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Advances in Applied Mathematics 39 (2007) 1–50

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Exact Euler–Maclaurin formulas for simple lattice polytopes ☆

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Received 15 March 2005; accepted 18 April 2006

Available online 24 January 2007

Abstract

Euler–Maclaurin formulas for a polytope express the sum of the values of a function over the lattice points in the polytope in terms of integrals of the function and its derivatives over faces of the polytope or its expansions. Exact Euler–Maclaurin formulas [A.G. Khovanskii, A.V. Pukhlikov, Algebra and Analysis 4 (1992) 188–216; S.E. Cappell, J.L. Shaneson, Bull. Amer. Math. Soc. 30 (1994) 62–69; C. R. Acad. Sci. Paris Sér. I Math. 321 (1995) 885–890; V. Guillemin, J. Differential Geom. 45 (1997) 53–73; M. Brion, M. Vergne, J. Amer. Math. Soc. 10 (2) (1997) 371–392] apply to exponential or polynomial functions; Euler–Maclaurin formulas with remainder [Y. Karshon, S. Sternberg, J. Weitsman, Proc. Natl. Acad. Sci. 100 (2) (2003) 426–433; Duke Math. J. 130 (3) (2005) 401–434] apply to more general smooth functions.

In this paper we review these results and present proofs of the exact formulas obtained by these authors, using elementary methods. We then use an algebraic formalism due to Cappell and Shaneson to relate the different formulas.

MSC: primary 65B15, 52B20

☆ This work was partially supported by United States–Israel Binational Science Foundation Grant number 2000352 (to Y.K. and J.W.), by the Connaught Fund (to Y.K.), and by National Science Foundation Grant DMS 99/71914 and DMS 04/05670 (to J.W.).

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0196-8858/$ – see front matter
doi:10.1016/j.aam.2006.04.003

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1. Introduction

Let f be a polynomial in one variable. The classical Euler–Maclaurin formula (see [31, Chapter XIV]) computes the sum of the values of f over the integer points in an interval in terms of the integral of f over the interval and the values of f and of its derivatives at the endpoints of the interval. The formula is almost three hundred years old [5,14,33]. We refer the readers to the treatments by Wirtinger [44] and by Bourbaki [6].

A version of this formula that was generalized by Khovanskii and Pukhlikov to higher dimensions (see [30]) involves variations of the interval. It reads

$$f(a)+f(a+1)+\cdots+f(b-1)+f(b) = \operatorname{Td}\left(\frac{\partial}{\partial h_1}\right)\operatorname{Td}\left(\frac{\partial}{\partial h_2}\right)\bigg|_{h_1=h_2=0}\int\limits_{a-h_2}^{b+h_1} f(x)\,dx, \tag{1.1}$$

where $a,b\in\mathbb{Z}$ and

$$\operatorname{Td}(D)=\frac{D}{1-e^{-D}}=1+\frac{1}{2}D+\frac{1}{12}D^2-\frac{1}{720}D^4+\cdots. \tag{1.2}$$

The right-hand side of (1.1) is well defined because $\int_{a-h_2}^{b+h_1} f(x)\,dx$, as a function of h_1 and h_2, is again a polynomial. We call this an *exact* formula, to distinguish it from Euler–Maclaurin formulas *with remainder*, which apply to more general smooth functions.

In higher dimensions, one replaces the interval $[a,b]$ by a *polytope*, that is, the convex hull of a finite set of points in a vector space, or, equivalently, a bounded finite intersection of closed half-spaces. We assume that our polytopes have nonempty interior. A polytope in $\mathbb{R}^n$ is called an *integral polytope*, or a *lattice polytope*, if its vertices are in the lattice $\mathbb{Z}^n$. It is called *simple* if exactly n edges emanate from each vertex. For example, a two dimensional polytope (a polygon) is always simple. A tetrahedron and a cube are simple; a square-based pyramid and an octahedron are not simple; see Fig. 1. A *polytope with a non-singular fan* is a simple polytope in which the

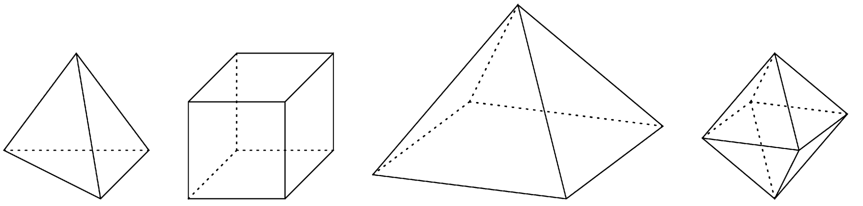

Fig. 1.

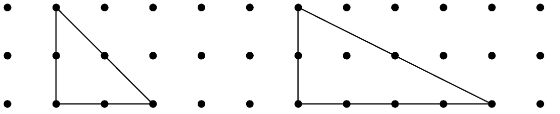

Fig. 2.

edges emanating from each vertex lie along vectors that generate the lattice $\mathbb{Z}^n$. For example, in Fig. 2, the triangle on the left has a non-singular fan, and that on the right does not (check its top vertex).

We refer the reader to [2,17,26,45] for general background on convex polytopes.

Remark 1.1. A *fan* in $\mathbb{R}^n$ is a set of convex polyhedral cones emanating from the origin, such that the intersection of any two cones in the set is a common face, and such that the face of any cone in the set is itself a cone in the set. The fan of a polytope $\Delta \subset \mathbb{R}^n$ consists of a set of cones associated to the faces of Δ; for each face we take the cone generated by the inward normals to the facets that meet at that face.

A convex polyhedral cone is *non-singular* if it can be generated by a set of vectors in $\mathbb{Z}^n$ which are part of a $\mathbb{Z}$-basis of $\mathbb{Z}^n$. A fan is non-singular if each cone in the fan is non-singular. The name "non-singular" comes from the theory of toric varieties; non-singular fans correspond to non-singular toric varieties.

The terminology in the literature is inconsistent. A non-singular cone is also called "smooth cone" and "unimodular cone." Polytopes with non-singular fans are also called "non-singular polytopes," "smooth polytopes," or "Delzant polytopes." In our previous papers [24,25] we used the terms "regular orthant" and "regular polytope" (not to be confused with the more common usage of this term as "platonic solid"). Other terms that have been suggested to us are "unimodular polytope" or "torsion-free polytope."

Khovanskii and Pukhlikov [29,30], following Khovanskii [27,28] (see also Kantor and Khovanskii [22,23]), gave a formula for the sum of the values of a quasi-polynomial (polynomial times exponential) function on the lattice points in a lattice polytope with non-singular fan. This formula was further generalized to *simple* polytopes by Cappell and Shaneson [9–11,38], and subsequently by Guillemin [18] and by Brion–Vergne [7,8]. Cappell–Shaneson and Brion–Vergne [8] also work with polytopes that are not simple; in this paper we will restrict ourselves to simple polytopes. Also see the explicit formulas in [39] and the survey [42]. When applied to the constant function $f \equiv 1$, these formulas compute the number of lattice points in a simple lattice polytope Δ in terms of the volumes of "expansions" of Δ. A sample of the literature on the problem of counting lattice points in convex polytopes is [4,12,13,21,23,34–37,40]; see the survey [3] and references therein.

Remark 1.2. Khovanskii's motivation came from algebraic geometry: a lattice polytope Δ with non-singular fan determines a toric variety M_Δ and a holomorphic line bundle $\mathbf{L}_\Delta \to M_\Delta$. The quantization $Q(M_\Delta)$ is interpreted as the space of holomorphic sections of L_Δ and is computed by the Hirzebruch–Riemann–Roch formula. The lattice points in Δ correspond to basis elements of $Q(M_\Delta)$. A *simple* polytope Δ still determines a toric variety, which now may have orbifold singularities. Cappell and Shaneson derived their formula from their theory of characteristic classes of singular algebraic varieties. Guillemin derived his formula by applying the Kawasaki–Riemann–Roch formula to symplectic toric orbifolds. Brion and Vergne's proof uses Fourier analysis and is closer to Khovanskii and Pukhlikov's original proof.

In this paper we present an elementary proof of the exact Euler–Maclaurin formulas that follows the lines of the original Khovanskii–Pukhlikov proof, through a decomposition of the polytope into an alternating sum of simple convex polyhedral cones. We then use an algebraic formalism due to Cappell and Shaneson to explain the equivalence of the different formulas.

The proof of the exact Euler–Maclaurin formula for a simple convex polyhedral cone involves the following ingredients: the summation of a geometric series, the change of variable formula for integration, and Frobenius' theorem that the average value of a non-trivial character of a finite group is zero. (See Section 4.) The "polar decomposition" of the polytope into simple convex polyhedral cones was proved in papers of Varchenko and Lawrence [32,40]. We present a short direct proof of it. (See Section 3.)

From a simple polytope Δ with d faces one gets an expanded polytope $\Delta(h)$, for $h = (h_1, \dots, h_d)$, by parallel translating the hyperplanes containing the facets, see Eq. (5.2) below. The integrals of a function f on $\Delta(h)$ and on its faces are functions of the d variables $h_1, \dots, h_d$. The formulas of Khovanskii–Pukhlikov, Guillemin, and Brion–Vergne involve an application of infinite order differential operators to these functions. The Cappell–Shaneson formula does not involve expansions of the polytope. It is stated through a formalism that we call the Cappell–Shaneson algebra. Their abstract formula translates to several different concrete formulas; each of these involves applying differential operators to the function and integrating over faces of the polytope. The relations in the Cappell–Shaneson algebra allow one to pass between the different concrete formulas. In Section 6 we show how to incorporate expansions in h into the Cappell–Shaneson formalism. In Section 7 we use a generalization of the "polar decomposition", which applies to polytopes with some facets removed, to prove that the Cappell–Shaneson formula is equivalent to the Khovanskii–Pukhlikov formula in the case of polytopes with non-singular fans.

2. Euler–Maclaurin formulas in one dimension

In this section we present Euler–Maclaurin formulas for a ray and for an interval, in order to illustrate arguments that generalize to higher dimensions.

The ODE for the exponential function. Let $D = \frac{\partial}{\partial h}$. Since

$$D^k e^{\xi h} = \xi^k e^{\xi h},$$

for any formal power series F in one variable we have

$$F(D) e^{\xi h} = F(\xi) e^{\xi h} \tag{2.1}$$

in the ring of power series in two variables. It follows that, for any non-negative integer N,

$$F(D)\frac{(\xi h)^N}{N!} = \big(F(\xi)e^{\xi h}\big)^{\langle N\rangle} \tag{2.2}$$

where the superscript $\langle N\rangle$ denotes the Nth term in the Taylor expansion in ξ. Under suitable convergence conditions, (2.1) is an equality of functions. See [6].

Euler–Maclaurin formula for a ray. To conform with the topological literature, let us define the *Todd function* by

$$\mathrm{Td}(S) := \frac{S}{1-e^{-S}} \tag{2.3}$$

and the corresponding *Todd operator* in one variable by

$$\mathrm{Td}(D).$$

Our general rule (2.1) gives

$$\mathrm{Td}(D)e^{\xi h} = \mathrm{Td}(\xi)e^{\xi h}$$

in the ring of formal power series. If $|\xi| < 2\pi$, so that the power series for $\mathrm{Td}(\xi)$ converges, we can regard this last equation as an equality of functions. Namely, the left-hand side is the limit of the functions obtained by applying the partial sums of the infinite series $\mathrm{Td}(D)$ to the exponential function. If $\xi \neq 0$, we can re-write this as

$$\mathrm{Td}(D)\frac{e^{\xi h}}{\xi} = \frac{e^{\xi h}}{1-e^{-\xi}}. \tag{2.4}$$

If $\xi > 0$, the geometric series expansion

$$1 + e^{-\xi} + e^{-2\xi} + e^{-3\xi} + \cdots = \frac{1}{1-e^{-\xi}}$$

converges, as does the integral

$$\int_{-\infty}^{h} e^{\xi x}\,dx = \frac{e^{\xi h}}{\xi},$$

so (2.4) gives

$$\mathrm{Td}\left(\frac{\partial}{\partial h}\right)\int_{-\infty}^{h} e^{\xi x}\,dx\,\bigg|_{h=0} = \sum_{n=-\infty}^{0} e^{\xi n}. \tag{2.5}$$

This is the Euler–Maclaurin formula for the ray $(-\infty, 0]$, with the function $f(x) = e^{\xi x}$.

Polar decomposition of an interval. In the one dimensional case, the "polar decomposition" becomes the relation

$$\mathbf{1}_I(x) = \mathbf{1}_{\mathbf{C}_b}(x) - \mathbf{1}_{\mathbf{C}_a^\sharp}(x) \tag{2.6}$$

between the characteristic functions of the interval $I = [a, b]$, the ray $\mathbf{C}_b = (-\infty, b]$, and the ray $\mathbf{C}_a^\sharp = (-\infty, a)$ (which is obtained from the ray $C_a = [a, \infty)$ by flipping its direction and removing its vertex).

Euler–Maclaurin on finite intervals. Let $I = [a, b]$ be a closed interval with integer endpoints. For $h = (h_1, h_2)$, consider the expanded interval

$$I(h) := [a - h_2, b + h_1].$$

Summation and integration of the function

$$f(x) = e^{\xi x}$$

gives

$$\mathcal{I}(h, \xi) := \int\limits_{I(h)} e^{\xi x}\, dx = \frac{e^{\xi(b+h_1)}}{\xi} - \frac{e^{\xi(a-h_2)}}{\xi} \tag{2.7}$$

for all ξ such that $\xi \neq 0$, and

$$\mathcal{S}(\xi) := \sum_{x \in I \cap \mathbb{Z}} e^{\xi x} = \frac{e^{\xi b}}{1 - e^{-\xi}} + \frac{e^{\xi a}}{1 - e^{\xi}} \tag{2.8}$$

for all $\xi \in \mathbb{C}$ such that $e^\xi \neq 1$. An indirect proof of (2.7) and (2.8), which generalizes to higher dimensions, uses the "polar decomposition" (2.6): if $\operatorname{Re} \xi > 0$, then

$$\sum_{k=-\infty}^{b} e^{\xi x}\, dx = \frac{e^{\xi b}}{1 - e^{-\xi}} \quad \text{and} \quad \sum_{k=-\infty}^{a-1} e^{\xi x}\, dx = \frac{e^{\xi(a-1)}}{1 - e^{-\xi}} = -\frac{e^{\xi a}}{1 - e^{\xi}}.$$

Since $\mathcal{S}(\xi)$ is the difference of these two infinite sums, (2.8) holds whenever $\operatorname{Re} \xi > 0$. Because the set $\{\xi \in \mathbb{C} \mid e^\xi \neq 1\}$ is connected, by analytic continuation (2.8) holds for *all* ξ in this set. A similar argument shows that (2.7) holds for all ξ in the set $\{\xi \in \mathbb{C} \mid \xi \neq 0\}$.

At this point one can proceed in several ways.

Formal approach. One can deduce an Euler–Maclaurin formula for polynomial functions directly from (2.7) and (2.8). This is the one dimensional case of the approach of Brion–Vergne. From (2.7) we get

$$\xi \mathcal{I}(h, \xi) = e^{\xi(b+h_1)} - e^{\xi(a-h_2)} \tag{2.9}$$

for all $\xi \neq 0$, and, by continuity, also for $\xi = 0$. From (2.8) we get

$$\xi \mathcal{S}(\xi) = \mathrm{Td}(\xi)e^{\xi b} - \mathrm{Td}(-\xi)e^{\xi a} \tag{2.10}$$

for all ξ such that $e^{\xi} \neq 1$ and, by continuity, also for $\xi = 0$. Comparing the Taylor coefficients with respect to ξ on the left- and right-hand sides of (2.9) and of (2.10), we get

$$\xi \int\limits_{I(h)} \frac{(\xi x)^N}{N!}\,dx = \frac{(\xi(b+h_1))^{N+1}}{(N+1)!} - \frac{(\xi(a-h_2))^{N+1}}{(N+1)!} \tag{2.11}$$

and

$$\xi \sum_{x \in I \cap \mathbb{Z}} \frac{(\xi x)^N}{N!} = \bigl(e^{\xi b}\mathrm{Td}(\xi) - e^{\xi a}\mathrm{Td}(-\xi)\bigr)^{\langle N+1\rangle}, \tag{2.12}$$

where the superscript $\langle N+1\rangle$ denotes the summand that is homogeneous of degree $N+1$ in ξ. Since $\mathrm{Td}(\frac{\partial}{\partial h_i}) = 1 +$ a multiple of $\frac{\partial}{\partial h_i}$,

$$\begin{aligned}
&\mathrm{Td}\biggl(\frac{\partial}{\partial h_1}\biggr)\mathrm{Td}\biggl(\frac{\partial}{\partial h_2}\biggr)\bigg|_{h=0} \xi \int\limits_{I(h)} \frac{(\xi x)^N}{N!}\,dx \\
&\quad = \mathrm{Td}\biggl(\frac{\partial}{\partial h_1}\biggr)\bigg|_{h_1=0} \frac{(\xi(b+h_1))^{N+1}}{(N+1)!} - \mathrm{Td}\biggl(\frac{\partial}{\partial h_2}\biggr)\bigg|_{h_2=0} \frac{(\xi(a-h_2))^{N+1}}{(N+1)!} \quad \text{by (2.11)} \\
&\quad = \bigl(\mathrm{Td}(\xi)e^{\xi b} - \mathrm{Td}(-\xi)e^{\xi a}\bigr)^{\langle N+1\rangle} \quad \text{by (2.2)} \\
&\quad = \xi \sum_{x \in I \cap \mathbb{Z}} \frac{(\xi x)^N}{N!} \quad \text{by (2.12).}
\end{aligned}$$

This gives the Euler–Maclaurin formula

$$\mathrm{Td}\biggl(\frac{\partial}{\partial h_1}\biggr)\mathrm{Td}\biggl(\frac{\partial}{\partial h_2}\biggr)\bigg|_{h=0} \int\limits_{I(h)} f = \sum_{x \in I \cap \mathbb{Z}} f \tag{2.13}$$

for the function $f(x) = \frac{\xi^{N+1}x^N}{N!}$, whenever $\xi \neq 0$. Because multiplication by $N!$ and division by the non-zero constant ξ^{N+1} commutes with summation, with integration, and with the infinite order differential operator $\mathrm{Td}(\frac{\partial}{\partial h_1})\mathrm{Td}(\frac{\partial}{\partial h_2})$, we deduce the Euler–Maclaurin formula (2.13) for the monomials $f(x) = x^N$, and hence for all polynomials.

