.B.,MAHEFFY,J.P., AND SIMPSON,J.T. ALGORITHMS FOR CLASSIC DUAL
C     TRIGONOMETRIC EQUATIONS,COMPUT.MATH.APPL.,1977
C  ITS USE IS DESCRIBED IN SEC.5 WHERE DEFINITIONS OF THE PARAMETERS IN THE
C  ARGUMENT LIST ARE GIVEN. THE MEANING OF PARAMETERS IN COMMON/BT/ IS OBVIOUS
C  FROM ASSIGNMENT STATEMENTS BELOW.
      COMMON/BT/GMA,PHI,PI,PSI,PSI SQ
      DIMENSION AP(1),CN(1)
      LOGICAL CASE,DEGNRT
      CASE(BOOL) = .NOT.BOOL
      DATA PY/3.14159265358967/
C  TEST INPUT PARAMETERS
      IERR = 0
      IF(KASE.LE.0 .OR. KASE.GE.4) IERR=IERR+1
      IF(NB TRMS.LT.0 ) IERR =IERR+2
      IF(K.LT.0) IERR=IERR+4
      IF(SMALL C .LT. 0.0 .OR. SMALL C .GT. PY) IERR = IERR + 8
      IF(IERR.NE.0) RETURN
C  END TESTING
      PI=PY
      C OVR 2 = SMALL C/2.0
      GMA=COS(SMALL C)
      PHI = COS(C OVR 2)
      PSI = SIN(C OVR 2)
      PSI SQ = PSI**2
      DEGNRT=SMALL C.EQ.0. .OR. (KASE.EQ.3.AND.K.EQ.0) .OR. PSI SQ.EQ.0.
      IF(CASE(DEGNRT))                                         GO TO 100
        CALL SET 0(C0,CN,NB TRMS)
        RETURN
C     ELSE, USE ALGORITHMS TO COMPUTE
  100                                                          CONTINUE
        CALL LGNDR(MAX0(NB TRMS,K ),GMA,AP)
        CALL COEFF(KASE,NB TRMS,C0,CN,AP,K)
        RETURN
      E N D
C
      FUNCTION AM(K,N,AP)
COMMENT. AM IS THE INTEGRAL M(K,N) GIVEN BY FORMULAS (2.4) AND (2.5)
      COMMON/BT/GMA,PHI,PI,PSI,PSI SQ
      DIMENSION AP(1)
      LOGICAL CASE,K EQ 0
      CASE(BOOL) = .NOT.BOOL
      AN=FLOAT(N)
      NHOLD = N
      KHOLD = K
      K EQ 0 = K.EQ.0
      IF(CASE(K EQ 0.AND.N.EQ.0))                              GO TO 10
        AM = -1.0 -GMA + 2.0*ALOG(2.0/(1.0-GMA))
        RETURN
   10                                                          CONTINUE
      IF(CASE(K EQ 0))                                         GO TO 20
        KHOLD=N
        NHOLD=0
        AN=0.
C     ELSE,K.NE.0
   20                                                          CONTINUE
      AM=(AN+AN+1.0)*R(KHOLD,NHOLD,AP)-AP(KHOLD+1)*SCHP(NHOLD,AP)
      AM =AM/BETA(KHOLD)
      RETURN
      E N D
C
      FUNCTION AN(K,N,AP)
COMMENT. AN IS THE INTEGRAL N(K,N) GIVEN BY FORMULA (2.6)

      COMMON/BT/GMA,PHI,PI,PSI,PSI SQ
      DIMENSION AP(1)
      LOGICAL CASE
      CASE(BOOL)=.NOT.BOOL
      AN = -AM(K,N,AP)
      IF(CASE(N.EQ.0))                                         GO TO 10
        AN = AN -2.0/BETA(K)
        RETURN
C     ELSE, N.NE.0
   10                                                          CONTINUE
      AN = AN +(2.0/BETA(N))*(DELTA(K,N)-DELTA(0,K))
      RETURN
      END
C
      FUNCTION BETA(K)
      AK=FLOAT(K)
      AK1=AK+1.0
      BETA=AK*AK1/(AK1+AK)
      RETURN
      END
```

```
C
      SUBROUTINE  COEFF(ICASE, NB TRMS,C0,C,P,K)
COMMENT: FOURIER COEFFICIENTS C(N),N=0,1,...,NB TRMS,ARE EVALUATED.