Approach through Euler–Maclaurin for exponentials. In the original approach of Khovanskii–Pukhlikov, one deduces an Euler–Maclaurin formula for polynomials, and, more generally, for (quasi-)polynomials, from a formula for exponentials. (A quasi-polynomial is a sum of products of exponentials by polynomials.) In the one dimensional case, the Euler–Maclaurin formula for exponentials asserts that

$$\mathrm{Td}\left(\frac{\partial}{\partial h_1}\right)\mathrm{Td}\left(\frac{\partial}{\partial h_2}\right)\bigg|_{h=0}\int\limits_{I(h)} e^{\xi x} = \sum_{I\cap\mathbb{Z}} e^{\xi x}, \tag{2.14}$$

or, equivalently, that

$$\mathrm{Td}\left(\frac{\partial}{\partial h_1}\right)\mathrm{Td}\left(\frac{\partial}{\partial h_2}\right)\bigg|_{h=0}\mathcal{I}(h,\xi) = \mathcal{S}(\xi).$$

This formula is true for all ξ such that $|\xi| < 2\pi$. For ξ in the punctured disk

$$\left\{\xi \in \mathbb{C} \;\middle|\; \xi \neq 0,\ |\xi| < 2\pi\right\}, \tag{2.15}$$

the formula follows immediately from (2.6)–(2.8), and from the facts that

$$\mathrm{Td}\left(\frac{\partial}{\partial h_1}\right)\bigg|_{h=0}\frac{e^{\xi(b+h_1)}}{\xi} = \mathrm{Td}(\xi)\frac{e^{\xi b}}{\xi} = \frac{e^{\xi b}}{1-e^{-\xi}} \tag{2.16}$$

and

$$\mathrm{Td}\left(\frac{\partial}{\partial h_2}\right)\bigg|_{h=0}\frac{e^{\xi(a-h_2)}}{\xi} = \mathrm{Td}(-\xi)\frac{e^{\xi a}}{\xi} = -\frac{e^{\xi a}}{1-e^{\xi}}. \tag{2.17}$$

(If $\mathrm{Re}\,\xi \neq 0$ then (2.16) is an Euler–Maclaurin formula for the ray $(-\infty, b]$ or $[b,\infty)$, and similarly for (2.17). However, (2.16) and (2.17) hold for *all* ξ in the set (2.15).)

In (2.16) and (2.17), the left-hand sides converge to the right-hand sides uniformly in ξ on compact subsets of the punctured disk (2.15). This is because the Taylor series of $\mathrm{Td}(\cdot)$ converges uniformly on compact subsets of the disk $\{|\xi| < 2\pi\}$, and the functions $\frac{e^{\xi b}}{\xi}$ and $\frac{e^{\xi a}}{\xi}$ are bounded on compact subsets of the punctured disk (2.15). It follows that in (2.14) the left-hand side converges to the right-hand side uniformly in ξ on compact subsets of (2.15). But the right-hand side and the partial sums of the left-hand side of (2.14) are analytic in ξ for *all* $|\xi| < 2\pi$. It follows from the Cauchy integral formula that the left-hand side of (2.14) converges to the right-hand side, uniformly on compact subsets, on *all* of $\{\xi \mid |\xi| < 2\pi\}$.

It further follows that the infinite sum on the left-hand side of (2.14) can be differentiated with respect to ξ term by term. Hence, the infinite order differential operator on the left-hand side of (2.14) commutes with differentiation with respect to ξ. Since

$$\frac{\partial^k}{\partial\xi^k}\int\limits_{I(h)} e^{\xi x}\,dx = \int\limits_{I(h)} x^k e^{\xi x}\,dx \quad\text{and}\quad \frac{\partial^k}{\partial\xi^k}\sum_{I\cap\mathbb{Z}} e^{\xi x} = \sum_{I\cap\mathbb{Z}} x^k e^{\xi x},$$

we get the Euler–Maclaurin formula (2.13) for the function $f(x) = x^k e^{\xi x}$ by differentiating the left- and right-hand sides of (2.14) k times with respect to ξ.

Approach through Euler–Maclaurin for rays. Yet another approach is to deduce the Euler–Maclaurin formula for an interval directly from the Euler–Maclaurin formula for a ray. See Appendix B.

3. Polar decomposition of a simple polytope

In this section we describe a decomposition of a simple polytope into simple convex polyhedral cones. These cones have apexes at the vertices of the polytope. Each is generated by flipping some of the edge vectors according to a choice of "polarization." so that they all point roughly in the same direction, and removing corresponding facets. For an illustration of this decomposition in the case of a triangle, see Fig. 3.

In this section we present a short direct proof of the polar decomposition of a simple polytope, similar to the one that we gave in [25]. In Section 7 (see (7.4)) we give a variant of this decomposition that applies to a polytope with some facets removed.

Let Δ be a polytope in an n dimensional vector space V and F a face of Δ. The *tangent cone* to Δ at F is

$$\mathbf{C}_F = \{ y + r(x-y) \mid r \geqslant 0,\ y \in F,\ x \in \Delta \}.$$

(Warning: other authors define the tangent cone as $\{r(x-y) \mid r \geqslant 0,\ y \in F,\ x \in \Delta\}$.)

Example 3.1. Consider the triangle in $\mathbb{R}^2$ with vertices $(0,0)$, $(2,0)$, and $(0,1)$. The tangent cone at the hypotenuse is the (closed) half-plane consisting of all points lying below the line extending the hypotenuse; the tangent cone at the top vertex consists of all rays subtended from this vertex and pointing in the direction of the triangle; if F is the face consisting of the triangle itself, then the tangent cone $\mathbf{C}_F$ is the whole plane. See Fig. 4.

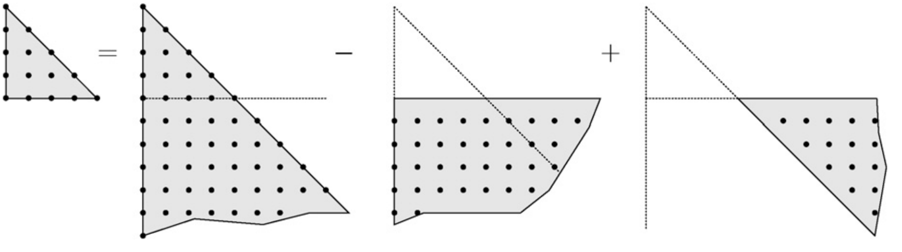

Fig. 3. The polar decomposition theorem.

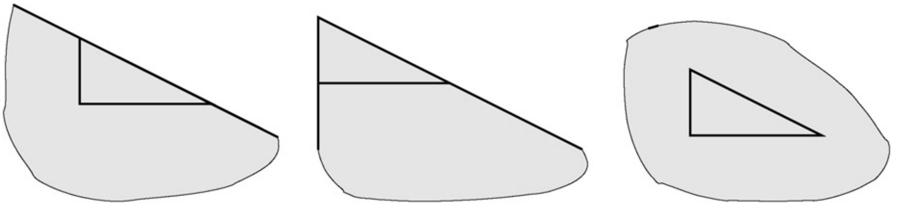

Fig. 4. Triangle and tangent cones.

Let $\sigma_1, \ldots, \sigma_d$ denote the facets (codimension one faces) of Δ. (Warning: Cappell and Shaneson use the symbols σ_i to denote the *dual* objects to the facets, namely, the one dimensional cones in the corresponding fan.) Assume that Δ is simple, so that exactly n facets intersect at each vertex. Let $\mathrm{Vert}(\Delta)$ denote the set of vertices of Δ. For each vertex $v \in \mathrm{Vert}(\Delta)$, let

$$I_v \subset \{1, \ldots, d\}$$

encode the set of facets that meet at v, so that

$$i \in I_v \quad \text{if and only if} \quad v \in \sigma_i.$$

Let $\alpha_{i,v}$, for $i \in I_v$, be edge vectors emanating from v; concretely, assume that $\alpha_{i,v}$ lies along the unique edge at v which is not contained in the facet σ_i. (At the moment, the $\alpha_{i,v}$ are only determined up to positive scalars.) In terms of the edge vectors, the tangent cone at a vertex v is

$$C_v = \left\{ v + \sum_{j \in I_v} x_j \alpha_{j,v} \;\middle|\; x_j \geqslant 0 \text{ for all } j \right\}.$$

The polar decomposition theorem relates the characteristic function of the polytope to the characteristic functions of convex polyhedral cones. As in the one dimensional case (2.6), we cannot just consider the tangent cones, but we must make two modifications. First, we must "polarize" the tangent cones by flipping some of the edge vectors. Second, we must remove some facets.

To carry this out, we choose a vector $\xi \in V^*$ such that the pairings $\langle \xi, \alpha_{i,v} \rangle$ are all non-zero; we call it a "polarizing vector" and think of it as defining the "upward" direction in V. We "polarize" the edge vectors so that they all point "down": for each vertex v of Δ and each edge vector $\alpha_{i,v}$ emanating from v, we define the corresponding *polarized edge vector* to be

$$\alpha^\#_{i,v} = \begin{cases} \alpha_{i,v} & \text{if } \langle \xi, \alpha_{i,v} \rangle < 0, \\ -\alpha_{i,v} & \text{if } \langle \xi, \alpha_{i,v} \rangle > 0. \end{cases} \tag{3.1}$$

Let

$$\varphi_v = \left\{ j \in I_v \;\middle|\; \langle \xi, \alpha_{j,v} \rangle > 0 \right\}$$

denote the set of "upward" edge vectors emanating from v, that is, those edge vectors that get flipped in the polarization process (3.1). The *polarized tangent cone* to Δ at v is obtained from the tangent cone $\mathbf{C}_v$ by flipping the jth edge and removing the jth facet for each $j \in \varphi_v$:

$$\mathbf{C}^\#_v = \left\{ v + \sum_{j \in I_v} x_j \alpha^\#_{j,v} \;\middle|\; \begin{array}{l} x_j \geqslant 0 \text{ if } j \in I_v \setminus \varphi_v, \text{ and} \\ x_j > 0 \text{ if } j \in \varphi_v \end{array} \right\}. \tag{3.2}$$

Recall that the characteristic function of a set $A \subset \mathbb{R}^n$ is

$$\mathbf{1}_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases}$$

Theorem 3.2 *(Lawrence, Varchenko).*

$$\mathbf{1}_{\Delta}(x) = \sum_{v} (-1)^{|\varphi_v|} \mathbf{1}_{\mathbf{C}_v^{\#}}(x). \tag{3.3}$$

This decomposition was proved by Lawrence and Varchenko; see [32,40,41]. A version for non-simple polytopes appeared in [20].

In preparation for our proof of Theorem 3.2 we introduce some notation.

Let $E_1, \ldots, E_N$ be all the different hyperspaces in V^* that are perpendicular to edges of Δ under the pairing between V and V^*. That is,

$$\{E_i \mid 1 \leqslant i \leqslant N\} = \left\{\ker \alpha_{j,v} \;\middle|\; v \in \mathrm{Vert}(\Delta),\ j \in I_v\right\}. \tag{3.4}$$

(For instance, if no two edges of Δ are parallel, then the number N of such hyperplanes is equal to the number of edges of Δ.) A vector ξ can be taken to be a "polarizing vector" if and only if it belongs to the complement

$$V^*_{\Delta} = V^* \setminus (E_1 \cup \cdots \cup E_N). \tag{3.5}$$

The connected components of this complement are called *chambers*. The signs of the pairings $\langle \xi, \alpha_{i,v} \rangle$ only depend on the chamber containing ξ.

Remark 3.3. For a Hamiltonian action of a torus T on a symplectic manifold M, one similarly obtains chambers in the Lie algebra $\mathfrak{t}$ of T from the isotropy weights $\alpha_{j,p}$ at the fixed points for the action. When M is a toric variety corresponding to the polytope Δ, this gives the same notion of chambers as we have just described. For a flag manifold $M \cong G/T$, where G is a compact Lie group and T is a maximal torus, the isotropy weights $\alpha_{j,p}$ are the roots of G, and the corresponding chambers are precisely the interiors of the Weyl chambers.

Now suppose that ξ belongs to exactly one of the "walls" in (3.4). Let e be an edge of Δ that is perpendicular to this wall. Let

$$I_e \subset \{1, \ldots, d\}$$

correspond to the facets whose intersection is e, so that

$$i \in I_e \quad \text{if and only if} \quad e \subset \sigma_i.$$

Let v be an endpoint of e. The edge vectors at v are $\alpha_{j,v}$, for $j \in I_e$, and an edge vector that lies along e, which we denote $\alpha_{e,v}$. Note that

$$\langle \xi, \alpha_{e,v} \rangle = 0.$$

Define

$$\varphi_e = \left\{ j \in I_e \;\middle|\; \langle \xi, \alpha_{j,v} \rangle > 0 \right\}$$

and, for each $j \in I_e$,

$$\alpha^{\#}_{j,v} = \begin{cases} \alpha_{j,v} & \text{if } \langle \xi, \alpha_{j,v} \rangle < 0, \\ -\alpha_{j,v} & \text{if } \langle \xi, \alpha_{j,v} \rangle > 0. \end{cases}$$

Define the *polarized tangent cone* to Δ at the edge e to be

$$\mathbf{C}^{\#}_e = \left\{ v + x_e \alpha_e + \sum_{j \in I_e} x_j \alpha^{\#}_{j,v} \;\middle|\; \begin{array}{l} x_e \in \mathbb{R}, \\ x_j \geqslant 0 \text{ if } j \in I_e \setminus \varphi_e, \text{ and} \\ x_j > 0 \text{ if } j \in \varphi_e. \end{array} \right\}$$

Let v' be the other endpoint of e. We can normalize the edge vectors such that $\alpha_{e,v'} = -\alpha_{e,v}$ and $\alpha_{j,v} - \alpha_{j,v'} \in \mathbb{R}\alpha_{e,v}$. The set φ_e and the cone $C^{\#}_e$ are independent of the choice of endpoint v of e.

Proof of Theorem 3.2. Pick any polarizing vector $\xi \in V^*_\Delta$. Let $v \in \mathrm{Vert}(\Delta)$ be the vertex for which $\langle \xi, v \rangle$ is maximal. Then none of the $\alpha_{j,v}$'s are flipped, and so $\mathbf{C}^{\sharp}_v = \mathbf{C}_v$. For any other vertex $u \in \mathrm{Vert}(\Delta)$, at least one of the $\alpha_{j,u}$'s is flipped, and so $\mathbf{C}^{\sharp}_u \cap \mathbf{C}_u = \emptyset$. So the polytope Δ is contained in the polarized tangent cone $\mathbf{C}^{\#}_v$ at v and is disjoint from the polarized tangent cone $\mathbf{C}^{\#}_u$ for all other $u \in \mathrm{Vert}(\Delta)$, and Eq. (3.3), when evaluated at $x \in \Delta$, reads $1 = 1$.

Suppose now that $x \notin \Delta$. The set of vectors ξ which separate x from Δ, that is, such that

$$\langle \xi, x \rangle > \max_{y \in \Delta} \langle \xi, y \rangle, \tag{3.6}$$

is open in V^*. Choose a polarizing vector $\xi \in V^*_\Delta$ that satisfies (3.6). Then x is not in the polarized tangent cone $\mathbf{C}^{\#}_v$ for any $v \in \mathrm{Vert}(\Delta)$. Equation (3.3) for the polarizing vector ξ, when evaluated at x, reads $0 = 0$.

We finish by showing that, when the polarizing vector ξ crosses a single wall E_j in V^*, the right-hand side of (3.3) does not change.

If E_j is not perpendicular to any of the edge vectors at v, the signs of $\langle \xi, \alpha_{j,v} \rangle$ do not change, so the polarized tangent cone $C^{\#}_v$ does not change as ξ crosses the wall. The vertices whose contributions to the right-hand side of (3.3) change as ξ crosses E_j come in pairs, because each edge of Δ that is perpendicular to E_j has exactly two endpoints.

For each such vertex v, denote by $\mathbf{S}_v(x)$ and $\mathbf{S}'_v(x)$ its contributions to the right-hand side of (3.3) before and after ξ crossed E_j. Let e be an edge perpendicular to E_j and v an endpoint of e. Let $\mathbf{S}^{\#}_e(x)$ be the characteristic function of the polarized tangent cone $C^{\#}_e$ corresponding to the value of ξ as it crosses E_j. The difference $\mathbf{S}_v(x) - \mathbf{S}'_v(x)$ is plus/minus $\mathbf{S}^{\#}_e(x)$. For the *other* endpoint v of e, the difference $\mathbf{S}_v(x) - \mathbf{S}'_v(x)$ is minus/plus $\mathbf{S}^{\#}_e(x)$. So the differences $\mathbf{S}_v(x) - \mathbf{S}'_v(x)$, for the two endpoints v of e, sum to zero. □

Remark 3.4. If we multiply both sides of (3.3) by Lebesgue measure dx we obtain a formula for $\mathbf{1}_\Delta\, dx$ supported on Δ in terms of an alternating sum of the measures $\mathbf{1}_{\mathbf{C}^{\#}_v}(x)\, dx$. This formula (which is a special case of "Filliman duality" [15]) allows us to express the integral of any compactly supported continuous function f over the polytope in terms of its integrals over the cone $\mathbf{C}^{\#}_v$. From the point of view of measure theory, the missing facets of $\mathbf{C}^{\#}_v$ are irrelevant as they have measure zero; what is important is the change in the direction of some of the edges at a vertex and the sign $(-1)^{|\varphi_v|}$ associated to the vertex.

Remark 3.5. If the Δ is a polytope with non-singular fan, then (up to factors of 2π) the measure $\mathbf{1}_\Delta\, dx$ is the Duistermaat–Heckman measure of the associated toric manifold, and the vertices of Δ are the images of the fixed points of the torus acting on this manifold. In this case, the equality (3.3) multiplied by Lebesgue measure dx becomes a special case of the Guillemin–Lerman–Sternberg (G-L-S) formula. The G-L-S formula expresses the Duistermaat–Heckman measure for a Hamiltonian torus action in terms of an alternating sum of the Duistermaat–Heckman measure associated to the linearized action along the components of the fixed point set. See [19, Section 3.3]. This formula, in turn, can be deduced from the fact that a Hamiltonian torus action is cobordant (in an appropriate sense) to a union of its linearized actions along the components of its fixed point set. See [16, Chapter 4]. The role of the polarizing vector $\xi \in V^*$ is played by a vector η in the Lie algebra $\mathbf{t}$ such that the η-component Φ^η of the moment map Φ is proper and bounded from below.

4. Sums and integrals over simple polytopes

Formulas for sums and integrals of exponential functions over simple polytopes appeared in [7,8]. In this section we deduce such formulas from the "polar decomposition."

We work in a vector space V with a lattice $V_\mathbb{Z}$. One may identify V with $\mathbb{R}^n$ and $V_\mathbb{Z}$ with $\mathbb{Z}^n$, but we prefer to use notation that is independent of the choice of a basis. The *dual* lattice is

$$V_\mathbb{Z}^* = \{u \in V^* \mid \langle u, \alpha\rangle \in \mathbb{Z} \text{ for all } \alpha \in V_\mathbb{Z}\}. \tag{4.1}$$

Remark 4.1. When our polytope is viewed as associated to a toric variety, the vector space V is the dual $\mathfrak{t}^*$ of the Lie algebra $\mathfrak{t}$ of a torus T, the lattice $V_\mathbb{Z}$ is the weight lattice in $\mathfrak{t}^*$, and $V_\mathbb{Z}^*$ is the kernel of the exponential map $\mathfrak{t} \to T$.

Let $\mathbf{C}_v$ be a simple convex polyhedral cone in V, that is, a set of the form

$$\mathbf{C}_v = \left\{ v + \sum_{j=1}^n x_j\alpha_j \;\middle|\; x_j \geqslant 0 \text{ for all } j \right\} \tag{4.2}$$

where $\alpha_1, \ldots, \alpha_n$ are a basis of V. Equivalently, we can write

$$\mathbf{C}_v = \{x \mid \langle u_i, x\rangle + \lambda_j \geqslant 0,\ j = 1, \ldots, n\}, \tag{4.3}$$

where $u_1, \ldots, u_n$ are a basis of V^*. We can pass from one description to another by setting $u_1, \ldots, u_n$ to be the dual basis to $\alpha_1, \ldots, \alpha_n$ and vice versa. Geometrically, the vectors $u_i \in V^*$ are the inward normal vectors to the facets of $\mathbf{C}_v$, and they encode the *slopes* of the facets; the real numbers λ_i then determine the *locations* of the facets; the "edge vectors" α_i generate the edges of $\mathbf{C}_v$.

A priori, the u_i's and the α_i's are only determined up to multiplication by positive scalars. To fix a particular normalization, we assume that the cone $\mathbf{C}_v$ is *rational*, that is, that the α_i's can be chosen to be elements of the lattice $V_\mathbb{Z}$, or, equivalently, the u_i's can be chosen to be elements of $V_\mathbb{Z}^*$. We choose the normal vectors u_i to be primitive lattice elements, which means that each u_i is in $V_\mathbb{Z}^*$ and is not equal to the product of an element of $V_\mathbb{Z}^*$ by an integer greater than one. We choose $\alpha_1, \ldots, \alpha_n$ to be the dual basis to $u_1, \ldots, u_n$.

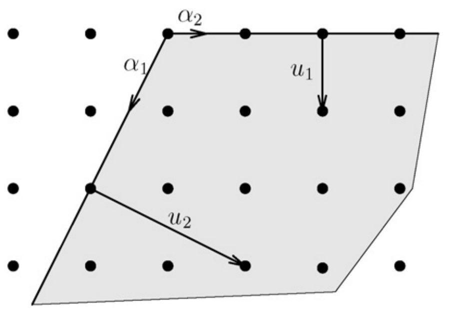

Fig. 5.

Although the u_i's are primitive lattice elements, they may generate a lattice that is coarser than $V_{\mathbb{Z}}^*$. The α_i's then generate a lattice that is *finer* than $V_{\mathbb{Z}}$; in particular, the α_i's themselves might not be in $V_{\mathbb{Z}}$. See Fig. 5. The cone $\mathbf{C}_v$ is *non-singular* if the u_i's generate the lattice $V_{\mathbb{Z}}^*$, or, equivalently, the α_i's generate $V_{\mathbb{Z}}$.

Let dx denote Lebesgue measure on V, normalized so that the measure of $V/V_{\mathbb{Z}}$ is one. Consider an exponential function $f: V \to \mathbb{C}$ given by

$$f(x) = e^{\langle \xi, x\rangle}, \tag{4.4}$$

where $\xi \in V_{\mathbb{C}}^*$ is such that

$$\operatorname{Re}\langle \xi, \alpha_i\rangle < 0 \quad \text{for all } i.$$

Then the integral of f over the cone $\mathbf{C}_v$ and the sum of f over the lattice points in $\mathbf{C}_v$ both converge.

If the cone $\mathbf{C}_v$ is non-singular and v is in the lattice $V_{\mathbb{Z}}$ then the map

$$(t_1, \ldots, t_n) \mapsto v + \sum t_j \alpha_j$$

sends the standard positive orthant

$$\mathbb{R}_+^n = \{(t_1, \ldots, t_n) \mid t_j \geqslant 0 \text{ for all } j\}$$

onto $\mathbf{C}_v$ and sends $\mathbb{R}_+^n \cap \mathbb{Z}^n$ onto $\mathbf{C}_v \cap V_{\mathbb{Z}}$. In particular, it takes the standard Lebesgue measure on $\mathbb{R}^n$ to the measure dx on V. So

$$\begin{aligned} \int_{\mathbf{C}_v} f(x)\,dx &= \int_0^\infty \cdots \int_0^\infty e^{\langle \xi, v + \sum t_j \alpha_j\rangle}\,dt_1 \cdots dt_n \\ &= e^{\langle \xi, v\rangle} \prod_{j=1}^n \int_0^\infty e^{t\langle \xi, \alpha_j\rangle}\,dt_j \\ &= e^{\langle \xi, v\rangle} \prod_{j=1}^n -\frac{1}{\langle \xi, \alpha_j\rangle} \end{aligned} \tag{4.5}$$

and

$$\sum_{\mathbf{C}_v \cap V_{\mathbb{Z}}} f = \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} e^{\langle \xi, v + \sum k_j \alpha_j \rangle}$$
$$= e^{\langle \xi, v \rangle} \prod_{j=1}^{n} \sum_{k_j=0}^{\infty} \left(e^{\langle \xi, \alpha_j \rangle}\right)^{k_j}$$
$$= e^{\langle \xi, v \rangle} \prod_{j=1}^{n} \frac{1}{1 - e^{\langle \xi, \alpha_j \rangle}}. \tag{4.6}$$

A crucial ingredient in extending the Khovanskii–Pukhlikov formula to the case of simple polytopes is an extension of the formulas (4.5) and (4.6) to the case that the cone $\mathbf{C}_v$ is not non-singular.

We associate to $\mathbf{C}_v$ the finite abelian group

$$\Gamma = V_{\mathbb{Z}}^* / \operatorname{span}_{\mathbb{Z}}\{u_i\}.$$

Note that the group Γ is trivial if and only if the cone $\mathbf{C}_v$ is non-singular. Also note that $e^{2\pi i \langle \gamma, x \rangle}$ is well defined whenever $\gamma \in \Gamma$ and $x \in \operatorname{span}_{\mathbb{Z}}\{\alpha_i\}$.

Remark. The toric variety associated to the cone is $\mathbb{C}^n/\Gamma$, where Γ acts on $\mathbb{C}^n$ through the homomorphism $\Gamma \to (S^1)^d$ given by $\gamma \mapsto (e^{2\pi i \langle \gamma, \alpha_1 \rangle}, \dots, e^{2\pi i \langle \gamma, \alpha_n \rangle})$.

We have the following generalization of formula (4.6). Suppose that the vertex of C_v satisfies $v \in \operatorname{span}_{\mathbb{Z}}\{\alpha_i\}$. Then

$$\sum_{x \in \mathbf{C}_v \cap V_{\mathbb{Z}}} e^{\langle \xi, x \rangle} = e^{\langle \xi, v \rangle} \cdot \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} e^{2\pi i \langle \gamma, v \rangle} \prod_{j=1}^{n} \frac{1}{1 - e^{2\pi i \langle \gamma, \alpha_j \rangle} e^{\langle \xi, \alpha_j \rangle}}. \tag{4.7}$$

Proof. The main step is to transform the left-hand side of (4.7) into a summation over elements of the finer lattice $\operatorname{span}_{\mathbb{Z}}\{\alpha_j\}$. For each $x \in \operatorname{span}_{\mathbb{Z}}\{\alpha_i\}$,

$$\gamma \mapsto e^{2\pi i \langle \gamma, x \rangle} \tag{4.8}$$

is a homomorphism from Γ to S^1, and it is trivial if and only if $x \in V_{\mathbb{Z}}$. Frobenius' theorem asserts that, for a finite group Γ, the sum of the values of a non-trivial homomorphism $\Gamma \to S^1$ is zero. It follows that, for $x \in \operatorname{span}_{\mathbb{Z}}\{\alpha_i\}$,

$$\frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} e^{2\pi i \langle \gamma, x \rangle} = \begin{cases} 1 & \text{if } x \in V_{\mathbb{Z}}; \\ 0 & \text{otherwise.} \end{cases} \tag{4.9}$$

By (4.9), the left-hand side of (4.7) is equal to

$$\sum_{x \in \mathbf{C}_v \cap \operatorname{span}_{\mathbb{Z}}\{\alpha_j\}} e^{\langle \xi, x \rangle} \cdot \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} e^{2\pi i \langle \gamma, x \rangle}. \tag{4.10}$$

Writing

$$x = v + \sum k_j \alpha_j,$$

this becomes

$$e^{\langle \xi, v\rangle} \cdot \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} e^{2\pi i \langle \gamma, v\rangle} \prod_{j=1}^{n} \sum_{k=0}^{\infty} e^{2\pi i k \langle \gamma, \alpha_j\rangle} e^{k\langle \xi, \alpha_j\rangle},$$

which is equal to the right-hand side of (4.7) by the formula for the sum of a geometric series. □

We also have the following generalization of formula (4.5).

$$\int_{\mathbf{C}_v} e^{\langle \xi, x\rangle}\, dx = e^{\langle \xi, v\rangle} \cdot \frac{1}{|\Gamma|} \prod_{j=1}^{n} -\frac{1}{\langle \xi, \alpha_j\rangle}. \tag{4.11}$$

Proof. We perform the change of variable $x = v + \sum_{j=1}^{n} t_j \alpha_j$. Then $x \in \mathbf{C}_v$ if and only if $t = (t_1, \ldots, t_n)$ belongs to the positive orthant $\mathbb{R}^n_+$. The inverse transformation is

$$t_j = \langle u_j, x - v\rangle;$$

its Jacobian is $[V^*_{\mathbb{Z}} : \mathrm{span}_{\mathbb{Z}}\, u_j] = |\Gamma|$. So

$$\begin{aligned}\int_{\mathbf{C}_v} e^{\langle \xi, x\rangle}\, dx &= \frac{1}{|\Gamma|} \int_{\mathbb{R}^n_+} e^{\langle \xi, v + \sum t_j \alpha_j\rangle}\, dt_1 \cdots dt_n \\ &= e^{\langle \xi, v\rangle} \cdot \frac{1}{|\Gamma|} \prod_{j=1}^{n} \int_0^{\infty} e^{t\langle \xi, \alpha_j\rangle}\, dt \\ &= e^{\langle \xi, v\rangle} \cdot \frac{1}{|\Gamma|} \prod_{j=1}^{n} -\frac{1}{\langle \xi, \alpha_j\rangle}. \qquad \square\end{aligned}$$

To apply formulas (4.7) and (4.11) to the "polar decomposition" of a polytope, we need to consider "polarized cones." Suppose that $\xi \in V^*_{\mathbb{C}}$ satisfies $\mathrm{Re}\langle \xi, \alpha_j\rangle \neq 0$ for all j. As in Section 3, let

$$\varphi_v = \{ j \mid \mathrm{Re}\langle \xi, \alpha_j\rangle > 0 \};$$

for each j, let

$$\alpha^\sharp_j = \begin{cases} \alpha_j & \text{if } j \notin \varphi_v, \\ -\alpha_j & \text{if } j \in \varphi_v; \end{cases} \tag{4.12}$$

and let

$$\mathbf{C}_v^\sharp = \left\{ v + \sum_{j=1}^{n} x_j \alpha_j^\sharp \;\middle|\; \begin{array}{l} x_j \geqslant 0 \text{ if } j \notin \varphi_v, \text{ and} \\ x_j > 0 \text{ if } j \in \varphi_v. \end{array} \right\}.$$

Then the integral of f over $\mathbf{C}_v^\sharp$ and the sum of f over the lattice points in $\mathbf{C}_v^\sharp$ converge. We compute this integral and this sum:

$$\int_{\mathbf{C}_v^\sharp} e^{\langle \xi, x\rangle}\, dx = e^{\langle \xi, v\rangle} \cdot \frac{1}{|\Gamma|} \prod_{j=1}^{n} -\frac{1}{\langle \xi, \alpha_j^\# \rangle} \quad \text{by (4.11)}$$

$$= (-1)^{|\varphi_v|} e^{\langle \xi, v\rangle} \cdot \frac{1}{|\Gamma|} \prod_{j=1}^{n} -\frac{1}{\langle \xi, \alpha_j \rangle} \quad \text{by (4.12).} \tag{4.13}$$

Because $v \in \operatorname{span}_{\mathbb{Z}}\{\alpha_j\}$,

$$\mathbf{C}_v^\sharp \cap \left(\operatorname{span}_{\mathbb{Z}}\{\alpha_j\}\right) = \left\{ v + \sum m_j \alpha_j^\sharp \;\middle|\; \begin{array}{l} m_j \in \mathbb{Z}, \\ m_j \geqslant 0 \text{ if } j \notin \varphi_v, \\ m_j > 0 \text{ if } j \in \varphi_v \end{array} \right\}$$

$$= \left\{ v_{\text{shift}} + \sum m_j \alpha_j^\sharp \;\middle|\; \begin{array}{l} m_j \in \mathbb{Z}, \\ m_j \geqslant 0 \text{ for all } j \end{array} \right\}$$

where

$$v_{\text{shift}} = v + \sum_{j \in \varphi_v} \alpha_j^\sharp. \tag{4.14}$$

So

$$\mathbf{C}_v^\sharp \cap V_{\mathbb{Z}} = \overline{\mathbf{C}}_{v,\text{shift}}^\sharp \cap V_{\mathbb{Z}} \tag{4.15}$$

where

$$\overline{\mathbf{C}}_{v,\text{shift}}^\sharp = \left\{ v_{\text{shift}} + \sum x_j \alpha_j^\sharp \;\middle|\; x_j \geqslant 0 \text{ for all } j \right\}.$$

We have

$$\sum_{x \in \mathbf{C}_v^\sharp \cap V_{\mathbb{Z}}} e^{\langle \xi, x\rangle} = \sum_{x \in \overline{\mathbf{C}}_{v,\text{shift}}^\sharp \cap V_{\mathbb{Z}}} e^{\langle \xi, x\rangle} \quad \text{by (4.15)}$$

$$= e^{\langle \xi, v_{\text{shift}}\rangle} \cdot \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} e^{2\pi i \langle \gamma, v_{\text{shift}}\rangle} \prod_{j=1}^{n} \frac{1}{1 - e^{2\pi i \langle \gamma, \alpha_j^\sharp\rangle} e^{\langle \xi, \alpha_j^\sharp\rangle}} \quad \text{by (4.7)}$$

$$= e^{\langle \xi, v\rangle} \cdot \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} e^{2\pi i \langle \gamma, v\rangle} \prod_{j \notin \varphi_v} \frac{1}{1 - e^{2\pi i \langle \gamma, \alpha_j\rangle} e^{\langle \xi, \alpha_j\rangle}}$$

$$\times \prod_{j\in\varphi_v} \frac{e^{-2\pi i\langle\gamma,\alpha_j\rangle} e^{-\langle\xi,\alpha_j\rangle}}{1-e^{-2\pi i\langle\gamma,\alpha_j\rangle} e^{-\langle\xi,\alpha_j\rangle}} \quad \text{by (4.14) and (4.12)}$$

$$= (-1)^{|\varphi_v|} e^{\langle\xi,v\rangle}\cdot\frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma} e^{2\pi i\langle\gamma,v\rangle}\prod\frac{1}{1-e^{2\pi i\langle\gamma,\alpha_j\rangle}e^{\langle\xi,\alpha_j\rangle}} \tag{4.16}$$

by applying the relation

$$\frac{e^x}{1-e^x} = -\frac{1}{1-e^{-x}}$$

to $x = -2\pi i\langle\gamma,\alpha_j\rangle - \langle\xi,\alpha_j\rangle$ for $j\in\varphi_v$.

We can now reproduce Brion–Vergne's formulas for simple polytopes. Let $\Delta\subset V$ be a simple polytope. Suppose that $\xi\in V^*_{\mathbb{C}}$ satisfies $\operatorname{Re}\langle\xi,\alpha_{j,v}\rangle\neq 0$ for all $v\in\operatorname{Vert}(\Delta)$ and all $j\in I_v$. With the notation of Section 3,

$$\int_\Delta e^{\langle\xi,x\rangle}\,dx = \sum_{v\in\operatorname{Vert}(\Delta)}(-1)^{|\varphi_v|}\int_{\mathbf{C}^\#_v} e^{\langle\xi,x\rangle}\,dx \quad\text{by (3.3)}$$

$$= \sum_{v\in\operatorname{Vert}(\Delta)} e^{\langle\xi,v\rangle}\cdot\frac{1}{|\Gamma_v|}\prod_{j\in I_v} -\frac{1}{\langle\xi,\alpha_{j,v}\rangle} \quad\text{by (4.13).} \tag{4.17}$$

Similar formulas appeared in [7, Proposition 3.10] and [8, p. 801, Theorem, part (ii)].

Also,

$$\sum_{x\in\Delta\cap V_{\mathbb{Z}}} e^{\langle\xi,x\rangle} = \sum_{v\in\operatorname{Vert}(\Delta)}(-1)^{|\varphi_v|}\sum_{x\in\mathbf{C}^\#_v\cap V_{\mathbb{Z}}} e^{\langle\xi,x\rangle} \quad\text{by (3.3)}$$

$$= \sum_{v\in\operatorname{Vert}(\Delta)} e^{\langle\xi,v\rangle}\cdot\frac{1}{|\Gamma_v|}\sum_{\gamma\in\Gamma_v} e^{2\pi i\langle\gamma,v\rangle}\prod_{j\in I_v}\frac{1}{1-e^{2\pi i\langle\gamma,\alpha_{j,v}\rangle}e^{\langle\xi,\alpha_{j,v}\rangle}} \quad\text{by (4.16)}$$

$$= \sum_{v\in\operatorname{Vert}(\Delta)} e^{\langle\xi,v\rangle}\operatorname{Td}_v\bigl(\{-\langle\xi,\alpha_{j,v}\rangle\}\bigr)\cdot\frac{1}{|\Gamma_v|}\prod_{j\in I_v} -\frac{1}{\langle\xi,\alpha_{j,v}\rangle} \tag{4.18}$$

where

$$\operatorname{Td}_v(S) = \sum_{\gamma\in\Gamma_v} e^{2\pi i\langle\gamma,v\rangle}\prod_{j\in I_v}\frac{S_j}{1-e^{2\pi i\langle\gamma,\alpha_{j,v}\rangle}e^{-S_j}} \quad \text{for } S=\{S_j\}_{j\in I_v}. \tag{4.19}$$

Similar formulas appeared in [7, Proposition 3.9] and [8, p. 801, Theorem, part (iii)].

Remark 4.2. We proved (4.17) for ξ outside the real hyperplanes

$$\operatorname{Re}\langle\xi,\alpha_{j,v}\rangle = 0, \quad v\in\operatorname{Vert}(\Delta),\ j\in I_v \tag{4.20}$$

in $V_{\mathbb{C}}^*$. However, the left-hand side of (4.17) is analytic for all $\xi \in V_{\mathbb{C}}^*$, and the right-hand side is analytic outside the *complex* hyperplanes

$$\langle \xi, \alpha_{j,v} \rangle = 0.$$

By analytic continuation (4.17) continues to hold for all ξ outside these complex hyperplanes. Similarly, we proved (4.18) for ξ outside the real hyperplanes (4.20), but by analytic continuation it remains true for all ξ outside the *complex* hyperplanes

$$\langle \xi, \alpha_{j,v} \rangle = 2\pi i \langle y, \alpha_{j,v} \rangle, \quad y \in V_{\mathbb{Z}}^*. \tag{4.21}$$

(Notice that these complex hyperplanes are contained in the real hyperplanes (4.20).) We are grateful to A. Khovanskii for calling our attention to this approach.