      COMMON/BT/GMA,PHI,PI,PSI,PSI SQ
      DIMENSION C(1),P(1)
      LOGICAL CASE
      CASE(BOOL)=.NOT.BOOL
      DATA IZERO/0/
      AK=FLOAT(K)
      AKHLF=AK+.5
      GO TO (1000,2000,3000 ),ICASE
 1000                                                           CONTINUE
        ELE = ELLIP E(PSI SQ)
        ELK = ELLIP K(PSI SQ)
        IF(CASE(K.EQ.0))                                        GO TO 1100
          C0=(.5/ELK)*(PSI SQ*ELK+2.0-(PI/4.0)*RHALF(IZERO,P,ELE,ELK))
          IF(NB TRMS.EQ.0) RETURN
          DO 1010 N=1,NB TRMS
            L=N+1
            C(N)=C0*P(L)-.5*(PSI SQ*P(L)+.5*R(IZERO,N,P))
 1010     CONTINUE
      RETURN
 1100                                                           CONTINUE
C     ELSE,K.GE.1
        C0= SCRP(K,P)*ELK+FLOAT((-1)**K)*PI*AKHLF*RHALF(K,P,ELE,ELK)
        C0=(.5/ELK)*(2.-AKHLF*C0)
        IF(NB TRMS.EQ.0) RETURN
        DO 1020 N=1,NB TRMS
          L=N+1
          C(N)=C0*P(L)+.5*AKHLF*BETA(N)*AN(K,N,P)
 1020   CONTINUE
      RETURN
 2000                                                           CONTINUE
        C0=AKHLF*.5*(DELTA(K,IZERO)/AKHLF-R(K,IZERO,P))
        IF(NB TRMS .EQ.0) RETURN
        DO 2010 N=1, NB TRMS
          ANHLF=FLOAT(N)+.5
          C(N)=AKHLF*ANHLF*(DELTA(K,N)/ANHLF-R(K,N,P))
 2010   CONTINUE
      RETURN
 3000                                                           CONTINUE
        C0=0.0
        IF(NB TRMS .EQ.0) RETURN
        K MNS 1 = K-1
        C(1)=(-AK/2.0)*(R(K,IZERO,P)+R(KMNS1,IZERO,P)-2.*DELTA(IZERO,
     1       KMNS1))
        IF(NB TRMS.EQ.1) RETURN
        NB MNS 1 = NB TRMS - 1
        DO 3010 N=1,NB MNS 1
          AN HLF = FLOAT(N) + .5
          TERM=DELTA(K,N)+DELTA(KMNS1,N)-ANHLF*(R(K,N,P)+R(KMNS1,N,P))
          C(N+1)=-C(N)+AK*TERM
 3010   CONTINUE
      RETURN
      E  N  D
C
      FUNCTION DELTA(K,N)
COMMENT: DELTA IS THE KRONECKER DELTA
                DELTA=0.0
      IF(K.EQ.N)DELTA=1.0
                RETURN
      END
C
      FUNCTION ELLIP E (ETA)
C ELLIP E COMPUTES THE  ELLIPTIC INTEGRAL E(SQRT(1 - ETA))
      DIMENSION A(7), B(7)
      DATA A(1),B(1)/ 4.43147 19346 7733 E-1, 2.49999 99844 8655 E-1 /
      DATA A(2),B(2)/ 5.68115 68105 3803 E-2, 9.37488 06209 8189 E-2/
      DATA A(3),B(3)/ 2.21862 20699 3846 E-2, 5.84950 29706 6166 E-2/
      DATA A(4),B(4)/ 1.56847 70023 9786 E-2, 4.09074 82159 3164 E-2   /
      DATA A(5),B(5)/ 1.92284 38902 2977 E-2, 2.35091 60256 4984 E-2/
      DATA A(6),B(6)/ 1.21819 48148 6695 E-2, 6.45682 24731 5060 E-3/
      DATA A(7),B(7)/ 1.55618 74474 5296 E-3, 3.78886 48734 9367 E-4/
      SUMA = 0.
      SUMB = 0.