Remark 4.3. Note that the function $\mathrm{Td}_v(S)$ is analytic on the polydisk

$$|S_j| < b_j, \quad j \in I_v,$$

in $\mathbb{C}^{I_v}$, where

$$b_j = \min_{\substack{y \in V_{\mathbb{Z}}^* \\ \langle y, \alpha_{j,v} \rangle \neq 0}} \big\{ \big| 2\pi \langle y, \alpha_{j,v} \rangle \big| \big\}.$$

5. Euler–Maclaurin formulas for a simple polytope

In this section we present Euler–Maclaurin formulas for simple lattice polytopes in arbitrary dimensions. As in the previous section, we work with an n dimensional vector space V with a lattice $V_{\mathbb{Z}}$. Let Δ be a convex polytope in V with d facets, given by

$$\Delta = \bigcap_{i=1}^{d} \big\{ x \mid \langle x, u_i \rangle + \lambda_i \geqslant 0 \big\}. \tag{5.1}$$

The vectors $u_i \in V^*$ are inward normal vectors to the facets, and they encode the slopes of the facets; the real numbers λ_i determine the location of the facets. As before, we assume that the slopes of the facets are rational, and we choose the normal vectors u_i to be primitive elements of the dual lattice $V_{\mathbb{Z}}^*$. We assume that the polytope Δ is simple, meaning that exactly n facets intersect at each vertex. We also assume that the λ_i's are integers.

Remark 5.1. If the vertices of Δ are lattice points, the λ_i's are integers. If Δ has a non-singular fan and the λ_i's are integers, then the vertices of Δ are lattice points. However, if Δ is simple but does not have a non-singular fan, its vertices might not be lattice points even if the λ_i are all integers.

For h near 0, the expanded polytope

$$\Delta(h) = \bigcap_{i=1}^{d} \big\{ x \in V \bigm| \langle x, u_i \rangle + \lambda_i + h_i \geqslant 0 \big\} \tag{5.2}$$

is obtained from Δ by shifting the half-spaces defining its facets without changing their slopes.

As before, we normalize Lebesgue measure on V so that a fundamental domain with respect to the lattice $V_{\mathbb{Z}}$ has measure one. The integral of a function f over the expanded polytope $\Delta(h)$ is a function of $h_1, \dots, h_d$.

For a polynomial P in d variables, the expression

$$P\left(\frac{\partial}{\partial h_1}, \dots, \frac{\partial}{\partial h_d}\right)\bigg|_{h=0} \int_{\Delta(h)} f \tag{5.3}$$

makes sense if the integral is a smooth function of h for h near 0. If P is a power series in d variables, the expression (5.3) makes sense if the resulting series converges. If P is an analytic function in d variables, we interpret the expression (5.3) by expanding P into its Taylor series about the origin.

In what follows, the function f can be taken to be the product of a polynomial function with an exponential function of the form $e^{\langle \xi, x\rangle}$ where $\xi \in V_{\mathbb{C}}^*$ is sufficiently small.

Khovanskii–Pukhlikov's formula for a polytope with a non-singular fan. The formula of Khovanskii and Pukhlikov (see Section 4 of [30]), translated to our notation, is the following formula:

$$\sum_{\Delta \cap V_{\mathbb{Z}}} f = \mathrm{Td}\left(\frac{\partial}{\partial h_1}\right) \cdots \mathrm{Td}\left(\frac{\partial}{\partial h_d}\right)\bigg|_{h=\lambda} \int_{\Delta(h)} f, \tag{5.4}$$

where the polytope Δ is integral and has a non-singular fan.

Finite groups associated to the faces of a simple rational polytope. The facets of the polytope Δ are

$$\sigma_i = \{x \in \Delta \mid \langle u_i, x\rangle + \lambda_i = 0\}, \quad i = 1, \dots, d.$$

Because the polytope Δ is simple, each face F of Δ can be uniquely described as an intersection of facets. We let $I_F \subset \{1, \dots, d\}$ denote the subset such that

$$F = \bigcap_{i \in I_F} \sigma_i.$$

The number of elements in I_F is equal to the codimension of F. The relation

$$F \mapsto I_F$$

is order (inclusion) reversing.

For each vertex v of Δ, the vectors

$$u_i, \quad i \in I_v,$$

form a basis of V^*. Let

$$\alpha_{i,v}, \quad i \in I_v,$$

be the dual basis. The $\alpha_{i,v}$'s are edge vectors at v, that is, they point in the directions of the edges emanating from v.

The vector space V_F normal to a face F is the quotient of V by $T_F = \{r(x-y) \mid x, y \in F, r \in \mathbb{R}\}$. Its dual is the subspace

$$V_F^* := \operatorname{span}\{u_j \mid j \in I_F\} \tag{5.5}$$

of V^*. Let $\alpha_{j,F}$, $j \in I_F$, be the basis of V_F that is dual to the basis u_j, $j \in I_F$, of V_F^*.

To each face F of Δ we associate a finite abelian group Γ_F in the following way. The lattice

$$\operatorname{span}_{\mathbb{Z}}\{u_i \mid i \in I_F\} \subset V_F^* \cap V_{\mathbb{Z}}^*$$

is a sublattice of $V_F^* \cap V_{\mathbb{Z}}^*$ of finite index. The finite abelian group associated to the face F is the quotient

$$\Gamma_F := \left(V_F^* \cap V_{\mathbb{Z}}^*\right) / \operatorname{span}_{\mathbb{Z}}\{u_i \mid i \in I_F\}. \tag{5.6}$$

For each $\gamma \in \Gamma_F$ and $j \in I_F$, the pairing $\langle \gamma, \alpha_{j,F} \rangle$ is well defined modulo 1, so

$$e^{2\pi i \langle \gamma, \alpha_{j,F} \rangle}$$

is well defined.

Remark 5.2. The group Γ_F measures the singularity of the toric variety associated to Δ along the stratum corresponding to F.

If $F \subseteq E$ are faces of Δ, so that $I_E \subseteq I_F$, then $\{u_i \mid i \in I_E\}$ is a subset of $\{u_i \mid i \in I_F\}$. Because these sets are bases of V_E^* and V_F^*, we have

$$V_E^* \subseteq V_F^*,$$

and

$$V_E^* \cap \operatorname{span}_{\mathbb{Z}}\{u_i \mid i \in I_F\} = \operatorname{span}_{\mathbb{Z}}\{u_i \mid i \in I_E\}.$$

Hence, the natural map from Γ_E to Γ_F is one-to-one, and provides us with a natural inclusion map:

$$\text{if } F \subseteq E \quad \text{then } \Gamma_E \subseteq \Gamma_F.$$

We define

$$\Gamma_F^\flat = \Gamma_F \setminus \bigcup_{\text{faces } E \text{ such that } E \supsetneq F} \Gamma_E. \tag{5.7}$$

Finally, note that, for each face F and element $\gamma \in \Gamma_F$, the number $e^{-2\pi i \langle \gamma, x \rangle}$ is the same for all $x \in F$; we denote this number

$$e^{-2\pi i \langle \gamma, F \rangle}.$$

Guillemin and Brion–Vergne formulas for a simple polytope. On any linear subspace A of V with rational slope we normalize Lebesgue measure so that a fundamental domain with respect to the lattice $A \cap V_{\mathbb{Z}}$ has measure one. We shift this measure to any affine translate of A. Integration over each face F of Δ is defined with respect to these measures.

For each face F of Δ, let

$$F(h) = \Delta(h) \cap \bigcap_{i \in I_F} \bigl\{ x \bigm| \langle u_i, x\rangle + \lambda_i + h_i = 0 \bigr\} \tag{5.8}$$

denote the corresponding face of the expanded polytope $\Delta(h)$.

Guillemin gives an Euler–Maclaurin formula for a polytope when the polytope is expressed as the set of solutions of the equation $k_1\alpha_1 + \cdots + k_d\alpha_d = \mu$, $k_1, \ldots, k_d \in \mathbb{R}_{\geqslant 0}$, for some fixed integral vectors $\alpha_1, \ldots, \alpha_d, \mu$. (See Theorem 1.3 and formula (3.28) of [18].) When translated to our setup, his formula becomes the following formula:

$$\sum_{\Delta \cap V_{\mathbb{Z}}} f = \sum_F \frac{1}{|\Gamma_F|} \sum_{\gamma \in \Gamma_F^\flat} e^{2\pi i \langle \gamma, F\rangle} \prod_{j \notin I_F} \frac{\frac{\partial}{\partial h_j}}{1 - e^{-\partial/\partial h_j}} \prod_{j \in I_F} \frac{1}{1 - e^{2\pi i \langle \gamma, \alpha_{j,F}\rangle} e^{-\partial/\partial h_j}} \bigg|_{h=0} \int_{F(h)} f, \tag{5.9}$$

where the polytope Δ is simple and is given by (5.1) where all the λ_i's are integers.

Finally, the formula of Brion and Vergne in our notation is

$$\sum_{\Delta \cap V_{\mathbb{Z}}} f = \sum_F \sum_{\gamma \in \Gamma_F^\flat} e^{2\pi i \langle \gamma, F\rangle} \prod_{j \notin I_F} \frac{\frac{\partial}{\partial h_j}}{1 - e^{-\partial/\partial h_j}} \prod_{j \in I_F} \frac{\frac{\partial}{\partial h_j}}{1 - e^{2\pi i \langle \gamma, \alpha_{j,F}\rangle} e^{-\partial/\partial h_j}} \bigg|_{h=0} \int_{\Delta(h)} f. \tag{5.10}$$

See [7, Theorem 2.15] (where Δ is simple and integral), and, more generally, [8].

Remark 5.3. If the polytope Δ is integral (which is a stronger requirement than the assumption that the λ_i's be integers) then $e^{2\pi i \langle \gamma, F\rangle} = 1$ for each face F and each $\gamma \in \Gamma_F$.

We now give a self-contained statement and an elementary proof of the Guillemin–Brion–Vergne formulas.

Theorem 1. *Let V be a vector space with a lattice $V_{\mathbb{Z}}$. Let $V_{\mathbb{Z}}^* \subset V^*$ be the dual lattice. Let $\Delta \subset V$ be a simple rational polytope with d facets. Let $u_1, \ldots, u_d \in V_{\mathbb{Z}}^*$ be the primitive inward normals to the facets of Δ. Let $\lambda_1, \ldots, \lambda_d$ be the real numbers so that*

$$\Delta = \bigcap_{i=1}^d \bigl\{ x \bigm| \langle u_i, x\rangle + \lambda_i \geqslant 0 \bigr\}.$$

Suppose that $\lambda_1, \ldots, \lambda_d$ are integers. For $h = (h_1, \ldots, h_d)$, let

$$\Delta(h) = \bigcap_{i=1}^d \bigl\{ x \bigm| \langle u_i, x\rangle + \lambda_i + h_i \geqslant 0 \bigr\}.$$

For each face F of Δ, let $I_F \subset \{1, \dots, d\}$ be the subset such that F consists of those $x \in \Delta$ for which $\langle u_i, x\rangle + \lambda_i = 0$ for all $i \in I_F$. Let $\alpha_{i,F}$, for $i \in I_F$, be the basis of $V/T_F = V/\mathbb{R}(F - F)$ that is dual to the basis u_i, $i \in I_F$, of $V_F^* = T_F^0$. (In particular, if $v \in \mathrm{Vert}(\Delta)$ and $i \in I_v$ then $\alpha_{i,v}$ are the edge vectors emanating from v.) Let

$$\Gamma_F = \left(V_F^* \cap V_{\mathbb{Z}}^*\right) / \operatorname{span}_{\mathbb{Z}}\{u_i \mid i \in I_F\}$$

be the finite group associated to the face F, and let $\Gamma_F^\flat = \Gamma_F \setminus \bigcup \Gamma_E$, where the union is over all faces E such that $E \supsetneq F$. Let

$$\mathrm{Td}_\Delta(S_1, \dots, S_d) = \sum_F \sum_{\gamma \in \Gamma_F^\flat} e^{2\pi i \langle \gamma, F\rangle} \prod_{j \notin I_F} \frac{S_j}{1 - e^{-S_j}} \prod_{j \in I_F} \frac{S_j}{1 - e^{2\pi i \langle \gamma, \alpha_{j,F}\rangle} e^{-S_j}}.$$

Let $f : V \to \mathbb{C}$ be a quasi-polynomial function, that is, a linear combination of functions of the form

$$f(x) = p(x) e^{\langle \xi, x\rangle}$$

where $p : V \to \mathbb{C}$ are polynomial functions and where the exponents $\xi \in V_{\mathbb{C}}^*$ satisfy

$$\left|\langle \xi, \alpha_{j,v}\rangle\right| < 2\pi \left|\langle y, \alpha_{j,v}\rangle\right| \tag{5.11}$$

for each vertex v, each edge vector $\alpha_{j,v}$, $j \in I_v$, and each $y \in V_{\mathbb{Z}}^*$ such that $\langle y, \alpha_{j,v}\rangle \neq 0$. (Each of the sets $\{\langle y, \alpha_{j,v}\rangle \mid y \in V_{\mathbb{Z}}^*\}$ is discrete, so the set of ξ's that satisfy these conditions is a neighborhood of the origin in $V_{\mathbb{C}}^*$.) Then

$$\sum_{x \in \Delta \cap V_{\mathbb{Z}}} f(x) = \mathrm{Td}_\Delta\left(\frac{\partial}{\partial h_1}, \dots, \frac{\partial}{\partial h_d}\right)\bigg|_{h=0} \int_{\Delta(h)} f(x)\, dx. \tag{5.12}$$

Remark 5.4. The right-hand side of (5.12) is an infinite sequence. The theorem asserts that this sequence converges to the left-hand side. In Appendix B we show that this convergence is uniform on compact subsets of (5.11).

The proof of Theorem 1 uses the following characterization of $\Gamma_F^\flat$.

Lemma 5.5. *Let F be a face of Δ.*

(1) *If $\gamma \in \Gamma_F$ and v is a vertex of Δ such that $v \in F$, then*

$$e^{2\pi i \langle \gamma, \alpha_{j,v}\rangle} = \begin{cases} e^{2\pi i \langle \gamma, \alpha_{j,F}\rangle} & \text{for all } j \in I_F, \\ 1 & \text{for all } j \in I_v \setminus I_F. \end{cases}$$

(2) *If $j \in I_F$ and $\gamma \in \Gamma_F^\flat$, then $e^{2\pi i \langle \gamma, \alpha_{j,F}\rangle} \neq 1$.*
(3) *For $\gamma \in \Gamma_F$,*

$$\gamma \in \Gamma_F^\flat \quad \text{if and only if} \quad e^{2\pi i \langle \gamma, \alpha_{j,F}\rangle} \neq 1 \quad \text{for all } j \in I_F.$$

Proof. Let $y \in V_F^* \cap V_{\mathbb{Z}}^*$ be a representative of γ (see (5.6)). Then, by definition, $e^{2\pi i\langle\gamma,\alpha_{j,F}\rangle} = e^{2\pi i\langle y,\alpha_{j,F}\rangle}$. Because $y \in V_F^*$, and by (5.5), there exist real numbers a_j, for $j \in I_F$, such that $y = \sum_{j\in I_F} a_j u_j$. Then

$$\langle y, \alpha_{j,F}\rangle = a_j \quad \text{for all } j \in I_F. \tag{5.13}$$

Defining $a_j = 0$ for $j \in I_v \setminus I_F$, we also have $y = \sum_{j\in I_v} a_j u_j$, and

$$\langle y, \alpha_{j,v}\rangle = a_j \quad \text{for all } j \in I_v. \tag{5.14}$$

In particular,

$$\langle y, \alpha_{j,v}\rangle = 0 \quad \text{for all } j \in I_v \setminus I_F. \tag{5.15}$$

Part (1) follows from (5.13)–(5.15).

Fix $j \in I_F$. Suppose $e^{2\pi i\langle\gamma,\alpha_{j,F}\rangle} = 1$. Then we can choose a representative $y \in V_F^* \cap V_{\mathbb{Z}}^*$ of γ such that $\langle y, \alpha_{j,F}\rangle = 0$. Writing $y = \sum_{l\in I_F} a_l u_l$, we have $a_j = \langle y, \alpha_{j,F}\rangle = 0$. Let E be the face of Δ such that $I_E = I_F \setminus \{j\}$. Then $y = \sum_{l\in I_E} a_l u_l$, so, by (5.5), y is in V_E^*, and so $\gamma \in \Gamma_E$. In particular, by (5.7), $\gamma \notin \Gamma_F^\flat$. This proves part (2).

Let $\gamma \in \Gamma_F$. By part (2), if $\gamma \in \Gamma_F^\flat$ then $e^{2\pi i\langle\gamma,\alpha_{j,F}\rangle} \neq 1$ for all $j \in I_F$. Conversely, suppose that $\Gamma \notin \Gamma_F^\flat$. Then, by (5.7), there exists a face E such that $\gamma \in \Gamma_E$ and $E \supsetneq F$. Let $j \in I_F \setminus I_E$. Let v be any vertex of F (and hence of E). Then $e^{2\pi i\langle\gamma,\alpha_{j,F}\rangle} = e^{2\pi i\langle\gamma,\alpha_{j,v}\rangle} = 1$, where the first equality follows from part (1) for the face F, and where the second equality follows from part (1) for the face E. This proves part (3). □

Claim. *For each* $v \in \mathrm{Vert}(\Delta)$,

$$\mathrm{Td}_\Delta(S_1,\dots,S_d) = \mathrm{Td}_v\big(\{S_j\}_{j\in I_v}\big) + \textit{multiples of } S_j \textit{ for } j \notin I_v. \tag{5.16}$$

Proof. Recall that

$$\mathrm{Td}_\Delta(S_1,\dots,S_d) = \sum_F \sum_{\gamma\in\Gamma_F^\flat} e^{2\pi i\langle\gamma,F\rangle} \prod_{j\notin I_F} \frac{S_j}{1-e^{-S_j}} \prod_{j\in I_F} \frac{S_j}{1-e^{2\pi i\langle\gamma,\alpha_{j,F}\rangle}e^{-S_j}}.$$

By part (2) of Lemma 5.5, for each $\gamma \in \Gamma_F^\flat$ and $j \in I_F$,

$$\frac{S_j}{1-e^{2\pi i\langle\gamma,\alpha_{j,F}\rangle}e^{-S_j}} = \text{a multiple of } S_j.$$

Because $v \notin F$ implies that there exists $j \in I_F$ such that $j \notin I_v$, and because $\frac{S_j}{1-e^{-S_j}} = 1 +$ a multiple of S_j,

$$\mathrm{Td}_\Delta(S_1,\dots,S_d) = \sum_{\substack{F \text{ such that} \\ v\in F}} \sum_{\gamma\in\Gamma_F^\flat} e^{2\pi i\langle\gamma,F\rangle} \prod_{j\in I_v\setminus I_F} \frac{S_j}{1-e^{-S_j}} \prod_{j\in I_F} \frac{S_j}{1-e^{2\pi i\langle\gamma,\alpha_{j,v}\rangle}e^{-S_j}}$$
$$+ \text{multiples of } S_j \text{ for } j \notin I_v.$$

By (5.7),

$$\Gamma_v = \bigsqcup_{\substack{F \text{ such that} \\ v \in F}} \Gamma_F^{\flat}.$$

Also, $e^{2\pi i\langle\gamma,F\rangle} = e^{2\pi i\langle\gamma,v\rangle}$ whenever $v \in F$. By this and part (1) of Lemma 5.5,

$$\mathrm{Td}_\Delta(S_1,\dots,S_d) = \sum_{\gamma\in\Gamma_v} e^{2\pi i\langle\gamma,v\rangle} \prod_{j\in I_v} \frac{S_j}{1-e^{2\pi i\langle\gamma,\alpha_{j,v}\rangle}e^{-S_j}} + \text{multiples of } S_j \text{ for } j \notin I_v.$$

By the definition (4.19) of Td_v, this exactly shows (5.16). □

Proof of Theorem 1 *(Khovanskii–Pukhlikov approach).* Let $\Omega \subset \mathbb{R}^d$ be the set of all $h \in \mathbb{R}^d$ that are sufficiently small so that the polytope $\Delta(h)$ has the same combinatorics as the polytope Δ (i.e., the same subsets $I_F \subset \{1,\dots,d\}$ correspond to faces). The vertices of the expanded polytope $\Delta(h)$ are then

$$v(h) = v - \sum_{j\in I_v} h_j\alpha_{j,v}, \tag{5.17}$$

and we have

$$\frac{\partial}{\partial h_j} e^{\langle\xi,v(h)\rangle} = -\langle\xi,\alpha_{j,v}\rangle e^{\langle\xi,v(h)\rangle} \tag{5.18}$$

for any $j \in I_v$. Let

$$\mathcal{I}_\Delta(h,v) = \int_{\Delta(h)} e^{\langle\xi,x\rangle}\,dx \quad \text{and} \quad \mathcal{S}_\Delta(\xi) = \sum_{x\in\Delta\cap V_{\mathbb{Z}}} e^{\langle\xi,x\rangle}.$$

By (4.17),

$$\mathcal{I}_\Delta(h,\xi) = \sum_{v\in\mathrm{Vert}(\Delta)} e^{\langle\xi,v(h)\rangle} \cdot \frac{1}{|\Gamma_v|} \prod_{j\in I_v} -\frac{1}{\langle\xi,\alpha_{j,v}\rangle} \tag{5.19}$$

for all $h \in \Omega$ and $\xi \in V_{\mathbb{C}}^*$ that lies outside the complex hyperplanes

$$\langle\xi,\alpha_{j,v}\rangle = 0, \quad v \in \mathrm{Vert}(\Delta), \quad j \in I_v. \tag{5.20}$$

By (4.18),

$$\mathcal{S}_\Delta(\xi) = \sum_{v\in\mathrm{Vert}(\Delta)} e^{\langle\xi,v\rangle}\mathrm{Td}_v\big(\{-\langle\xi,\alpha_{j,v}\rangle\}_{j\in I_v}\big) \cdot \frac{1}{|\Gamma_v|} \prod_{j\in I_v} -\frac{1}{\langle\xi,\alpha_{j,v}\rangle}, \tag{5.21}$$

for all $\xi \in V_{\mathbb{C}}^*$ that lie outside the hyperplanes (4.21). See Remark 4.2.

By (5.16), (5.17), and (5.19),

$$\begin{aligned}
&\mathrm{Td}_\Delta\left(\frac{\partial}{\partial h_1},\ldots,\frac{\partial}{\partial h_d}\right)\mathcal{I}_\Delta(h,\xi) \\
&\quad=\sum_{v\in\mathrm{Vert}(\Delta)}\mathrm{Td}_v\left(\left\{\frac{\partial}{\partial h_j}\right\}_{j\in I_v}\right)e^{\langle\xi,v(h)\rangle}\cdot\frac{1}{|\Gamma_v|}\prod_{j\in I_v}-\frac{1}{\langle\xi,\alpha_{j,v}\rangle}. \qquad (5.22)
\end{aligned}$$

So the partial sums of the series (5.22) are

$$\sum_v P_{m,v}\big(\{-\langle\xi,\alpha_{j,v}\rangle\}_{j\in I_v}\big)e^{\langle\xi,v(h)\rangle}\cdot\frac{1}{|\Gamma_v|}\prod_{j\in I_v}-\frac{1}{\langle\xi,\alpha_{j,v}\rangle} \qquad (5.23)$$

where $P_{m,v}$ are the Taylor polynomials of Td_v. By Remark 4.3, $P_{m,v}(\{S_j\}_{j\in I_v})$ converges to $\mathrm{Td}_v(\{S_j\}_{j\in I_v})$ uniformly on compact subsets of the polydisk

$$\big\{S\in\mathbb{C}^{I_v}\ \big|\ |S_j|<b_{j,v}\text{ for all } j\in I_v\big\}, \qquad (5.24)$$

where

$$b_{j,v}=\min_{\substack{y\in V_{\mathbb{Z}}^*\\ \langle y,\alpha_{j,v}\rangle\neq 0}} 2\pi\langle y,\alpha_{j,v}\rangle. \qquad (5.25)$$

Because the functions

$$e^{\langle\xi,v\rangle}\cdot\frac{1}{|\Gamma_v|}\prod_{j\in I_v}-\frac{1}{\langle\xi,\alpha_{j,v}\rangle}$$

are (continuous, hence) bounded on compact subsets of the set of ξ's that lie outside the complex hyperplanes given by (5.20), the partial sums (5.23) of the series (5.22) converge to

$$\sum_{v\in\mathrm{Vert}(\Delta)}\mathrm{Td}_v\big(\{-\langle\xi,\alpha_{j,v}\rangle\}_{j\in I_v}\big)e^{\langle\xi,v(h)\rangle}\cdot\frac{1}{|\Gamma_v|}\prod_{j\in I_v}-\frac{1}{\langle\xi,\alpha_{j,v}\rangle}$$

uniformly on compact subsets of the set of $(h,\xi)\in\Omega\times V_{\mathbb{C}}^*$ such that ξ is outside the hyperplanes (5.20) and satisfies

$$\big|\langle\xi,\alpha_{j,v}\rangle\big|<b_{j,v}\quad\text{for all } v\in\mathrm{Vert}(\Delta)\text{ and } j\in I_v. \qquad (5.26)$$

Setting $h=0$, by (5.21), we get

$$\mathrm{Td}_\Delta\left(\frac{\partial}{\partial h_1},\ldots,\frac{\partial}{\partial h_d}\right)\bigg|_{h=0}\mathcal{I}_\Delta(h,\xi)=\mathcal{S}_\Delta(\xi), \qquad (5.27)$$

and that the left-hand side of (5.27) converges to the right-hand side of (5.27) uniformly in ξ on compact subsets of the set of $\xi\in V_{\mathbb{C}}^*$ that lie outside the hyperplanes (5.20) and in the set (5.26).

However, the right-hand side of (5.27) and the partial sums of the left-hand side of (5.27) are analytic in ξ for *all* ξ.

Recall that, as a consequence of Cauchy's integral formula, if $g_\nu(\xi)$ is a sequence of complex analytic functions on an open subset U of $\mathbb{C}^n$, $g(\xi)$ is a complex analytic function on U, $g_\nu(\xi)$ converges to $g(\xi)$ in $U \setminus E$ where E is a locally finite union of complex hyperplanes, and this convergence is uniform on compact subsets of $U \setminus E$, then $g_\nu(\xi)$ converges to $g(\xi)$ for *all* $\xi \in U$, uniformly on compact subsets of U.

It follows that (5.27) holds for *all* $\xi \in V_{\mathbb{C}}^*$ that satisfy (5.26), and, moreover, the left-hand side of (5.27) converges to the right-hand side uniformly in ξ on compact subsets of (5.26). This gives the Euler–Maclaurin formula for exponential functions $e^{\langle \xi, x\rangle}$ for *all* ξ in the set (5.26). It also shows that the limit on the left-hand side of (5.27) commutes with differentiations with respect to ξ. Applying such differentiations to the left- and right-hand sides of (5.27), we get

$$\mathrm{Td}_\Delta\left(\frac{\partial}{\partial h_1}, \ldots, \frac{\partial}{\partial h_d}\right)\bigg|_{h=0} \int_{\Delta(h)} P(x) e^{\langle \xi, x\rangle}\, dx = \sum_{x \in \Delta \cap V_{\mathbb{Z}}} P(x) e^{\langle \xi, x\rangle}\, dx$$

whenever $P(x)$ is a polynomial and ξ is in the set (5.26).

In particular, for $\xi = 0$, we get the Euler–Maclaurin formula for polynomials. □

Proof of Theorem 1 for polynomial functions *(Brion–Vergne approach).* The terms in (5.19) and (5.21) are functions of ξ whose products with $\prod_{j,v}\langle \xi, \alpha_{j,v}\rangle$ extend to analytic functions near $\xi = 0$. Comparing the Taylor expansions in ξ of the left- and right-hand terms of these products, we get

$$\int_\Delta \frac{\langle \xi, x\rangle^k}{k!}\, dx = \sum_{v \in \mathrm{Vert}(\Delta)} \frac{\langle \xi, v\rangle^{k+n}}{(k+n)!} \cdot \frac{1}{|\Gamma_v|} \prod_{j \in I_v} -\frac{1}{\langle \xi, \alpha_{j,v}\rangle} \tag{5.28}$$

and

$$\sum_{x \in \Delta \cap V_{\mathbb{Z}}} \frac{\langle \xi, x\rangle^k}{k!} = \sum_{v \in \mathrm{Vert}(\Delta)} \left(e^{\langle \xi, v\rangle} \mathrm{Td}_v\left(\left\{-\langle \xi, \alpha_{j,v}\rangle\right\}_{j \in I_v}\right)\right)^{\langle k+n\rangle} \cdot \frac{1}{|\Gamma_v|} \prod_{j \in I_v} -\frac{1}{\langle \xi, \alpha_{j,v}\rangle} \tag{5.29}$$

where the superscript $\langle k+n\rangle$ denotes the homogeneous term of degree $k+n$ in ξ.

Recall that the vertices of $\Delta(h)$ are

$$v(h) = v - \sum_{j \in I_v} h_j \alpha_{j,v}.$$

$\langle \xi, v(h)\rangle^k$ is a polynomial of degree k in the h_j's that only depends on h_j for $j \in I_v$. For all $j \in I_v$,

$$\frac{\partial}{\partial h_j} \frac{\langle \xi, v(h)\rangle^k}{k!} = -\langle \xi, \alpha_{j,v}\rangle \frac{\langle \xi, v(h)\rangle^{k-1}}{(k-1)!}.$$

So for any homogeneous polynomial $T(\cdot)$ of degree ℓ in the variables S_j, $j \in I_v$,

$$T\left(\left\{\frac{\partial}{\partial h_j}\right\}_{j \in I_v}\right)\bigg|_{h=0} \frac{\langle \xi, v(h)\rangle^k}{k!} = T\left(\left\{-\langle \xi, \alpha_{j,v}\rangle\right\}_{j \in I_v}\right) \frac{\langle \xi, v(h)\rangle^{k-\ell}}{(k-\ell)!}. \tag{5.30}$$

We have

$$\begin{aligned}
&\mathrm{Td}_{\Delta}\left(\frac{\partial}{\partial h_1},\dots,\frac{\partial}{\partial h_d}\right)\int\limits_{\Delta(h)}\frac{\langle\xi,x\rangle^k}{k!}\,dx\\
&\quad=\sum_{v\in\mathrm{Vert}(\Delta)}\mathrm{Td}_v\left(\left\{\frac{\partial}{\partial h_j}\right\}_{j\in I_v}\right)\frac{\langle\xi,v(h)\rangle^{k+n}}{(k+n)!}\cdot\frac{1}{|\Gamma_v|}\prod_{j\in I_v}-\frac{1}{\langle\xi,\alpha_{j,v}\rangle}\quad\text{by (5.28) and (5.16)}\\
&\quad=\sum_{v\in\mathrm{Vert}(\Delta)}\sum_{0\leqslant\ell\leqslant k+n}\mathrm{Td}_v^{\langle\ell\rangle}\big(\{-\langle\xi,\alpha_{j,v}\rangle\}_{j\in I_v}\big)\frac{\langle\xi,v(h)\rangle^{k+n-\ell}}{(k+n-\ell)!}\cdot\frac{1}{|\Gamma_v|}\prod_{j\in I_v}-\frac{1}{\langle\xi,\alpha_{j,v}\rangle}\quad\text{by (5.30).}
\end{aligned}$$

When $h=0$, by (5.29), this is equal to

$$\sum_{x\in\Delta\cap V_{\mathbb{Z}}}\frac{\langle\xi,x\rangle^k}{k!}.\qquad\square$$

6. The Stokes formula for polytopes and the Cappell–Shaneson algebra

Khovanskii and Pukhlikov work with derivatives $\frac{\partial}{\partial h_i}$ associated to the "expansion" $\Delta(h)\subset V$ of the polytope. In this section we give two results that relate such derivatives to differential operators on V. The first result, Proposition 6.1, is the Stokes formula for polynomials. The second result, Proposition 6.2, is that integration over faces can be replaced by differentiations with respect to corresponding h_j's. A similar argument appears in [8, Section 3.6]. We use these results to define the "Cappell–Shaneson algebra," a formalism used by Cappell and Shaneson to express their formulas. These results play a key role in relating the Cappell–Shaneson formula to the original Khovanskii–Pukhlikov formula; we do this in Section 7. The two results can also be used to derive the Euler–Maclaurin formula for polynomials from the formula for exponentials, as we do in Appendix B.

The Stokes formula for polytopes.

Proposition 6.1. *Let V be a vector space with a lattice $V_{\mathbb{Z}}$. Normalize Lebesgue measure on V so that the measure of a fundamental domain for the lattice $V_{\mathbb{Z}}$ is one. Let Δ be a rational polytope in V. Let $u_1,\dots,u_d$ denote the inward normals to its facets, normalized so that they are primitive elements of the dual lattice $V_{\mathbb{Z}}^*$. For any $v\in V$, let D_v denote the directional derivative in the direction of v. Then, for any $f\in C^\infty(V)$,*

$$\int\limits_{\Delta}D_v f=-\sum_{i=1}^{d}\langle u_i,v\rangle\int\limits_{\sigma_i}f.\tag{6.1}$$

Proof. The formula is an immediate consequence of the general Stokes formula.

Alternatively, it follows directly from

$$\Delta\big(h_1+\langle u_1,v\rangle,\dots,h_n+\langle u_n,v\rangle\big)=\Delta(h)-v.\qquad\square$$

Integration over faces.

Proposition 6.2. *Let Δ be a simple polytope and let F be a face of Δ. Let $\Delta(h)$ be the expanded polytope and $F(h)$ the corresponding face of $\Delta(h)$. (See* (5.1), (5.2), *and* (5.8).*) Then, for any smooth function $f \in C^\infty(V)$, the integral of f on $\Delta(h)$ is a smooth function of h for h near* 0, *and*

$$\int_{F(h)} f = |\Gamma_F| \prod_{i \in I_F} \frac{\partial}{\partial h_i} \int_{\Delta(h)} f \tag{6.2}$$

where Γ_F is the finite abelian group associated to the face F. (See (5.6).*)*

In particular,

$$\frac{\partial}{\partial h_{i_1}} \cdots \frac{\partial}{\partial h_{i_k}} \int_{\Delta(h)} f = 0 \quad \text{if } \sigma_{i_1} \cap \cdots \cap \sigma_{i_k} = \emptyset, \tag{6.3}$$

and

$$\frac{\partial}{\partial h_i} \int_{\Delta(h)} f = \int_{\sigma_i(h)} f. \tag{6.4}$$

Proof. Choose a polarizing vector $\xi \in V_\Delta^*$ such that if $v \in \mathrm{Vert}(\Delta)$, $v \notin F$, and $x \in F$, then $\langle \xi, x \rangle > \langle \xi, v \rangle$. (For instance, we may take ξ' such that the restriction of the linear functional $\langle \xi', \cdot \rangle$ to Δ attains its maximum along the face F, and take ξ to be a perturbation of ξ' which is in V_Δ^*.) Then the edge vectors that are based at a vertex of F but are not contained in F (that is, $\alpha_{j,v}$ for $v \in F$ and $j \in I_F$) are not flipped in the polarization process (3.1).

Let P_F denote the affine plane generated by the face F. After possibly multiplying f by a cut-off function which is equal to one near F, we may assume that $\langle \xi, x \rangle > \langle \xi, v \rangle$ for every vertex v which is not in F and every $x \in P_F \cap \mathrm{supp}(f)$, where $\mathrm{supp}(f)$ is the support of f.

Then for every vertex v which is not contained in F we have

$$\int_{P_F \cap \mathbf{C}_v^\#} f = 0.$$

Similarly,

$$\int_{P_{F(h)} \cap \mathbf{C}_v^\#(h)} f = 0 \quad \text{if } v \notin F \text{ and } h \text{ is sufficiently small}, \tag{6.5}$$

where $P_{F(h)}$ is the affine plane generated by $F(h)$ and where $\mathbf{C}_v^\#(h)$ are the cones that occur in the polar decomposition

$$\mathbf{1}_{\Delta(h)} = \sum_v (-1)^{|\varphi_v|} \mathbf{1}_{C_v^\#(h)}. \tag{6.6}$$

From (6.5) and (6.6) we get

$$\int_{F(h)} f = \int_{P_F(h)} f \cdot \mathbf{1}_{\Delta(h)} = \int_{P_F(h)} f \cdot \sum_{v\in F}(-1)^{|\varphi_v|}\mathbf{1}_{\mathbf{C}_v^{\#}(h)} = \sum_{v\in F}(-1)^{|\varphi_v|} \int_{P_{F(h)}\cap \mathbf{C}_v^{\#}(h)} f.$$

It remains to show that

$$\int_{P_{F(h)}\cap \mathbf{C}_v^{\#}(h)} f = |\Gamma_F| \prod_{i\in I_F} \frac{\partial}{\partial h_i} \int_{\mathbf{C}_v^{\#}(h)} f \tag{6.7}$$

for each $v \in \mathrm{Vert}(\Delta)$ such that $v \in F$. Assume without loss of generality that $h_j = 0$ for all $j \in I_v \setminus I_F$.

Let

$$T_F = \{ r(x-y) \mid x, y \in F \}$$

denote the tangent space to the face F. Consider the affine change of variable map

$$\varphi : T_F \times \mathbb{R}^{I_F} \to V$$

given by

$$\varphi(y,t) = v + y + \sum_{j\in I_F} t_j \alpha_{j,v}.$$

Let

$$F_0^{\#} = \sum_{j\in I_v\setminus I_F} \mathbb{R}_+ \alpha_{j,v}^{\#} \quad \text{and} \quad \mathbb{R}_+^{I_F}(h) = \prod_{j\in I_F} [-h_j, \infty).$$

The map φ sends $F_0^{\#} \times \mathbb{R}_+^{I_F}$ onto $\overline{\mathbf{C}}_v^{\#}(h)$ and sends $F_0^{\#} \times \{(-h_j)_{j\in I_F}\}$ onto $F_v^{\#}(h)$. Lebesgue measure in $F_0^{\#} \subset T_F$ is normalized so that the measure of a fundamental chamber for the lattice $T_F \cap V_{\mathbb{Z}}$ is one. Clearly,

$$\int_{F_0^{\#}} f(\varphi(y,-h))\,dy = \prod_{i\in I_F} \frac{\partial}{\partial h_i} \int_{F_0^{\#}\times \mathbb{R}_+^{I_F}(h)} f(\varphi(y,t))\,dy\,dt.$$

To conclude (6.7) it remains to show that $|\det d\varphi| = \frac{1}{|\Gamma_F|}$.