      DO 10  I = 1, 7
      J = 8 - I
      SUMA = (SUMA + A(J)) * ETA
      SUMB = (SUMB + B(J)) * ETA
   10 CONTINUE
      ELLIP E = 1. + SUMA - ALOG(  ETA ) * SUMB
      RETURN
      E  N  D
C
      FUNCTION ELLIP K (ETA)
C ELLIP K COMPUTES THE ELLIPTIC INTEGRAL K(SQRT(1 - ETA))
      DIMENSION A(7), B(7)
      DATA A(1),B(1)/9.65736 02051 6771 E-02, 1.24999 99858 5309 E-1   /
      DATA A(2),B(2)/3.06909 63386 1795 E-2, 7.03114 10585 3296 E-2    /
      DATA A(3),B(3)/1.52618 32062 2534 E-2, 4.87379 51094 5218 E-2    /
      DATA A(4),B(4)/1.25565 69354 3211 E-2, 3.57218 44300 7327 E-2    /
      DATA A(5),B(5)/1.68695 68596 7517 E-2, 2.09657 67733 6790 E-2    /
      DATA A(6),B(6)/1.09423 81068 8623 E-2, 5.81807 96187 1996 E-3    /
      DATA A(7),B(7)/1.40704 91549 6101 E-3, 3.42805 71922 9748 E-4    /
      DATA ALOG4/1.38629 43611 1969                            /
      SUMA = 0.
      SUMB = 0.
      DO 10  I = 1, 7
      J = 8 - I
      SUMA = (SUMA + A(J)) * ETA
      SUMB = (SUMB + B(J)) * ETA
   10 CONTINUE
      ELLIP K = ALOG4 + SUMA - ALOG(  ETA)  * (.5 + SUMB)
      RETURN
      E  N  D
C
      SUBROUTINE LGNDR(N,X,AP)
COMMENT. GENERATES LEGENDRE POLYNOMIALS OF ORDER N=0,1,...N+1. STORES IN AP
      DIMENSION AP(1)
      AP(1)=1.0
      AP(2)=X
      N2=N+2

      K=2
   10 CONTINUE
        RK=2.0-1.0/FLOAT(K)
        AP(K+1)=X*RK*AP(K)-(RK-1.0)*AP(K-1)
        K=K+1
      IF(K.LE.N2) GO TO 10
      RETURN
      E N D
```

```
C
      FUNCTION R(K,N,P)
COMMENT. R IS INTEGRAL EVALUATED USING FORMULAS(2.1) AND (2.2) .
      COMMON/BT/GMA,PHI,PI,PSI,PSI SQ
      DIMENSION P(1)
      LOGICAL CASE
      CASE(BOOL)=.NOT.BOOL
      AK=FLOAT(K)
      AN=FLOAT(N)
      AK2=AK+AK
      L=K+1
      M=N+1
      IF(CASE(K.NE.N))                                      GO TO 100
        R=BETA(N)*SCRP(N,P)*P(L)-BETA(K)*SCRP(K,P)*P(M)
        R=R/(AN*(AN+1.0)-AK*(AK+1.0))
        RETURN
C     ELSE,K.EQ.N
  100                                                       CONTINUE
        DENOM=AK2+1.0
        IF(K.EQ.0) PK=0.0
        IF(K.GT.0) PK=P(K)
        R=1.0+GMA*P(L)**2 - ((AK2-2.0)/(AK2-1.0))*P(L)*PK
        R=R/DENOM
        IF(K.LE.1)RETURN
C     K.EQ.N .AND. K.GE.2,SO FORM SUM
        SUM=0.0
        KK=K
  104   CONTINUE
          AKK2=FLOAT(2*KK)
          SUM=SUM+(P(KK   )*P(KK-1))/((AKK2-1.0)*(AKK2-3.0))
          KK=KK-1
        IF(KK.GE.2) GO TO 104
        R=R + (2.*SUM)/DENOM
        RETURN
      E N D
C
      FUNCTION RHALF(N,P,ELE,ELK)
COMMENTS: RHALF(N) IS THE INTEGRAL EVALUATED USING FORMULA (2.3) WITH GMA
C  REPLACED BY -GMA.
      COMMON/BT/GMA,PHI,PI,PSI,PSI SQ
      DIMENSION P(1)
      M=N+1
      IF(N.EQ.0) GO TO 30
                 GO TO 40
   30                                                         CONTINUE
      RHALF=-1.0+ ELE- ( PSI SQ )*ELK
      RHALF = -8.*RHALF/PI
      RETURN
   40                                                         CONTINUE
C     IF(N.GE.1)
      RN=FLOAT(N)
      TERM=( PSI SQ   - RN*GMA)* P(M) + RN*P(N)
      TERM=TERM*ELK
      TERM=TERM - P(M)*ELE + 1.0
      RHALF = 2.0*((-1.)**N)/(PI*((RN+.5)**2))
      RHALF=RHALF*TERM
      RETURN
      E N D
```