The map $d\varphi$ sends the subspace $TF \times \{0\} \subset TF \times \mathbb{R}^{I_F}$ to the subspace $TF \subset V$ and respects the lattices in these subspaces. So its determinant is equal to that of the induced map on quotients. Recall that $V/TF = V_F$. The induced map on quotients is the map

$$\bar{\varphi} : \mathbb{R}^{I_F} \to V_F$$

given by $\overline{\varphi}((t_j)_{j\in I_F}) = \sum_{j\in I_F} t_j \alpha_{j,v}$. Its inverse,

$$\psi : V_F \to \mathbb{R}^{I_F},$$

is

$$\psi(\beta) = \big(\langle u_j, \beta\rangle\big)_{j\in I_F}.$$

The dual $\psi^* : \mathbb{R}^{I_F} \to V_F^*$ sends the standard basis element e_j to u_j for each $j \in I_F$. Finally,

$$\det d\varphi = \det \overline{\varphi} = (\det \psi)^{-1} = (\det \psi^*)^{-1} = \big[\operatorname{span}_{\mathbb{Z}}\{u_j\}\colon\ V_F^* \cap V_{\mathbb{Z}}^*\big]^{-1} = |\Gamma_F|^{-1},$$

as desired. □

By (6.2), the formulas of Guillemin (5.9) and of Brion–Vergne (5.10) are equivalent.

The Cappell–Shaneson algebra. Let V be a vector space with a lattice $V_{\mathbb{Z}}$ and $\Delta \subset V$ a simple lattice polytope. Let $\mathcal{D}$ denote the ring of infinite order constant coefficient differential operators on V. Consider the algebra $\mathcal{D}[\![[\sigma_1], \ldots, [\sigma_d]]\!]$ of power series in the formal variables $[\sigma_i]$, corresponding to the facets, and with coefficients in $\mathcal{D}$. A general element of this algebra can be written as

$$A = \sum_\alpha p_\alpha \prod_{i=1}^d [\sigma_i]^{\alpha(i)}$$

where $p_\alpha \in \mathcal{D}$ for each $\alpha : \{1, \ldots, d\} \to \mathbb{Z}_{\geqslant 0}$. Each such element A defines a linear functional $\int A$ which associates to each polynomial f on V the number

$$\int A(f) := \sum_\alpha \prod_{i=1}^d \left(\frac{\partial}{\partial h_i}\right)^{\alpha(i)} \bigg|_{h=\lambda} \int_{\Delta(h)} p_\alpha(f). \tag{6.8}$$

$\int A(f)$ is also defined for every smooth function f for which the right-hand side of (6.8) is absolutely convergent. By (6.3),

$$\int [\sigma_{i_1}] \cdots [\sigma_{i_k}] = 0 \quad \text{if } \sigma_{i_1} \cap \cdots \cap \sigma_{i_k} = \emptyset. \tag{6.9}$$

By (6.1) and (6.4),

$$\int \left(D_v + \sum_{i=1}^d \langle v, u_i\rangle [\sigma_i] \right) = 0 \quad \text{for all } v \in V. \tag{6.10}$$

Following Cappell and Shaneson, we consider the quotient $Q(\Delta)$ of $\mathcal{D}[\![[\sigma_1], \ldots, [\sigma_d]]\!]$ by the equivalence relations

$$[\sigma_{i_1}] \cdots [\sigma_{i_k}] = 0 \tag{6.11}$$

for each set $\sigma_{i_1}, \ldots, \sigma_{i_k}$ of facets where $\sigma_{i_1} \cap \cdots \cap \sigma_{i_k} = \emptyset$, and, for each $v \in V$,

$$D_v + \sum_{i=1}^{d} \langle v, u_i \rangle [\sigma_i] = 0. \tag{6.12}$$

We call $Q(\Delta)$ the *Cappell–Shaneson algebra*. (Cappell and Shaneson denote this algebra $Q(\Sigma)$, where Σ is the corresponding fan.)

Remark 6.3. The quotient of the polynomial algebra $\mathcal{D}[\![[\sigma_1], \ldots, [\sigma_d]]\!]$ by the relation (6.11) is the Stanley–Reisner ring (the face ring) of Δ with coefficients in $\mathcal{D}$.

By (6.9) and (6.10), the definition of $\int A(f)$ descends to the Cappell–Shaneson algebra $Q(\Delta)$: for $T \in Q(\Delta)$ that is represented by $A \in \mathcal{D}[\![[\sigma_1], \ldots, [\sigma_d]]\!]$, we can define

$$\int T(f) = \int A(f). \tag{6.13}$$

Remark 6.4. Cappell and Shaneson do not consider variations $\Delta(h)$ of the polytope Δ. They consider the free $\mathcal{D}$-module $P(\Delta)$ with basis elements $[F]$ corresponding to the faces F of Δ and the $\mathcal{D}$-module map $\rho : P(\Delta) \to Q(\Delta)$ defined by

$$\rho\big([F]\big) = |\Gamma_F| \prod_{i \in I_F} [\sigma_i].$$

For $T = \rho(\Omega)$, with $\Omega = \sum_F p_F[F] \in P(\Delta)$, Eq. (6.2) implies

$$\int T(f) = \sum_F \int_F p_F f. \tag{6.14}$$

Cappell and Shaneson *define* $\int T(f)$ by (6.14). Comparing with 6.13, we see that this is well defined.

Define the *summation functional* by

$$S(f) = \sum_{\Delta \cap V_{\mathbb{Z}}} f.$$

Using the Cappell–Shaneson algebra, the Khovanskii–Pukhlikov formula (5.4) for an integral polytope with non-singular fan reads

$$S = \int \prod_{i=1}^{d} \frac{[\sigma_i]}{1 - e^{-[\sigma_i]}},$$

and the Guillemin–Brion–Vergne formula for an integral simple polytope reads

$$S = \int \sum_F \sum_{\gamma \in \Gamma_F^\flat} \prod_{j \notin I_F} \frac{[\sigma_j]}{1 - e^{-[\sigma_j]}} \prod_{j \in I_F} \frac{[\sigma_j]}{1 - e^{2\pi i \langle \gamma, \alpha_{j,F} \rangle} e^{-[\sigma_j]}}.$$

7. The Cappell–Shaneson formula

The main differences between the formulas of Cappell and Shaneson and those of Khovanskii–Pukhlikov, Guillemin, and Brion–Vergne, are these:

- The latter authors work with expansions of the polytope. Cappell and Shaneson work with derivatives of the function on V.
- Cappell and Shaneson express their formula in terms of what we call the "Cappell–Shaneson algebra."
- Cappell and Shaneson derive their formula for the sum $\sum_{\Delta\cap V_{\mathbb{Z}}} f$ from formulas for the weighted sum

$$\sum_{\Delta\cap V_{\mathbb{Z}}}{}' f := \sum_F \left(\frac{1}{2}\right)^{\operatorname{codim} F} \sum_{\text{rel-int}(F)\cap V_{\mathbb{Z}}} f.$$

Cappell and Shaneson's exact formulas for simple lattice polytopes, when applied to polytopes with non-singular fans, become the following formulas:

Theorem 2. *Let Δ be an integral lattice polytope in a vector space V with a lattice $V_{\mathbb{Z}}$. Let f be a polynomial function on V. Then*

$$\sum_{\Delta\cap V_{\mathbb{Z}}} f = \int T(f), \tag{7.1}$$

$$\sum_{\text{interior}(\Delta)\cap V_{\mathbb{Z}}} f = \int \hat{T}(f), \tag{7.2}$$

and

$$\sum_{\Delta\cap V_{\mathbb{Z}}} f - \frac{1}{2}\sum_{\partial\Delta\cap V_{\mathbb{Z}}} f = \int \frac{1}{2}(T+\hat{T})(f), \tag{7.3}$$

where

$$T = \sum_F \prod_{i\in I_F} \frac{[\sigma_i]}{2} \prod_{i\notin I_F} \frac{[\sigma_i]/2}{\tanh([\sigma_i]/2)}$$

and

$$\hat{T} = \sum_F (-1)^{\operatorname{codim} F} \prod_{i\in I_F} \frac{[\sigma_i]}{2} \prod_{i\notin I_F} \frac{[\sigma_i]/2}{\tanh([\sigma_i]/2)}.$$

The relation of these formula to the Khovanskii–Pukhlikov formula goes through similar formulas that apply to a polytope with some facets removed. We first need a corresponding polar decomposition.

As before, let Δ be a polytope with d facets. For a subset $L \subseteq \{1, \dots, d\}$, denote by Δ^L the set obtained by removing from Δ the facets σ_i, $i \in L$. In particular, for $L = \emptyset$, $\Delta^L = \Delta$, and for $L = \{1, \dots, d\}$, $\Delta^L = \text{interior}(\Delta)$.

Recall that $\Delta = H_1 \cap \dots \cap H_d$ where each H_j is a half-spaces whose boundary is the affine span of the facet σ_j. Let

$$H_j^L = \begin{cases} H_j & \text{if } j \in L, \\ \text{interior}(H_j) & \text{if } j \notin L. \end{cases}$$

Then we have

$$\Delta^L = \bigcap_j H_j^L.$$

Fix a polarizing vector ξ for Δ. Recall that this determines a subset φ_v of I_v for each vertex v. Consider the cones

$$C_v^{\#,L} = \bigcap_{j \in I_v} H_{j,v}^{\#,L}$$

where

$$H_{j,v}^{\#,L} = \begin{cases} H_j^L & \text{if } j \in I_v \setminus \varphi_v, \\ (H_j^L)^c & \text{if } j \in \varphi_v. \end{cases}$$

We have the following polar decomposition for Δ^L:

Proposition 7.1 *(Polar decomposition with some facets removed).*

$$\mathbf{1}_{\Delta^L}(x) = \sum_v (-1)^{|\varphi_v|} \mathbf{1}_{C_v^{\#,L}}(x). \tag{7.4}$$

Proof. We shift the bounding hyperplanes of H_j inward or outward according to whether $j \in L$ or $j \notin L$. That is, we shift by an h which belongs to the set

$$\text{Orth}^L := \big\{(h_1, \dots, h_d) \bigm| h_j < 0 \text{ for } j \in L \text{ and } h_j > 0 \text{ for } j \notin L\big\}.$$

We have the pointwise limits

$$\mathbf{1}_{\Delta^L}(x) = \lim_{\substack{h \to 0 \\ h \in \text{Orth}^L}} \mathbf{1}_{\Delta(h)}(x)$$

and

$$\mathbf{1}_{C_v^{\#,L}}(x) = \lim_{\substack{h \to 0 \\ h \in \text{Orth}^L}} \mathbf{1}_{C_v^{\#}(h)}(x)$$

for all x. The proposition follows immediately from these limits and from the polar decomposition theorem for $\Delta(h)$. □

We have the following variant of the Khovanskii–Pukhlikov formula for a polytope with some facets removed:

Proposition 7.2. *Let V be a vector space with a lattice $V_{\mathbb{Z}}$. Let Δ be a lattice polytope in V with facets $\sigma_1, \ldots, \sigma_d$ and f a polynomial function on V. Let $L \subset \{1, \ldots, d\}$ be any subset. Suppose that Δ is a polytope with a non-singular fan. Then*

$$\sum_{\Delta^L \cap V_{\mathbb{Z}}} f = \int \prod_{i \in L} \frac{[\sigma_j] e^{-[\sigma_j]}}{1 - e^{-[\sigma_j]}} \prod_{i \notin L} \frac{[\sigma_j]}{1 - e^{-[\sigma_j]}} (f).$$

Proof. The proof follows exactly the same lines as the proof of the Khovanskii–Pukhlikov formula, (5.4), using the polar decomposition (7.4) for Δ^L. We leave the details to the reader. □

We derive the following formula for the weighted sum:

Proposition 7.3. *Suppose that Δ is an integral polytope with a non-singular fan and f is a polynomial. Then*

$$\int \prod_{i=1}^{d} \frac{[\sigma_i]/2}{\tanh([\sigma_i]/2)} (f) = \sum_{\Delta \cap V_{\mathbb{Z}}}' f. \tag{7.5}$$

Proof. Consider the left-hand side of Eq. (7.5):

$$\int \prod_{j=1}^{d} \frac{[\sigma_j/2]}{\tanh[\sigma_j/2]} (f). \tag{7.6}$$

Since

$$\frac{D/2}{\tanh(D/2)} = (D/2) \frac{e^{D/2} + e^{-D/2}}{e^{D/2} - e^{-D/2}} = \frac{1}{2}\big(1 + e^{-D}\big) \frac{D}{1 - e^{-D}},$$

(7.6) is equal to

$$\int \prod_{j=1}^{d} \frac{1}{2}\big(1 + e^{-[\sigma_j]}\big) \frac{[\sigma_j]}{1 - e^{-[\sigma_j]}} (f).$$

Expanding, this becomes

$$\left(\frac{1}{2}\right)^d \sum_{L \subseteq \{1, \ldots, d\}} \int \prod_{j \in L} \frac{[\sigma_j] e^{-[\sigma_j]}}{1 - e^{-[\sigma_j]}} \prod_{j \notin L} \frac{[\sigma_j]}{1 - e^{-[\sigma_j]}} (f),$$

which, by Proposition 7.2, is equal to

$$\left(\frac{1}{2}\right)^d \sum_L \sum_{\Delta^L \cap V_{\mathbb{Z}}} f. \tag{7.7}$$

For each face F of Δ, the relative interior of F is contained in Δ^L if and only if $I_F \cap L = \emptyset$. The number of subsets L which satisfy this condition is $2^{d-\operatorname{codim} F}$. Therefore, (7.7) is equal to

$$\sum_F \left(\frac{1}{2}\right)^{\operatorname{codim} F} \sum_{\operatorname{interior}(F)\cap V_{\mathbb{Z}}} f,$$

which is the right-hand side of (7.5). □

Lemma 7.4 relates weighted sums to non-weighted sums. For a face F of the simple polytope Δ, let

$$\sum_{\Delta\cap V_{\mathbb{Z}}}{}' f$$

denote the weighted sum with respect to the affine span of F, that is,

$$\sum_{\Delta\cap V_{\mathbb{Z}}}{}' f = \sum_{\substack{E \\ E\subseteq F}} \left(\frac{1}{2}\right)^{\dim F - \dim E} \sum_{\operatorname{rel-int}(E)\cap V_{\mathbb{Z}}} f.$$

Here, the faces of F are exactly the faces E of Δ that are contained in F, and the exponent $\dim F - \dim E$ is the codimension of E in the affine span of F.

Lemma 7.4.

$$\sum_{\Delta\cap V_{\mathbb{Z}}} f = \sum_F \left(\frac{1}{2}\right)^{\operatorname{codim} F} \sum_{F\cap V_{\mathbb{Z}}}{}' f,$$

and

$$\sum_{\operatorname{interior}(\Delta)\cap V_{\mathbb{Z}}} f = \sum_F \left(-\frac{1}{2}\right)^{\operatorname{codim} F} \sum_{F\cap V_{\mathbb{Z}}}{}' f.$$

Proof.

$$\begin{aligned}\sum_F \left(\pm\frac{1}{2}\right)^{\operatorname{codim} F} \sum_{F\cap V_{\mathbb{Z}}}{}' f &= \sum_F \left(\pm\frac{1}{2}\right)^{\operatorname{codim} F} \sum_{\substack{E \\ E\subseteq F}} \left(\frac{1}{2}\right)^{\dim F - \dim E} \sum_{\operatorname{rel-int}(E)\cap V_{\mathbb{Z}}} f \\ &= \sum_E \left(\frac{1}{2}\right)^{\operatorname{codim} E} \left(\sum_{\substack{F \\ F\supseteq E}} (\pm 1)^{\operatorname{codim} F} \right) \sum_{\operatorname{rel-int}(E)\cap V_{\mathbb{Z}}} f. \end{aligned} \tag{7.8}$$

Because Δ is simple,

$$\sum_{\substack{F\\ F\supseteq E}} 1 = 2^{\operatorname{codim} E} \quad \text{and} \quad \sum_{\substack{F\\ F\supseteq E}} (-1)^{\operatorname{codim} F} = \begin{cases} 0 & E \subsetneq \Delta, \\ 1 & E = \Delta. \end{cases}$$

Substituting this in (7.8) gives the lemma. □

We are now ready to derive the Cappell–Shaneson formula.

For each face F of Δ, we have $i \in I_F$ if and only if $\sigma_i \supseteq F$. For $i \notin I_F$, the intersection $\sigma_i \cap F$ is either empty or is equal to a face of Δ which is a facet of F. We denote

$$\Sigma_F = \{i \mid \sigma_i \cap F \text{ is a facet of } F\}.$$

Since

$$\frac{[\sigma_i]/2}{\tanh([\sigma_i]/2)} = 1 + \text{a multiple of } [\sigma_i],$$

and by (6.9), we get

$$T = \sum_F \prod_{i\in I_F} \frac{[\sigma_i]}{2} \prod_{i\in\Sigma_F} \frac{[\sigma_i]/2}{\tanh([\sigma_i]/2)} \tag{7.9}$$

and

$$\hat{T} = \sum_F (-1)^{\operatorname{codim} F} \prod_{i\in I_F} \frac{[\sigma_i]}{2} \prod_{i\in\Sigma_F} \frac{[\sigma_i]/2}{\tanh([\sigma_i]/2)} \tag{7.10}$$

as elements of the Cappell–Shaneson algebra $Q(\Delta)$. By (7.9) and (6.2), followed by Proposition 7.3 applied to the face F, and further followed by Lemma 7.4, we get

$$\int T(f) = \sum_F \left(\frac{1}{2}\right)^{\operatorname{codim} F} \int_F \prod_{i\in\Sigma_F} \frac{[\sigma_i]/2}{\tanh([\sigma_i]/2)}(f) = \sum_F \left(\frac{1}{2}\right)^{\operatorname{codim} F} \sum_{F\cap V_{\mathbb{Z}}}{}' f = \sum_{\Delta\cap V_{\mathbb{Z}}} f.$$

Similarly, from (7.10) we get

$$\int \hat{T}(f) = \sum_F \left(-\frac{1}{2}\right)^{\operatorname{codim} F} \int_F \prod_{i\in\Sigma_F} \frac{[\sigma_i]/2}{\tanh([\sigma_i]/2)}(f) = \sum_F \left(-\frac{1}{2}\right)^{\operatorname{codim} F} \sum_{F\cap V_{\mathbb{Z}}}{}' f$$
$$= \sum_{\operatorname{interior}(\Delta)\cap V_{\mathbb{Z}}} f.$$

This proves (7.1) and (7.2). The equality (7.3) clearly follows from these.

Remark 7.5. We expect that a similar argument will show that the Cappell–Shaneson formula for simple polytopes is equivalent to the Guillemin–Brion–Vergne formula.

Remark 7.6. Formulas with more general weightings have been developed in [1].

Acknowledgments

We would like to thank Itai Bar-Natan, S. Cappell, V. Guillemin, G. Kalai, G. Kuperberg, and J. Shaneson for helpful discussions. We are particularly grateful to A. Khovanskii and to M. Vergne for explaining to us their proofs.

Appendix A. Relation to remainder formulas

Another exact formula for polynomial functions on simple polytopes appeared in our recent paper [25]. There we proved an Euler–Maclaurin formula *with remainder* for simple polytopes and gave estimates on the remainder. From this we deduced an exact formula for polynomials directly, without passing through formulas for exponential functions. Let us describe our exact formula from [25] in our current notation.

Let λ be a complex root of unity, say

$$\lambda^N = 1.$$

Define a sequence of functions $Q_{m,\lambda}(x)$ on $\mathbb{R}$ recursively, as follows. For $m = 1$, set

$$Q_{1,\lambda}(x) = \frac{\lambda}{1-\lambda} \sum_{n\in\mathbb{Z}} \lambda^n \mathbf{1}_{[n,n+1)}(x).$$

Given the function $Q_{m-1,\lambda}(x)$, define the function $Q_{m,\lambda}(x)$ by the conditions

$$\frac{d}{dx} Q_{m,\lambda}(x) = Q_{m-1,\lambda}(x) \quad \text{and} \quad \int_0^N Q_{m,\lambda}(x)\,dx = 0.$$

Consider the polynomial

$$\mathbf{M}^{k,\lambda}(S) = \left(\frac{1}{2} + \frac{\lambda}{1-\lambda}\right) S + Q_{2,\lambda}(0) S^2 + \cdots + Q_{k,\lambda}(0) S^k.$$

Let V be a vector space with a lattice $V_{\mathbb{Z}}$. Let

$$\Delta = \bigl\{x \bigm| \langle u_i, x\rangle + \mu_i \geqslant 0,\ i = 1, \ldots, d\bigr\}$$

be a simple lattice polytope in V, where $u_1, \ldots, u_d \in V^*$ are the normals to the facets of Δ, normalized so that they are primitive elements of the lattice $V^*_{\mathbb{Z}}$. Let

$$\Delta(h) = \bigl\{x \bigm| \langle u_i, x\rangle + \mu_i + h_i \geqslant 0,\ i = 1, \ldots, d\bigr\}.$$

For a face F of Δ, an element γ of Γ_F, and an index $1 \leqslant j \leqslant d$, let

$$\lambda_{\gamma,j,F} = \begin{cases} e^{2\pi i \langle \gamma, \alpha_{j,F}\rangle} & j \in I_F, \\ 1 & j \notin I_F, \end{cases}$$

and consider the differential operators

$$\mathbf{M}^k_{\gamma,F} = \prod_{j=1}^{d} \mathbf{M}^{k,\lambda_{\gamma,j,F}}\left(\frac{\partial}{\partial h_j}\right).$$

Let

$$\sum_{\Delta\cap V_{\mathbb{Z}}}{}' f := \sum_F (1/2)^{\operatorname{codim} F} \sum_{\text{rel-int}(F)\cap V_{\mathbb{Z}}} f,$$

summing over the faces F of Δ. Then for any polynomial function f on Δ, for sufficiently large k,

$$\sum_{\Delta\cap V_{\mathbb{Z}}}{}' f = \sum_F \sum_{\gamma\in\Gamma_F^\flat} \mathbf{M}^k_{\gamma,F} \int_{\Delta(h)} f \bigg|_{h=0}. \tag{A.1}$$

For comparison, the Euler–Maclaurin formula (5.12) for simple lattice polytopes can be written as

$$\sum_{\Delta\cap V_{\mathbb{Z}}} f = \sum_F \sum_{\gamma\in\Gamma_F^\flat} \mathbf{T}_{\gamma,F} \int_{\Delta(h)} f \bigg|_{h=0}$$

with

$$\mathbf{T}_{\gamma,F} = \prod_{j=1}^{d} T^{\lambda_{\gamma,j,F}}\left(\frac{\partial}{\partial h_j}\right) \quad \text{and} \quad \mathbf{T}^\lambda(S) = \frac{S}{1-\lambda e^{-S}}.$$

A similar argument (see below) gives

$$\sum_{\Delta\cap V_{\mathbb{Z}}}{}' f = \sum_F \sum_{\gamma\in\Gamma_F^\flat} \mathbf{L}_{\gamma,F} \int_{\Delta(h)} f \bigg|_{h=0} \tag{A.2}$$

with

$$\mathbf{L}_{\gamma,F} = \prod_{j=1}^{d} \mathbf{L}^{\lambda_{\gamma,j,F}}\left(\frac{\partial}{\partial h_j}\right)$$

and

$$\mathbf{L}^\lambda(S) = \frac{S}{2}\cdot\frac{1+\lambda e^{-S}}{1-\lambda e^{-S}} = s\cdot\left(\frac{1}{2} + \lambda e^{-S} + \lambda^2 e^{-2S} + \lambda^3 e^{-3S} + \cdots\right).$$

As observed by Michèle Vergne [43], the equivalence of formulas (A.1) and (A.2) is seen from

Lemma A.1. $\mathbf{M}^{k,\lambda}(S)$ *is the kth Taylor polynomial of* $\mathbf{L}^{\lambda}(S)$.

We complete this section by giving the proofs of (A.2) and of Lemma A.1.

Proof of (A.2). We have the following analogue of (4.7):

$$\sum_{x\in\mathbf{C}_v\cap V_{\mathbb{Z}}}{}' e^{\langle\xi,x\rangle}=e^{\langle\xi,v\rangle}\cdot\frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}e^{2\pi i\langle\gamma,v\rangle}\prod_{j=1}^{n}\frac{1}{2}\cdot\frac{1+e^{2\pi i\langle\gamma,\alpha_j\rangle}e^{\langle\xi,\alpha_j\rangle}}{1-e^{2\pi i\langle\gamma,\alpha_j\rangle}e^{\langle\xi,\alpha_j\rangle}}. \tag{A.3}$$

Indeed, applying

$$\frac{1}{2}\cdot\frac{1+\lambda e^{-S}}{1-\lambda e^{-S}}=\frac{1}{2}+\lambda e^{-S}+\lambda^2e^{-2S}+\cdots$$

to $\lambda=e^{2\pi i\langle\gamma,v\rangle}$ and $e^{-S}=e^{\langle\xi,\alpha_j\rangle}$ and rearranging the terms, the right-hand side of (A.3) is equal to

$$\sum_{(k_1,\dots,k_n)\in\mathbb{Z}^n_{\geqslant 0}}{}' e^{\langle\xi,v+\sum k_j\alpha_j\rangle}\cdot\frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}e^{2\pi i\langle\gamma,v+\sum k_j\alpha_j\rangle}$$

which, by (4.9), is equal to the right-hand side of (A.3).

From this we get the following analogue of (4.16):

$$\begin{aligned}\sum_{x\in\overline{\mathbf{C}}_v^{\sharp}\cap V_{\mathbb{Z}}}{}' e^{\langle\xi,x\rangle}&=e^{\langle\xi,v_{\mathrm{shift}}\rangle}\cdot\frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}e^{2\pi i\langle\gamma,v\rangle}\prod_{j=1}^{n}\frac{1}{2}\frac{1+e^{2\pi i\langle\gamma,\alpha_j^{\sharp}\rangle}e^{\langle\xi,\alpha_j^{\sharp}\rangle}}{1-e^{2\pi i\langle\gamma,\alpha_j^{\sharp}\rangle}e^{\langle\xi,\alpha_j^{\sharp}\rangle}}\quad\text{by (A.3)}\\&=e^{\langle\xi,v\rangle}\cdot\frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}e^{2\pi i\langle\gamma,v\rangle}\prod_{j\notin\varphi_v}\frac{1}{2}\frac{1+e^{2\pi i\langle\gamma,\alpha_j\rangle}e^{\langle\xi,\alpha_j\rangle}}{1-e^{2\pi i\langle\gamma,\alpha_j\rangle}e^{\langle\xi,\alpha_j\rangle}}\\&\quad\times\prod_{j\in\varphi_v}\frac{1}{2}\frac{1+e^{-2\pi i\langle\gamma,\alpha_j\rangle}e^{-\langle\xi,\alpha_j\rangle}}{1-e^{-2\pi i\langle\gamma,\alpha_j\rangle}e^{-\langle\xi,\alpha_j\rangle}}\quad\text{by (4.12)}\\&=(-1)^{|\varphi_v|}e^{\langle\xi,v\rangle}\cdot\frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}e^{2\pi i\langle\gamma,v\rangle}\prod\frac{1}{2}\frac{1+e^{2\pi i\langle\gamma,\alpha_j\rangle}e^{\langle\xi,\alpha_j\rangle}}{1-e^{2\pi i\langle\gamma,\alpha_j\rangle}e^{\langle\xi,\alpha_j\rangle}}\end{aligned} \tag{A.4}$$

by applying the relation

$$\frac{1+e^x}{1-e^x}=-\frac{1+e^{-x}}{1-e^{-x}}$$

to $x=-2\pi i\langle\gamma,\alpha_j\rangle-\langle\xi,\alpha_j\rangle$ for $j\in\varphi_v$.

Let $\mathbf{1}^w_\Delta(x)$ denote the weighted characteristic function, given by $\mathbf{1}^w_\Delta(x) = (\frac{1}{2})^{\operatorname{codim} F}$ if x lies in the relative interior of a face F, and $\mathbf{1}^w_\Delta(x) = 0$ if $x \notin \Delta$. Define $\mathbf{1}^w_C(x)$ in a similar manner whenever C is a convex polyhedral cone. We have the following analogue of (3.3):

$$\mathbf{1}^w_\Delta(x) = \sum_v (-1)^{|\varphi_v|} \mathbf{1}^w_{\overline{\mathbf{C}}^\sharp_v}(x). \tag{A.5}$$

This can be proved directly (see [25, Section 3]), or it can be deduced from (7.4) using the formulas

$$\mathbf{1}^w_\Delta(x) = \frac{1}{2^d} \sum_{L \subseteq \{1,\dots,d\}} \mathbf{1}_{\Delta^L}(x) \quad \text{and} \quad \mathbf{1}^w_{\overline{\mathbf{C}}^\sharp_v}(x) = \frac{1}{2^d} \mathbf{1}_{\mathbf{C}^{\sharp,L}_v}(x).$$

From this we get the following analogue of (4.18):

$$\begin{aligned}
\sum_{x \in \Delta \cap V_{\mathbb{Z}}}{}' e^{\langle \xi, x \rangle} &= \sum_{v \in \operatorname{Vert}(\Delta)} (-1)^{|\varphi_v|} \sum_{x \in \overline{\mathbf{C}}^\sharp_v \cap V_{\mathbb{Z}}}{}' e^{\langle \xi, x \rangle} \quad \text{by (A.5)} \\
&= \sum_{v \in \operatorname{Vert}(\Delta)} e^{\langle \xi, v \rangle} \cdot \frac{1}{|\Gamma_v|} \sum_{\gamma \in \Gamma_v} e^{2\pi i \langle \gamma, v \rangle} \prod_{j \in I_v} \frac{1}{2} \cdot \frac{1 + e^{2\pi i \langle \gamma, \alpha_{j,v} \rangle} e^{\langle \xi, \alpha_{j,v} \rangle}}{1 - e^{2\pi i \langle \gamma, \alpha_{j,v} \rangle} e^{\langle \xi, \alpha_{j,v} \rangle}} \quad \text{by (A.4)} \\
&= \sum_{v \in \operatorname{Vert}(\Delta)} e^{\langle \xi, v \rangle} \mathbf{L}_v\big(\{-\langle \xi, \alpha_{j,v} \rangle\}\big) \cdot \frac{1}{|\Gamma_v|} \prod_{j \in I_v} -\frac{1}{\langle \xi, \alpha_{j,v} \rangle}
\end{aligned} \tag{A.6}$$

where

$$\mathbf{L}_v(S) = \sum_{\gamma \in \Gamma_v} e^{2\pi i \langle \gamma, v \rangle} \prod_{j \in I_v} \mathbf{L}^{\lambda_{\gamma,j,v}}(S_j). \tag{A.7}$$

Note that $\mathbf{L}_v(S)$ is analytic on the polydisk $\{|S_j| < b_j,\ j \in I_v\}$ that is described in Remark (4.3).

The operator that appears in (A.2) can be written as $\mathbf{L}_\Delta(\frac{\partial}{\partial h_1}, \dots, \frac{\partial}{\partial h_d})$ where

$$\mathbf{L}_\Delta(S_1, \dots, S_d) = \sum_F \sum_{\gamma \in \Gamma^\flat_F} \prod_{j=1}^d \mathbf{L}^{\lambda_{\gamma,j,F}}(S_j).$$

We have the following analogue of (5.16): for each $v \in \operatorname{Vert}(\Delta)$,

$$\mathbf{L}_\Delta(S_1, \dots, S_d) = \mathbf{L}_v\big(\{S_j\}_{j \in I_v}\big) + \text{multiples of } S_j \text{ for } j \notin I_v. \tag{A.8}$$

This is shown exactly like (5.16), using the facts that if $\lambda \neq 1$ then $\mathbf{L}^\lambda(S)$ is a multiple of S and if $\lambda = 1$ then $\mathbf{L}^\lambda(S) = 1 +$ a multiple of S.

By (5.17) and (5.19),

$$\int_{\Delta(h)} e^{\langle \xi, x \rangle}\, dx = \sum_{v \in \operatorname{Vert}(\Delta)} e^{\langle \xi, v - \sum_{j \in I_v} h_j \alpha_{j,v} \rangle} \cdot \frac{1}{|\Gamma_v|} \prod_{j \in I_v} -\frac{1}{\langle \xi, \alpha_{j,v} \rangle}. \tag{A.9}$$

(A.2) follows from (A.9) and (A.8) by the same arguments as in the proof of Theorem 1 in Section 5. □

Proof of Lemma A.1. Suppose that $\lambda \neq 1$ and $\lambda^N = 1$. An argument similar to those in Section 2 gives

$$\left(\mathbf{L}^{\lambda}\left(\frac{\partial}{\partial h_1}\right) + \mathbf{L}^{\lambda^{-1}}\left(\frac{\partial}{\partial h_2}\right)\right)\bigg|_{h_1=h_2=0} \int_{-h_1}^{N+h_2} f(x)\,dx$$
$$= \frac{1}{2}f(0) + \lambda f(1) + \lambda^2 f(2) + \cdots + \lambda^{N-2} f(N-2) + \lambda^{N-1} f(N-1) + \frac{1}{2}f(N) \tag{A.10}$$

for all polynomial functions $f(x)$. Indeed, direct computation of $\frac{1}{2} + \lambda e^{\xi} + \lambda^2 e^{2\xi} + \cdots + \lambda^{N-1}e^{(N-1)\xi} + \frac{1}{2}$, followed by multiplication by ξ and taking the Nth degree term in the Taylor expansion, gives the following analogue of (2.12):

$$\xi \sum_{x\in[0,N]\cap\mathbb{Z}}{}' (\xi) = \left(\mathbf{L}^{\lambda^{-1}}(\xi)\cdot e^{\xi N} - \mathbf{L}^{\lambda}(-\xi)\cdot 1\right)^{\langle N+1\rangle} \tag{A.11}$$

whenever $\lambda e^{\xi} \neq 1$, where the superscript $\langle N+1\rangle$ denotes the $(N+1)$th term in the Taylor expansion. On the other hand,

$$\mathbf{L}^{\lambda}\left(\frac{\partial}{\partial h_2}\right) + \mathbf{L}^{\lambda^{-1}}\left(\frac{\partial}{\partial h_1}\right)\bigg|_{h=0} \xi \int_{-h_2}^{N+h_1} \frac{(\xi x)^N}{N!}\,dx$$
$$= \mathbf{L}^{\lambda^{-1}}\left(\frac{\partial}{\partial h_1}\right)\bigg|_{h_1=0} \frac{(\xi(N+h_1))^{N+1}}{(N+1)!} - \mathbf{L}^{\lambda}\left(\frac{\partial}{\partial h_2}\right)\bigg|_{h_2=0} \frac{(-\xi h_2)^{N+1}}{(N+1)!} \quad \text{by (2.11)}$$
$$= \left(\mathbf{L}^{\lambda^{-1}}(\xi)e^{\xi N} - \mathbf{L}^{\lambda}(-\xi)\right)^{\langle N+1\rangle} \quad \text{by (2.2).}$$

By direct computation, the first Taylor coefficient of $\mathbf{L}^{\lambda}(S)$ is $\frac{1}{2} + \frac{\lambda}{1-\lambda}$ and that of $\mathbf{L}^{\lambda^{-1}}(S)$ is $\frac{1}{2} + \frac{\lambda^{-1}}{1-\lambda^{-1}} = \frac{1}{2} - \frac{1}{1-\lambda}$. Let a_m denote the mth Taylor coefficient of $\mathbf{L}^{\lambda}(S)$. Since $\mathbf{L}^{\lambda}(-S) = \mathbf{L}^{\lambda^{-1}}(S)$, the mth Taylor coefficient of $\mathbf{L}^{\lambda^{-1}}(S)$ is $(-1)^m a_m$. Taking $F(x)$ to be a polynomial of degree $\leqslant k+1$ and $f(x) = F'(x)$, the left-hand side of (A.10) becomes

$$\left(\frac{1}{2} + \frac{\lambda}{1-\lambda}\right)f(0) + \left(\frac{1}{2} - \frac{1}{1-\lambda}\right)f(N)$$
$$+ \sum_{m=2}^{\infty}\left(a_m\left(\frac{\partial}{\partial h_1}\right)^m + (-1)^m a_m\left(\frac{\partial}{\partial h_2}\right)^m\right)\bigg|_{h_1=h_2=0}\left(F(N) - F(0)\right)$$
$$= \left(\frac{1}{2} + \frac{\lambda}{1-\lambda}\right)f(0) + \left(\frac{1}{2} - \frac{1}{1-\lambda}\right)f(N) + \sum_{m=2}^{k}(-1)^m a_m\left(F^{(m)}(N) - F^{(m)}(0)\right). \tag{A.12}$$

On the other hand, the right-hand side of (A.10) is equal to

$$\left(\frac{1}{2}+\frac{\lambda}{1-\lambda}\right)f(0)+\left(\frac{1}{2}-\frac{1}{1-\lambda}\right)f(N)+\frac{\lambda}{1-\lambda}\sum_{n=0}^{N-1}\bigl(f(n+1)-f(n)\bigr)$$

$$=\left(\frac{1}{2}+\frac{\lambda}{1-\lambda}\right)f(0)+\left(\frac{1}{2}-\frac{1}{1-\lambda}\right)f(N)+\frac{\lambda}{1-\lambda}\int_0^N Q_{1,\lambda}(x)f'(x)\,dx$$

$$=\left(\frac{1}{2}+\frac{\lambda}{1-\lambda}\right)f(0)+\left(\frac{1}{2}-\frac{1}{1-\lambda}\right)f(N)+\sum_{m=2}^{k}(-1)^m Q_{m,\lambda}(x)f^{(m-1)}(x)\Big|_0^N$$

$$+(-1)^{k+1}\int_0^N Q_{k,\lambda}(x)f^{(k)}(x)\,dx \tag{A.13}$$

for any $k \geqslant 2$, by repeated integration by parts as in the proof of Proposition 27 of [25]. Recalling that $Q_{m,\lambda}(0)=Q_{m,\lambda}(N)$ and that $\int_0^N Q_{k,\lambda}(x)\,dx=0$, taking $f(x)=F'(x)$ where F is polynomial of degree $\leqslant k+1$, the right-hand side of (A.13) becomes

$$\left(\frac{1}{2}+\frac{\lambda}{1-\lambda}\right)f(0)+\left(\frac{1}{2}-\frac{1}{1-\lambda}\right)f(N)+\sum_{m=2}^{k}(-1)^m Q_{m,\lambda}(0)\bigl(F^{(m)}(N)-F^{(m)}(0)\bigr).$$

Comparing this with (A.12) for the monomials $F(x)=x^3,x^4,x^5,\ldots$ we deduce, by induction on m, that the coefficients $Q_{m,\lambda}(0)$ in $\mathbf{M}^{k,\lambda}$ are equal to the Taylor coefficients a_m of $\mathbf{L}^{\lambda}(S)$ for $m=2,3,4,\ldots.$ □

Appendix B. From exponentials to quasi-polynomials: Alternative approach

One can also deduce the Euler–Maclaurin formula for a polytope directly from an Euler–Maclaurin formula for a cone. This approach is a bit longer than the approaches taken in Section 5. Here we outline this approach and include a lemma that may be of independent interest.

The exact Euler–Maclaurin formula for an exponential function on a non-singular convex polyhedral cone is this. Let V be a vector space with a lattice $V_{\mathbb{Z}}$, and let $V_{\mathbb{Z}}^* \subset V^*$ be the dual lattice. Let $u_1,\ldots,u_n$ be primitive elements of $V_{\mathbb{Z}}^*$ which form a basis for V^*, and let $\alpha_1,\ldots,\alpha_n \in V$ be the dual basis. Take any $\lambda_1,\ldots,\lambda_n \in \mathbb{Z}$ and let $v=-\sum_{j=1}^n \lambda_j\alpha_j$. Consider the finite abelian group

$$\Gamma = V_{\mathbb{Z}}^*/\operatorname{span}_{\mathbb{Z}}\{u_i\}.$$

Consider the cone

$$\mathbf{C}_v=\bigl\{x \bigm| \langle u_j,x\rangle+\lambda_j \geqslant 0,\ j=1,\ldots,n\bigr\}=\Bigl\{v+\sum t_j\alpha_j \Bigm| t_j \geqslant 0,\ j=1,\ldots,n\Bigr\},$$

and its expansions, given by

$$\mathbf{C}_v(h)=\bigl\{x \bigm| \langle u_j,x\rangle+\lambda_j+h_j \geqslant 0,\ j=1,\ldots,n\bigr\}$$

for h near 0. Let

$$f(x) = e^{\langle \xi, x \rangle},$$

where $\xi \in V_{\mathbb{C}}^*$ satisfies, for each $j = 1, \ldots, n$,

(a) $\operatorname{Re}(\langle \xi, \alpha_j \rangle) < 0$, and
(b) $|\langle \xi, \alpha_j \rangle| < 2\pi |\langle y, \alpha_j \rangle|$ for all $y \in V_{\mathbb{Z}}^*$ such that $\langle y, \alpha_j \rangle \neq 0$.

(The set of ξ's that satisfy (b) is a neighborhood of the origin in $V_{\mathbb{C}}^*$.) Then

$$\sum_{\mathbf{C}_v \cap V_{\mathbb{Z}}} f = \sum_{\gamma \in \Gamma} e^{2\pi i \langle \gamma, v \rangle} \prod_{j=1}^{n} \frac{\frac{\partial}{\partial h_j}}{1 - e^{2\pi i \langle \gamma, \alpha_j \rangle} e^{-\frac{\partial}{\partial h_j}}} \Bigg|_{h=0} \int_{\mathbf{C}_v(h)} f. \tag{B.1}$$

This formula follows directly from (4.7), (4.11), and (5.18).

By applying this formula to the polarized cones $\mathbf{C}_v^\sharp$ that occur in the polar decomposition (Section 3), together with some bookkeeping, one deduces the exact Euler–Maclaurin formula on a polytope,

$$\sum_{x \in \Delta \cap V_{\mathbb{Z}}} e^{\langle \xi, x \rangle} = \operatorname{Td}_\Delta\left(\frac{\partial}{\partial h_1}, \ldots, \frac{\partial}{\partial h_d}\right) \int_{\Delta(h)} e^{\langle \xi, x \rangle} \, dx \tag{B.2}$$

where $\xi \in V^*$ is sufficiently small and is "polarizing," i.e., belongs to the complement of a finite union of (real!) hyperplanes through the origin.

One would like to obtain a similar formula for polynomial functions by taking the derivatives of (B.2) with respect to ξ and taking the limit as $\xi \to 0$, (or, alternatively, by comparing the coefficients in the Taylor expansions in ξ of the left- and right-hand sides of (B.2)). For this one needs to show that the infinite order differential operator $\operatorname{Td}_\Delta(\frac{\partial}{\partial h_1}, \ldots, \frac{\partial}{\partial h_d})$, applied to $\int_{\Delta(h)} e^{\langle \xi, x \rangle} \, dx$, commutes with derivatives and limits with respect to ξ. This follows from the following lemma, which may be of independent interest.

Lemma B.1. *Consider the exponential function*

$$f(\xi, x) = e^{\langle \xi, x \rangle} \tag{B.3}$$

where $x \in V$ and $\xi \in V_{\mathbb{C}}^*$. Let $b_1, \ldots, b_d$ be positive numbers, and let $T_\Delta(S_1, \ldots, S_d)$ be a formal power series that converges on the multi-disk

$$|S_i| < b_i, \quad i = 1, \ldots, d. \tag{B.4}$$

Then the series

$$T_\Delta\left(\frac{\partial}{\partial h_1}, \ldots, \frac{\partial}{\partial h_d}\right) \int_{\Delta(h)} f(\xi, x) \, dx \tag{B.5}$$

is absolutely convergent whenever $\xi \in V_{\mathbb{C}}^*$ satisfies the inequalities

$$\left|\langle \xi, \alpha_{i,v}\rangle\right| < b_i \quad \textit{for all } v \in \operatorname{Vert}(\Delta) \textit{ and all } i \in I_v, \tag{B.6}$$

and this convergence is uniform on compact subsets of the domain (B.6).

Proof. By general properties of power series, $T_\Delta(S_1, \dots, S_d)$ is absolutely convergent on the multi-disk (B.4), and this convergence is uniform on any strictly smaller multi-disk. Let

$$T_\Delta(S_1, \dots, S_d) = \sum_F \sum_{\substack{\vec{m}=(m_i)_{i\in I_F} \\ m_i \text{ non-negative integers}}} C_{F,\vec{m}} \prod_{i\in I_F} S_i^{1+m_i} + \text{terms that involve other monomials.} \tag{B.7}$$

The terms that involve other monomials make zero contribution to (B.5), by (6.3).

Because the series (B.7) is absolutely convergent on the multi-disk (B.4), so is the series

$$\sum_F \sum_{\substack{\vec{n}=(n_i)_{i\in I_F} \\ n_i \text{ non-negative integers}}} |C_{F,\vec{n}+\vec{\delta}}| \prod_{i\in I_F} S_i^{n_i}$$

for each $\vec{\delta} \in \mathbb{Z}^{I_F}$, where we set $C_{F,\vec{m}} = 0$ if $m_i < 0$ for some $i \in I_F$. For $\vec{n} = (n_i)_{i\in I_F}$ and $\vec{m} = (m_i)_{i\in I_F}$ we write $\vec{n} \leqslant \vec{m}$ to mean $n_i \leqslant m_i$ for all $i \in I_F$, and, for such $\vec{n}$, we write $|\vec{m} - \vec{n}| = \sum m_i - n_i$. Then the series

$$\sum_F \sum_{\vec{n}} \Bigg(\sum_{\substack{\vec{m} \text{ such that} \\ \vec{n}\leqslant\vec{m} \text{ and} \\ |\vec{m}-\vec{n}|\leqslant \dim V}} |C_{F,\vec{m}}| \Bigg) \prod_{i\in I_F} S_i^{n_i}$$

is also absolutely convergent on the multi-disk (B.4).

Let K be any compact subset of the set of ξ's that satisfy (B.6). Choose positive numbers λ and b_i' such that $0 < b_i' < b_i$ and $0 < \lambda < 1$ and such that

$$\left|\langle \xi, \alpha_{i,v}\rangle\right| < \lambda b_i' \quad \text{for all } v \in \operatorname{Vert}(\Delta) \text{ and all } i \in I_v \tag{B.8}$$

for every $\xi \in K$. To prove the lemma, we will show that the series (B.5) is dominated on K by a multiple of the converging positive series

$$\sum_F \sum_{\vec{n}} \Bigg(\sum_{\substack{\vec{m} \text{ such that} \\ \vec{n}\leqslant\vec{m} \text{ and} \\ |\vec{m}-\vec{n}|\leqslant \dim V}} |C_{F,\vec{m}}| \Bigg) \prod_{i\in I_F} \left(b_i'\right)^{n_i}.$$

For each face F of Δ and each $i \in I_F$ we choose an element $\tilde{\alpha}_{i,F} \in V$ to be equal to $\alpha_{i,v}$ for an arbitrary vertex $v \in F$. Then the elements $\tilde{\alpha}_{i,F}$ of V have the following properties:

(1)
$$\langle u_l, \tilde{\alpha}_{i,F}\rangle = \begin{cases} 1 & \text{if } l = i, \\ 0 & \text{if } l \in I_F \setminus \{i\}. \end{cases} \tag{B.9}$$

(2) By (B.8), for every $\xi \in K$,

$$\left|\langle \xi, \tilde{\alpha}_{i,F}\rangle\right| < \lambda b_i' \quad \text{for each face } F \text{ and index } i \in I_F.$$

The Stokes formula (6.1) and part (1) of (B.9) imply that for each face F of Δ and each $i \in I_F$

$$\int_{\sigma_i} f = -\sum_{l \notin I_F} \langle u_l, \tilde{\alpha}_{i,F}\rangle \int_{\sigma_l} f - \int_{\Delta} D_{\tilde{\alpha}_{i,F}} f. \tag{B.10}$$

The exponential function (B.3) satisfies $D_\alpha f = \langle \xi, \alpha\rangle f$ for any $\alpha \in V$. Combining these facts with (6.4) and (B.10), we get

$$\frac{\partial}{\partial h_i} \int_{\Delta(h)} f = -\sum_{l \notin I_F} \langle u_l, \tilde{\alpha}_{i,F}\rangle \frac{\partial}{\partial h_l} \int_{\Delta(h)} f - \langle \xi, \tilde{\alpha}_{i,F}\rangle \int_{\Delta(h)} f. \tag{B.11}$$

Applying $\prod_{i \in I_F} \frac{\partial}{\partial h_i}$ to (B.11), using the fact that the $\frac{\partial}{\partial h_i}$'s commute, and applying (6.2), we get

$$\frac{\partial}{\partial h_i} \frac{1}{|\Gamma_F|} \int_{F(h)} f = -\sum_{\substack{l \text{ such that} \\ E := F \cap \sigma_l \\ \text{satisfies } \emptyset \neq E \subsetneq F}} \langle u_l, \tilde{\alpha}_{i,F}\rangle \frac{1}{|\Gamma_E|} \int_{E(h)} f - \langle \xi, \tilde{\alpha}_{i,F}\rangle \frac{1}{|\Gamma_F|} \int_{F(h)} f.$$

Iterating this formula we get, for any $i_1, \ldots, i_k \in I_F$,

$$\begin{aligned}
&\prod_{j=1}^{k} \frac{\partial}{\partial h_{i_j}} \frac{1}{|\Gamma_F|} \int_{F(h)} f \\
&= (-1)^k \sum_{\substack{l_1,\ldots,l_s \text{ such that} \\ E_r := F \cap \sigma_{l_1} \cap \cdots \cap \sigma_{l_r} \\ \text{satisfy} \\ F = E_0 \supsetneq \cdots \supsetneq E_s \neq \emptyset}} \sum_{1 \leqslant j_1 < \cdots < j_s \leqslant k} \left(\prod_{j=1}^{j_1 - 1} \langle \xi, \tilde{\alpha}_{i_j,F}\rangle \right) \langle u_{l_1}, \tilde{\alpha}_{i_{j_1},F}\rangle \\
&\quad \cdot \left(\prod_{j=j_1+1}^{j_2-1} \langle \xi, \tilde{\alpha}_{i_j,E_1}\rangle \right) \langle u_{l_2}, \tilde{\alpha}_{i_{j_2},E_1}\rangle \cdots \langle u_{l_s}, \tilde{\alpha}_{i_{j_s},E_{s-1}}\rangle \left(\prod_{j=j_s+1}^{k} \langle \xi, \tilde{\alpha}_{i_j,E_s}\rangle \right) \frac{1}{|\Gamma_{E_s}|} \int_{E_s(h)} f.
\end{aligned} \tag{B.12}$$

Let $B \geqslant 1$ be such that

$$\left|\langle u_l, \tilde{\alpha}_{i,E}\rangle\right| \leqslant B \quad \text{for all } l, i, \text{ and } E.$$

By this and (B.8), the term on the right-hand side of (B.12) that corresponds to some fixed $l_1, \dots, l_s$ and some fixed $j_1, \dots, j_s$ is bounded by

$$B^s \lambda^{k-s} \prod_{\substack{j=1,\dots,k \\ j \notin \{j_1,\dots,j_s\}}} b'_{i_j} \cdot \frac{1}{|\Gamma_{E_s}|} \left| \int\limits_{E_s(h)} f \right| \tag{B.13}$$

where $E_s = F \cap \sigma_{l_1} \cap \dots \cap \sigma_{l_s}$. Let B_1 be strictly greater than $B^s \lambda^{-s} \frac{1}{|\Gamma_E|} | \int_E f |$ for all $0 \leqslant s \leqslant \dim V$ and all faces E of Δ. Then, for h near 0, the bound (B.13) is less than or equal to

$$B_1 \lambda^k \prod_{\substack{j=1,\dots,k \\ j \notin \{j_1,\dots,j_s\}}} b'_{i_j}. \tag{B.14}$$

Let m_i be the number of times that i occurs in $(i_1, \dots, i_k)$. Then the left-hand side of (B.12) can be rewritten as

$$\prod_{i \in I_F} \left(\frac{\partial}{\partial h_i} \right)^{1+m_i} \int\limits_{\Delta(h)} f.$$

Let n_i be the number of times that i occurs among i_j for $j \in \{1, \dots, k\} \setminus \{j_1, \dots, j_s\}$. Then the bound (B.14) can be rewritten as

$$B_1 \lambda^k \prod_{i \in I_F} \left(b'_i\right)^{n_i}. \tag{B.15}$$

Denote $\vec{m} = (m_i,\ i \in I_F)$ and $\vec{n} = (n_i,\ i \in I_F)$. Then $n_i \leqslant m_i$ for all i, which we write as $\vec{n} \leqslant \vec{m}$, and $\sum(m_i - n_i) \leqslant s \leqslant \dim F$, which we write as $|\vec{m} - \vec{n}| \leqslant \dim F$. Then we can further bound (B.15) by the following number which depends on $\vec{m}$ and not on $\vec{n}$:

$$B_1 \lambda^k \max_{\substack{\vec{n} \text{ such that} \\ \vec{n} \leqslant \vec{m} \text{ and} \\ |\vec{m}-\vec{n}| \leqslant \dim F}} \prod_{i \in I_F} \left(b'_i\right)^{n_i}. \tag{B.16}$$

This bound was for the summand of (B.12) which corresponds to a fixed choice of $l_1, \dots, l_s$ and of $j_1, \dots, j_s$. The number of possible choices of $l_1, \dots, l_s$ is bounded by d^s, which is bounded by $d^{\dim F}$, and, further, by $d^{\dim V}$. The number of possible choices for $j_1, \dots, j_s$ is $\binom{k}{s}$, which is bounded by $k^{\dim F}$, and, further, by $k^{\dim V}$. It follows that the right-hand side of (B.12) is bounded by

$$d^{\dim V} k^{\dim V} B_1 \lambda^k \max_{\substack{\vec{n} \text{ such that} \\ \vec{n} \leqslant \vec{m} \text{ and} \\ |\vec{m}-\vec{n}| \leqslant \dim F}} \prod_{i \in I_F} \left(b'_i\right)^{n_i}. \tag{B.17}$$

Since $k^{\dim V}\lambda^k \xrightarrow[k\to\infty]{} 0$ and since the maximum among positive numbers is bounded by their sum, there exists B_2 such that (B.17) is bounded by

$$B_2 \sum_{\substack{\vec{n} \text{ such that} \\ \vec{n}\leqslant\vec{m} \text{ and} \\ |\vec{m}-\vec{n}|\leqslant \dim F}} \prod_{i\in I_F} \left(b_i'\right)^{n_i}. \tag{B.18}$$

To conclude, we found B_2 such that

$$\left|\prod_{i\in I_F}\left(\frac{\partial}{\partial h_i}\right)^{1+m_i} \int_{\Delta(h)} f\right| \leqslant B_2 \sum_{\substack{\vec{n} \text{ such that} \\ \vec{n}\leqslant\vec{m} \text{ and} \\ |\vec{m}-\vec{n}|\leqslant \dim F}} \prod_{i\in I_F} \left(b_i'\right)^{n_i}.$$

The series (B.5) can be written as

$$\sum_F \sum_{\vec{m}} C_{F,\vec{m}} \prod_{i\in I_F}\left(\frac{\partial}{\partial h_i}\right)^{1+m_i} \int_{\Delta(h)} f.$$

By what we have shown, for ξ in the compact set K, this series is dominated by the positive series

$$\sum_F \sum_{\vec{m}} |C_{F,\vec{m}}| B_2 \sum_{\substack{\vec{n} \text{ such that} \\ \vec{n}\leqslant\vec{m} \text{ and} \\ |\vec{m}-\vec{n}|\leqslant \dim F}} \prod_{i\in I_F} \left(b_i'\right)^{n_i},$$

which can also be re-written as

$$B_2 \sum_F \sum_{\vec{n}} \left(\sum_{\substack{\vec{m} \text{ such that} \\ \vec{m}\geqslant\vec{n} \text{ and} \\ |\vec{m}-\vec{n}|\leqslant \dim V}} |C_{F,\vec{m}}| \right) \prod_{i\in I_F} \left(b_i'\right)^{n_i}.$$

As we have shown, this series is convergent. □

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